July 11, 2015

Debrecen University Symposium
Győry 75
July 10-11, 2015

Effective upper bounds for the solutions of a family of Thue equations involving powers of units of the simplest cubic fields.

Michel Waldschmidt
Joint work with *Claude Levesque*.

The pdf file of this talk can be downloaded at URL
http://www.imj-prg.fr/~michel.waldschmidt/

Boldog születésnapot, Kálmán !

Abstract

Emery Thomas was one of the first to solve an infinite family of Thue equations, when he considered the forms

$$F_n(X, Y) = X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3$$

and the family of equations $F_n(x, y) = \pm 1$, $n \in \mathbf{N}$, $x, y \in \mathbf{Z}$. This family is associated to the family of the simplest cubic fields $\mathbf{Q}(\lambda)$ of D. Shanks, λ being a root of $F_n(X, 1)$.
We introduce in this family a second parameter by replacing the roots of the minimal polynomial $F_n(X, 1)$ of λ by the a-th powers of the roots and we effectively solve the family of Thue equations that we obtain and which depends now on the two parameters n and a.

Thue equation

Thue (1908) : there are only finitely many integer solutions of

$$F(x, y) = m,$$

when F is homogeneous irreducible form over $\mathbf{Q}$ of degree ≥ 3.

Baker – Fel'dman Effective upper bounds for the solutions.

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Gel'fond–Baker method

While Thue's method was based on the non effective Thue–Siegel–Roth Theorem, Baker and Fel'dman followed an effective method introduced by A.O. Gel'fond, involving *lower bounds for linear combinations of logarithms of algebraic numbers with algebraic coefficients.*

Lower bounds for linear combinations of logarithms

A lower bound for a nonvanishing difference

$$\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1$$

is essentially the same as a lower bound for a nonvanishing number of the form

$$b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n,$$

since $e^z - 1 \sim z$ for $z \to 0$.
The first nontrivial lower bounds were obtained by A.O. Gel'fond. His estimates were effective only for $n = 2$: for $n \geq 3$, he needed to use estimates related to the Thue–Siegel–Roth Theorem.

Explicit version of Gel'fond's estimates

A. Schinzel (1968) computed explicitly the constants introduced by A.O. Gel'fond. in his lower bound for

$$\left|\alpha_1^{b_1}\alpha_2^{b_2} - 1\right|.$$

He deduced explicit Diophantine results using the approach introduced by A.O. Gel'fond.

Alan Baker

In 1968, A. Baker succeeded to extend to any $n \geq 2$ the transcendence method used by A.O. Gel'fond for $n = 2$. As a consequence, effective upper bounds for the solutions of Thue's equations have been derived.

Thue equations and the Siegel unit equation

The main idea behind the Gel'fond–Baker approach for solving Thue equations is to exploit Siegel's unit equation.
Assume $\alpha_1, \alpha_2, \alpha_3$ are algebraic integers and x, y rational integers such that

$$(x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y) = 1.$$

Then the three numbers

$$u_1 = x - \alpha_1 y, \quad u_2 = x - \alpha_2 y, \quad u_3 = x - \alpha_3 y,$$

are units. Eliminating x and y, one deduces *Siegel's unit equation*

$$u_1(\alpha_2 - \alpha_3) + u_2(\alpha_3 - \alpha_1) + u_3(\alpha_1 - \alpha_2) = 0.$$

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Siegel's unit equation

Write Siegel's unit equation

$$u_1(\alpha_2 - \alpha_3) + u_2(\alpha_3 - \alpha_1) + u_3(\alpha_1 - \alpha_2) = 0$$

in the form

$$\frac{u_1(\alpha_2 - \alpha_3)}{u_2(\alpha_1 - \alpha_3)} - 1 = \frac{u_3(\alpha_1 - \alpha_2)}{u_2(\alpha_1 - \alpha_3)}.$$

The quotient

$$\frac{u_1(\alpha_2 - \alpha_3)}{u_2(\alpha_1 - \alpha_3)}$$

is the quantity

$$\alpha_1^{b_1} \cdots \alpha_n^{b_n}$$

in Gel'fond–Baker Diophantine inequality.

Siegel's unit equation

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Work on Baker's method :

A. Baker (1968), N.I. Fel'dman (1971), V.G. Sprindžuk and H.M. Stark (1973), K. Győry and Z.Z. Papp (1983), E. Bombieri (1993), Y. Bugeaud and K. Győry (1996), Y. Bugeaud (1998)...

