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Representation of integers by cyclotomic binary forms

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Abstract

The homogeneous form $\Phi_n(X,Y)$ of degree $\varphi(n)$ which is associated with the cyclotomic polynomial $\phi_n(t)$ is dubbed a cyclotomic binary form. A positive integer $m \geq 1$ is said to be representable by a cyclotomic binary form if there exist integers n, x, y with $n \geq 3$ and $\max\{|x|, |y|\} \geq 2$ such that $\Phi_n(x,y) = m$. These definitions give rise to a number of questions that we plan to address.

This is a joint work with Étienne Fouvry and Claude Levesque

Étienne Fouvry

Claude Levesque

Representation of integers by cyclotomic binary forms.
Acta Arithmetica, 20p. Online First March 2018.
`arXiv: 712.09019 [math.NT]`

Cyclotomic polynomials

Definition by induction :

$$\phi_1(t) = t-1, \qquad t^n - 1 = \prod_{d|n} \phi_d(t).$$

For p prime,

$$t^p - 1 = (t-1)(t^{p-1} + t^{p-2} + \cdots + t + 1) = \phi_1(t)\phi_p(t),$$

so

$$\phi_p(t) = t^{p-1} + t^{p-2} + \cdots + t + 1.$$

For instance

$$\phi_2(t) = t+1, \quad \phi_3(t) = t^2+t+1, \quad \phi_5(t) = t^4+t^3+t^2+t+1.$$

Cyclotomic polynomials

$$\phi_n(t) = \frac{t^n - 1}{\prod_{\substack{d \neq n \\ d \mid n}} \phi_d(t)}.$$

For instance

$$\phi_4(t) = \frac{t^4-1}{t^2-1} = t^2+1 = \phi_2(t^2),$$

$$\phi_6(t) = \frac{t^6-1}{(t^3-1)(t+1)} = \frac{t^3+1}{t+1} = t^2-t+1 = \phi_3(-t).$$

The degree of $\phi_n(t)$ is $\varphi(n)$, where φ is the Euler totient function.

Cyclotomic polynomials and roots of unity

For $n \geq 1$, if ζ is a primitive n–th root of unity,

$$\phi_n(t) = \prod_{\gcd(j,n)=1} (t - \zeta^j).$$

For $n \geq 1$, $\phi_n(t)$ is the irreducible polynomial over $\mathbb{Q}$ of the primitive n–th roots of unity,

Let K be a field and let n be a positive integer. Assume that K has characteristic either 0 or else a prime number p prime to n. Then the polynomial $\phi_n(t)$ is separable over K and its roots in K are exactly the primitive n–th roots of unity which belong to K.

Properties of $\phi_n(t)$

- For $n \geq 2$ we have

$$\phi_n(t) = t^{\varphi(n)} \phi_n(1/t)$$

- Let $n = 2^{e_0} p_1^{e_1} \cdots p_r^{e_r}$ where $p_1, \ldots, p_r$ are different odd primes, $e_0 \geq 0$, $e_i \geq 1$ for $i = 1, \ldots, r$ and $r \geq 1$. Denote by R the radical of n, namely

$$R = \begin{cases} 2p_1 \cdots p_r & \text{if } e_0 \geq 1, \\ p_1 \cdots p_r & \text{if } e_0 = 0. \end{cases}$$

Then,

$$\phi_n(t) = \phi_R(t^{n/R}).$$

- Let $n = 2m$ with m odd ≥ 3. Then

$$\phi_n(t) = \phi_m(-t).$$

$\phi_n(1)$

For $n \geq 2$, we have $\phi_n(1) = e^{\Lambda(n)}$, where the von Mangoldt function is defined for $n \geq 1$ as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^r \text{ with } p \text{ prime and } r \geq 1\,; \\ 0 & \text{otherwise.} \end{cases}$$

