TIFR Center, Bangalore *April 26, 2011*

Ramachandra's contributions to transcendental number theory

Michel Waldschmidt

http://www.math.jussieu.fr/∼miw/

Abstract

In 1968, K. Ramachandra published two seminal papers "Contributions to the theory of transcendental numbers. I, II" [Acta Arith. 14 (1967/68), 65–72, 73–88 ; MR0224566 (37 # 165)]. In these papers he raised some questions which are still unanswered. The results he proved almost half a century ago, and the method he developed there, have a lasting influence on the development of the theory. They are at the origin of many research papers. We shall discuss some of them.

http://www.math.jussieu.fr/∼miw/

Lecture by K. Srinivas at IMSc in 2011

Ramachandra,
some professional and personal reminiscences

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Lecture at MathScience in 2011

K. Ramachandra (1933–2011)

K. Ramachandra (1933–2011)

- Without his efforts, perhaps Analytic Number Theory would have become extinct in India back in the mid 1970's
- During student days won a prize : *Ramanujan : Twelve Lectures on Subjects Suggested by His Life and Work.*
- Ramachandra's *taxi-cab* number : Only no. with the property $3435 = 3^3 + 4^4 + 3^3 + 5^5$
- TIFR Mumbai (1958-1995) : Ph D in 1965 Advisor : K. G. Ramanathan.
- Ramachandra believed that as a mathematician one not only has to contribute to the subject but also guide the next generation of mathematicians.
- He put up a brave fight, defended both number theory and the rights of interested students to take up the subject.

K. Ramachandra as an advisor

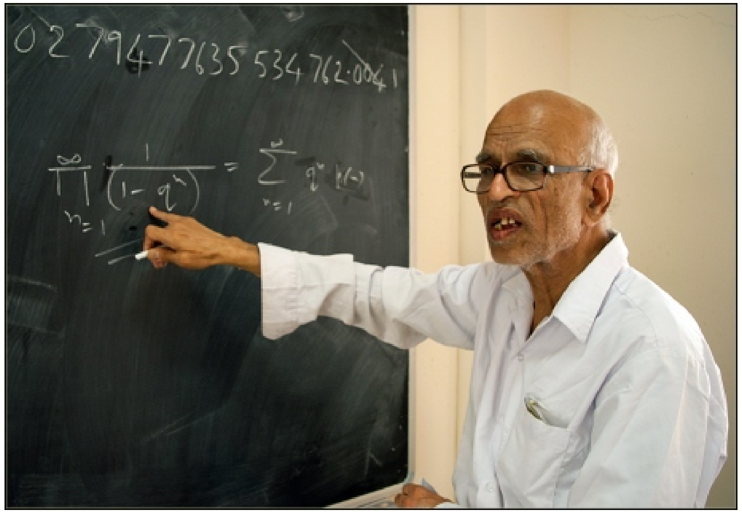
- Ramachandra acted as the doctoral advisor for eight students :
  1. S. Srinivasan
  2. T. N. Shorey
  3. M. Narlikar
  4. R. Balasubramanian
  5. V. V. Rane
  6. A. Sankaranarayanan
  7. K. Srinivas
  8. Kishor Bhat (thesis submitted)

K. Ramachandra and his students

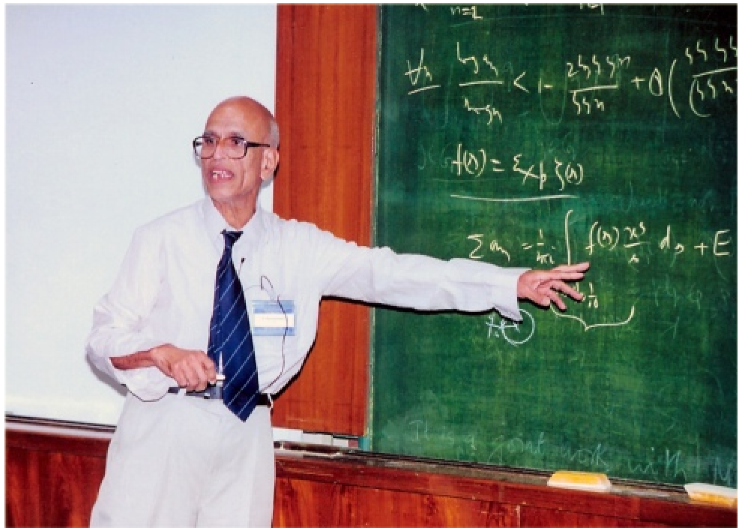
K. Ramachandra (1933–2011)

- Ramachandra's collaborated with many leading number theorists of the world and also invited many of the leading mathematicians to TIFR, including the legendary Paul Erdős, who stayed as a guest in his house
- Ramachandra is one of the few Indian mathematicians with Erdős Number 1. He published two joint papers with Erdős
- Wrote more than 200 papers
- Books : Mean-value and Omega theorems for the Riemann zeta function, Introduction to Number theory, Transcendental Number theory, Riemann zeta-function,$\cdots$
- Founder of Hardy-Ramanujan Journal
- Very honest, sincere, unpretentious, apolitical, extremely caring. He was a man of high integrity and moral character

Mathematics was his Life!



