Master of Science in Mathematics *Michel Waldschmidt* Master Training Program
Royal University of Phnom Penh RUPP URPP - Université Royale de Phnom Penh
Centre International de Mathématiques Pures et Appliquées CIMPA
Coopération Mathématique Interuniversitaire Cambodge France

Linear Algebra

Final Exam, October 20, 2010 (*3 hours*)

Exercise 1. Determine whether each of the following subsets of $\mathbf{R}^2$ is a subspace of $\mathbf{R}^2$. For each of the subsets which is a subspace of $\mathbf{R}^2$, give the dimension of the subspace, find a basis for the subspace, and then extend it to a basis of $\mathbf{R}^2$.
a) $E = \{(x, y) \in \mathbf{R}^2 \; ; \; xy = 0\}$.
b) $F = \{(x, y) \in \mathbf{R}^2 \; ; \; x + 2y = 0\}$.
c) $G = \{(x, y) \in \mathbf{R}^2 \; ; \; x + y = 1\}$.
d) $H = \{(x, y) \in \mathbf{R}^2 \; ; \; x^2 + y^2 = 0\}$.

Exercise 2. Write the characteristic polynomial of the 4×4 matrix:

$$\begin{pmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}.$$

Find all eigenvalues and describe the eigenspaces. Is this matrix diagonalizable?

Exercise 3. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be real numbers.
a) Consider the diagonal 4×4 matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$

Give a necessary and sufficient condition for the matrix D to be invertible. When D is invertible, compute D^{-1}.
b) Let V be a vector space of dimension 4 and $(\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4)$ be a basis of V. Show that there is a unique linear operator on V such that

$$T(\underline{u}_1) = \lambda_1 \underline{u}_1, \quad T(\underline{u}_2) = \lambda_2 \underline{u}_2, \quad T(\underline{u}_3) = \lambda_3 \underline{u}_3, \quad T(\underline{u}_4) = \lambda_4 \underline{u}_4.$$

c) Find all eigenvalues and describe the eigenspaces of T.
d) Give a necessary and sufficient condition for T to be one–to–one. When T is one–to–one, compute T^{-1}, then find all eigenvalues and describe the eigenspaces of T^{-1}.

Exercise 4. For each of the following 2×2 matrices, find the characteristic polynomial, find all eigenvalues and describe the eigenspaces. If possible, diagonalize the matrix:

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 3 & 5 \\ -1 & -3 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}.$$

Exercise 5. Let V be a vector space of dimension 5 and W a vector space of dimension 4. Let $\mathcal{B} = (\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4, \underline{u}_5)$ be a basis of V and $\mathcal{C} = (\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4)$ be a basis of W. Let $T : V \to W$ be the unique linear transformation satisfying

$$T(\underline{u}_1) = 0,\ T(\underline{u}_2) = \underline{v}_1,\ T(\underline{u}_3) = 2\underline{v}_1+5\underline{v}_2,\ T(\underline{u}_4) = 3\underline{v}_1+6\underline{v}_2+8\underline{v}_3,\ T(\underline{u}_5) = 4\underline{v}_1+7\underline{v}_2+9\underline{v}_3.$$

a) Write the matrix for the linear transformation T with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.
b) Check that $T(\underline{u}_2), T(\underline{u}_3), T(\underline{u}_4)$ are linearly independent.
c) The range of T is $R(T) = \{T(\underline{u})\ ;\ \underline{u} \in V\}$. What is the dimension of $R(T)$? Give a basis of $R(T)$.
d) The kernel of T is $\ker(T) = \{\underline{u} \in V\ ;\ T(\underline{u}) = 0\}$. What is the dimension of $\ker(T)$? Give a basis of $\ker(T)$.

Exercise 6. Denote by $\mathcal{P}_3$ the space of polynomials of degree ≤ 3 together with the zero polynomial:

$$\mathcal{P}_3 = \{a_0 + a_1X + a_2X^2 + a_3X^3\ ;\ (a_0, a_1, a_2, a_3) \in \mathbf{R}^4)\}$$

Denote by $\mathcal{B} = \{1, X, X^2, X^3\}$ the standard basis of $\mathcal{P}_3$. Consider the linear operator $T : \mathcal{P}_3 \to \mathcal{P}_3$ defined by $T(p) = p + p' + p'' + p'''$, where p' denotes the derivative of p, while p'' denotes the derivative of p' and p''' denotes the derivative of p''.
a) Write the matrix for the operator T with respect to the basis $\mathcal{B}$.
b) Show that the operator T is invertible and write the matrix for T^{-1} with respect to the basis $\mathcal{B}$.

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Final Exam, October 20, 2010— solutions

Solution exercise 1

a) The set E is the union of two lines, hence it is not a subspace. The points $(1,0)$ and $(0,1)$ are in E, not their sum $(1,1)$.

b) The set F is a subspace of $\mathbf{R}^2$, it is the line containing $(0,0)$ and $(2,-1)$, a basis is $\{(2,-1)\}$. We can extend this basis to a basis $(\underline{u}_1, \underline{u}_2)$ of $\mathbf{R}^2$ by taking $\underline{u}_1 = (2,-1)$, $\underline{u}_2 = (1,0)$ for instance.

c) The set G is a line which does not contain $(0,0)$, hence it is not a subspace of $\mathbf{R}^2$ (all subspaces of a vector space contain the origin).

d) The set H is $\{(0,0)\}$, hence it is a subspace of $\mathbf{R}^2$ of dimension 0, a basis is the empty set with 0 elements, we extend it to a basis of $\mathbf{R}^2$ by taking any basis of $\mathbf{R}^2$, for instance the canonical basis $\{(1,0),(0,1)\}$.

