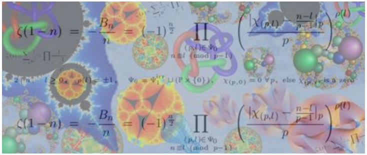
November 27, 2018

Department of Mathematics, Ramakrishna Mission Vivekananda University (RKMVU), Belur Math, Howrah, Kolkata (India).



On the Landau–Ramanujan constant

Michel Waldschmidt

Sorbonne Université, Institut de Mathématiques de Jussieu
http://www.imj-prg.fr/~michel.waldschmidt/

Abstract

The Landau–Ramanujan constant α is defined as follows : for $N \to \infty$, the number of positive integers $\leq N$ which are sums of two squares is asymptotically

$$\alpha \frac{N}{\sqrt{\log N}}.$$

In a joint work with Etienne Fouvry and Claude Levesque, we replace the quadratic form $\Phi_4(X,Y) = X^2 + Y^2$, which is the homogeneous version of the cyclotomic polynomial $\phi_4(t) = t^2 + 1$, with other binary forms.

This is a joint work with Étienne Fouvry and Claude Levesque

Étienne Fouvry

Claude Levesque

Representation of integers by cyclotomic binary forms.
Acta Arithmetica, **184**.1 (2018), 67 - 86.
Dedicated to Rob Tijdeman. `arXiv: 712.09019 [math.NT]`

November 6, 2017

Lecture on *Representation of positive integers by binary cyclotomic forms*
Joint work with Claude Levesque, in progress

Science Faculty, Mahidol University (Phrayathai campus), Bangkok (Thailand)
Invited by Chatchawan Panraksa

Chatchawan Panraksa

November 6, 2017

ALGEBRA AND NUMBER THEORY 7:5 (2013)
dx.doi.org/10.2140/ant.2013.7.1207

On binary cyclotomic polynomials

Étienne Fouvry

We study the number of nonzero coefficients of cyclotomic polynomials Φ_m, where m is the product of two distinct primes.

Joint work with Claude Levesque :
Representation of positive integers by binary cyclotomic forms

Étienne Fouvry

November 10-12, 2017 : ICMMEDC 2017

Mandalay (Myanmar)
The Tenth International Conference on Science and Mathematics Education in Developing Countries.

Claude Levesque

N.B. : The 11th International Conference on Mathematics and Mathematics Education in Developing Countries (ICMMEDC 2018) took place in Vientiane (Laos), October 31 - November 4, 2018.

The Landau–Ramanujan constant

Edmund Landau

1877 – 1938

Srinivasa Ramanujan

1887 – 1920

The number of positive integers $\leq N$ which are sums of two squares is asymptotically $\mathsf{C}_{\Phi_4} N (\log N)^{-\frac{1}{2}}$, where

$$\mathsf{C}_{\Phi_4} = \frac{1}{2^{\frac{1}{2}}} \cdot \prod_{p \equiv 3 \bmod 4} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

Online Encyclopedia of Integer Sequences https://oeis.org/A064533

[OEIS A064533] `Decimal expansion of Landau-Ramanujan constant.`

$$C_{\Phi_4} = 0.764\,223\,653\,589\,220\ldots$$

- Ph. Flajolet and I. Vardi, Zeta function expansions of some classical constants, Feb 18 1996.
- Xavier Gourdon and Pascal Sebah, Constants and records of computation.
- David E. G. Hare, $125\,079$ digits of the Landau-Ramanujan constant.

The Landau–Ramanujan constant

References : https://oeis.org/A064533

- B. C. Berndt, Ramanujan's notebook part IV, Springer-Verlag, 1994
- S. R. Finch, Mathematical Constants, Cambridge, 2003, pp. 98-104.
- G. H. Hardy, "Ramanujan, Twelve lectures on subjects suggested by his life and work", Chelsea, 1940.
- Institute of Physics, Constants - Landau-Ramanujan Constant
- Simon Plouffe, Landau Ramanujan constant
- Eric Weisstein's World of Mathematics, Ramanujan constant
- https://en.wikipedia.org/wiki/Landau-Ramanujan_constant

Sums of two squares

A prime number is a sum of two squares if and only if it is either 2 or else congruent to 1 modulo 4.

Pierre de Fermat

1607 (?) – 1665

Identity of Brahmagupta :

$$(a^2 + b^2)(c^2 + d^2) = e^2 + f^2$$

with

$$e = ac - bd, \; f = ad + bc.$$

Brahmagupta

598 – 668

Sums of two squares

If a and q are two integers, we denote by $N_{a,q}$ any integer ≥ 1 satisfying the condition

$$p \mid N_{a,q} \Longrightarrow p \equiv a \bmod q.$$

An integer $m \geq 1$ can be written as

$$m = \Phi_4(x, y) = x^2 + y^2$$

if and only if there exist integers $a \geq 0$, $N_{3,4}$ and $N_{1,4}$ such that

$$m = 2^a \, N_{3,4}^2 \, N_{1,4}.$$

Sums of two squares

If a and q are two integers, we denote by $N_{a,q}$ any integer ≥ 1 satisfying the condition

$$p \mid N_{a,q} \Longrightarrow p \equiv a \bmod q.$$

An integer $m \geq 1$ can be written as

$$m = \Phi_4(x, y) = x^2 + y^2$$

if and only if there exist integers $a \geq 0$, $N_{3,4}$ and $N_{1,4}$ such that

$$m = 2^a \, N_{3,4}^2 \, N_{1,4}.$$

Positive definite quadratic forms

Let $F \in \mathbb{Z}[X, Y]$ be a positive definite quadratic form. There exists a positive constant C_F such that, for $N \to \infty$, the number of positive integers $m \in \mathbb{Z}$, $m \leq N$ which are represented by F is asymptotically $\mathsf{C}_F N (\log N)^{-\frac{1}{2}}$.

Paul Bernays
1888 – 1977

P. Bernays, *Über die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht quadratischen Diskriminante*, Ph.D. dissertation, Georg-August-Universität, Göttingen, Germany, 1912.

http://www.ethlife.ethz.ch/archive_articles/120907_bernays_fm/

Paul Bernays (1888 – 1977)

https://www.thefamouspeople.com/profiles/paul-bernays-7244.php

- 1912, Ph.D. in mathematics, University of Göttingen, *On the analytic number theory of binary quadratic forms* (Advisor : Edmund Landau).
- 1913, Habilitation, University of Zürich, *On complex analysis and Picard's theorem*, advisor Ernst Zermelo.
- 1912 – 1917, Zürich ; work with Georg Pólya, Albert Einstein, Hermann Weyl.
- 1917 – 1933, Göttingen, with David Hilbert. Studied with Emmy Noether, Bartel Leendert van der Waerden, Gustav Herglotz.
- 1935 – 1936, Institute for Advanced Study, Princeton. Lectures on mathematical logic and axiomatic set theory.
- 1936 —, ETH Zürich.
- With David Hilbert, "Grundlagen der Mathematik" (1934 – 39) 2 vol. — Hilbert–Bernays paradox.
- Axiomatic Set Theory (1958). — Von Neumann–Bernays–Gödel set theory.

Specific binary forms

- Sums of cubes, biquadrates,...

Notice that $X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2)$

We start with the quadratic form $\Phi_3(X, Y) = X^2 + XY + Y^2$ which is the homogeneous version of the cyclotomic polynomial $\phi_3(t) = t^2 + t + 1$.
Notice that

$$\Phi_6(X, Y) = \Phi_3(X, -Y) = X^2 - XY + Y^2$$

Also

$$\Phi_8(X, Y) = X^4 + Y^4.$$

Specific binary forms

- Sums of cubes, biquadrates,...

Notice that $X^3 + Y^3 = (X+Y)(X^2 - XY + Y^2)$

We start with the quadratic form $\Phi_3(X,Y) = X^2 + XY + Y^2$ which is the homogeneous version of the cyclotomic polynomial $\phi_3(t) = t^2 + t + 1$.
Notice that

$$\Phi_6(X,Y) = \Phi_3(X,-Y) = X^2 - XY + Y^2$$

Also

$$\Phi_8(X,Y) = X^4 + Y^4.$$

Specific binary forms

- Sums of cubes, biquadrates,...

Notice that $X^3+Y^3=(X+Y)(X^2-XY+Y^2)$

We start with the quadratic form $\Phi_3(X,Y)=X^2+XY+Y^2$ which is the homogeneous version of the cyclotomic polynomial $\phi_3(t)=t^2+t+1$.
Notice that

$$\Phi_6(X,Y)=\Phi_3(X,-Y)=X^2-XY+Y^2$$

Also

$$\Phi_8(X,Y)=X^4+Y^4.$$

Specific binary forms

- Sums of cubes, biquadrates,...

Notice that $X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2)$

We start with the quadratic form $\Phi_3(X, Y) = X^2 + XY + Y^2$ which is the homogeneous version of the cyclotomic polynomial $\phi_3(t) = t^2 + t + 1$.
Notice that

$$\Phi_6(X, Y) = \Phi_3(X, -Y) = X^2 - XY + Y^2$$

Also

$$\Phi_8(X, Y) = X^4 + Y^4.$$

Specific binary forms

- Sums of cubes, biquadrates,...

Notice that $X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2)$

We start with the quadratic form $\Phi_3(X, Y) = X^2 + XY + Y^2$ which is the homogeneous version of the cyclotomic polynomial $\phi_3(t) = t^2 + t + 1$.
Notice that

$$\Phi_6(X, Y) = \Phi_3(X, -Y) = X^2 - XY + Y^2$$

Also

$$\Phi_8(X, Y) = X^4 + Y^4.$$

The quadratic form $x^2 + xy + y^2$

A prime number is represented by the quadratic form $x^2 + xy + y^2$ if and only if it is either 3 or else congruent to 1 modulo 3.