Blog : `http://hardyramanujan.wordpress.com/`



SRINIVASA RAMANUJAN MEDAL (1997)

TO

PROFESSOR KANAKANAHALLI RAMACHANDRA, FNA

CITATION

The Srinivasa Ramanujan Medal is a subjectwise medal of the Academy awarded once in three years for outstanding contributions in Mathematics or a related subject. The award is made to an eminent scientist whose work has had an impact over a considerable length of time.

The award for the year 1997 has been made to Professor Kanakanahalli Ramachandra, FNA, Hon Visiting Professor, NIAS, Indian Institute of Science Campus, Bangalore for his outstanding contributions to classical number theory, especially algebraic number theory, analytical number theory and transcendental numbers.

Professor Ramachandra improved upon the results of Siegel on the approximation of algebraic numbers by algebraic Numbers of a given degree and height. He has made many significant contributions to Transcendental Number Theory by extending the methods of Baker on Linear forms of Logarithms of algebraic numbers and gave interesting applications of these on gaps between numbers with a large prime factor.

Professor Ramachandra has proved important results dealing with mean Values of Titchmarsh Series, mean values of Riemann Zeta Function, Zeros of Riemann function and omega theorems for Zeta function. He has also proved many stability theorems for some generalized dirichlet Series.

Professor Ramachandra was President of Ramanujan Mathematical Society (1988-90). He is a Life Member, Indian Mathematical Society and National Academy of Sciences (India); Fellow, Indian Academy of Sciences and Biographical Academy of Commonwealth. He received Meghnad Saha Award (UGC) (1976) and was the UGC National Lecturer (1978).

Professor Ramachandra was elected to the Fellowship of the Indian National Science Academy in the year 1974.

Personal reminiscences

My first visit to India : TIFR Bombay, October-December 1976, invited by K. Ramachandra (3 months course)
Visits of Bangalore, Mysore (Belur and Halebid), Madras and Pondicherry, Goa, Ajanta and Ellora, Delhi, Chandigarh (R.P. Bambah)

Learning English with Ramachandra

Transcendental numbers and algebraic groups

Personal reminiscences

My next visits to India : September 1985 (Bambah's conference in Chandirgarh postponed), December 1987 (Ramanujan Centenary : Chennai, Kumbakonam).

My visit to India with my family in August 1988 : Bombay (TIFR), Madras (MathScience), Pondicherry (Sinnou David)
Visit of the *Prince of Wales Museum* in Bombay with Ramachandra

Pondicherry, August 1988

Mahabalipuram, August 1988

Hélène went back to India in 2000 and 2001.

More recent meetings with Ramachandra

Ramachandra's sixtieth (1993) and seventieth (2003) birthdays

January 25 – February 4, 2005, Bangalore, CIMPA Research School Security for Computer Systems and Networks, K. Gopinath (IISc, India) and J-J. Levy (INRIA, France). Discussions with Ramachandra in the Indian Institute of Science Campus.

My last meeting with K. Ramachandra : my visits to Mysore (Yogananda) and Bangalore in January 2010 (Kishor Bhat).

Ramachandra's sixty and seventy birthdays

The author wishes to convey his best thanks to the organizer of the Madras Conference of July 1993 in honor of Professor Ramachandra's 60th birthday, R. Balasubramanian, for his invitation to participate, which provided him the opportunity to write this paper. Next he is grateful to the organizer of the Bangalore Conference of December 2003 in honor of Professor Ramachandra's 70th birthday, K. Srinivas, for his invitation to participate, which provided him the opportunity to publish this paper. He is also glad to express his deep gratitude to Professor K. Ramachandra for the inspiring role of his work and for his invitation to the Tata Institute as early as 1976.

Ramachandra's Contributions to the theory of transcendental numbers

[1968].– *Contributions to the theory of transcendental numbers* (I) ; Acta Arith., **14** (1968), 65–72 ; (II), id., 73–88.

http://matwbn.icm.edu.pl/tresc.php?wyd=6& tom=14

ACTA ARITHMETICA
XIV (1968)

Contributions to the theory of transcendental numbers (I)

by

K. Ramachandra (Bombay)

Dedicated to the memory of
Jacques Hadamard (1865-1963)

§ 1. Introduction. In this paper we prove the main theorem relating to the set (or a subset) of complex numbers at which a given set of algebraically independent meromorphic functions assume values in a fixed algebraic number field (we actually prove a more general result which may be useful elsewhere). We state a few deductions in § 2 and it is interesting to note that the Main Theorem gives significant results in the case (overlooked by Gelfond) *where the functions concerned do not satisfy algebraic differential equations of the first order with algebraic number coefficients.* Since some of the deductions require lengthy preparations we postpone the proofs of these and other deductions to part II, which is a continuation of this paper. We give a brief history of this theorem in this section.