Solving Thue equations :
A. Pethő and R. Schulenberg (1987), B. de Weger (1987), N. Tzanakis and B. de Weger (1989), Y. Bilu and G. Hanrot (1996), (1999)...

Solving Thue–Mahler equations :
J.H. Coates (1969), S.V. Kotov and V.G. Sprindžuk (1973), A. Bérczes–Yu Kunrui– K. Győry (2006)...

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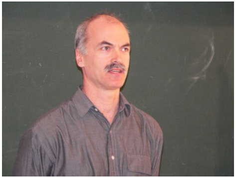
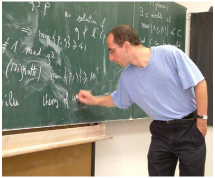
Diophantine equations

A.O. Gel'fond, A. Baker, V. Sprindžuk, R. Tijdeman, C.L. Stewart, M. Mignotte, M. Bennett, P. Voutier, Y. Bugeaud, T.N. Shorey. . .






Mathematical genealogy of Kálmán Győry

Kálmán Győry

MathSciNet

Ph.D. University of Debrecen 1966

Dissertation: *Contributions to the Theory of Diophantine Equations*

Mathematics Subject Classification: 11—Number theory

Advisor: Pál Turán

Students:
Click here to see the students ordered by family name.

Name	School	Year	Descendants
Béla Kovács	University of Debrecen	1973	
Péter Kiss	University of Debrecen	1976	2
Attila Pethő	Kossuth University	1976	9
Zoltán Papp	University of Debrecen	1977	
Sándor Turjányi	University of Debrecen	1977	
János Rimán	University of Debrecen	1978	
Béla Brindza	University of Debrecen	1985	
István Gaál	University of Debrecen	1987	3
Ákos Pintér	University of Debrecen	1996	
Lajos Hajdu	University of Debrecen	1998	1
Attila Bérczes	University of Debrecen	2001	1
Csaba Rakaczki	University of Debrecen	2005	
István Pink	University of Debrecen	2006	

According to our current on-line database, Kálmán Győry has 13 students and 29 descendants.

Kálmán Győry and his School

K. Győry

A. Pethő

I. Gaál,

Á. Pintér

L. Hajdu

A. Bérczes

Families of Thue equations

The first families of Thue equations having only trivial solutions were introduced by A. Thue himself.

$$(a+1)X^n - aY^n = 1.$$

He proved that the only solution in positive integers x, y is $x = y = 1$ for n prime and a sufficiently large in terms of n. For $n = 3$ this equation has only this solution for $a \geq 386$.

M. Bennett (2001) proved that this is true for all a and n with $n \geq 3$ and $a \geq 1$.

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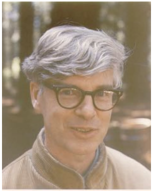
E. Thomas's family of Thue equations

E. Thomas in 1990 studied
the families of Thue equations
$x^3 - (n-1)x^2y - (n+2)xy^2 - y^3 = 1$



Set

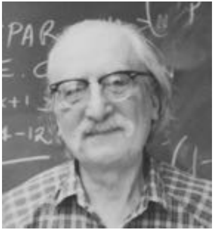
$$F_n(X,Y) = X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3.$$

The cubic fields $\mathbf{Q}(\lambda)$ generated by a root λ of $F_n(X,1)$ are called by D. Shanks the *simplest cubic fields.* The roots of the polynomial $F_n(X,1)$ can be described via homographies of degree 3.

D. Shanks's simplest cubic fields $\mathbf{Q}(\lambda)$.

Let λ be one of the three roots of

$$F_n(X,1) = X^3 - (n-1)X^2 - (n+2)X - 1.$$



Then $\mathbf{Q}(\lambda)$ is a Galois cubic field.

Write

$$F_n(X,Y) = (X - \lambda_0 Y)(X - \lambda_1 Y)(X - \lambda_2 Y)$$

with

$$\lambda_0 > 0 > \lambda_1 > -1 > \lambda_2.$$

Then

$$\lambda_1 = -\frac{1}{\lambda_0 + 1} \quad \text{and} \quad \lambda_2 = -\frac{\lambda_0 + 1}{\lambda_0}.$$

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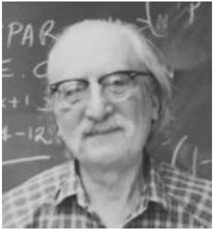

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Simplest fields.

