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Families of cubic and quartic Thue Diophantine equations related with the simplest fields of D. Shanks.

by

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1. Cubic forms associated with the simplest cubic fields of Shanks.

For $n \geq 0$, let $F_n(X,Y)$ be the cubic form

$$F_n(X,Y) = X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3.$$

Denote by $\lambda_{1,n}, \lambda_{2,n}, \lambda_{3,n}$ the roots of the polynomial $F_n(t,1) \in \mathbf{Z}[t]$:

$$F_n(X,Y) = (X - \lambda_{1,n}Y)(X - \lambda_{2,n}Y)(X - \lambda_{3,n}Y).$$

In [2], it is proved that the set of $(n,a,x,y) \in \mathbf{Z}^4$ with $n \geq 0$, $a \geq 0$, $\max\{|x|,|y|\} \geq 2$ and

$$(x - \lambda_{1,n}^a y)(x - \lambda_{2,n}^a y)(x - \lambda_{3,n}^a y) = \pm 1$$

is finite, and 37 solutions are given, all of them have $n \leq 4$, $a \leq 5$, $|x| \leq 14$, $|y| \leq 9$.
Question 1. *Are there other solutions?*

2. Quartic forms associated with the simplest quartic fields of Marie-Nicole Gras.

For $n \geq 0$, let $G_n(X,Y)$ be the quartic form

$$G_n(X,Y) = X^4 - nX^3Y - 6X^2Y^2 - nXY^3 + Y^4.$$

Denote by $\mu_{1,n}, \mu_{2,n}, \mu_{3,n}, \mu_{4,n}$ the roots of the polynomial $G_n(t,1) \in \mathbf{Z}[t]$:

$$G_n(X,Y) = (X - \mu_{1,n}Y)(X - \mu_{2,n}Y)(X - \mu_{3,n}Y)(X - \mu_{4,n}Y).$$

Question 2. *Is-it true that the set of* $(n,a,x,y) \in \mathbf{Z}^4$ *with* $n \geq 0$, $a \geq 0$, $\max\{|x|,|y|\} \geq 2$ *and*

$$(x - \mu_{1,n}^a y)(x - \mu_{2,n}^a y)(x - \mu_{3,n}^a y)(x - \mu_{4,n}^a y) = \pm 1$$

is finite?

Further related open problems can be proposed following [1] and [3]

References

[1] T. Hilgart and V. Ziegler *On a conjecture of Levesque and Waldschmidt.*
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[2] C. Levesque and M. Waldschmidt, *A family of Thue equations involving powers of units of the simplest cubic fields*, J. Théor. Nombres Bordx. **27**, No. 2 (2015), 537–563.
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[3] — , *Quest for bounds for the solutions of families of Thue equations*, Friendly workshop on Diophantine Equations and related problems (workshop FWDERP2019), Bursa Uludag University, 2019. Problem session (edited by Alain Togbé).
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