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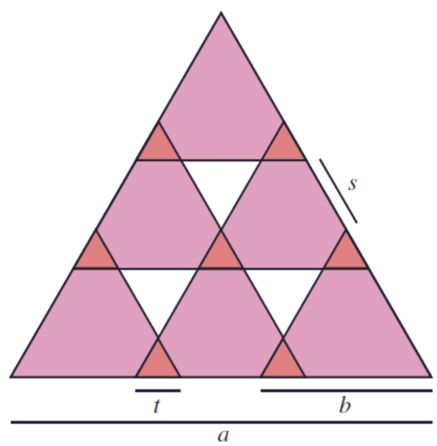
A course on linear recurrent sequences
African Institute for Mathematical Sciences (AIMS)

Michel Waldschmidt, Sorbonne Université

Assignment

• **1.** Prove the irrationality of $\sqrt{6}$ using the following picture



Explain the connection with question 1 of the first tutorial.

• **2.** Define a sequence $(u_n)_{n\geq 0}$ of numbers by the condition

$$\sum_{n\geq 0} u_n z^n = \frac{1}{(1-z)^2}\cdot$$

Let

$$\varphi(z) = \sum_{n\geq 0} u_n \frac{z^n}{n!}\cdot$$

Show that φ is a solution of a differential equation. Give all the solution of this differential equation.

• **3.** A triangular number is a positive integer of the form $m(m+1)/2$. The sequence of triangular numbers starts with

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, \ldots$$

Let $(u_n)_{n\ge 0}$ be the sequence of integers such that u_n^2 is a triangular number. Check that

$$u_1 = 1 \quad (u_1^2 = 1,\, m = 1) \quad \text{and} \quad u_2 = 6 \quad (u_2^2 = 36,\ m = 8).$$

For $n \ge 3$, write u_n as a linear combination of u_{n-1} and u_{n-2}. Compute u_3 and u_4.

• **4.** Let $a \ge 1$ be a positive integer. Let

$$\theta = \frac{a + \sqrt{a^2+4}}{2}$$

be the positive root of the quadratic polynomial $X^2 - aX - 1$.
Write the continued fraction expansion of θ.
Define a recurrence linear sequence $(u_n)_{n\ge 0}$ by $u_n = au_{n-1} + u_{n-2}$ for $n \ge 2$ with the initial conditions $u_0 = 0$ and $u_1 = 1$. Check that u_n is the nearest integer to

$$\frac{\theta^n}{\sqrt{a^2+4}}\cdot$$

Write the rational fraction having the Taylor expansion at the origin

$$\sum_{n\ge 0} u_n z^n.$$

Write a differential equation satisfied by the power series

$$\varphi(z) = \sum_{n\ge 0} u_n \frac{z^n}{n!}\cdot$$

Give all the solutions of this differential equation.