Master of Science in Mathematics — *Michel Waldschmidt* — Master Training Program
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Real Analysis

Second Assignment, October 11, 2010 (*1 hour*)

Exercise 1. We denote by f a function defined on the open interval $(-1, 1)$.
a) Can you give an example where f is uniformly continous and not continuous?
b) Can you give an example where f is continuous and not monotonic?
c) Can you give an example where f is uniformly continuous and strictly monotonic?
d) Can you give an example where f is strictly monotonic and not monotonic?
e) Can you give an example where $\lim_{x\to 0} f(x) = f(0)$ and f is not continuous at $x = 0$?
f) Can you give an example where $\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) \neq f(0)$?
g) Can you give an example where f is monotonic and continuous but not bounded above?
h) Can you give an example where f is bounded and $\limsup_{x\to 0} f(x) = +\infty$?
i) Can you give an example where f is continuous and $\limsup_{x\to 0} f(x) = +\infty$?

Exercise 2. Let f be a function which is increasing on $[0, 1]$.
a) Prove
$$\limsup_{x\to 1-} f(x) \le f(1).$$
b) Can you give an example where $\limsup_{x\to 1-} f(x) < f(1)$?
c) Give the values of $\sup_{0\le x\le 1} f(x)$ and $\inf_{0\le x\le 1} f(x)$.
d) Does there exist bounded functions g on $[0, 1]$ for which $\sup_{0\le x\le 1} g(x) = \inf_{0\le x\le 1} g(x)$?

Exercise 3.
a) Is the function $\sin(1/x)$ continuous on $(0, +\infty)$? Is–it bounded above? Is–it bounded below? Compute $\limsup_{x\to 0+} \sin(1/x)$ and $\liminf_{x\to 0+} \sin(1/x)$.
b) Is the function $(1/x) \sin(1/x)$ continuous on $(0, +\infty)$? Is–it bounded above? Is–it bounded below? Compute
$$\limsup_{x\to 0+}(1/x)\sin(1/x) \quad \text{and} \quad \liminf_{x\to 0+}(1/x)\sin(1/x).$$

Exercise 4. Give examples of functions f_1, f_2, f_3, f_4 and g_1, g_2, g_3, g_4, which are continuous on $(0, 1]$, tend to 0 as $x \to 0+$, such that $g_i(x) \neq 0$ for $x \in (0, 1]$ and for $i = 1, 2, 3, 4$, and such that
a) $\lim_{x\to 0+} f_1(x)/g_1(x) = 1$.
b) $\lim_{x\to 0+} f_2(x)/g_2(x) = 0$.
c) $\lim_{x\to 0+} f_3(x)/g_3(x) = +\infty$.
d) $f_4(x)/g_4(x)$ has no limit as $x \to 0+$.

Exercise 5. Is the function
$$\frac{x^3 + 2x - 1}{x^2 + 1} e^{-3x^2} \sin\sqrt{x}$$
continuous on $[0, 1]$? Is-it uniformly continuous on $[0, 1]$?

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Second Assignement, October 11, 2010 — solutions

Solution exercise 1
a) No: if f is uniformly continuous, then f is continuous.
b) An example where f is continuous and not monotonic is $f(x) = |x|$.
c) An example where f is uniformly continuous and strictly monotonic is $f(x) = x$.
d) No: if f is strictly monotonic, then f is monotonic..
e) No: If f is continuous at $x = 0$ then $\lim_{x\to 0} f(x) = f(0)$.
f) The function defined by $f(x) = 0$ for $x \neq 0$ and $f(1) = 1$ satisfies

$$lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = 0 \neq f(0).$$

g) An example where f is monotonic and continuous but not bounded above is $f(x) = 1/(x-1)$.
h) No: if f is bounded, say $f(x) \leq M$ for all $x \in (-1,1)$, then $\limsup_{x\to 0} f(x) \leq M$.
i) No: if f is continuous, then f is bounded on any closed interval contained in $(0,1)$.

Solution exercise 2
Since the function f is increasing on $[0,1]$, we have $f(x) \leq f(1)$ for all $x \in [0,1]$, hence $\limsup_{x\to 1-} f(x) \leq f(1)$. An example where $\limsup_{x\to 1-} f(x) < f(1)$ is the function f defined by $f(x) = x$ for $0 \leq x < 1$ and $f(1) = 2$.
c) We have $\sup_{0\leq x\leq 1} f(x) = f(1)$ and $\inf_{0\leq x\leq 1} f(x) = f(0)$.
d) The bounded functions g on $[0,1]$ for which $\sup_{0\leq x\leq 1} g(x) = \inf_{0\leq x\leq 1} g(x)$ are the constants. They are not strictly monotonic.

Solution exercise 3
a) The function $\sin(1/x)$ continuous on $(0,+\infty)$, bounded above by 1 and below by -1. We have $\limsup_{x\to 0+} \sin(1/x) = 1$ and $\liminf_{x\to 0+} \sin(1/x) = -1$.
b) The function $(1/x)\sin(1/x)$ is continuous on $(0,+\infty)$, it is not bounded above and not bounded below. We have

$$\limsup_{x\to 0+}(1/x)\sin(1/x) = +\infty \quad \text{and} \quad \liminf_{x\to 0+}(1/x)\sin(1/x) = -\infty.$$

Solution exercise 4
a) For $f_1(x) = g_1(x) = x$, we have $\lim_{x\to 0+} f_1(x)/g_1(x) = 1$.
b) For $f_1(x) = x^2$ and $g_1(x) = x$, we have $\lim_{x\to 0+} f_2(x)/g_2(x) = 0$.
c) For $f_1(x) = x$ and $g_1(x) = x^2$, we have $\lim_{x\to 0+} f_3(x)/g_3(x) = +\infty$.
d) For $f_4(x) = x\sin(1/x)$ and $g_4(x) = x$, the quotient $f_4(x)/g_4(x) = \sin(1/x)$ has no limit as $x \to 0+$. In this example $f_4(x)/g_4(x) = \sin(1/x)$ is bounded on $(0,1]$. With the same f_4 and with $g_4(x) = x^2$, the function $f_4(x)/g_4(x) = (1/x)\sin(1/x)$ is not bounded on $(0,1]$ (see exercise 3).

Solution exercise 5
The sum and the product of continuous functions is continuous. This is true also for the quotient of two functions when the denominator has no zero. The composition of continuous functions is continuous. Hence the given function is continuous on $[0,1]$. Since this interval is closed and bounded, this function is uniformly continuous.