Product of two numbers represented by the quadratic form $x^2 + xy + y^2$:

$$(a^2 + ab + b^2)(c^2 + cd + d^2) = e^2 + ef + f^2$$

with

$$e = ac - bd, \; f = ad + bd + bc.$$

The quadratic cyclotomic field $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, $1 + \zeta_3 + {\zeta_3}^2 = 0$:

$$a^2 + ab + b^2 = \mathrm{Norm}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(a - \zeta_3 b).$$

The quadratic form $x^2 + xy + y^2$

A prime number is represented by the quadratic form $x^2 + xy + y^2$ if and only if it is either 3 or else congruent to 1 modulo 3.

Product of two numbers represented by the quadratic form $x^2 + xy + y^2$:

$$(a^2 + ab + b^2)(c^2 + cd + d^2) = e^2 + ef + f^2$$

with

$$e = ac - bd, \; f = ad + bd + bc.$$

The quadratic cyclotomic field $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, $1 + \zeta_3 + {\zeta_3}^2 = 0$:

$$a^2 + ab + b^2 = \mathrm{Norm}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(a - \zeta_3 b).$$

The quadratic form $x^2 + xy + y^2$

A prime number is represented by the quadratic form $x^2 + xy + y^2$ if and only if it is either 3 or else congruent to 1 modulo 3.

Product of two numbers represented by the quadratic form $x^2 + xy + y^2$:

$$(a^2 + ab + b^2)(c^2 + cd + d^2) = e^2 + ef + f^2$$

with

$$e = ac - bd, \; f = ad + bd + bc.$$

The quadratic cyclotomic field $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, $1 + \zeta_3 + {\zeta_3}^2 = 0$:

$$a^2 + ab + b^2 = \mathrm{Norm}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(a - \zeta_3 b).$$

Loeschian numbers : $m = x^2 + xy + y^2$

An integer $m \geq 1$ can be written as

$$m = \Phi_3(x, y) = \Phi_6(x, -y) = x^2 + xy + y^2$$

if and only if there exist integers $b \geq 0$, $N_{2,3}$ and $N_{1,3}$ such that

$$m = 3^b \, N_{2,3}^2 \, N_{1,3}.$$

The number of positive integers $\leq N$ which are represented by the quadratic form $x^2 + xy + y^2$ is asymptotically $\mathrm{C}_{\Phi_3} N (\log N)^{-\frac{1}{2}}$, where

$$\mathrm{C}_{\Phi_3} = \frac{1}{2^{\frac{1}{2}} 3^{\frac{1}{4}}} \cdot \prod_{p \equiv 2 \bmod 3} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}$$

Loeschian numbers : $m = x^2 + xy + y^2$

An integer $m \geq 1$ can be written as

$$m = \Phi_3(x, y) = \Phi_6(x, -y) = x^2 + xy + y^2$$

if and only if there exist integers $b \geq 0$, $N_{2,3}$ and $N_{1,3}$ such that

$$m = 3^b \, N_{2,3}^2 \, N_{1,3}.$$

The number of positive integers $\leq N$ which are represented by the quadratic form $x^2 + xy + y^2$ is asymptotically $\mathsf{C}_{\Phi_3} N (\log N)^{-\frac{1}{2}}$, where

$$\mathsf{C}_{\Phi_3} = \frac{1}{2^{\frac{1}{2}} 3^{\frac{1}{4}}} \cdot \prod_{p \equiv 2 \bmod 3} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}$$

Online Encyclopedia of Integer Sequences OEIS A301429

[OEIS A301429] `Decimal expansion of an analog of the Landau-Ramanujan constant for Loeschian numbers.`

The first decimal digits of C_{Φ_3} are

$$C_{\Phi_3} = 0.638\,909\,405\,44\ldots$$

April 17-21, 2018 : Roma (Italia)

Lecture on
Representation of integers by cyclotomic binary forms

4th Mini Symposium of the Roman Number Theory Association

http://www.rnta.eu/ms.html

Zeta function expansions of some classical constants, Feb 18 1996.

Philippe Flajolet

Ilan Vardi

Bill Allombert

$$C_{\Phi_3} = 0.63890940544534388\,22549426749282450937\,54975508029123345421\,69236570807631002764\,96582468971791125286\,64388141687519107424\ldots$$

Loeschian numbers which are sums of two squares

An integer $m \geq 1$ is simultaneously of the forms

$$m = \Phi_4(x,y) = x^2 + y^2 \text{ and } m = \Phi_3(u,v) = u^2 + uv + v^2$$

if and only if there exist integers $a,\, b \geq 0$, $N_{5,12}$, $N_{7,12}$, $N_{11,12}$ and $N_{1,12}$ such that

$$m = \left(2^a\, 3^b\, N_{5,12}\, N_{7,12}\, N_{11,12}\right)^2 N_{1,12}.$$

The number of Loeschian integers $\leq N$ which are sums of two squares is asymptotically $\beta N (\log N)^{-3/4}$, where

$$\beta = \frac{3^{\frac{1}{4}}}{2^{\frac{5}{4}}} \cdot \pi^{\frac{1}{2}} \cdot (\log(2+\sqrt{3}))^{\frac{1}{4}} \cdot \frac{1}{\Gamma(1/4)} \cdot \prod_{p \equiv 5,7,11 \bmod 12} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

Loeschian numbers which are sums of two squares

An integer $m \geq 1$ is simultaneously of the forms

$$m = \Phi_4(x, y) = x^2 + y^2 \text{ and } m = \Phi_3(u, v) = u^2 + uv + v^2$$

if and only if there exist integers $a,\, b \geq 0$, $N_{5,12}$, $N_{7,12}$, $N_{11,12}$ and $N_{1,12}$ such that

$$m = \left(2^a\, 3^b\, N_{5,12}\, N_{7,12}\, N_{11,12}\right)^2 N_{1,12}.$$

The number of Loeschian integers $\leq N$ which are sums of two squares is asymptotically $\beta N (\log N)^{-3/4}$, where

$$\beta = \frac{3^{\frac{1}{4}}}{2^{\frac{5}{4}}} \cdot \pi^{\frac{1}{2}} \cdot (\log(2+\sqrt{3}))^{\frac{1}{4}} \cdot \frac{1}{\Gamma(1/4)} \cdot \prod_{p \equiv 5,\, 7,\, 11 \bmod 12} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

OEIS A301430 $\beta = 0.302\,316\,142\,35\ldots$

[OEIS A301430] Decimal expansion of an analog of the Landau-Ramanujan constant for Loeschian numbers which are sums of two squares.

$$\beta = \frac{3^{\frac{1}{4}}}{2^{\frac{5}{4}}} \cdot \pi^{\frac{1}{2}} \cdot (\log(2+\sqrt{3}))^{\frac{1}{4}} \cdot \frac{1}{\Gamma(1/4)} \cdot \prod_{p \equiv 5,7,11 \bmod 12} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}$$

Bill Allombert

$$\begin{aligned} \beta = \; & 0.30231614235706563794 \\ & 77699004801997156024 12 \\ & 79518936964545886784 12 \\ & 88865448752410510899 48 \\ & 74678139792727085677 65 \\ & 91327259106668371358 63\ldots \end{aligned}$$

Cyclotomic polynomials

Definition by induction :

$$\phi_1(t) = t - 1, \qquad t^n - 1 = \prod_{d|n} \phi_d(t).$$

For p prime,

$$t^p - 1 = (t-1)(t^{p-1} + t^{p-2} + \cdots + t + 1) = \phi_1(t)\phi_p(t),$$

so

$$\phi_p(t) = t^{p-1} + t^{p-2} + \cdots + t + 1.$$

For instance

$$\phi_2(t) = t+1, \quad \phi_3(t) = t^2+t+1, \quad \phi_5(t) = t^4+t^3+t^2+t+1.$$

Cyclotomic polynomials

Definition by induction :

$$\phi_1(t) = t - 1, \qquad t^n - 1 = \prod_{d|n} \phi_d(t).$$

For p prime,

$$t^p - 1 = (t-1)(t^{p-1} + t^{p-2} + \cdots + t + 1) = \phi_1(t)\phi_p(t),$$

so

$$\phi_p(t) = t^{p-1} + t^{p-2} + \cdots + t + 1.$$

For instance

$$\phi_2(t) = t+1, \quad \phi_3(t) = t^2+t+1, \quad \phi_5(t) = t^4+t^3+t^2+t+1.$$

Cyclotomic polynomials

Definition by induction :

$$\phi_1(t) = t - 1, \qquad t^n - 1 = \prod_{d|n} \phi_d(t).$$

For p prime,

$$t^p - 1 = (t-1)(t^{p-1} + t^{p-2} + \cdots + t + 1) = \phi_1(t)\phi_p(t),$$

so

$$\phi_p(t) = t^{p-1} + t^{p-2} + \cdots + t + 1.$$

For instance

$$\phi_2(t) = t+1, \quad \phi_3(t) = t^2+t+1, \quad \phi_5(t) = t^4+t^3+t^2+t+1.$$

Cyclotomic polynomials

$$\phi_n(t) = \frac{t^n - 1}{\prod_{\substack{d \neq n \\ d | n}} \phi_d(t)}.$$

For instance

$$\phi_4(t) = \frac{t^4 - 1}{t^2 - 1} = t^2 + 1 = \phi_2(t^2),$$

$$\phi_6(t) = \frac{t^6 - 1}{(t^3 - 1)(t + 1)} = \frac{t^3 + 1}{t + 1} = t^2 - t + 1 = \phi_3(-t).$$

The degree of $\phi_n(t)$ is $\varphi(n)$, where φ is the Euler totient function.

Cyclotomic polynomials

$$\phi_n(t) = \frac{t^n - 1}{\prod_{\substack{d \neq n \\ d \mid n}} \phi_d(t)}.$$

For instance

$$\phi_4(t) = \frac{t^4 - 1}{t^2 - 1} = t^2 + 1 = \phi_2(t^2),$$

$$\phi_6(t) = \frac{t^6 - 1}{(t^3 - 1)(t + 1)} = \frac{t^3 + 1}{t + 1} = t^2 - t + 1 = \phi_3(-t).$$

The degree of $\phi_n(t)$ is $\varphi(n)$, where φ is the Euler totient function.