In other terms we have

$$\phi_n(1) = \begin{cases} p & \text{if } n = p^r \text{ with } p \text{ prime and } r \geq 1\,; \\ 1 & \text{otherwise.} \end{cases}$$

$\phi_n(-1)$

For $n \geq 3$,

$$\phi_n(-1) = \begin{cases} 1 & \text{if } n \text{ is odd ;} \\ \phi_{n/2}(1) & \text{if } n \text{ is even.} \end{cases}$$

In other terms, for $n \geq 3$,

$$\phi_n(-1) = \begin{cases} p & \text{if } n = 2p^r \text{ with } p \text{ a prime and } r \geq 1 \text{ ;} \\ 1 & \text{otherwise.} \end{cases}$$

Hence $\phi_n(-1) = 1$ when n is odd or when $n = 2m$ where m has at least two distinct prime divisors.

Lower bound for $\phi_n(t)$

For $n \geq 3$, the polynomial $\phi_n(t)$ has real coefficients and no real root, hence it takes only positive values (and its degree $\varphi(n)$ is even).
For $n \geq 3$ and $t \in \mathbb{R}$, we have

$$\phi_n(t) \geq 2^{-\varphi(n)}.$$

Consequence : from

$$\phi_n(t) = t^{\varphi(n)}\phi_n(1/t)$$

we deduce, for $n \geq 3$ and $t \in \mathbb{R}$,

$$\phi_n(t) \geq 2^{-\varphi(n)} \max\{1, |t|\}^{\varphi(n)}.$$

$\phi_n(t) \geq 2^{-\varphi(n)}$ for $n \geq 3$ and $t \in \mathbb{R}$

Proof.

Let ζ_n be a primitive n-th root of unity in $\mathbb{C}$;

$$\phi_n(t) = \mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(t - \zeta_n) = \prod_\sigma (t - \sigma(\zeta_n)),$$

where σ runs over the embeddings $\mathbb{Q}(\zeta_n) \to \mathbb{C}$. We have

$$|t - \sigma(\zeta_n)| \geq |\Im\mathrm{m}(\sigma(\zeta_n))| > 0,$$

$$(2i)\Im\mathrm{m}(\sigma(\zeta_n)) = \sigma(\zeta_n) - \overline{\sigma(\zeta_n)} = \sigma(\zeta_n - \overline{\zeta_n}).$$

Now $(2i)\Im\mathrm{m}(\zeta_n) = \zeta_n - \overline{\zeta_n} \in \mathbb{Q}(\zeta_n)$ is an algebraic integer :

$$2^{\varphi(n)}\phi_n(t) \geq |\mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}((2i)\Im\mathrm{m}(\zeta_n))| \geq 1.$$

The cyclotomic binary forms

For $n \geq 2$, define

$$\Phi_n(X, Y) = Y^{\varphi(n)}\phi_n(X/Y).$$

This is a binary form in $\mathbb{Z}[X, Y]$ of degree $\varphi(n)$.
Consequence of the lower bound for $\phi_n(t)$: for $n \geq 3$ and $(x, y) \in \mathbb{Z}^2$,

$$\Phi_n(x, y) \geq 2^{-\varphi(n)} \max\{|x|, |y|\}^{\varphi(n)}.$$

Therefore, if $\Phi_n(x, y) = m$, then

$$\max\{|x|, |y|\} \leq 2m^{1/\varphi(n)}.$$

If $\max\{|x|, |y|\} \geq 3$, then n is bounded :

$$\varphi(n) \leq \frac{\log m}{\log(3/2)}.$$

Generalization to CM fields (Győry, 1977)

Let K be a CM field of degree d over $\mathbb{Q}$. Let $\alpha \in K$ be such that $K = \mathbb{Q}(\alpha)$; let f be the irreducible polynomial of α over $\mathbb{Q}$ and let $F(X,Y) = Y^d f(X/Y)$ the associated homogeneous binary form :

$$f(t) = a_0 t^d + a_1 t^{d-1} + \cdots + a_d,$$

$$F(X,Y) = a_0 X^d + a_1 X^{d-1} Y + \cdots + a_d Y^d.$$

For $(x,y) \in \mathbb{Z}^2$ we have

$$x^d \leq 2^d a_d^{d-1} F(x,y) \quad \text{and} \quad y^d \leq 2^d a_0^{d-1} F(x,y).$$