When the following polynomials are irreducible for $s, t \in \mathbf{Z}$, the fields $\mathbf{Q}(\omega)$ generated by a root ω of respectively

$$\begin{cases} sX^3 - tX^2 - (t+3s)X - s, \\ sX^4 - tX^3 - 6sX^2 + tX + s, \\ sX^6 - 2tX^5 - (5t+15s)X^4 - 20sX^3 + 5tX^2 + (2t+6s)X + s, \end{cases}$$

are cyclic over $\mathbf{Q}$ of degree 3, 4 and 6 respectively.
For $s = 1$, they are called *simplest fields* by many authors.
For $s \geq 1$, I. Wakabayashi call them *simplest fields*.

In each of the three cases, the roots of the polynomials can be described via homographies of $PSL_2(\mathbf{Z})$ of degree 3, 4 and 6 respectively.

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E. Thomas's family of Thue equations

In 1990, E. Thomas proved in some effective way that the set of $(n, x, y) \in \mathbf{Z}^3$ with

$$n \geq 0, \quad \max\{|x|, |y|\} \geq 2 \quad \text{and} \quad F_n(x, y) = \pm 1$$

is finite.

In his paper, he completely solved the equation $F_n(x, y) = 1$ for $n \geq 1.365 \cdot 10^7$: the only solutions are $(0, -1)$, $(1, 0)$ and $(-1, +1)$.

Since $F_n(-x, -y) = -F_n(x, y)$, the solutions to $F_n(x, y) = -1$ are given by $(-x, -y)$ where (x, y) are the solutions to $F_n(x, y) = 1$.

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Exotic solutions found by E. Thomas in 1990

$F_0(X,Y) = X^3 + X^2Y - 2XY^2 - Y^3$

Solutions (x,y) to $F_0(x,y) = 1$:

$(-9,5)$, $(-1,2)$, $(2,-1)$, $(4,-9)$, $(5,4)$

$F_1(X,Y) = X^3 - 3XY^2 - Y^3$

Solutions (x,y) to $F_1(x,y) = 1$:

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$F_3(X,Y) = X^3 - 2X^2Y - 5XY^2 - Y^3$

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M. Mignotte's work on E. Thomas's family

In 1993, M. Mignotte completed the work of E. Thomas by solving the problem for each n.

For $n \geq 4$ and for $n = 2$, the only solutions to $F_n(x, y) = 1$ are $(0, -1)$, $(1, 0)$ and $(-1, +1)$, while for the cases $n = 0, 1, 3$, the only nontrivial solutions are the ones found by E. Thomas.

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E. Thomas's family of Thue equations

For the same family

$$F_n(X,Y) = X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3,$$

given $m \neq 0$, M. Mignotte, A. Pethő and F. Lemmermeyer (1996) studied the family of Diophantine equations $F_n(X,Y) = m$.

M. Mignotte, A. Pethő and F. Lemmermeyer (1996)

For $n \geq 2$, when x, y are rational integers verifying

$$0 < |F_n(x, y)| \leq m,$$

then

$$\log |y| \leq c(\log n)(\log n + \log m)$$

with an effectively computable absolute constant c.

One would like an upper bound for $\max\{|x|, |y|\}$ depending only on m, not on n. This is still open.

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Besides, M. Mignotte, A. Pethő and F. Lemmermeyer found all solutions of the Thue inequality $|F_n(X, Y)| \leq 2n + 1$.

As a consequence, when m is a given positive integer, there exists an integer n_0 depending upon m such that the inequality $|F_n(x, y)| \leq m$, with $n \geq 0$ and $|y| > \sqrt[3]{m}$, implies $n \leq n_0$.

Note that for $0 < |t| \leq \sqrt[3]{m}$, $(-t, t)$ and $(t, -t)$ are solutions. Therefore, the condition $|y| > \sqrt[3]{m}$ cannot be omitted.

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E. Thomas's family of Thue inequations

In 1996, for the family of Thue inequations

$$0 < |F_n(x, y)| \leq m,$$

Chen Jian Hua has given a bound for n by using Padé's approximations. This bound was highly improved in 1999 by G. Lettl, A. Pethő and P. Voutier.



