Yogyakarta, CIMPA School UGM, February 27, 2020

# Linear recurrence sequences,

*Michel Waldschmidt*

Sorbonne University, Paris
Institut de Mathématiques de Jussieu

http://www.imj-prg.fr/~michel.waldschmidt/



## Abstract

Linear recurrence sequences are ubiquitous. They occur in biology, economics, computer science (analysis of algorithms), digital signal processing and number theory. We give a survey of this subject, together with connections with linear combinations of powers, with powers of matrices and with linear differential equations.
We first work over a field of any characteristic. Next we consider linear recurrence sequences over finite fields.



## Applications of linear recurrence sequences

Combinatorics

Elimination

Symmetric functions

Hypergeometric series

Language

Communication, shift registers

Finite difference equations

Logic

Approximation

Pseudo–random sequences



## Applications of linear recurrence sequences

- Biology (Integrodifference equations, spatial ecology).

- Computer science (analysis of algorithms).

- Digital signal processing (infinite impulse response (IIR) digital filters).

- Economics (time series analysis).

https://en.wikipedia.org/wiki/Recurrence_relation

## Linear recurrence sequences : definitions

A *linear recurrence sequence* is a sequence of numbers $\mathbf{u} = (u_0, u_1, u_2, \dots)$ for which there exist a positive integer $d$ together with numbers $a_1, \dots, a_d$ with $a_d \neq 0$ such that, for $n \geq 0$,

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

Here, a *number* means an element of a field $\mathbb{K}$.

Given $\underline{a} = (a_1, \dots, a_d) \in \mathbb{K}^d$, the set $E_{\underline{a}}$ of linear recurrence sequences $\mathbf{u} = (u_n)_{n\geq 0}$ satisfying $(\star)$ is a $\mathbb{K}$–vector subspace of dimension $d$ of the space $\mathbb{K}^{\mathbb{N}}$ of all sequences.

A basis of this space is obtained by taking for the initial $d$ values $(u_0, u_1, \dots, u_{d-1})$ the elements of the canonical basis of $\mathbb{K}^d$.



## Generating series, characteristic polynomial

The generating series is the formal series

$$\sum_{n\geq 0} u_n X^n.$$

Let $\gamma \in K^\times$ ; the sequence $(\gamma^n)_{n\geq 0}$ satisfies the linear recurrence

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

if and only if $\gamma^d = a_1\gamma^{d-1} + \cdots + a_d$.
The characteristic (or companion) polynomial of the linear recurrence is

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

Recall that $0$ is not a root of this polynomial ($a_d \neq 0$).



## Linear recurrence sequences : examples

- Constant sequence : $u_n = u_0$.
Linear recurrence sequence of order $1$ : $u_{n+1} = u_n$.
Characteristic polynomial : $f(X) = X - 1$.
Generating series :

$$\sum_{n\geq 0} u_0 X^n = \frac{u_0}{1-X}.$$

- Geometric progression : $u_n = u_0\gamma^n$.
Linear recurrence sequence of order $1$ : $u_n = \gamma u_{n-1}$.
Characteristic polynomial $f(X) = X - \gamma$.
Generating series :

$$\sum_{n\geq 0} u_0 \gamma^n X^n = \frac{u_0}{1-\gamma X}.$$



## Linear recurrence sequences : examples

- $u_n = n$. This is a linear recurrence sequence of order $2$ :

$$n + 2 = 2(n+1) - n.$$

Characteristic polynomial

$$f(X) = X^2 - 2X + 1 = (X-1)^2.$$

Generating series

$$\sum_{n\geq 0} nX^n = \frac{1}{1 - 2X + X^2}.$$

Power of matrices :

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}^n = \begin{pmatrix} -n+1 & n \\ -n & n+1 \end{pmatrix}.$$

## Linear recurrence sequences : examples

• $u_n = p(n)$, where $p$ is a polynomial of degree $d$. This is a linear recurrence sequence of order $d+1$.

**Proof.** The sequences

$$(p(n))_{n\geq 0}, \quad (p(n+1))_{n\geq 0}, \quad \cdots, \quad (p(n+k))_{n\geq 0}$$

are $\mathbb{K}$–linearly independent in $\mathbb{K}^{\mathbb{N}}$ for $k = d-1$ and linearly dependent for $k = d$.

A basis of the space of polynomials of degree $d$ is given by the $d+1$ polynomials

$$p(X),\ p(X+1),\ \ldots,\ p(X+d).$$

`Question` : *which is the characteristic polynomial ?*



## Linear sequences which are ultimately recurrent

The sequence

$$(1, 0, 0, \ldots)$$

is not a linear recurrence sequence.