Kálmán Győry, László Lovász

K. Győry

L. Lovász

K. Győry & L. Lovász, *Representation of integers by norm forms II*, Publ. Math. Debrecen **17**, 173–181, (1970).
K. Győry, *Représentation des nombres entiers par des formes binaires*, Publ. Math. Debrecen **24** , 363–375, (1977).

Best possible for CM fields

Let $n \geq 3$, not of the form p^a nor $2p^a$ with p prime and $a \geq 1$, so that $\phi_n(1) = \phi_n(-1) = 1$.
Then the binary form

$$F_n(X,Y) = \Phi_n(X, Y - X)$$

has degree $d = \varphi(n)$ and $a_0 = a_d = 1$. For $x \in \mathbb{Z}$ we have $F_n(x, 2x) = \Phi_n(x,x) = x^d$.
Hence, for $y = 2x$, we have

$$y^d = 2^d a_0^{d-1} F(x,y).$$

Binary cyclotomic forms (EF–CL–MW 2018)

Let m be a positive integer and let n, x, y be rational integers satisfying $n \geq 3$, $\max\{|x|,|y|\} \geq 2$ and $\Phi_n(x,y) = m$.Then

$$\max\{|x|,|y|\} \leq \frac{2}{\sqrt{3}} m^{1/\varphi(n)}, \quad \text{hence} \quad \varphi(n) \leq \frac{2}{\log 3} \log m.$$

These estimates are optimal, since for $\ell \geq 1$,

$$\Phi_3(\ell, -2\ell) = 3\ell^2.$$

If we assume $\varphi(n) > 2$, namely $\varphi(n) \geq 4$, then

$$\varphi(n) \leq \frac{4}{\log 11} \log m$$

which is best possible since $\Phi_5(1,-2) = 11$.

Lower bound for the cyclotomic polynomials

The upper bound

$$\max\{|x|,|y|\} \le \frac{2}{\sqrt{3}} m^{1/\varphi(n)}$$

for $\Phi_n(x,y) = m$ is equivalent to the following result :

For $n \ge 3$ and $t \in \mathbb{R}$,

$$\phi_n(t) \ge \left(\frac{\sqrt{3}}{2}\right)^{\varphi(n)}.$$

The sequence $(c_n)_{n\ge 3}$

$$c_n = \inf_{t\in\mathbb{R}} \phi_n(t) \qquad (n \ge 3).$$

Let $n \ge 3$. Write

$$n = 2^{e_0} p_1^{e_1} \cdots p_r^{e_r}$$

where $p_1, \dots, p_r$ are odd primes with $p_1 < \cdots < p_r$, $e_0 \ge 0$, $e_i \ge 1$ for $i = 1, \dots, r$ and $r \ge 0$.
(i) For $r = 0$, we have $e_0 \ge 2$ and $c_n = c_{2^{e_0}} = 1$.
(ii) For $r \ge 1$ we have

$$c_n = c_{p_1\cdots p_r} \ge p_1^{-2^{r-2}}.$$

End of the proof of $\phi_n(t) \ge \left(\frac{\sqrt{3}}{2}\right)^{\varphi(n)}$.