Cyclotomic polynomials

$$\phi_n(t) = \frac{t^n - 1}{\prod\limits_{\substack{d \neq n \\ d | n}} \phi_d(t)}.$$

For instance

$$\phi_4(t) = \frac{t^4 - 1}{t^2 - 1} = t^2 + 1 = \phi_2(t^2),$$

$$\phi_6(t) = \frac{t^6 - 1}{(t^3 - 1)(t + 1)} = \frac{t^3 + 1}{t + 1} = t^2 - t + 1 = \phi_3(-t).$$

The degree of $\phi_n(t)$ is $\varphi(n)$, where φ is the Euler totient function.

Cyclotomic polynomials and roots of unity

For $n \geq 1$, if ζ is a primitive n–th root of unity,

$$\phi_n(t) = \prod_{\gcd(j,n)=1} (t - \zeta^j).$$

For $n \geq 1$, $\phi_n(t)$ is the irreducible polynomial over $\mathbb{Q}$ of the primitive n–th roots of unity,

Let K be a field and let n be a positive integer. Assume that K has characteristic either 0 or else a prime number p prime to n. Then the polynomial $\phi_n(t)$ is separable over K and its roots in K are exactly the primitive n–th roots of unity which belong to K.

Cyclotomic polynomials and roots of unity

For $n \geq 1$, if ζ is a primitive n–th root of unity,

$$\phi_n(t) = \prod_{\gcd(j,n)=1} (t - \zeta^j).$$

For $n \geq 1$, $\phi_n(t)$ is the irreducible polynomial over $\mathbb{Q}$ of the primitive n–th roots of unity,

Let K be a field and let n be a positive integer. Assume that K has characteristic either 0 or else a prime number p prime to n. Then the polynomial $\phi_n(t)$ is separable over K and its roots in K are exactly the primitive n–th roots of unity which belong to K.

Cyclotomic polynomials and roots of unity

For $n \geq 1$, if ζ is a primitive n–th root of unity,

$$\phi_n(t) = \prod_{\gcd(j,n)=1} (t - \zeta^j).$$

For $n \geq 1$, $\phi_n(t)$ is the irreducible polynomial over $\mathbb{Q}$ of the primitive n–th roots of unity,

Let K be a field and let n be a positive integer. Assume that K has characteristic either 0 or else a prime number p prime to n. Then the polynomial $\phi_n(t)$ is separable over K and its roots in K are exactly the primitive n–th roots of unity which belong to K.

Properties of $\phi_n(t)$

- For $n \geq 2$ we have

$$\phi_n(t) = t^{\varphi(n)}\phi_n(1/t)$$

- Let $n = 2^{e_0}p_1^{e_1}\cdots p_r^{e_r}$ where $p_1, \ldots, p_r$ are different odd primes, $e_0 \geq 0$, $e_i \geq 1$ for $i = 1, \ldots, r$ and $r \geq 1$. Denote by R the radical of n, namely

$$R = \begin{cases} 2p_1\cdots p_r & \text{if } e_0 \geq 1, \\ p_1\cdots p_r & \text{if } e_0 = 0. \end{cases}$$

Then,

$$\phi_n(t) = \phi_R(t^{n/R}).$$

- Let $n = 2m$ with m odd ≥ 3. Then

$$\phi_n(t) = \phi_m(-t).$$

Properties of $\phi_n(t)$

- For $n \geq 2$ we have

$$\phi_n(t) = t^{\varphi(n)}\phi_n(1/t)$$

- Let $n = 2^{e_0}p_1^{e_1}\cdots p_r^{e_r}$ where $p_1, \ldots, p_r$ are different odd primes, $e_0 \geq 0$, $e_i \geq 1$ for $i = 1, \ldots, r$ and $r \geq 1$. Denote by R the radical of n, namely

$$R = \begin{cases} 2p_1\cdots p_r & \text{if } e_0 \geq 1, \\ p_1\cdots p_r & \text{if } e_0 = 0. \end{cases}$$

Then,

$$\phi_n(t) = \phi_R(t^{n/R}).$$

- Let $n = 2m$ with m odd ≥ 3. Then

$$\phi_n(t) = \phi_m(-t).$$

Properties of $\phi_n(t)$

- For $n \geq 2$ we have

$$\phi_n(t) = t^{\varphi(n)}\phi_n(1/t)$$

- Let $n = 2^{e_0}p_1^{e_1}\cdots p_r^{e_r}$ where $p_1, \ldots, p_r$ are different odd primes, $e_0 \geq 0$, $e_i \geq 1$ for $i = 1, \ldots, r$ and $r \geq 1$. Denote by R the radical of n, namely

$$R = \begin{cases} 2p_1\cdots p_r & \text{if } e_0 \geq 1, \\ p_1\cdots p_r & \text{if } e_0 = 0. \end{cases}$$

Then,

$$\phi_n(t) = \phi_R(t^{n/R}).$$

- Let $n = 2m$ with m odd ≥ 3. Then

$$\phi_n(t) = \phi_m(-t).$$

$\phi_n(1)$

For $n \geq 2$, we have $\phi_n(1) = e^{\Lambda(n)}$, where the von Mangoldt function is defined for $n \geq 1$ as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^r \text{ with } p \text{ prime and } r \geq 1\,; \\ 0 & \text{otherwise.} \end{cases}$$

In other terms we have

$$\phi_n(1) = \begin{cases} p & \text{if } n = p^r \text{ with } p \text{ prime and } r \geq 1\,; \\ 1 & \text{otherwise.} \end{cases}$$

$\phi_n(1)$

For $n \geq 2$, we have $\phi_n(1) = e^{\Lambda(n)}$, where the von Mangoldt function is defined for $n \geq 1$ as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^r \text{ with } p \text{ prime and } r \geq 1\,; \\ 0 & \text{otherwise.} \end{cases}$$

In other terms we have

$$\phi_n(1) = \begin{cases} p & \text{if } n = p^r \text{ with } p \text{ prime and } r \geq 1\,; \\ 1 & \text{otherwise.} \end{cases}$$

$\phi_n(-1)$

For $n \geq 3$,

$$\phi_n(-1) = \begin{cases} 1 & \text{if } n \text{ is odd}\,; \\ \phi_{n/2}(1) & \text{if } n \text{ is even.} \end{cases}$$

In other terms, for $n \geq 3$,

$$\phi_n(-1) = \begin{cases} p & \text{if } n = 2p^r \text{ with } p \text{ a prime and } r \geq 1\,; \\ 1 & \text{otherwise.} \end{cases}$$

Hence $\phi_n(-1) = 1$ when n is odd or when $n = 2m$ where m has at least two distinct prime divisors.

$\phi_n(-1)$

For $n \geq 3$,

$$\phi_n(-1) = \begin{cases} 1 & \text{if } n \text{ is odd ;} \\ \phi_{n/2}(1) & \text{if } n \text{ is even.} \end{cases}$$

In other terms, for $n \geq 3$,

$$\phi_n(-1) = \begin{cases} p & \text{if } n = 2p^r \text{ with } p \text{ a prime and } r \geq 1 \text{ ;} \\ 1 & \text{otherwise.} \end{cases}$$

Hence $\phi_n(-1) = 1$ when n is odd or when $n = 2m$ where m has at least two distinct prime divisors.

Lower bound for $\phi_n(t)$

For $n \geq 3$, the polynomial $\phi_n(t)$ has real coefficients and no real root, hence it takes only positive values (and its degree $\varphi(n)$ is even).

For $n \geq 3$ and $t \in \mathbb{R}$, we have

$$\phi_n(t) \geq 2^{-\varphi(n)}.$$

Consequence : from

$$\phi_n(t) = t^{\varphi(n)} \phi_n(1/t)$$

we deduce, for $n \geq 3$ and $t \in \mathbb{R}$,

$$\phi_n(t) \geq 2^{-\varphi(n)} \max\{1, |t|\}^{\varphi(n)}.$$

Lower bound for $\phi_n(t)$

For $n \geq 3$, the polynomial $\phi_n(t)$ has real coefficients and no real root, hence it takes only positive values (and its degree $\varphi(n)$ is even).

For $n \geq 3$ and $t \in \mathbb{R}$, we have

$$\phi_n(t) \geq 2^{-\varphi(n)}.$$

Consequence : from

$$\phi_n(t) = t^{\varphi(n)} \phi_n(1/t)$$

we deduce, for $n \geq 3$ and $t \in \mathbb{R}$,

$$\phi_n(t) \geq 2^{-\varphi(n)} \max\{1, |t|\}^{\varphi(n)}.$$

Lower bound for $\phi_n(t)$

For $n \geq 3$, the polynomial $\phi_n(t)$ has real coefficients and no real root, hence it takes only positive values (and its degree $\varphi(n)$ is even).

For $n \geq 3$ and $t \in \mathbb{R}$, we have

$$\phi_n(t) \geq 2^{-\varphi(n)}.$$

Consequence : from

$$\phi_n(t) = t^{\varphi(n)} \phi_n(1/t)$$

we deduce, for $n \geq 3$ and $t \in \mathbb{R}$,

$$\phi_n(t) \geq 2^{-\varphi(n)} \max\{1, |t|\}^{\varphi(n)}.$$

$\phi_n(t) \geq 2^{-\varphi(n)}$ for $n \geq 3$ and $t \in \mathbb{R}$

Proof.

Let ζ_n be a primitive n-th root of unity in $\mathbb{C}$;

$$\phi_n(t) = \mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(t - \zeta_n) = \prod_{\sigma}(t - \sigma(\zeta_n)),$$

where σ runs over the embeddings $\mathbb{Q}(\zeta_n) \to \mathbb{C}$. We have

$$|t - \sigma(\zeta_n)| \geq |\Im\mathrm{m}(\sigma(\zeta_n))| > 0,$$

$$(2i)\Im\mathrm{m}(\sigma(\zeta_n)) = \sigma(\zeta_n) - \overline{\sigma(\zeta_n)} = \sigma(\zeta_n - \overline{\zeta_n}).$$

Now $(2i)\Im\mathrm{m}(\zeta_n) = \zeta_n - \overline{\zeta_n} \in \mathbb{Q}(\zeta_n)$ is an algebraic integer :

$$2^{\varphi(n)}\phi_n(t) \geq |\mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}((2i)\Im\mathrm{m}(\zeta_n))| \geq 1.$$

$\phi_n(t) \geq 2^{-\varphi(n)}$ for $n \geq 3$ and $t \in \mathbb{R}$

Proof.