Homogeneous variant of E. Thomas's family

I. Wakabayashi, using again the approximants of Padé, extended these results to the families of forms, depending upon two parameters,

$$sX^3 - tX^2Y - (t+3s)XY^2 - sY^3,$$

which includes the family of Thomas for $s = 1$ (with $t = n - 1$).

May 2010, Rio de Janeiro What were we doing on the beach of Rio ?

Suggestion of Claude Levesque

Consider Thomas's family of cubic Thue equations $F_n(X, Y) = \pm 1$ with

$$F_n(X, Y) = X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3.$$

Write

$$F_n(X, Y) = (X - \lambda_{0n}Y)(X - \lambda_{1n}Y)(X - \lambda_{2n}Y)$$

where λ_{in} are units in the totally real cubic field $\mathbf{Q}(\lambda_{0n})$. Twist these equations by introducing a new parameter $a \in \mathbf{Z}$:

$$F_{n,a}(X, Y) = (X - \lambda_{0n}^a Y)(X - \lambda_{1n}^a Y)(X - \lambda_{2n}^a Y) \in \mathbf{Z}[X, Y].$$

Then we get a family of cubic Thue equations depending on two parameters (n, a) :

$$F_{n,a}(x, y) = \pm 1.$$

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Write

$$F_n(X, Y) = (X - \lambda_{0n}Y)(X - \lambda_{1n}Y)(X - \lambda_{2n}Y)$$

where λ_{in} are units in the totally real cubic field $\mathbf{Q}(\lambda_{0n})$. Twist these equations by introducing a new parameter $a \in \mathbf{Z}$:

$$F_{n,a}(X, Y) = (X - \lambda_{0n}^a Y)(X - \lambda_{1n}^a Y)(X - \lambda_{2n}^a Y) \in \mathbf{Z}[X, Y].$$

Then we get a family of cubic Thue equations depending on two parameters (n, a) :

$$F_{n,a}(x, y) = \pm 1.$$

Suggestion of Claude Levesque

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Thomas's family with two parameters

Joint work with Claude Levesque

Main result (2014) : *there is an effectively computable absolute constant* $c > 0$ *such that, if* (x, y, n, a) *are nonzero rational integers with* $\max\{|x|, |y|\} \geq 2$ *and*

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then

$$\max\{|n|, |a|, |x|, |y|\} \leq c.$$

For all $n \geq 0$, trivial solutions with $a \geq 2$:

$(\pm 1, 0)$, $(0, \pm 1)$

$(\pm 1, \pm 1)$ for $a = 2$

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Exotic solutions to $F_{n,a}(x, y) = 1$ with $a \geq 2$

(n, a)	(x, y)					
$(0, 2)$	$(-14, -9)$	$(-3, -1)$	$(-2, -1)$	$(1, 5)$	$(3, 2)$	$(13, 4)$
$(0, 3)$	$(2, 1)$					
$(0, 5)$	$(-3, -1)$	$(19, -1)$				
$(1, 2)$	$(-7, -2)$	$(-3, -1)$	$(2, 1)$	$(7, 3)$		
$(2, 2)$	$(-7, -1)$	$(-2, -1)$				
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No further solution in the range

$$0 \leq n \leq 10, \qquad 2 \leq a \leq 70, \qquad -1000 \leq x, y \leq 1000.$$

Open question : are there further solutions ?

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Computer search by specialists

Further Diophantine results on the family $F_{n,a}(x, y)$

Let $m \geq 1$. There exists an absolute effectively computable constant κ such that, if there exists $(n, a, m, x, y) \in \mathbf{Z}^5$ with $a \neq 0$ verifying

$$0 < |F_{n,a}(x, y)| \leq m,$$

then

$$\log \max\{|x|, |y|\} \leq \kappa\mu$$

with

$$\mu = \begin{cases} (\log m + |a| \log |n|)(\log |n|)^2 \log \log |n| & \text{for } |n| \geq 3, \\ \log m + |a| & \text{for } n = 0, \pm 1, \pm 2. \end{cases}$$

For $a = 1$, this follows from the above mentioned result of M. Mignotte, A. Pethő and F. Lemmermeyer.