Lemma. For any odd squarefree integer $n = p_1 \cdots p_r$ with $p_1 < p_2 < \cdots < p_r$ satisfying $n \ge 11$ and $n \ne 15$, we have

$$\varphi(n) > 2^{r+1} \log p_1.$$

The sequence $(c_n)_{n\ge 3}$

$$\Phi_n(x,y) \ge c_n \max\{|x|,|y|\}^{\varphi(n)}.$$

$$c_n \ge \left(\frac{\sqrt{3}}{2}\right)^{\varphi(n)}.$$

- $\liminf_{n\to\infty} c_n = 0$ and $\limsup_{n\to\infty} c_n = 1$.
- The sequence $(c_p)_{p \text{ odd prime}}$ is decreasing from $3/4$ to $1/2$.
- For p_1 and p_2 primes, $c_{p_1p_2} \ge \frac{1}{p_1}\cdot$
- For any prime p_1, $\lim_{p_2\to\infty} c_{p_1p_2} = \frac{1}{p_1}\cdot$

The sequence $(a_m)_{m\geq 1}$

For each integer $m \geq 1$, the set

$$\{(n,x,y) \in \mathbb{N}\times\mathbb{Z}^2 \ \mid \ n \geq 3, \ \max\{|x|,|y|\} \geq 2, \ \Phi_n(x,y) = m\}$$

is finite. Let a_m the number of its elements.

The sequence of integers $m \geq 1$ such that $a_m \geq 1$ starts with the following values of a_m

m	3	4	5	7	8	9	10	11	12	13	16	17
a_m	8	16	8	24	4	16	8	8	12	40	40	16

OEIS A299214

https://oeis.org/A299214
Number of representations of integers by cyclotomic binary forms.

The sequence $(a_m)_{m\geq 1}$ starts with
$0, 0, 8, 16, 8, 0, 24, 4, 16, 8, 8, 12, 40, 0, 0, 40, 16, 4, 24, 8, 24, 0, 0, 0, 24, 8, 12, 24, 8, 0, 32, 8, 0, 8, 0, 16, 32, 0, 24, 8, 8, 0, 32, 0, 8, 0, 0, 12, 40, 12, 0, 32, 8, 0, 8, 0, 32, 8, 0, 0, 48, 0, 24, 40, 16, 0, 24, 8, 0, 0, 0, 4, 48, 8, 12, 24, \ \dots$

OEIS A296095

https://oeis.org/A296095
Integers represented by cyclotomic binary forms.

$a_m \neq 0$ for $m =$
$3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 21, 25, 26, 27, 28, 29, 31, 32, 34, 36, 37, 39, 40, 41, 43, 45, 48, 49, 50, 52, 53, 55, 57, 58, 61, 63, 64, 65, 67, 68, 72, 73, 74, 75, 76, 79, 80, 81, 82, 84, 85, 89, 90, 91, 93, 97, 98, 100, 101, 103, 104, 106, 108, 109, 111, 112, 113, 116, 117, 121, 122, \ \dots$

OEIS A293654

https://oeis.org/A293654
Integers not represented by cyclotomic binary forms.

$a_m = 0$ for $m =$
$1, 2, 6, 14, 15, 22, 23, 24, 30, 33, 35, 38, 42, 44, 46, 47, 51, 54, 56, 59, 60, 62, 66, 69, 70, 71, 77, 78, 83, 86, 87, 88, 92, 94, 95, 96, 99, 102, 105, 107, 110, 114, 115, 118, 119, 120, 123, 126, 131, 132, 134, 135, 138, 140, 141, 142, 143, 150, \ \dots$

Integers represented by cyclotomic binary forms

For $N \geq 1$, let $\mathcal{A}(N)$ be the number of $m \leq N$ which are represented by cyclotomic binary forms :

$$\mathcal{A}(N) = \#\{m \in \mathbb{N} \ \mid \ m \leq N, \ a_m \neq 0\}.$$

We have

$$\mathcal{A}(N) = \alpha \frac{N}{(\log N)^{\frac{1}{2}}} - \beta \frac{N}{(\log N)^{\frac{3}{4}}} + O\left(\frac{N}{(\log N)^{\frac{3}{2}}}\right)$$

as $N \to \infty$.