Let ζ_n be a primitive n-th root of unity in $\mathbb{C}$;

$$\phi_n(t) = \mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(t - \zeta_n) = \prod_\sigma (t - \sigma(\zeta_n)),$$

where σ runs over the embeddings $\mathbb{Q}(\zeta_n) \to \mathbb{C}$. We have

$$|t - \sigma(\zeta_n)| \geq |\Im\mathrm{m}(\sigma(\zeta_n))| > 0,$$

$$(2i)\Im\mathrm{m}(\sigma(\zeta_n)) = \sigma(\zeta_n) - \overline{\sigma(\zeta_n)} = \sigma(\zeta_n - \overline{\zeta_n}).$$

Now $(2i)\Im\mathrm{m}(\zeta_n) = \zeta_n - \overline{\zeta_n} \in \mathbb{Q}(\zeta_n)$ is an algebraic integer :

$$2^{\varphi(n)}\phi_n(t) \geq |\mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}((2i)\Im\mathrm{m}(\zeta_n))| \geq 1.$$

$$\phi_n(t) \geq 2^{-\varphi(n)} \text{ for } n \geq 3 \text{ and } t \in \mathbb{R}$$

Proof.

Let ζ_n be a primitive n-th root of unity in $\mathbb{C}$;

$$\phi_n(t) = \mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(t - \zeta_n) = \prod_\sigma (t - \sigma(\zeta_n)),$$

where σ runs over the embeddings $\mathbb{Q}(\zeta_n) \to \mathbb{C}$. We have

$$|t - \sigma(\zeta_n)| \geq |\Im\mathrm{m}(\sigma(\zeta_n))| > 0,$$

$$(2i)\Im\mathrm{m}(\sigma(\zeta_n)) = \sigma(\zeta_n) - \overline{\sigma(\zeta_n)} = \sigma(\zeta_n - \overline{\zeta_n}).$$

Now $(2i)\Im\mathrm{m}(\zeta_n) = \zeta_n - \overline{\zeta_n} \in \mathbb{Q}(\zeta_n)$ is an algebraic integer :

$$2^{\varphi(n)}\phi_n(t) \geq |\mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}((2i)\Im\mathrm{m}(\zeta_n))| \geq 1.$$

Generalization to CM fields

K. Győry

L. Lovász

K. Győry & L. Lovász, *Representation of integers by norm forms II*, Publ. Math. Debrecen **17**, 173–181, (1970).
K. Győry, *Représentation des nombres entiers par des formes binaires*, Publ. Math. Debrecen **24**, 363–375, (1977).

Refinement (FLW)

Let $c_n = \inf_{t\in\mathbb{R}} \phi_n(t)$.
Refinement of the lower bound $c_n \geq 2^{-\varphi(n)}$:

For $n \geq 3$

$$c_n \geq \left(\frac{\sqrt{3}}{2}\right)^{\varphi(n)} .$$

Equality for $n = 3$ and $n = 6$.

For n a power of 2, $c_n = 1$.
Otherwise, if n has r distinct primes $p_1, \ldots, p_r$ with p_1 the smallest, then

$$c_n = c_{p_1 \cdots p_r} \geq {p_1}^{-2^{r-2}} .$$

Refinement (FLW)

Let $c_n = \inf_{t\in\mathbb{R}} \phi_n(t)$.
Refinement of the lower bound $c_n \geq 2^{-\varphi(n)}$:

For $n \geq 3$

$$c_n \geq \left(\frac{\sqrt{3}}{2}\right)^{\varphi(n)}.$$

Equality for $n = 3$ and $n = 6$.

For n a power of 2, $c_n = 1$.
Otherwise, if n has r distinct primes $p_1, \ldots, p_r$ with p_1 the smallest, then

$$c_n = c_{p_1\cdots p_r} \geq {p_1}^{-2^{r-2}}.$$

Refinement (FLW)

Let $c_n = \inf_{t\in\mathbb{R}} \phi_n(t)$.
Refinement of the lower bound $c_n \geq 2^{-\varphi(n)}$:

For $n \geq 3$

$$c_n \geq \left(\frac{\sqrt{3}}{2}\right)^{\varphi(n)} .$$

Equality for $n = 3$ and $n = 6$.

For n a power of 2, $c_n = 1$.
Otherwise, if n has r distinct primes $p_1, \dots, p_r$ with p_1 the smallest, then

$$c_n = c_{p_1 \cdots p_r} \geq {p_1}^{-2^{r-2}} .$$

The cyclotomic binary forms

For $n \geq 2$, define

$$\Phi_n(X,Y) = Y^{\varphi(n)}\phi_n(X/Y).$$

This is a binary form in $\mathbb{Z}[X,Y]$ of degree $\varphi(n)$.

Consequence of the lower bound $c_n \geq 2^{-\varphi(n)}$:
for $n \geq 3$ and $(x,y) \in \mathbb{Z}^2$,

$$\Phi_n(x,y) \geq 2^{-\varphi(n)} \max\{|x|,|y|\}^{\varphi(n)}.$$

Therefore, if $\Phi_n(x,y) = m$, then

$$\max\{|x|,|y|\} \leq 2m^{1/\varphi(n)}.$$

If $\max\{|x|,|y|\} \geq 3$, then n is bounded :

$$\varphi(n) \leq \frac{\log m}{\log(3/2)}.$$

The cyclotomic binary forms

For $n \geq 2$, define

$$\Phi_n(X,Y) = Y^{\varphi(n)}\phi_n(X/Y).$$

This is a binary form in $\mathbb{Z}[X,Y]$ of degree $\varphi(n)$.
Consequence of the lower bound $c_n \geq 2^{-\varphi(n)}$:
for $n \geq 3$ and $(x,y) \in \mathbb{Z}^2$,

$$\Phi_n(x,y) \geq 2^{-\varphi(n)} \max\{|x|,|y|\}^{\varphi(n)}.$$

Therefore, if $\Phi_n(x,y) = m$, then

$$\max\{|x|,|y|\} \leq 2m^{1/\varphi(n)}.$$

If $\max\{|x|,|y|\} \geq 3$, then n is bounded :

$$\varphi(n) \leq \frac{\log m}{\log(3/2)}.$$

The cyclotomic binary forms

For $n \geq 2$, define

$$\Phi_n(X,Y) = Y^{\varphi(n)} \phi_n(X/Y).$$

This is a binary form in $\mathbb{Z}[X,Y]$ of degree $\varphi(n)$.
Consequence of the lower bound $c_n \geq 2^{-\varphi(n)}$:
for $n \geq 3$ and $(x,y) \in \mathbb{Z}^2$,

$$\Phi_n(x,y) \geq 2^{-\varphi(n)} \max\{|x|,|y|\}^{\varphi(n)}.$$

Therefore, if $\Phi_n(x,y) = m$, then

$$\max\{|x|,|y|\} \leq 2m^{1/\varphi(n)}.$$

If $\max\{|x|,|y|\} \geq 3$, then n is bounded :

$$\varphi(n) \leq \frac{\log m}{\log(3/2)}.$$

Binary cyclotomic forms (EF–CL–MW 2018)

Let m be a positive integer and let n, x, y be rational integers satisfying $n \geq 3$, $\max\{|x|, |y|\} \geq 2$ and $\Phi_n(x, y) = m$. Then

$$\max\{|x|, |y|\} \leq \frac{2}{\sqrt{3}} m^{1/\varphi(n)}, \quad \text{hence} \quad \varphi(n) \leq \frac{2}{\log 3} \log m.$$

These estimates are optimal, since for $\ell \geq 1$,

$$\Phi_3(\ell, -2\ell) = 3\ell^2.$$

If we assume $\varphi(n) > 2$, namely $\varphi(n) \geq 4$, then

$$\varphi(n) \leq \frac{4}{\log 11} \log m$$

which is best possible since $\Phi_5(1, -2) = 11$.

Binary cyclotomic forms (EF–CL–MW 2018)

Let m be a positive integer and let n, x, y be rational integers satisfying $n \geq 3$, $\max\{|x|, |y|\} \geq 2$ and $\Phi_n(x, y) = m$. Then

$$\max\{|x|, |y|\} \leq \frac{2}{\sqrt{3}} m^{1/\varphi(n)}, \quad \text{hence} \quad \varphi(n) \leq \frac{2}{\log 3} \log m.$$

These estimates are optimal, since for $\ell \geq 1$,

$$\Phi_3(\ell, -2\ell) = 3\ell^2.$$

If we assume $\varphi(n) > 2$, namely $\varphi(n) \geq 4$, then

$$\varphi(n) \leq \frac{4}{\log 11} \log m$$

which is best possible since $\Phi_5(1, -2) = 11$.

Binary cyclotomic forms (EF–CL–MW 2018)

Let m be a positive integer and let n, x, y be rational integers satisfying $n \geq 3$, $\max\{|x|, |y|\} \geq 2$ and $\Phi_n(x, y) = m$. Then

$$\max\{|x|, |y|\} \leq \frac{2}{\sqrt{3}} m^{1/\varphi(n)}, \quad \text{hence} \quad \varphi(n) \leq \frac{2}{\log 3} \log m.$$

These estimates are optimal, since for $\ell \geq 1$,

$$\Phi_3(\ell, -2\ell) = 3\ell^2.$$

If we assume $\varphi(n) > 2$, namely $\varphi(n) \geq 4$, then

$$\varphi(n) \leq \frac{4}{\log 11} \log m$$

which is best possible since $\Phi_5(1, -2) = 11$.

The sequence $(a_m)_{m\geq 1}$

For each integer $m \geq 1$, the set

$$\{(n,x,y) \in \mathbb{N}\times\mathbb{Z}^2 \mid n \geq 3,\ \max\{|x|,|y|\} \geq 2,\ \Phi_n(x,y) = m\}$$

is finite. Let a_m the number of its elements.