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with $n \geq 0$, $a \geq 1$ and $|y| \geq 2\sqrt[3]{m}$, then

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$$\mu' = \begin{cases} (\log m + \log n)(\log n) \log\log n & \text{for } n \geq 3, \\ 1 + \log m & \text{for } n = 0, 1, 2. \end{cases}$$

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with $xy \neq 0$, $n \geq 0$ and $a \geq 1$, then

$$a \leq \kappa \max\left\{1,\ (1 + \log|x|) \log\log(n+3),\ \log|y|,\ \frac{\log m}{\log(n+2)}\right\}.$$

Conjecture on the family $F_{n,a}(x, y)$

Assume that there exists $(n, a, m, x, y) \in \mathbf{Z}^5$ with $xy \neq 0$ and $|a| \geq 2$ verifying

$$0 < |F_{n,a}(x, y)| \leq m.$$

We conjecture the upper bound

$$\max\{\log |n|, |a|, \log |x|, \log |y|\} \leq \kappa(1 + \log m).$$

For $m > 1$ we cannot give an upper bound for $|n|$.

Since the rank of the units of $\mathbf{Q}(\lambda_0)$ is 2, one may expect a more general result as follows :

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Conjecture on a family $F_{n,s,t}(x,y)$

Conjecture. For s, t and n in $\mathbf{Z}$, define

$$F_{n,s,t}(X,Y) = (X - \lambda_{0n}^s \lambda_{1n}^t Y)(X - \lambda_{1n}^s \lambda_{2n}^t Y)(X - \lambda_{2n}^s \lambda_{0n}^t Y).$$

There exists an effectively computable positive absolute constant κ with the following property : If n, s, t, x, y, m are integers satisfying

$$\max\{|x|, |y|\} \geq 2, \quad (s,t) \neq (0,0) \quad \text{and} \quad 0 < |F_{n,s,t}(x,y)| \leq m,$$

then

$$\max\{\log|n|, |s|, |t|, \log|x|, \log|y|\} \leq \kappa(1 + \log m).$$

Sketch of proof

We want to prove the **Main result** : *there is an effectively computable absolute constant* $c > 0$ *such that, if* (x, y, n, a) *are nonzero rational integers with* $\max\{|x|, |y|\} \geq 2$ *and*

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Sketch of proof (continued)

Write λ_i for λ_{in}, $(i = 0, 1, 2)$:

$$\begin{aligned} F_n(X, Y) &= X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3 \\ &= (X - \lambda_0 Y)(X - \lambda_1 Y)(X - \lambda_2 Y). \end{aligned}$$

We have

$$\begin{cases} n + \frac{1}{n} & \leq \lambda_0 \leq & n + \frac{2}{n}, \\ -\frac{1}{n+1} & \leq \lambda_1 \leq & -\frac{1}{n+2}, \\ -1 - \frac{1}{n} & \leq \lambda_2 \leq & -1 - \frac{1}{n+1}. \end{cases}$$

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Sketch of proof (continued)

Define

$$\gamma_i = x - \lambda_i^a y, \quad (i = 0, 1, 2)$$

so that $F_{n,a}(x, y) = \pm 1$ becomes $\gamma_0 \gamma_1 \gamma_2 = \pm 1$.

One γ_i, say γ_{i_0}, has a small absolute value, namely

$$|\gamma_{i_0}| \leq \frac{m}{y^2 \lambda_0^a},$$

the two others, say $\gamma_{i_1}, \gamma_{i_2}$, have large absolute values :

$$\min\{|\gamma_{i_1}|, |\gamma_{i_2}|\} > y|\lambda_2|^a.$$

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Sketch of proof (continued)

Use λ_0, λ_2 as a basis of the group of units of $\mathbf{Q}(\lambda_0)$: there exist $\delta = \pm 1$ and rational integers A and B such that

$$\begin{cases} \gamma_{0,a} &= \delta \lambda_0^A \lambda_2^B, \\ \gamma_{1,a} &= \delta \lambda_1^A \lambda_0^B = \delta \lambda_0^{-A+B} \lambda_2^{-A}, \\ \gamma_{2,a} &= \delta \lambda_2^A \lambda_1^B = \delta \lambda_0^{-B} \lambda_2^{A-B}. \end{cases}$$

We can prove

$$|A| + |B| \le \kappa \left(\frac{\log y}{\log \lambda_0} + a \right).$$

Sketch of proof (continued)