$\alpha = \alpha_3 + \alpha_4$

The number of positive integers $\leq N$ represented by Φ_4 (namely the sums of two squares) is

$$\alpha_4 \frac{N}{(\log N)^{\frac{1}{2}}} + O\left(\frac{N}{(\log N)^{\frac{3}{2}}}\right).$$

The number of positive integers $\leq N$ represented by Φ_3 (namely $x^2 + xy + y^2$: Loeschian numbers) is

$$\alpha_3 \frac{N}{(\log N)^{\frac{1}{2}}} + O\left(\frac{N}{(\log N)^{\frac{3}{2}}}\right).$$

The number of positive integers $\leq N$ represented by Φ_4 and by Φ_3 is

$$\beta \frac{N}{(\log N)^{\frac{3}{4}}} + O\left(\frac{N}{(\log N)^{\frac{7}{4}}}\right).$$

The Landau–Ramanujan constant

Edmund Landau
1877–1938

Srinivasa Ramanujan
1887–1920

The number of positive integers $\leq N$ which are sums of two squares is asymptotically $\alpha_4 N(\log N)^{-1/2}$, where

$$\alpha_4 = \frac{1}{2^{\frac{1}{2}}} \cdot \prod_{p \equiv 3 \bmod 4} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

OEIS A064533

OEIS A064533 Decimal expansion of Landau-Ramanujan constant.

$$\alpha_4 = 0.764\,223\,653\,589\,220\ \ldots$$

- Ph. Flajolet and I. Vardi, Zeta function expansions of some classical constants, Feb 18 1996.
- Xavier Gourdon and Pascal Sebah, Constants and records of computation.
- David E. G. Hare, $125\,079$ digits of the Landau-Ramanujan constant.

The Landau–Ramanujan constant

References : https://oeis.org/A064533

- B. C. Berndt, Ramanujan's notebook part IV, Springer-Verlag, 1994.
- S. R. Finch, Mathematical Constants, Cambridge, 2003, pp. 98-104.
- G. H. Hardy, "Ramanujan, Twelve lectures on subjects suggested by his life and work", Chelsea, 1940.
- Institute of Physics, Constants - Landau-Ramanujan Constant.
- Simon Plouffe, Landau Ramanujan constant.
- Eric Weisstein's World of Mathematics, Ramanujan constant.
- https://en.wikipedia.org/wiki/Landau-Ramanujan_constant.

Sums of two squares

If a and q are two integers, we denote by $N_{a,q}$ any integer ≥ 1 satisfying the condition

$$p \mid N_{a,q} \Longrightarrow p \equiv a \bmod q.$$

An integer $m \geq 1$ is of the form

$$m = \Phi_4(x,y) = x^2 + y^2$$

if and only if there exist integers $a \geq 0$, $N_{3,4}$ and $N_{1,4}$ such that

$$m = 2^a \, N_{3,4}^2 \, N_{1,4}.$$

Loeschian numbers : $m = x^2 + xy + y^2$

An integer $m \geq 1$ is of the form

$$m = \Phi_3(x,y) = \Phi_6(x,-y) = x^2 + xy + y^2$$

if and only if there exist integers $b \geq 0$, $N_{2,3}$ and $N_{1,3}$ such that

$$m = 3^b \, N_{2,3}^2 \, N_{1,3}.$$

The number of positive integers $\leq N$ which are represented by the quadratic form $X^2 + XY + Y^2$ is asymptotically $\alpha_3 N(\log N)^{-1/2}$ where

$$\alpha_3 = \frac{1}{2^{\frac{1}{2}}3^{\frac{1}{4}}} \cdot \prod_{p \equiv 2 \bmod 3} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

OEIS A301429

OEIS A301429 Decimal expansion of an analog of the Landau-Ramanujan constant for Loeschian numbers.

$$\alpha_3 = \frac{1}{2^{\frac{1}{2}}3^{\frac{1}{4}}} \cdot \prod_{p \equiv 2 \bmod 3} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

$$\alpha_3 = 0.638\,909\,405\,44 \ \ldots$$

$$\alpha = \alpha_3 + \alpha_4 = 1.403\,133\,059 \ \ldots$$

Zeta function expansions of some classical constants, Feb 18 1996.