The sequence of integers $m \geq 1$ such that $a_m \geq 1$ starts with the following values of a_m

m	3	4	5	7	8	9	10	11	12	13	16	17
a_m	8	16	8	24	4	16	8	8	12	40	40	16

The sequence $(a_m)_{m\geq 1}$

For each integer $m \geq 1$, the set

$$\{(n,x,y) \in \mathbb{N}\times\mathbb{Z}^2 \ \mid \ n \geq 3,\ \max\{|x|,|y|\} \geq 2,\ \Phi_n(x,y) = m\}$$

is finite. Let a_m the number of its elements.

The sequence of integers $m \geq 1$ such that $a_m \geq 1$ starts with the following values of a_m

m	3	4	5	7	8	9	10	11	12	13	16	17
a_m	8	16	8	24	4	16	8	8	12	40	40	16

OEIS A299214

https://oeis.org/A299214
Number of representations of integers by cyclotomic binary forms.

The sequence $(a_m)_{m\geq 1}$ starts with
$0, 0, 8, 16, 8, 0, 24, 4, 16, 8, 8, 12, 40, 0, 0, 40, 16, 4, 24, 8, 24,$
$0, 0, 0, 24, 8, 12, 24, 8, 0, 32, 8, 0, 8, 0, 16, 32, 0, 24, 8, 8, 0, 32,$
$0, 8, 0, 0, 12, 40, 12, 0, 32, 8, 0, 8, 0, 32, 8, 0, 0, 48, 0, 24, 40,$
$16, 0, 24, 8, 0, 0, 0, 4, 48, 8, 12, 24, \ \ldots$

OEIS A296095

https://oeis.org/A296095
Integers represented by cyclotomic binary forms.

$a_m \neq 0$ for $m =$
$3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 21, 25, 26, 27, 28, 29, 31, 32, 34, 36, 37, 39, 40, 41, 43, 45, 48, 49, 50, 52, 53, 55, 57, 58, 61, 63, 64, 65, 67, 68, 72, 73, 74, 75, 76, 79, 80, 81, 82, 84, 85, 89, 90, 91, 93, 97, 98, 100, 101, 103, 104, 106, 108, 109, 111, 112, 113, 116, 117, 121, 122, \ldots$

OEIS A293654

https://oeis.org/A293654
Integers not represented by cyclotomic binary forms.

$a_m = 0$ for $m =$
$1, 2, 6, 14, 15, 22, 23, 24, 30, 33, 35, 38, 42, 44, 46, 47, 51, 54,$
$56, 59, 60, 62, 66, 69, 70, 71, 77, 78, 83, 86, 87, 88, 92, 94, 95,$
$96, 99, 102, 105, 107, 110, 114, 115, 118, 119, 120, 123, 126,$
$131, 132, 134, 135, 138, 140, 141, 142, 143, 150, \ \ldots$

Numbers represented by a cyclotomic binary form of degree ≥ 2

For $N \geq 1$, the number of $m \leq N$ for which there exists $n \geq 3$ and $(x, y) \in \mathbb{Z}^2$ with $\max(|x|, |y|) \geq 2$ and $m = \Phi_n(x, y)$, is asymptotically

$$(\mathsf{C}_{\Phi_4} + \mathsf{C}_{\Phi_3}) \frac{N}{(\log N)^{\frac{1}{2}}} - \beta \frac{N}{(\log N)^{\frac{3}{4}}} + O\left(\frac{N}{(\log N)^{\frac{3}{2}}} \right)$$

as $N \to \infty$.

$\mathsf{C}_{\Phi_4} + \mathsf{C}_{\Phi_3} = 1.403\,133\,059\,034\ \ldots \qquad \beta = 0.302\,316\,142\,35\ldots$

Etienne Fouvry, Claude Levesque & M.W.; *Representation of integers by cyclotomic binary forms.* Acta Arithmetica, **184**.1 (2018), 67 - 86.

Dedicated to Rob Tijdeman. arXiv: 712.09019 [math.NT]

Numbers represented by a cyclotomic binary form of degree ≥ 2

For $N \geq 1$, the number of $m \leq N$ for which there exists $n \geq 3$ and $(x, y) \in \mathbb{Z}^2$ with $\max(|x|, |y|) \geq 2$ and $m = \Phi_n(x, y)$, is asymptotically

$$(\mathsf{C}_{\Phi_4} + \mathsf{C}_{\Phi_3}) \frac{N}{(\log N)^{\frac{1}{2}}} - \beta \frac{N}{(\log N)^{\frac{3}{4}}} + O\left(\frac{N}{(\log N)^{\frac{3}{2}}} \right)$$

as $N \to \infty$.

$\mathsf{C}_{\Phi_4} + \mathsf{C}_{\Phi_3} = 1.403\,133\,059\,034\ \ldots \qquad \beta = 0.302\,316\,142\,35\ldots$

Etienne Fouvry, Claude Levesque & M.W.; *Representation of integers by cyclotomic binary forms.* Acta Arithmetica, **184**.1 (2018), 67 - 86.

Dedicated to Rob Tijdeman. arXiv: 712.09019 [math.NT]

Numbers represented by a cyclotomic binary form of degree ≥ 2

For $N \geq 1$, the number of $m \leq N$ for which there exists $n \geq 3$ and $(x, y) \in \mathbb{Z}^2$ with $\max(|x|, |y|) \geq 2$ and $m = \Phi_n(x, y)$, is asymptotically

$$(\mathsf{C}_{\Phi_4} + \mathsf{C}_{\Phi_3}) \frac{N}{(\log N)^{\frac{1}{2}}} - \beta \frac{N}{(\log N)^{\frac{3}{4}}} + O\left(\frac{N}{(\log N)^{\frac{3}{2}}}\right)$$

as $N \to \infty$.

$\mathsf{C}_{\Phi_4} + \mathsf{C}_{\Phi_3} = 1.403\,133\,059\,034\ \dots \qquad \beta = 0.302\,316\,142\,35\dots$

Etienne Fouvry, Claude Levesque & M.W. ; *Representation of integers by cyclotomic binary forms.* Acta Arithmetica, **184**.1 (2018), 67 - 86.

Dedicated to Rob Tijdeman. `arXiv: 712.09019 [math.NT]`

Higher degree

The situation for quadratic forms of degree ≥ 3 is different for several reasons.

• If a positive integer m is represented by a positive definite quadratic form, it usually has many such representations ; while if a positive integer m is represented by an irreducible binary form of degree $d \geq 3$, it usually has few such representations.

• If F is a positive definite quadratic form, the number of (x, y) with $F(x, y) \leq N$ is asymptotically a constant times N, but the number of $F(x, y)$ is much smaller.

• If F is an irreducible binary form of degree $d \geq 3$, the number of (x, y) with $F(x, y) \leq N$ is asymptotically a constant times $N^{\frac{2}{d}}$, the number of $F(x, y)$ is also asymptotically a constant times $N^{\frac{2}{d}}$.

Higher degree

The situation for quadratic forms of degree ≥ 3 is different for several reasons.

• If a positive integer m is represented by a positive definite quadratic form, it usually has many such representations ; while if a positive integer m is represented by an irreducible binary form of degree $d \geq 3$, it usually has few such representations.

• If F is a positive definite quadratic form, the number of (x, y) with $F(x, y) \leq N$ is asymptotically a constant times N, but the number of $F(x, y)$ is much smaller.

• If F is an irreducible binary form of degree $d \geq 3$, the number of (x, y) with $F(x, y) \leq N$ is asymptotically a constant times $N^{\frac{2}{d}}$, the number of $F(x, y)$ is also asymptotically a constant times $N^{\frac{2}{d}}$.

Higher degree

The situation for quadratic forms of degree ≥ 3 is different for several reasons.

• If a positive integer m is represented by a positive definite quadratic form, it usually has many such representations ; while if a positive integer m is represented by an irreducible binary form of degree $d \geq 3$, it usually has few such representations.

• If F is a positive definite quadratic form, the number of (x, y) with $F(x, y) \leq N$ is asymptotically a constant times N, but the number of $F(x, y)$ is much smaller.

• If F is an irreducible binary form of degree $d \geq 3$, the number of (x, y) with $F(x, y) \leq N$ is asymptotically a constant times $N^{\frac{2}{d}}$, the number of $F(x, y)$ is also asymptotically a constant times $N^{\frac{2}{d}}$.

Higher degree

A quadratic form has infinitely many automorphisms, an irreducible binary form of higher degree has a finite group of automorphisms.

Stanley Yao Xiao

S. Yao Xiao, *On the representation of integers by binary quadratic forms.*
arXiv:1704.00221

Higher degree

A quadratic form has infinitely many automorphisms, an irreducible binary form of higher degree has a finite group of automorphisms.

Stanley Yao Xiao

S. Yao Xiao, *On the representation of integers by binary quadratic forms*.
arXiv:1704.00221

Sums of k–th powers

If a positive integer m is a sum of two squares, there are many such representations.
Indeed, the number of (x, y) in $\mathbb{Z} \times \mathbb{Z}$ with $x^2 + y^2 \leq N$ is asymptotic to πN, while the number of values $\leq N$ taken by the quadratic form Φ_4 is asymptotic to $\mathsf{C}_{\Phi_4} N/\sqrt{\log N}$ where C_{Φ_4} is the Landau–Ramanujan constant. Hence Φ_4 takes each of these values with a high multiplicity, on the average $(\pi/\mathsf{C}_{\Phi_4})\sqrt{\log N}$.
On the opposite, given an integer $k \geq 3$, that a positive integer is a sum of two k–th powers in more than one way (not counting symmetries) is

- rare for $k = 3$,
- extremely rare for $k = 4$,
- maybe impossible for $k \geq 5$.