The condition

$$u_{n+1} = u_n$$

is satisfied only for $n \geq 1$.

The relation

$$u_{n+2} = u_{n+1} + 0u_n$$

with $d = 2$, $a_d = 0$ does not fulfil the requirement $a_d \neq 0$.



## Order of a linear recurrence sequence

If $\mathbf{u} = (u_n)_{n\geq 0}$ satisfies the linear recurrence, the characteristic polynomial of which is $f$, then, for any monic polynomial $g \in \mathbb{K}[X]$ with $g(0) \neq 0$, this sequence $\mathbf{u}$ also satisfies the linear recurrence, the characteristic polynomial of which is $fg$.
`Example` : for $g(X) = X - \gamma$ with $\gamma \neq 0$, from

$$(\star) \qquad u_{n+d} - a_1 u_{n+d-1} - \cdots - a_d u_n = 0$$

we deduce

$$u_{n+d+1} - a_1 u_{n+d} - \cdots - a_d u_{n+1} - \gamma(u_{n+d} - a_1 u_{n+d-1} - \cdots - a_d u_n) = 0.$$

The *order* of a linear recurrence sequence is the smallest $d$ such that $(\star)$ holds for all $n \geq 0$.



## Generating series of a linear recurrence sequence

Let $\mathbf{u} = (u_n)_{n\geq 0}$ be a linear recurrence sequence

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

with characteristic polynomial

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

Denote by $f^-$ the reciprocal polynomial of $f$ :

$$f^-(X) = X^d f(X^{-1}) = 1 - a_1 X - \cdots - a_d X^d.$$

Then

$$\sum_{n=0}^{\infty} u_n X^n = \frac{r(X)}{f^-(X)},$$

where $r$ is a polynomial of degree less than $d$ determined by the initial values of $\mathbf{u}$.

## Generating series of a linear recurrence sequence

Assume

$$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0.$$

Then

$$\sum_{n=0}^{\infty} u_n X^n = \frac{r(X)}{f^-(X)}.$$

**Proof**. Comparing the coefficients of $X^n$ for $n \geq d$ shows that

$$f^-(X) \sum_{n=0}^{\infty} u_n X^n$$

is a polynomial of degree less than $d$.



## Taylor coefficients of rational functions

Conversely, the sequence of coefficients in the Taylor expansion of any rational fraction $a(X)/b(X)$ with $\deg a < \deg b$ and $b(0) \neq 0$ satisfies the recurrence relation with characteristic polynomial $f \in K[X]$ given by $f(X) = b^-(X)$.

Therefore a sequence $\mathbf{u} = (u_n)_{n\geq 0}$ satisfies the recurrence relation $(\star)$ with characteristic polynomial $f \in K[X]$ if and only if

$$\sum_{n=0}^{\infty} u_n X^n = \frac{r(X)}{f^-(X)},$$

where $r$ is a polynomial of degree less than $d$ determined by the initial values of $\mathbf{u}$.



## Linear differential equations

Given a sequence $(u_n)_{n\geq 0}$ of numbers, its exponential generating power series is

$$\psi(z) = \sum_{n\geq 0} u_n \frac{z^n}{n!}.$$

For $k \geq 0$, the $k$-the derivative $\psi^{(k)}$ of $\psi$ satisfies

$$\psi^{(k)}(z) = \sum_{n\geq 0} u_{n+k} \frac{z^n}{n!}.$$

Hence the sequence satisfies the linear recurrence relation

$(\star)$ $$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

if and only if $\psi$ is a solution of the homogeneous linear differential equation

$$y^{(d)} = a_1 y^{(d-1)} + \cdots + a_{d-1} y' + a_d y.$$



## Matrix notation for a linear recurrence sequence

The linear recurrence sequence

$(\star)$ $$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for} \quad n \geq 0$$

can be written

$$\begin{pmatrix} u_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+d} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}.$$

## Matrix notation for a linear recurrence sequence

$$U_{n+1} = AU_n$$

with

$$U_n = \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix}.$$

The determinant of $I_d X - A$ (the characteristic polynomial of $A$) is nothing but

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d,$$

the characteristic polynomial of the linear recurrence sequence.
By induction

$$U_n = A^n U_0.$$



## Powers of matrices

Let $A = (a_{ij})_{1 \le i,j \le d} \in \mathrm{GL}_{d \times d}(\mathbb{K})$ be a $d \times d$ matrix with coefficients in $\mathbb{K}$ and nonzero determinant. For $n \ge 0$, define

$$A^n = \left(a_{ij}^{(n)}\right)_{1 \le i,j \le d}.$$

Then each of the $d^2$ sequences $\left(a_{ij}^{(n)}\right)_{n \ge 0}$, $(1 \le i,j \le d)$ is a linear recurrence sequence. The roots of the characteristic polynomial of these linear recurrences are the eigenvalues of $A$.