In the year 1929, A. O. Gelfond made the important discovery that $a^b = e^{b \log a}$ is transcendental for every imaginary quadratic irrationality b and every algebraic a different from zero except for $\log a = 0$ (¹). Assuming the result to be false Gelfond applied the interpolation formula

Other papers by Ramachandra on transcendental number theory

[1969].–*Lectures on transcendental numbers*; The Ramanujan Institute, Univ. of Madras, 1969, 72 p.

[1969].– *A note on Baker's method*; J. Austral. Math. Soc., **10** (1969), 197–203.

[1973].– *Application of Baker's theory to two problems considered by Erdős and Selfridge*; J. Indian Math. Soc., **37** (1973), 25–34.

[1987].– *An easy transcendence measure for e*; J. Indian Math. Soc., **51** (1987), 111–116.

Joint papers by Ramachandra on transcendental number theory

[with T.N. Shorey, 1973].– *On gaps between numbers with a large prime factor*; Acta Arith., **24** (1973), 99–111.

[with R. Balasubramanian, 1982].– *Transcendental numbers and a lemma in combinatorics*; Proc. Sem. Combinatorics and Applications, Indian Stat. Inst., (1982), 57–59.

[with Srinivasan, 1983].– *A note to a paper by Ramachandra on transcendental numbers*; Hardy-Ramanujan Journal, **6** (1983), 37–44.

On Ramachandra's contributions to transcendental number theory (M.W.)

Ramanujan Mathematical Society, Lecture Notes Series Number 2.
The Riemann Zeta function and related themes : papers in honour of Professor K. Ramachandra
Proceedings of International Conference held at National Institute of Advanced Studies, Bangalore 13-15 December, 2003
Ed. R. Balasubramanian, K. Srinivas (2006), 155–179.

http://www.math.jussieu.fr/~miw/articles/ps/ramachandra.ps

Srinivasa Ramanujan

On highly composite and similar numbers
(Proc. London Math. Soc. 1915)

$n =$	2	4	6	12	24	36	48	60	120 . . .
$d(n) =$	2	3	4	6	8	9	10	12	16 . . .

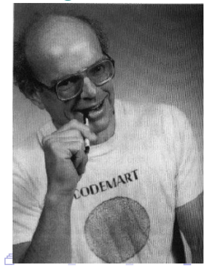
The On-Line Encyclopedia of Integer Sequences

1, 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, . . .

Number of divisors

1, 2, 3, 4, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 32, . . .

http://oeis.org/A002182 http://oeis.org/A002183



Neil J. A. Sloane

Alaoglu and Erdős

Ramanujan (1915) listed 102 highly composite numbers up to 6 746 328 388 800, but omitted 293 318 625 600.

Alaoglu and Erdős,
On highly composite and similar numbers, Trans. AMS **56** (3), 1944, 448–469.
highly abundant numbers, super abundant numbers, colossally abundant numbers.

Alaoglu and Erdős

THEOREM 10. *If n_ϵ is the colossally abundant number associated with ϵ, and if $k_q(\epsilon)$ is the exponent of the prime q, then*

$$k_q(\epsilon) = [\log \{(q^{1+\epsilon} - 1)/(q^\epsilon - 1)\}/\log q] - 1.$$

This shows that the error term in Theorem 4 is nearly the best possible. Here $[x]$ denotes the greatest integer less than x.

The numbers n_ϵ and $k_q(\epsilon)$ do not decrease as ϵ decreases. Since $\log \{(q^{1+\epsilon}-1)/(q^\epsilon-1)\}/\log q$ is a continuous function of ϵ, $k_q(\epsilon)$ will increase by steps of at most 1, and this will occur when $\log \{(q^{1+\epsilon}-1)/(q^\epsilon-1)\}/\log q$ is an integer. But this makes q^ϵ rational. It is very likely that q^x and p^x can not be rational at the same time except if x is an integer. This would show that the quotient of two consecutive colossally abundant numbers is a prime. At present we can not show this. Professor Siegel has communicated to us the result that q^x, r^x and s^x can not be simultaneously rational except if x is an integer. Hence *the quotient of two consecutive colossally abundant numbers is either a prime or the product of two distinct primes.*

2^t may be algebraic and t not an integer

Example : For $t = \dfrac{\log 1729}{\log 2}$, we have $2^t = 1729 \in \mathbf{Z}$, but

$$3^t = \exp((\log 3)(\log 1729)/\log 2) = 135451.447153\ldots$$

is not an integer.