1729 : the taxicab number

The smallest positive integer which is sum of two cubes in two essentially different ways :

$$1729 = 10^3 + 9^3 = 12^3 + 1^3.$$

Godfrey Harold Hardy
1877–1947

Srinivasa Ramanujan
1887 – 1920

1657 : Frénicle de Bessy (1605 ? – 1675)

The sequence of Taxicab numbers

[OEIS A001235] `Taxi-cab numbers: sums of` 2 `cubes in more than` 1 `way.`

$$1729 = 10^3 + 9^3 = 12^3 + 1^3, \quad 4104 = 2^3 + 16^3 = 9^3 + 15^3, \ldots$$

$1729, 4104, 13832, 20683, 32832, 39312, 40033, 46683, 64232,$
$65728, 110656, 110808, 134379, 149389, 165464, 171288, 195841,$
$216027, 216125, 262656, 314496, 320264, 327763, 373464, 402597,$
$439101, 443889, 513000, 513856, 515375, 525824, 558441, 593047, \ldots$

If n is in this sequence, then nk^3 also, hence this sequence is infinite.

The sequence of Taxicab numbers

[OEIS A001235] `Taxi-cab numbers: sums of` 2 `cubes in more than` 1 `way.`

$$1729 = 10^3 + 9^3 = 12^3 + 1^3, \quad 4104 = 2^3 + 16^3 = 9^3 + 15^3, \ldots$$

$$1729, 4104, 13832, 20683, 32832, 39312, 40033, 46683, 64232, 65728, 110656, 110808, 134379, 149389, 165464, 171288, 195841, 216027, 216125, 262656, 314496, 320264, 327763, 373464, 402597, 439101, 443889, 513000, 513856, 515375, 525824, 558441, 593047, \ldots$$

If n is in this sequence, then nk^3 also, hence this sequence is infinite.

The sequence of Taxicab numbers

[OEIS A001235] `Taxi-cab numbers: sums of` 2 `cubes in more than` 1 `way.`

$$1729 = 10^3 + 9^3 = 12^3 + 1^3, \quad 4104 = 2^3 + 16^3 = 9^3 + 15^3, \ldots$$

$$1729, 4104, 13832, 20683, 32832, 39312, 40033, 46683, 64232, 65728, 110656, 110808, 134379, 149389, 165464, 171288, 195841, 216027, 216125, 262656, 314496, 320264, 327763, 373464, 402597, 439101, 443889, 513000, 513856, 515375, 525824, 558441, 593047, \ldots$$

If n is in this sequence, then nk^3 also, hence this sequence is infinite.

Another sequence of Taxicab numbers (Fermat)

[OEIS A011541] Hardy-Ramanujan numbers: the smallest number that is the sum of 2 positive integral cubes in n ways.

http://mathworld.wolfram.com/TaxicabNumber.html

$T_a(1) = 2,$

$T_a(2) = 1729 = 10^3 + 9^3 = 12^3 + 1^3,$

$T_a(3) = 87\,539\,319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3,$

$T_a(4) = 6\,963\,472\,309\,248 = 2421^3 + 19\,083^3 = 5436^3 + 18\,948^3 = 10\,200^3 + 18\,072^3 = 13\,322^3 + 16\,630^3,$

$T_a(5) = 48\,988\,659\,276\,962\,496,$

$T_a(6) = 24\,153\,319\,581\,254\,312\,065\,344.$

Another sequence of Taxicab numbers (Fermat)

[OEIS A011541] Hardy-Ramanujan numbers: the smallest number that is the sum of 2 positive integral cubes in n ways.

http://mathworld.wolfram.com/TaxicabNumber.html

$T_a(1) = 2,$

$T_a(2) = 1729 = 10^3 + 9^3 = 12^3 + 1^3,$

$T_a(3) = 87\,539\,319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3,$

$T_a(4) = 6\,963\,472\,309\,248 = 2421^3 + 19\,083^3 = 5436^3 + 18\,948^3 = 10\,200^3 + 18\,072^3 = 13\,322^3 + 16\,630^3,$

$T_a(5) = 48\,988\,659\,276\,962\,496,$

$T_a(6) = 24\,153\,319\,581\,254\,312\,065\,344.$

Another sequence of Taxicab numbers (Fermat)

[OEIS A011541] Hardy-Ramanujan numbers: the smallest number that is the sum of 2 positive integral cubes in n ways.

http://mathworld.wolfram.com/TaxicabNumber.html

$T_a(1) = 2,$

$T_a(2) = 1729 = 10^3 + 9^3 = 12^3 + 1^3,$

$T_a(3) = 87\,539\,319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3,$

$T_a(4) = 6\,963\,472\,309\,248 = 2421^3 + 19\,083^3 = 5436^3 + 18\,948^3 = 10\,200^3 + 18\,072^3 = 13\,322^3 + 16\,630^3,$

$T_a(5) = 48\,988\,659\,276\,962\,496,$

$T_a(6) = 24\,153\,319\,581\,254\,312\,065\,344.$

Another sequence of Taxicab numbers (Fermat)

[OEIS A011541] Hardy-Ramanujan numbers: the smallest number that is the sum of 2 positive integral cubes in n ways.
http://mathworld.wolfram.com/TaxicabNumber.html
$T_a(1) = 2$,

$T_a(2) = 1729 = 10^3 + 9^3 = 12^3 + 1^3$,

$T_a(3) = 87\,539\,319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3$,

$T_a(4) = 6\,963\,472\,309\,248 = 2421^3 + 19\,083^3 = 5436^3 + 18\,948^3 = 10\,200^3 + 18\,072^3 = 13\,322^3 + 16\,630^3$,

$T_a(5) = 48\,988\,659\,276\,962\,496$,
$T_a(6) = 24\,153\,319\,581\,254\,312\,065\,344$.

Another sequence of Taxicab numbers (Fermat)

[OEIS A011541] Hardy-Ramanujan numbers: the smallest number that is the sum of 2 positive integral cubes in n ways.

http://mathworld.wolfram.com/TaxicabNumber.html

$T_a(1) = 2,$

$T_a(2) = 1729 = 10^3 + 9^3 = 12^3 + 1^3,$

$T_a(3) = 87\,539\,319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3,$

$T_a(4) = 6\,963\,472\,309\,248 = 2421^3 + 19\,083^3 = 5436^3 + 18\,948^3 = 10\,200^3 + 18\,072^3 = 13\,322^3 + 16\,630^3,$

$T_a(5) = 48\,988\,659\,276\,962\,496,$
$T_a(6) = 24\,153\,319\,581\,254\,312\,065\,344.$

Hardy and Wright, An Introduction of Theory of Numbers

Fermat proved that numbers expressible as a sum of two positive integral cubes in n different ways exist for any n.

Pierre de Fermat

1607 (?) – 1665

2003 : C. S. Calude, E. Calude and M. J. Dinneen,
With high probability,

$$T_a(6) = 24153319581254312065344.$$

Cubefree taxicab numbers

$$15\,170\,835\,645 = 517^3 + 2468^3 = 709^3 + 2456^3 = 1733^3 + 2152^3.$$

The smallest cubefree taxicab number with three representations was discovered by Paul Vojta (unpublished) in 1981 while he was a graduate student.

Paul Vojta

Cubefree taxicab numbers

Stuart Gascoigne and Duncan Moore (2003) :
$1\,801\,049\,058\,342\,701\,083 = 92227^3 + 1216500^3 = 136635^3 + 1216102^3 = 341995^3 + 1207602^3 = 600259^3 + 1165884^3$

[OEIS A080642] Cubefree taxicab numbers: the smallest cubefree number that is the sum of 2 cubes in n ways.

https://en.wikipedia.org/wiki/Taxicab_number

Cubefree taxicab numbers

Stuart Gascoigne and Duncan Moore (2003) :
$1\,801\,049\,058\,342\,701\,083 = 92227^3 + 1216500^3 = 136635^3 + 1216102^3 = 341995^3 + 1207602^3 = 600259^3 + 1165884^3$

[OEIS A080642] Cubefree taxicab numbers: the smallest cubefree number that is the sum of 2 cubes in n ways.

https://en.wikipedia.org/wiki/Taxicab_number

Taxicabs and Sums of Two Cubes

If the sequence (a_n) of cubefree taxicab numbers with n representations is infinite, then the Mordell-Weil rank of the elliptic curve $x^3 + y^3 = a_n$ tends to infinity with n.

Joseph Silverman

J. H. Silverman, Taxicabs and Sums of Two Cubes, Amer. Math. Monthly, **100** (1993), 331-340.

$$635\,318\,657 = 158^4 + 59^4 = 134^4 + 133^4.$$

Leonhard Euler

1707 – 1783

The smallest integer represented by $x^4 + y^4$ in two essentially different ways was found by Euler, it is $635\,318\,657 =$ $41 \times 113 \times 241 \times 569$.

[OEIS A216284] Number of solutions to the equation $x^4 + y^4 = n$ with $x \geq y > 0$.

An infinite family with one parameter is known for non trivial solutions to $x_1^4 + x_2^4 = x_3^4 + x_4^4$.

http://mathworld.wolfram.com/DiophantineEquation4thPowers.html

Sums of k–th powers

One conjectures that given $k \geq 5$, if an integer can be written as $x^k + y^k$, there is essentially a unique such representation. But there is no value of k for which this has been proved.

Binary cyclotomic forms of higher degree

The situation for binary cyclotomic forms is different when the degree is 2 or when it is > 2 also for the following reason.

A necessary and sufficient condition for a number m to be represented by one of the quadratic forms Φ_3, Φ_4, is given by a congruence.

By contrast, consider the quartic binary form $\Phi_8(X, Y) = X^4 + Y^4$. On the one hand, an odd integer represented by Φ_8 is of the form

$$N_{1,8}(N_{3,8}N_{5,8}N_{7,8})^4.$$

On the other hand, there are many integers of this form which are not represented by Φ_8.

[OEIS A004831] Numbers that are the sum of at most 2 nonzero 4th powers.

$$0, 1, 2, 16, 17, 32, 81, 82, 97, 162, 256, 257, 272, 337, 512, 625, \ldots$$

Binary cyclotomic forms of higher degree

The situation for binary cyclotomic forms is different when the degree is 2 or when it is > 2 also for the following reason. A necessary and sufficient condition for a number m to be represented by one of the quadratic forms Φ_3, Φ_4, is given by a congruence.