Philippe Flajolet

Ilan Vardi

Bill Allombert

$$\alpha_3 = \begin{array}{l} 0.63890940544534388 \\ 225494267492824509 37 \\ 549755080291233454 21 \\ 692365708076310027 64 \\ 965824689717911252 86 \\ 643881416875191074 24 \ldots \end{array}$$

OEIS A301430

OEIS A301430 Decimal expansion of an analog of the Landau-Ramanujan constant for Loeschian numbers which are sums of two squares.

$$\beta = \frac{3^{\frac{1}{4}}}{2^{\frac{5}{4}}} \cdot \pi^{\frac{1}{2}} \cdot (\log(2+\sqrt{3}))^{\frac{1}{4}} \cdot \frac{1}{\Gamma(1/4)} \cdot \prod_{p \equiv 5,7,11 \bmod 12} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

$$\beta = 0.302\,316\,142\,35 \;\; \ldots$$

Only 11 digits after the decimal point are known.

Zeta function expansions of some classical constants, Feb 18 1996.

Philippe Flajolet

Ilan Vardi

Bill Allombert

$$\beta = \begin{array}{l} 0.3023161423570656 37 \\ 947769900480199715 60 \\ 241279518936964545 88 \\ 678412888654487524 10 \\ 510899487467813979 27 \\ 270856776591327259 10 \ldots \end{array}$$

Further developments

- Prove similar estimates for the number of integers represented by other binary forms (done for quadratic forms) ; e.g. : prove similar estimates for the number of integers which are sums of two cubes, two biquadrates,. . .

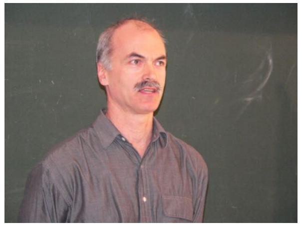
- Prove similar estimates for the number of integers which are represented by Φ_n for a given n.

- Prove similar estimates for the number of integers which are represented by Φ_n for some n with $\varphi(n) \geq d$.

Stewart - Xiao

Let F be a binary form of degree $d \geq 3$ with nonzero discriminant.
There exists a positive constant $C_F > 0$ such that the number of integers of absolute value at most N which are represented by $F(X, Y)$ is asymptotic to $C_F N^{2/d}$.

Cam Stewart and Stanley Yao Xiao



Cam Stewart

Stanley Yao Xiao

C.L. Stewart and S. Yao Xiao, *On the representation of integers by binary forms*,
`arXiv:1605.03427v2` (March 23, 2018).

K. Mahler (1933)

Let F be a binary form of degree $d \geq 3$ with nonzero discriminant.
Denote by A_F the area (Lebesgue measure) of the domain

$$\{(x, y) \in \mathbb{R}^2 \mid F(x, y) \leq 1\}.$$

For $Z > 0$ denote by $N_F(Z)$ the number of $(x, y) \in \mathbb{Z}^2$ such that $0 < |F(x, y)| \leq Z$.
Then

$$N_F(Z) = A_F Z^{2/d} + O(Z^{1/(d-1)})$$

as $Z \to \infty$.

Kurt Mahler

Kurt Mahler
1903 – 1988

Über die mittlere Anzahl der Darstellungen grosser Zahlen durch binäre Formen,
Acta Math. **62** (1933), 91-166.
https://carma.newcastle.edu.au/mahler/biography.html

Higher degree

The situation for positive definite forms of degree ≥ 3 is different for the following reason :

• If a positive integer m is represented by a positive definite quadratic form, it usually has many such representations ; while if a positive integer m is represented by a positive definite binary form of degree $d \geq 3$, it usually has few such representations.

If F is a positive definite quadratic form, the number of (x, y) with $F(x, y) \leq N$ is asymptotically a constant times N, but the number of $F(x, y)$ is much smaller.