In particular the sequence $\left(\mathrm{Tr}(A^n)\right)_{n \ge 0}$ satisfies the linear recurrence, the characteristic polynomial of which is the characteristic polynomial of the matrix $A$.



## Conversely :

Given a linear recurrence sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$, there exist an integer $d \ge 1$ and a matrix $A \in \mathrm{GL}_d(\mathbb{K})$ such that, for each $n \ge 0$,

$$u_n = a_{11}^{(n)}.$$

The characteristic polynomial of $A$ is the characteristic polynomial of the linear recurrence sequence.

Everest G., van der Poorten A., Shparlinski I., Ward T. – *Recurrence sequences,* Mathematical Surveys and Monographs (AMS, 2003), volume 104.



## Linear recurrence sequences : simple roots

A basis of $E_{\underline{a}}$ over $\mathbb{K}$ is obtained by attributing to the initial values $u_0, \ldots, u_{d-1}$ the values given by the canonical basis of $\mathbb{K}^d$.
Given $\gamma$ in $\mathbb{K}^\times$, a necessary and sufficient condition for a sequence $(\gamma^n)_{n \ge 0}$ to satisfy $(\star)$ is that $\gamma$ is a root of the characteristic polynomial

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d.$$

If this polynomial has $d$ distinct roots $\gamma_1, \ldots, \gamma_d$ in $\mathbb{K}$,

$$f(X) = (X - \gamma_1) \cdots (X - \gamma_d), \quad \gamma_i \ne \gamma_j,$$

then a basis of $E_{\underline{a}}$ over $\mathbb{K}$ is given by the $d$ sequences $({\gamma_i}^n)_{n \ge 0}$, $i = 1, \ldots, d$.

## Linear recurrence sequences : double roots

The characteristic polynomial of the linear recurrence $u_n = 2\gamma u_{n-1} - \gamma^2 u_{n-2}$ is $X^2 - 2\gamma X + \gamma^2 = (X-\gamma)^2$ with a double root $\gamma$.

The sequence $(n\gamma^n)_{n\geq 0}$ satisfies

$$n\gamma^n = 2\gamma(n-1)n\gamma^{n-1} - \gamma^2(n-2)\gamma^{n-2}.$$

A basis of $E_{\underline{a}}$ for $a_1 = 2\gamma$, $a_2 = -\gamma^2$ is given by the two sequences $(\gamma^n)_{n\geq 0}$, $(n\gamma^n)_{n\geq 0}$.

Given $\gamma \in \mathbb{K}^\times$, a necessary and sufficient condition for the sequence $n\gamma^n$ to satisfy the linear recurrence relation $(\star)$ is that $\gamma$ is a root of multiplicity $\geq 2$ of $f(X)$.



## Linear recurrence sequences : multiple roots

In general, when the characteristic polynomial splits as

$$X^d - a_1 X^{d-1} - \cdots - a_d = \prod_{i=1}^{\ell} (X - \gamma_i)^{t_i},$$

a basis of $E_{\underline{a}}$ is given by the $d$ sequences

$$(n^k {\gamma_i}^n)_{n\geq 0}, \qquad 0 \leq k \leq t_i - 1, \quad 1 \leq i \leq \ell.$$



## Polynomial combinations of powers

The sum and the product of any two linear recurrence sequences are linear recurrence sequences.

The set $\cup_{\underline{a}} E_{\underline{a}}$ of all linear recurrence sequences with coefficients in $\mathbb{K}$ is a sub–$\mathbb{K}$–algebra of $\mathbb{K}^{\mathbb{N}}$.

Given polynomials $p_1, \ldots, p_\ell$ in $\mathbb{K}[X]$ and elements $\gamma_1, \ldots, \gamma_\ell$ in $\mathbb{K}^\times$, the sequence

$$\bigl(p_1(n){\gamma_1}^n + \cdots + p_\ell(n)\gamma_\ell^n\bigr)_{n\geq 0}$$

is a linear recurrence sequence.