By contrast, consider the quartic binary form $\Phi_8(X,Y) = X^4 + Y^4$. On the one hand, an odd integer represented by Φ_8 is of the form

$$N_{1,8}(N_{3,8}N_{5,8}N_{7,8})^4.$$

On the other hand, there are many integers of this form which are not represented by Φ_8.

[OEIS A004831] Numbers that are the sum of at most 2 nonzero 4th powers.

$$0, 1, 2, 16, 17, 32, 81, 82, 97, 162, 256, 257, 272, 337, 512, 625, \dots$$

Binary cyclotomic forms of higher degree

The situation for binary cyclotomic forms is different when the degree is 2 or when it is > 2 also for the following reason.

A necessary and sufficient condition for a number m to be represented by one of the quadratic forms Φ_3, Φ_4, is given by a congruence.

By contrast, consider the quartic binary form $\Phi_8(X,Y) = X^4 + Y^4$. On the one hand, an odd integer represented by Φ_8 is of the form

$$N_{1,8}(N_{3,8}N_{5,8}N_{7,8})^4.$$

On the other hand, there are many integers of this form which are not represented by Φ_8.

[OEIS A004831] Numbers that are the sum of at most 2 nonzero 4th powers.

$$0, 1, 2, 16, 17, 32, 81, 82, 97, 162, 256, 257, 272, 337, 512, 625, \ldots$$

Binary cyclotomic forms of higher degree

The situation for binary cyclotomic forms is different when the degree is 2 or when it is > 2 also for the following reason.
A necessary and sufficient condition for a number m to be represented by one of the quadratic forms Φ_3, Φ_4, is given by a congruence.
By contrast, consider the quartic binary form
$\Phi_8(X,Y) = X^4 + Y^4$. On the one hand, an odd integer represented by Φ_8 is of the form

$$N_{1,8}(N_{3,8}N_{5,8}N_{7,8})^4.$$

On the other hand, there are many integers of this form which are not represented by Φ_8.

[OEIS A004831] `Numbers that are the sum of at most` 2 `nonzero` 4`th powers.`

$$0, 1, 2, 16, 17, 32, 81, 82, 97, 162, 256, 257, 272, 337, 512, 625, \ldots$$

Quartan primes

[OEIS A002645] Quartan primes: primes of the form $x^4 + y^4$, $x > 0$, $y > 0$.

The list of prime numbers represented by Φ_8 start with $2, 17, 97, 257, 337, 641, 881, 1297, 2417, 2657, 3697, 4177, 4721, 6577, 10657, 12401, 14657, 14897, 15937, 16561, 28817, 38561, 39041, 49297, 54721, 65537, 65617, 66161, 66977, 80177, 83537, 83777, 89041, 105601, 107377, 119617, \ldots$

It is not known whether this list is finite or not.

The largest known quartan prime is currently the largest known generalized Fermat prime: The $1\,353\,265$-digit $(145\,310^{65\,536})^4 + 1^4$.

Quartan primes

[OEIS A002645] Quartan primes: primes of the form $x^4 + y^4$, $x > 0$, $y > 0$.

The list of prime numbers represented by Φ_8 start with
$2, 17, 97, 257, 337, 641, 881, 1297, 2417, 2657, 3697, 4177, 4721, 6577, 10657, 12401, 14657, 14897, 15937, 16561, 28817, 38561, 39041, 49297, 54721, 65537, 65617, 66161, 66977, 80177, 83537, 83777, 89041, 105601, 107377, 119617, \ldots$

It is not known whether this list is finite or not.

The largest known quartan prime is currently the largest known generalized Fermat prime: The $1\,353\,265$-digit $(145\,310^{65\,536})^4 + 1^4$.

Quartan primes

[OEIS A002645] Quartan primes: primes of the form $x^4 + y^4$, $x > 0$, $y > 0$.

The list of prime numbers represented by Φ_8 start with $2, 17, 97, 257, 337, 641, 881, 1297, 2417, 2657, 3697, 4177, 4721, 6577, 10657, 12401, 14657, 14897, 15937, 16561, 28817, 38561, 39041, 49297, 54721, 65537, 65617, 66161, 66977, 80177, 83537, 83777, 89041, 105601, 107377, 119617, \dots$

It is not known whether this list is finite or not.

The largest known quartan prime is currently the largest known generalized Fermat prime: The $1\,353\,265$-digit $(145\,310^{65\,536})^4 + 1^4$.

Quartan primes

[OEIS A002645] `Quartan primes: primes of the form` $x^4 + y^4$, $x > 0$, $y > 0$.

The list of prime numbers represented by Φ_8 start with
$2, 17, 97, 257, 337, 641, 881, 1297, 2417, 2657, 3697, 4177,$
$4721, 6577, 10657, 12401, 14657, 14897, 15937, 16561,$
$28817, 38561, 39041, 49297, 54721, 65537, 65617, 66161,$
$66977, 80177, 83537, 83777, 89041, 105601, 107377, 119617, \dots$

It is not known whether this list is finite or not.

`The largest known quartan prime is currently the largest known generalized Fermat prime: The` $1\,353\,265$`-digit` $(145\,310^{65\,536})^4 + 1^4$.

Primes of the form $x^{2^k} + y^{2^k}$

[OEIS A002313] primes of the form $x^2 + y^2$,
[OEIS A002645] primes of the form $x^4 + y^4$,
[OEIS A006686] primes of the form $x^8 + y^8$,
[OEIS A100266] primes of the form $x^{16} + y^{16}$,
[OEIS A100267] primes of the form $x^{32} + y^{32}$.

Primes of the form $X^2 + Y^4$

John Friedlander

Henryk Iwaniec

However, it is known that there are infinitely many prime numbers of the form $X^2 + Y^4$.

Friedlander, J. & Iwaniec, H. *The polynomial* $X^2 + Y^4$ *captures its primes*, Ann. of Math. (2) **148** (1998), no. 3, 945–1040.

`https://arxiv.org/pdf/math/9811185.pdf` [A028916]

K. Mahler (1933)

Let F be a binary form of degree $d \geq 3$ with nonzero discriminant.
Denote by A_F the area (Lebesgue measure) of the domain

$$\{(x, y) \in \mathbb{R}^2 \ | \ F(x, y) \leq 1\}.$$

For $Z > 0$ denote by $N_F(Z)$ the number of $(x, y) \in \mathbb{Z}^2$ such that $0 < |F(x, y)| \leq Z$.
Then

$$N_F(Z) = A_F Z^{\frac{2}{d}} + O(Z^{\frac{1}{d-1}})$$

as $Z \to \infty$.

Kurt Mahler

Kurt Mahler

1903 – 1988

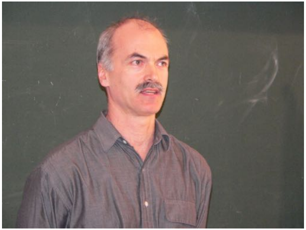
Über die mittlere Anzahl der Darstellungen grosser Zahlen durch binäre Formen,
Acta Math. **62** (1933), 91-166.

https://carma.newcastle.edu.au/mahler/biography.html

C.L. Stewart - S.Y. Xiao

Let F be a binary form of degree $d \geq 3$ with nonzero discriminant.
There exists a positive constant $\mathsf{C}_F > 0$ such that the number of integers of absolute value at most N which are represented by $F(X, Y)$ is asymptotic to $\mathsf{C}_F N^{\frac{2}{d}} + O(N^{\beta_d})$ with $\beta_d < \frac{2}{d}\cdot$

Cam Stewart and Stanley Yao Xiao

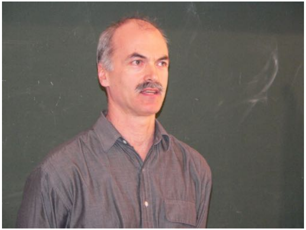


Cam Stewart

Stanley Yao Xiao

C.L. Stewart and S. Yao Xiao, *On the representation of integers by binary forms*, arXiv:1605.03427v2

Cam Stewart and Stanley Yao Xiao



Cam Stewart

Stanley Yao Xiao

C.L. Stewart and S. Yao Xiao, *On the representation of integers by binary forms*, `arXiv:1605.03427v2`

Cyclotomic binary forms of degree 4

(*Joint work with Étienne Fouvry - in progress*).

$\Phi_5(X,Y) = X^4 + X^3Y + X^2Y^2 + XY^3 + Y^4$.
$\Phi_8(X,Y) = X^4 + Y^4$.
$\Phi_{12}(X,Y) = X^4 - X^2Y^2 + Y^4$.

Also
$\Phi_{10}(X,Y) = \Phi_5(X,-Y) = X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4$.

For $n \in \{5, 8, 12\}$, the number of positive integers $m \le N$ which can be written as $m = \Phi_n(x,y)$ is asymptotic to $C_{\Phi_n} N^{\frac{1}{2}}$.

Cyclotomic binary forms of degree 4

(*Joint work with Étienne Fouvry - in progress*).

$\Phi_5(X,Y) = X^4 + X^3Y + X^2Y^2 + XY^3 + Y^4$.
$\Phi_8(X,Y) = X^4 + Y^4$.
$\Phi_{12}(X,Y) = X^4 - X^2Y^2 + Y^4$.

Also
$\Phi_{10}(X,Y) = \Phi_5(X,-Y) = X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4$.

For $n \in \{5, 8, 12\}$, the number of positive integers $m \le N$ which can be written as $m = \Phi_n(x,y)$ is asymptotic to $\mathsf{C}_{\Phi_n} N^{\frac{1}{2}}$.

Numbers represented by two cyclotomic binary forms of degree 4

The number of integers $\leq N$ which are represented by two of the three quartic cyclotomic binary forms Φ_5, Φ_8 and Φ_{12} is bounded by $O_\epsilon(N^{\frac{3}{8}+\epsilon})$.

Consequence : the number of integers $\leq N$ which are represented by a cyclotomic binary form of degree 4 is asymptotic to

$$C_4 N^{\frac{1}{2}} + O_\epsilon(N^{\frac{3}{8}+\epsilon}),$$

where

$$C_4 = \mathsf{C}_{\Phi_5} + \mathsf{C}_{\Phi_8} + \mathsf{C}_{\Phi_{12}}.$$

Numbers represented by two cyclotomic binary forms of degree 4

The number of integers $\leq N$ which are represented by two of the three quartic cyclotomic binary forms Φ_5, Φ_8 and Φ_{12} is bounded by $O_\epsilon(N^{\frac{3}{8}+\epsilon})$.