The Siegel equation

$$\gamma_{i_0,a}(\lambda^a_{i_1} - \lambda^a_{i_2}) + \gamma_{i_1,a}(\lambda^a_{i_2} - \lambda^a_{i_0}) + \gamma_{i_2,a}(\lambda^a_{i_0} - \lambda^a_{i_1}) = 0$$

leads to the identity

$$\frac{\gamma_{i_1,a}(\lambda^a_{i_2} - \lambda^a_{i_0})}{\gamma_{i_2,a}(\lambda^a_{i_1} - \lambda^a_{i_0})} - 1 = -\frac{\gamma_{i_0,a}(\lambda^a_{i_1} - \lambda^a_{i_2})}{\gamma_{i_2,a}(\lambda^a_{i_1} - \lambda^a_{i_0})}$$

and the estimate

$$0 < \left|\frac{\gamma_{i_1,a}(\lambda^a_{i_2} - \lambda^a_{i_0})}{\gamma_{i_2,a}(\lambda^a_{i_1} - \lambda^a_{i_0})} - 1\right| \leq \frac{2}{y^3\lambda^a_0}.$$

End of the proof when n is large

We complete the proof when n is large by means of a lower bound for a linear form in logarithms of algebraic numbers (Baker's method).

Next we need to consider the case where n is bounded. We have results which are valid not only for the Thue equations of the family of Thomas. The next result completes the proof of our main theorem.

Twists of a given cubic Thue equation

Consider a monic irreducible cubic polynomial $f(X) \in \mathbf{Z}[X]$ with $f(0) = \pm 1$ and write

$$F(X,Y) = Y^3 f(X/Y) = (X - \epsilon_1 Y)(X - \epsilon_2 Y)(X - \epsilon_3 Y).$$

For $a \in \mathbf{Z}$, $a \neq 0$, define

$$F_a(X,Y) = (X - \epsilon_1^a Y)(X - \epsilon_2^a Y)(X - \epsilon_3^a Y).$$

Then there exists an effectively computable constant $\kappa > 0$, depending only on f, such that, for any $m \geq 2$, any (x,y,a) in the set

$$\{(x,y,a) \in \mathbf{Z}^2 \times \mathbf{Z} \mid xya \neq 0, \ \max\{|x|,|y|\} \geq 2, \ |F_a(x,y)| \leq m\}$$

satisfies

$$\max\{|x|,|y|,e^{|a|}\} \leq m^\kappa.$$

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Twists of a given Thue equation

Much more general results can be proved for the twists of a given Thue equation. In particular :

Let α *be an algebraic number of degree* $n \geq 3$ *and* K *be the field* $\mathbf{Q}(\alpha)$. *When* ε *is a unit of* K *such that* $\alpha\varepsilon$ *has degree* n, *let* $f_\varepsilon(X)$ *be the irreducible polynomial of* $\alpha\varepsilon$ *and let* $F_\varepsilon(X, Y)$ *be its homogeneous version. Then for all but finitely many of these units, Thue equation* $F_\varepsilon(x, y) = \pm 1$ *has only the trivial solutions* x, y *in* $\mathbf{Z}$ *where* $xy = 0$.

This last result rests on Schmidt's subspace Theorem and is not effective.

A conjecture

The goal is to obtain effective results.

Conjecture. *There exists a constant* $\kappa > 0$, *depending only on* α, *such that, for any* $m \geq 2$, *all solutions* (x, y, ε) *in* $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_K^\times$ *of the inequality*

$$|F_\varepsilon(x,y)| \leq m, \text{ with } xy \neq 0 \text{ and } [\mathbf{Q}(\alpha\varepsilon) : \mathbf{Q}] \geq 3,$$

satisfy

$$\max\{|x|,\ |y|,\ e^{\mathrm{h}(\alpha\varepsilon)}\} \leq m^\kappa.$$

Twists of a given Thue equation

With Claude Levesque we obtained effective partial results in several cases :

• Our first paper (J. Austral. Math. Soc. 2013) was dealing with non totally real cubic fields.

• Our second one (to appear) was dealing with Thue equations attached to a number field having at most one real embedding.

• In the third paper (MJCNT, 2013), for each (irreducible) binary form attached to an algebraic number field, which is not a totally real cubic field, we exhibited an infinite family of equations twisted by units for which Baker's method provides effective bounds for the solutions.

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July 11, 2015

Debrecen University Symposium
Győry 75
July 10-11, 2015

Effective upper bounds for the solutions of a family of Thue equations involving powers of units of the simplest cubic fields.

Michel Waldschmidt
Joint work with *Claude Levesque*.

The pdf file of this talk can be downloaded at URL
http://www.imj-prg.fr/~michel.waldschmidt/