Conversely, any linear recurrence sequence is of this form.



## Consequence

• When $p$ is a polynomial of degree $< d$, the characteristic polynomial of the sequence $u_n = p(n)$ divides $(X-1)^d$.

`Proof.`
Set

$$A = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = I_d + N$$

where $I_d$ is the $d \times d$ identity matrix and $N$ is nilpotent : $N^d = 0$.

## Consequence

The characteristic polynomial of $A$ is $(X-1)^d$. Hence for $1 \leq i, j \leq d$, the sequence $u_n$ of the coefficient $a_{ij}^{(n)}$ of $A^n$ satisfies the linear recurrence relation

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n,$$

that is

$$u_{n+d} = d u_{n+d-1} - \binom{d}{2} u_{n+d-2} + \cdots + (-1)^{d-2} d u_{n+1} + (-1)^{d-1} u_n.$$

The characteristic polynomial of this recurrence relation is $(X-1)^d$.



## Characteristic polynomial of the recurrence sequence $p(n)$.

Since, for $1 \leq i, j \leq d$ and $n \geq 0$, we have

$$a_{ij}^{(n)} = \binom{n}{j-i}$$

(where we agree that $\binom{n}{k} = 0$ for $k < 0$ and for $k > n$, while $\binom{d}{0} = \binom{d}{d} = 1$), we deduce that each of the $d$ polynomials

$$1, \quad \frac{X(X+1)\cdots(X+k-1)}{k!} \qquad k = 1, 2, \ldots, d-1$$

namely

$$1, X, \frac{X(X+1)}{2}, \ldots, \frac{X(X+1)\cdots(X+d-2)}{(d-1)!},$$

satisfies the recurrence $(\star)$. These $d$ polynomials constitute a basis of the space of polynomials of degree $< d$.



## Sum of polynomial combinations of powers

If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two linear recurrence sequences of characteristic polynomials $f_1$ and $f_2$ respectively, then $\mathbf{u}_1 + \mathbf{u}_2$ satisfies the linear recurrence, the characteristic polynomial of which is

$$\frac{f_1 f_2}{\gcd(f_1, f_2)}.$$



## Product of polynomial combinations of powers

If the characteristic polynomials of the two linear recurrence sequences $\mathbf{u}_1$ and $\mathbf{u}_2$ are respectively

$$f_1(T) = \prod_{j=1}^{\ell} (T - \gamma_j)^{t_j} \quad \text{and} \quad f_2(T) = \prod_{k=1}^{\ell'} (T - \gamma'_k)^{t'_k},$$

then $\mathbf{u}_1 \mathbf{u}_2$ satisfies the linear recurrence, the characteristic polynomial of which is

$$\prod_{j=1}^{\ell} \prod_{k=1}^{\ell'} (T - \gamma_j \gamma'_k)^{t_j + t'_k - 1}.$$

## Linear recurrence sequences and Brahmagupta–Pell–Fermat Equation

Let $d$ be a positive integer, not a square. The solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of the Brahmagupta–Pell–Fermat Equation

$$x^2 - dy^2 = \pm 1$$

form a sequence $(x_n, y_n)_{n\in\mathbb{Z}}$ defined by

$$x_n + \sqrt{d}y_n = (x_1 + \sqrt{d}y_1)^n.$$

From

$$2x_n = (x_1 + \sqrt{d}y_1)^n + (x_1 - \sqrt{d}y_1)^n$$

we deduce that $(x_n)_{n\geq 0}$ is a linear recurrence sequence. Same for $y_n$, and also for $n \leq 0$.



## Doubly infinite linear recurrence sequences

A sequence $(u_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ is a linear recurrence sequence if it satisfies

$$(\star) \qquad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

for all $n \in \mathbb{Z}$.

Recall $a_d \neq 0$.

Such a sequence is determined by $d$ consecutive values.



## Discrete version of linear differential equations

A sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$ can be viewed as a linear map $\mathbb{N} \longrightarrow \mathbb{K}$. Define the discrete derivative $\mathcal{D}$ by

$$\begin{array}{rccc} \mathcal{D}\mathbf{u}: & \mathbb{N} & \longrightarrow & \mathbb{K} \\ & n & \longmapsto & u_{n+1} - u_n. \end{array}$$

A sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$ is a linear recurrence sequence if and only if there exists $Q \in \mathbb{K}[T]$ with $Q(1) \neq 1$ such that

$$Q(\mathcal{D})\mathbf{u} = 0.$$

Linear recurrence sequences are a discrete version of linear differential equations with constant coefficients.