Consequence : the number of integers $\leq N$ which are represented by a cyclotomic binary form of degree 4 is asymptotic to

$$C_4 N^{\frac{1}{2}} + O_\epsilon(N^{\frac{3}{8}+\epsilon}),$$

where

$$C_4 = \mathsf{C}_{\Phi_5} + \mathsf{C}_{\Phi_8} + \mathsf{C}_{\Phi_{12}}.$$

Numbers represented by a cyclotomic binary form of degree $\geq d$

Any prime number p is represented by a cyclotomic binary form : $\Phi_p(1,1) = p$.

Given an integer $d \geq 2$, we consider the set of positive integers m which can be written as $m = \Phi_n(x,y)$ with $n \geq d$ and $(x,y) \in \mathbb{Z}^2$ satisfying $\max(|x|,|y|) \geq 2$.

Numbers represented by a cyclotomic binary form of degree $\geq d$

Any prime number p is represented by a cyclotomic binary form : $\Phi_p(1,1) = p$.

Given an integer $d \geq 2$, we consider the set of positive integers m which can be written as $m = \Phi_n(x,y)$ with $n \geq d$ and $(x,y) \in \mathbb{Z}^2$ satisfying $\max(|x|,|y|) \geq 2$.

Numbers represented by a cyclotomic binary form of degree $\geq d$

Any prime number p is represented by a cyclotomic binary form : $\Phi_p(1,1) = p$.

Given an integer $d \geq 2$, we consider the set of positive integers m which can be written as $m = \Phi_n(x,y)$ with $n \geq d$ and $(x,y) \in \mathbb{Z}^2$ satisfying $\max(|x|,|y|) \geq 2$.

Numbers represented by a cyclotomic binary form of degree $\geq d$

Let $d \geq 6$. The number of integers $m \leq N$ which can be written $m = \Phi_n(x, y)$ with $n \geq d$ and $(x, y) \in \mathbb{Z}^2$ satisfying $\max(|x|, |y|) \geq 2$ is asymptotic to

$$C_d N^{\frac{2}{d}} + O_d(N^{\frac{2}{d+2}}),$$

with

$$C_d = \sum_n \mathsf{C}_{\Phi_n},$$

where the sum is over the set of integers n such that $\varphi(n) = d$ and n is not congruent to 2 modulo 4.

Isomorphic cyclotomic binary forms

Recall that the cyclotomic polynomials $\phi_n(t) \in \mathbb{Z}[t]$ satisfy $\phi_{2n}(t) = \phi_n(-t)$ for odd $n \geq 3$.

For n_1 and n_2 positive integers with $n_1 < n_2$, the following conditions are equivalent :
(1) $\varphi(n_1) = \varphi(n_2)$ and the two binary forms Φ_{n_1} et Φ_{n_2} are isomorphic.
(2) The two binary forms Φ_{n_1} and Φ_{n_2} represent the same integers.
(3) n_1 is odd and $n_2 = 2n_1$.

Isomorphic cyclotomic binary forms

Recall that the cyclotomic polynomials $\phi_n(t) \in \mathbb{Z}[t]$ satisfy $\phi_{2n}(t) = \phi_n(-t)$ for odd $n \geq 3$.

For n_1 and n_2 positive integers with $n_1 < n_2$, the following conditions are equivalent :
(1) $\varphi(n_1) = \varphi(n_2)$ and the two binary forms Φ_{n_1} et Φ_{n_2} are isomorphic.
(2) The two binary forms Φ_{n_1} and Φ_{n_2} represent the same integers.
(3) n_1 is odd and $n_2 = 2n_1$.

Even integers not represented by Euler totient function

The list of even integers which are not values of Euler φ function (i.e., for which $C_d = 0$) starts with

$14, 26, 34, 38, 50, 62, 68, \mathbf{74}, \mathbf{76}, 86, 90, 94, 98, 114, 118, \mathbf{122}, \mathbf{124}, 134, 142, 146, \mathbf{152}, \mathbf{154}, 158, 170, 174, 182, \mathbf{186}, \mathbf{188}, 194, 202, 206, 214, 218, 230, \mathbf{234}, \mathbf{236}, \mathbf{242}, \mathbf{244}, \mathbf{246}, \mathbf{248}, 254, 258, 266, 274, 278, \mathbf{284}, \mathbf{286}, 290, 298, \mathbf{302}, \mathbf{304}, 308, 314, 318, \ldots$

[OEIS A005277] `Nontotients: even` n `such that` $\varphi(m) = n$ `has no solution.`

Numbers represented by two cyclotomic binary forms of the same degree

Given two binary cyclotomic forms of the same degree and not isomorphic, and given $\epsilon > 0$, for $N \to \infty$ the number of positive integers $\leq N$ which are represented by these two forms is bounded by

$$\begin{cases} O_\epsilon(N^{\frac{3}{d\sqrt{d}}+\epsilon}) \text{ for } d = 4, 6, 8, \\ \\ O_{d,\epsilon}(N^{\frac{1}{d}+\epsilon}) \text{ for } d \geq 10. \end{cases}$$

A weak but uniform bound

For $d \geq 2$ and $N \to \infty$, the number of $m \leq N$ for which there exists $n \geq d$ and $(x, y) \in \mathbb{Z}^2$ with $\max(|x|, |y|) \geq 2$ and $m = \Phi_n(x, y)$ is bounded by

$$29N^{\frac{2}{d}}(\log N)^{1.161}.$$

Further developments (work in progress)

Representation of integers by other binary forms

- Representation of integers by the binary forms $X^n + Y^n$, $X^n - Y^n$ and $F_n(X,Y)$, where

$$F_n(X,Y) = X^n + X^{n-1}Y + \cdots + XY^{n-1} + Y^n.$$

Further developments (work in progress)

Representation of integers by other binary forms

- Representation of integers by the binary forms $X^n + Y^n$, $X^n - Y^n$ and $F_n(X, Y)$, where

$$F_n(X, Y) = X^n + X^{n-1}Y + \cdots + XY^{n-1} + Y^n.$$

Suggestion of *Florian Luca* (RNTA 2018)

Study the representation of integers by the polynomials
Dickson polynomials of the first and second kind

• The sequence of *Dickson polynomials of the first kind* $(D_n)_{n\geq 0}$ (resp. *second kind* $(E_n)_{n\geq 0}$) is defined by

$$D_n(X+Y, XY) = X^n + Y^n$$

(resp.

$$E_n(X+Y, XY) = F_n(X, Y)).$$

Dickson polynomials : representation of integers by $X^n + Y^n$ and $X^n - Y^n$ when $x+y$ and xy are integers (x and y are quadratic integers).

Suggestion of *Florian Luca* (RNTA 2018)

Study the representation of integers by the polynomials
Dickson polynomials of the first and second kind

• The sequence of *Dickson polynomials of the first kind* $(D_n)_{n\geq 0}$ (resp. *second kind* $(E_n)_{n\geq 0}$) is defined by

$$D_n(X+Y, XY) = X^n + Y^n$$

(resp.

$$E_n(X+Y, XY) = F_n(X, Y)).$$

Dickson polynomials : representation of integers by $X^n + Y^n$ and $X^n - Y^n$ when $x+y$ and xy are integers (x and y are quadratic integers).

Suggestion of *Florian Luca* (RNTA 2018)

Study the representation of integers by the polynomials *Dickson polynomials of the first and second kind*

- The sequence of *Dickson polynomials of the first kind* $(D_n)_{n\geq 0}$ (resp. *second kind* $(E_n)_{n\geq 0}$) is defined by

$$D_n(X+Y, XY) = X^n + Y^n$$

(resp.

$$E_n(X+Y, XY) = F_n(X, Y)).$$

Dickson polynomials : representation of integers by $X^n + Y^n$ and $X^n - Y^n$ when $x + y$ and xy are integers (x and y are quadratic integers).

Cyclotomic Dickson polynomials

- For $n \geq 2$, define

$$\Psi_n(X+Y, XY) = \Phi_n(X, Y).$$

Study the representation of integers by the polynomials Ψ_n.

Representation of integers by $\Phi_n(X, Y)$ where $x+y$ and xy are integers.

Dickson polynomials are not homogeneous.

Work in progress...

Cyclotomic Dickson polynomials

- For $n \geq 2$, define

$$\Psi_n(X+Y, XY) = \Phi_n(X, Y).$$

Study the representation of integers by the polynomials Ψ_n.

Representation of integers by $\Phi_n(X, Y)$ where $x+y$ and xy are integers.

Dickson polynomials are not homogeneous.

Work in progress. . .

Cyclotomic Dickson polynomials

- For $n \geq 2$, define

$$\Psi_n(X+Y, XY) = \Phi_n(X, Y).$$

Study the representation of integers by the polynomials Ψ_n.

Representation of integers by $\Phi_n(X, Y)$ where $x+y$ and xy are integers.

Dickson polynomials are not homogeneous.

Work in progress. . .

Cyclotomic Dickson polynomials

- For $n \geq 2$, define

$$\Psi_n(X+Y, XY) = \Phi_n(X, Y).$$

Study the representation of integers by the polynomials Ψ_n.

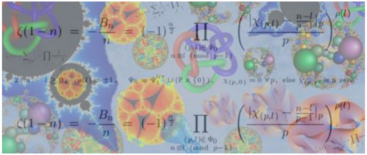
Representation of integers by $\Phi_n(X, Y)$ where $x+y$ and xy are integers.

Dickson polynomials are not homogeneous.

Work in progress...

November 27, 2018

Department of Mathematics, Ramakrishna Mission Vivekananda University (RKMVU), Belur Math, Howrah, Kolkata (India).



On the Landau–Ramanujan constant

Michel Waldschmidt

Sorbonne Université, Institut de Mathématiques de Jussieu

http://www.imj-prg.fr/~michel.waldschmidt/