Solution exercise 2

The characteristic polynomial is $(1-X)(3-X)(6-X)(10-X)$, its roots are $1,3,6,10$, hence these are the eigenvalues. Since they are distinct the matrix is diagonalizable. The eigenspace corresponding to the eigenvalue 1 is the x_1 axis, which is the line of $\mathbf{R}^4$ containing $(0,0,0,0)$ and $(1,0,0,0)$, the eigenspace corresponding to the eigenvalue 3 is the line of $\mathbf{R}^4$ containing $(0,0,0,0)$ and $(1,1,0,0)$, the eigenspace corresponding to the eigenvalue 6 is the line of $\mathbf{R}^4$ containing $(0,0,0,0)$ and $(22,25,15,0)$, the eigenspace corresponding to the eigenvalue 10 is the line of $\mathbf{R}^4$ containing $(0,0,0,0)$ and $(86,99,81,36)$.

Solution exercise 3

a) The eigenvalues of D are the elements in the diagonal, and the matrix is invertible if and only if 0 is not an eigenvalue. Hence the condition is that λ_1, λ_2, λ_3, λ_4 are all non–zero, which can be written $\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0$. If D is invertible, then the inverse matrix is

$$D^{-1} = \begin{pmatrix} \lambda_1^{-1} & 0 & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 & 0 \\ 0 & 0 & \lambda_3^{-1} & 0 \\ 0 & 0 & 0 & \lambda_4^{-1} \end{pmatrix}.$$

b) The operator T is defined by

$$T(x_1\underline{u}_1 + x_2\underline{u}_2 + x_3\underline{u}_3 + x_4\underline{u}_4) = \lambda_1\underline{u}_1 + \lambda_2\underline{u}_2 + \lambda_3\underline{u}_3 + \lambda_4\underline{u}_4.$$

c) The eigenvalues, eigenvectors and eigenspaces of T are exactly the same as those of D: the eigenvalues are λ_1, λ_2, λ_3, λ_4, the eigenspaces are the axes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ respectively.

d) A necessary and sufficient condition for T to be one–to–one is that D is invertible. In this case the matrix for T^{-1} with respect to the basis $\mathcal{B}$ is D^{-1}, the eigenvalues, eigenvectors and eigenspaces of T^{-1} are exactly the same as those of D^{-1}: the eigenvalues are λ_1^{-1}, λ_2^{-1}, λ_3^{-1}, λ_4^{-1}, the eigenspaces are the axes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ respectively.

Solution exercise 4

a) The characteristic polynomial of the matrix A_1 is X^2, there is a single eigenvalue which is 0, the eigenspace is $\mathbf{R}^2$, the matrix is diagonal.

b) The characteristic polynomial of the matrix A_2 is $(X+1)(X-3)$, there are two eigenvalues -1 and 3, the eigenspaces are the lines $y = 0$ and $x = 0$ respectively, the matrix is diagonal.

c) The characteristic polynomial of the matrix A_3 is $X^2 - 4$, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$, the eigenspace corresponding to λ_1 is the line defined by the eigenvector $\underline{v}_1$, the eigenspace corresponding to λ_2 is the line defined by the eigenvector $\underline{v}_2$, with

$$\underline{v}_1 = \begin{pmatrix} 5 \\ -1 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The matrices

$$P = \begin{pmatrix} 5 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{satisfy} \quad P^{-1}A_3P = D,$$

as we check with

$$PD = \begin{pmatrix} 10 & 2 \\ -2 & -2 \end{pmatrix} = A_3P. \quad \text{Also} \quad P^{-1} = \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 5/4 \end{pmatrix}.$$

d) The characteristic polynomial of the matrix A_4 is $(X-1)^2$, there is a unique eigenvalue $\lambda = 1$, the corresponding eigenspace is the line defined by the eigenvector $\underline{v}_1 = (2, -1)$, hence the matrix is not diagonalizable.

Remark. If we set

$$\underline{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad \text{then} \quad P^{-1}A_4P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Solution exercise 5

a) The matrix for the linear transformation T with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ is the 4×5 matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) The linear independence of $T(\underline{u}_2), T(\underline{u}_3), T(\underline{u}_4)$ follows from the fact that the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{pmatrix}.$$

has rank 3.

c) The space spanned by $T(\underline{u}_2), T(\underline{u}_3), T(\underline{u}_4)$ is contained in $R(T)$ and has dimension 3. Also $R(T)$ is contained in the subspace spanned by $\underline{v}_1, \underline{v}_2, \underline{v}_3$, hence $R(T)$ has dimension ≤ 3. Therefore $R(T)$ has dimension 3, and a basis is $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$. Another basis is $T(\underline{u}_2), T(\underline{u}_3), T(\underline{u}_4)$.

d) Since V has dimension 5 and $R(T)$ dimension 3, the kernel of T has dimension $5 - 3 = 2$. Clearly $\underline{u}_1$ is in the kernel. Another element in the kernel, linearly independent of $\underline{u}_1$, is obtained by solving the homogeneous linear system of equations, and one finds that $49\underline{u}_2 + 22\underline{u}_3 + 9\underline{u}_4 - 8\underline{u}_5$ is in the kernel.

Solution exercise 6

a) The matrix of T in the basis $\mathcal{B}$ is

$$A = \begin{pmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

b) The determinant is 1, hence T is one–to–one. The inverse of T is

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$