The condition $Q(1) \neq 0$ reflects $a_d \neq 0$ – otherwise one gets *ultimately* recurrent sequences.



## Conclusion

The same mathematical object occurs in a different guise :

- Linear recurrence sequences

$$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

- Linear combinations with polynomial coefficients of powers

$$p_1(n){\gamma_1}^n + \cdots + p_\ell(n)\gamma_\ell^n.$$

- Taylor coefficients of rational functions.
- Coefficients of power series which are solutions of homogeneous linear differential equations.
- Sequence of coefficients of powers of a matrix.

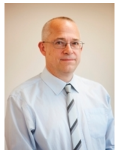
## Reference

Everest, Graham ; van der Poorten, Alf ; Shparlinski, Igor ; Ward, Tom – *Recurrence sequences,* Mathematical Surveys and Monographs (AMS, 2003), volume 104. `1290 references.`


Graham Everest


Alf van der Poorten


Igor Shparlinski


Tom Ward



## Linear recurrence sequences over finite fields

`Reference:` Chapter 8 : *Linear recurring sequences* of

Lidl, Rudolf ; Niederreiter, Harald.
*Finite fields.* Paperback reprint of the hardback 2nd edition 1996. (English)
Encyclopedia of Mathematics and Its Applications 20.
Cambridge University Press (ISBN 978-0-521-06567-2/pbk).
xiv, 755 p. (2008).



Harald Niederreiter



## Linear recurring sequences

Given $a, a_0, \ldots, a_{k-1}$ in a finite field $\mathbb{F}_q$, consider a $k$*–th order linear recurrence relation* : for $n = 0, 1, 2, \ldots,$

$$u_{n+k} = a_{k-1}u_{n+k-1} + a_{k-2}u_{n+k-2} + \cdots + a_1u_{n+1} + a_0u_n + a$$

Homogeneous : $a = 0$.

Initial values : $u_0, u_1, \ldots, u_{k-1}$.

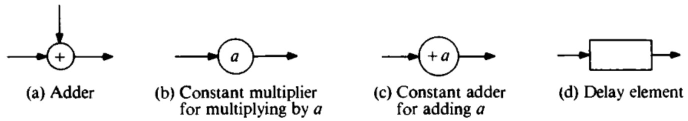
State vector : $\mathbf{u}_n = (u_n, u_{n+1}, \ldots, u_{n+k-1})$.

Initial state vector : $\mathbf{u}_0 = (u_0, u_1, \ldots, u_{k-1})$.

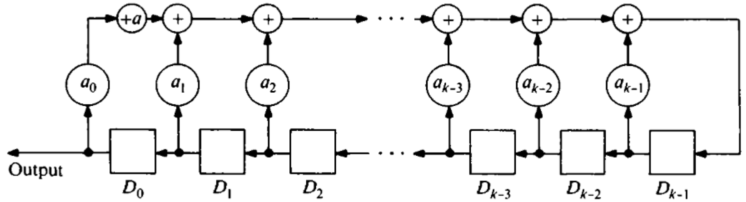


## Feedback shift register

Electronic switching circuit : adder, constant multiplier, constant adder, delay element (*flip-flop*)




$$u_{n+k} = a_{k-1}u_{n+k-1} + a_{k-2}u_{n+k-2} + \cdots + a_1u_{n+1} + a_0u_n + a$$



## The least period of a linear recurrence sequence

Since $\mathbb{F}_q$ is finite, any linear recurrence sequence $(u_n)_{n\geq 0}$ in $\mathbb{F}_q$ is *ultimately periodic* : there exists $r>0$ and $n_0\geq 0$ such that $u_n=u_{n+r}$ for $n\geq n_0$. The least $n_0$ for which this relation holds is the *preperiod*.

Any period is a multiple of the least period.

A linear recurrence sequence $(u_n)_{n\geq 0}$ is periodic if there exists a period $r>0$ such that $u_n=u_{n+r}$ for $n\geq 0$. In this case this relation holds for the least period ; the preperiod is $0$. If $a_0\neq 0$, then the sequence is periodic.

The least period $r$ of a (homogeneous) linear recurrence sequence in $\mathbb{F}_q$ of order $k$ satisfies $r\leq q^k-1$.