If F is a positive definite binary form of degree $d \geq 3$, the number of (x, y) with $F(x, y) \leq N$ is asymptotically a constant times $N^{1/d}$, the number of $F(x, y)$ is also asymptotically a constant times $N^{1/d}$.

Sums of k–th powers

If a positive integer m is a sum of two squares, there are many such representations.

Indeed, the number of (x, y) in $\mathbb{Z} \times \mathbb{Z}$ with $x^2 + y^2 \leq N$ is asymptotic to πN, while the number of values $\leq N$ taken by the quadratic form Φ_4 is asymptotic to $\alpha_4 N/\sqrt{\log N}$ where α_4 is the Landau–Ramanujan constant. Hence Φ_4 takes each of these values with a high multiplicity, on the average $(\pi/\alpha)\sqrt{\log N}$.

On the opposite, it is extremely rare that a positive integer is a sum of two biquadrates in more than one way (not counting symmetries).

$635\,318\,657 = 158^4 + 59^4 = 134^4 + 133^4$.

Leonhard Euler

1707 – 1783

The smallest integer represented by $x^4 + y^4$ in two essentially different ways was found by Euler, it is $635318657 = 41 \times 113 \times 241 \times 569$.

[OEIS A216284] `Number of solutions to the equation` $x^4 + y^4 = n$ `with` $x \geq y > 0$.

An infinite family with one parameter is known for non trivial solutions to $x_1^4 + x_2^4 = x_3^4 + x_4^4$.

http://mathworld.wolfram.com/DiophantineEquation4thPowers.html

Sums of k–th powers

One conjectures that given $k \geq 5$, if an integer is of the form $x^k + y^k$, there is essentially a unique such representation. But there is no value of k for which this has been proved.

Higher degree

The situation for positive definite forms of degree ≥ 3 is different also for the following reason.

A necessary and sufficient condition for a number m to be represented by one of the quadratic forms Φ_3, Φ_4, is given by a congruence.

By contrast, consider the quartic binary form $\Phi_8(X, Y) = X^4 + Y^4$. On the one hand, an integer represented by Φ_8 is of the form

$$N_{1,8}(N_{3,8}N_{5,8}N_{7,8})^4.$$

On the other hand, there are many integers of this form which are not represented by Φ_8.

Quartan primes

[OEIS A002645] `Quartan primes: primes of the form` $x^4 + y^4$, $x > 0$, $y > 0$.

The list of prime numbers represented by Φ_8 start with $2, 17, 97, 257, 337, 641, 881, 1297, 2417, 2657, 3697, 4177, 4721, 6577, 10657, 12401, 14657, 14897, 15937, 16561, 28817, 38561, 39041, 49297, 54721, 65537, 65617, 66161, 66977, 80177, 83537, 83777, 89041, 105601, 107377, 119617, \dots$

It is not known whether this list is finite or not.

`The largest known quartan prime is currently the largest known generalized Fermat prime: The` 1353265`-digit` $(145310^{65536})^4 + 1^4$.

Primes of the form $x^{2^k} + y^{2^k}$

[OEIS A002313] `primes of the form` $x^2 + y^2$.
[OEIS A002645] `primes of the form` $x^4 + y^4$,
[OEIS A006686] `primes of the form` $x^8 + y^8$,
[OEIS A100266] `primes of the form` $x^{16} + y^{16}$,
[OEIS A100267] `primes of the form` $x^{32} + y^{32}$.

Primes of the form $X^2 + Y^4$

John Friedlander

Étienne Fouvry

But it is known that there are infinitely many prime numbers of the form $X^2 + Y^4$.
Friedlander, J. & Iwaniec, H. *The polynomial $X^2 + Y^4$ captures its primes*, Ann. of Math. (2) **148** (1998), no. 3, 945–1040.
`https://arxiv.org/pdf/math/9811185.pdf`