## The companion matrix

The linear recurrence sequence

$$u_{n+k}=a_{k-1}u_{n+k-1}+\cdots+a_0u_n \quad \text{for} \quad n\geq 0$$

can be written

$$\mathbf{u}_n=\mathbf{u}_0A^n$$

where

$$A=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0\\ 1 & 0 & 0 & \cdots & 0 & a_1\\ 0 & 1 & 0 & \cdots & 0 & a_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & a_{k-1}\end{pmatrix}.$$



## The least period

Assume $a_0\neq 0$

The least period of the linear recurrence sequence divides the order of the matrix $A$ in the general linear group $\mathrm{GL}_k(\mathbb{F}_q)$.

The *impulse response sequence* is the linear recurrence sequence with the initial state $(0,0,\ldots,0,1)$.

The least period of a linear recurrence sequence divides the least period of the corresponding impulse response sequence.



## Further examples of linear recurrence sequences

- Fibonacci
- Lucas
- Perrin
- Padovan
- Narayana

References
Linear recurrence sequences : an introduction.
http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/LinearRecurrenceSequencesIntroduction.pdf
Linear recurrence sequences, exponential polynomials and Diophantine approximation.
http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/LinRecSeqDiophAppxVI.pdf

## Leonardo Pisano (Fibonacci)

Fibonacci sequence $(F_n)_{n\geq 0}$,

$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,$

$34,\ 55,\ 89,\ 144,\ 233,\ldots$

is defined by

$$F_0 = 0,\ F_1 = 1,$$

$$F_{n+2} = F_{n+1} + F_n \quad \text{for} \quad n \geq 0.$$

http://oeis.org/A000045

Leonardo Pisano (Fibonacci)
(1170–1250)



## Lucas sequence http://oeis.org/000032

The Lucas sequence $(L_n)_{n\geq 0}$ satisfies the same recurrence relation as the Fibonacci sequence, namely

$$L_{n+2} = L_{n+1} + L_n \quad \text{for} \quad n \geq 0,$$

only the initial values are different :

$$L_0 = 2,\ L_1 = 1.$$

The sequence of Lucas numbers starts with

$$2,\ 1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123,\ 199,\ 322,\ldots$$

A closed form involving the Golden ratio $\Phi$ is

$$L_n = \Phi^n + (-\Phi)^{-n},$$

from which it follows that for $n \geq 2$, $L_n$ is the nearest integer to $\Phi^n$.



## Perrin sequence http://oeis.org/A001608

The Perrin sequence (also called *skiponacci sequence*) is the linear recurrence sequence $(P_n)_{n\geq 0}$ defined by

$$P_{n+3} = P_{n+1} + P_n \quad \text{for} \quad n \geq 0,$$

with the initial conditions

$$P_0 = 3,\ P_1 = 0,\ P_2 = 2.$$

It starts with

$$3,\ 0,\ 2,\ 3,\ 2,\ 5,\ 5,\ 7,\ 10,\ 12,\ 17,\ 22,\ 29,\ 39,\ 51,\ 68,\ldots$$

François Olivier Raoul Perrin (1841-1910) :
https://en.wikipedia.org/wiki/Perrin_number



## Narayana sequence https://oeis.org/A000930

Narayana sequence is defined by the recurrence relation

$$C_{n+3} = C_{n+2} + C_n$$

with the initial values $C_0 = 2$, $C_1 = 3$, $C_2 = 4$.
It starts with

$$2,\ 3,\ 4,\ 6,\ 9,\ 13,\ 19,\ 28,\ 41,\ 60,\ 88,\ 129,\ 189,\ 277,\ldots$$

Real root of $x^3 - x^2 - 1$

$$\frac{\sqrt[3]{\dfrac{29+3\sqrt{93}}{2}} + \sqrt[3]{\dfrac{29-3\sqrt{93}}{2}} + 1}{3} = 1.465571231876768\ldots$$

## Padovan sequence https://oeis.org/A000931

The Padovan sequence $(p_n)_{n\geq 0}$ satisfies the same recurrence

$$p_{n+3} = p_{n+1} + p_n$$

as the Perrin sequence but has different initial values :

$$p_0 = 1, \quad p_1 = p_2 = 0.$$

It starts with

$$1,\ 0,\ 0,\ 1,\ 0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 3,\ 4,\ 5,\ 7,\ 9,\ 12,\ 16, \dots$$

Richard Padovan
http://mathworld.wolfram.com/LinearRecurrenceEquation.html



Yogyakarta, CIMPA School UGM, February 27, 2020

**Linear recurrence sequences,**

*Michel Waldschmidt*

Sorbonne University, Paris
Institut de Mathématiques de Jussieu
http://www.imj-prg.fr/~michel.waldschmidt/