Also 3^t is an integer for some values of t not in $\mathbf{Z}$ – but can they be the same ?

Let t be a real number such that 2^t and 3^t are integers. Does it follow that t is a positive integer ?

The origin of the four exponentials conjecture

Conjecture : *Let p and q be two distinct prime numbers. If t is a real number such that p^t and q^t are both algebraic, then t is a rational number.*

Theorem : *If the three numbers p^t, q^t and r^t are algebraic for three distinct primes p, q, r, then t is a rational number.*

Corollary : *The quotient of two consecutive colossally abundant numbers is either a prime or a product of two distinct primes.*

The six exponentials theorem

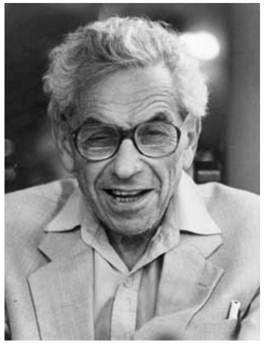
Siegel, Selberg, Lang, Ramachandra

At the invitation of Norwegian mathematician Atle Selberg, Ramachandra went to the Institute of Advanced Study at Princeton, USA , as a visiting professor and spent a period of six months.

This was Ramachandra first foreign trip and years later when Ramachandra constructed his house in Bangalore, he named it “Selberg House” in honor of Atle Selberg.

“With Siegel’s blessings, I was able to prove some results,” Ramachandra said, referring to the German mathematician C. L. Siegel, as he spoke about his theorems in transcendental number theory.

Pál Erdős and Atle Selberg



Paul Erdős
1913 - 1996

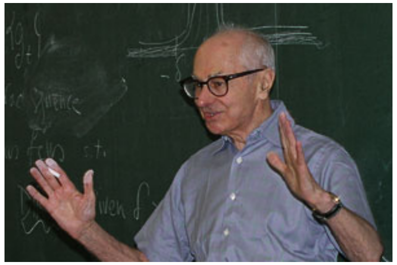
Atle Selberg
1917 - 2007

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Erdos.html

Mathematicians/Selberg.html

Carl Ludwig Siegel and Serge Lang

Carl Ludwig Siegel
(1896 - 1981)



Serge Lang
(1927 - 2005)

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Siegel.html

Mathematicians/Lang.html

Four exponentials Conjecture

Set $2^t = a$ and $3^t = b$. Then the determinant

$$\begin{vmatrix} \log 2 & \log 3 \\ \log a & \log b \end{vmatrix}$$

vanishes.

Four exponentials Conjecture. *Let*

$$\begin{pmatrix} \log \alpha_1 & \log \alpha_2 \\ \log \beta_1 & \log \beta_2 \end{pmatrix}$$

be a 2×2 matrix whose entries are logarithms of algebraic numbers. Assume the two columns are $\mathbf{Q}$ -linearly independent and the two rows are also $\mathbf{Q}$ -linearly independent. Then the matrix is regular.

Four exponentials Conjecture and first problem of Schneider

Problem 1 :

$$\frac{\log \alpha_1}{\log \alpha_2} = \frac{\log \alpha_3}{\log \alpha_4}$$

Problem 8 : *One at least of the two numbers*

$$e^e, \; e^{e^2}.$$

is transcendental.

Introduction aux Nombres Transcendants

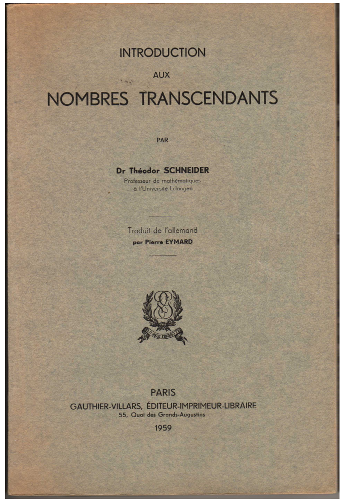
Theodor Schneider

Einführung in die Transzendenten Zahlen

Springer, 1957.

Traduction française par Pierre Eymard

Gauthier-Villars, 1959



Four exponentials Conjecture and Six exponentials Theorem

Conjecture. *Let* x_1, x_2 *be* $\mathbf{Q}$*–linearly independent complex numbers and* y_1, y_2 *be also* $\mathbf{Q}$*–linearly independent complex numbers. Then one at least of the four numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_2y_1},\ e^{x_2y_2}$$

is transcendental.

Theorem. *Let* d *and* ℓ *be positive integers with* $d\ell > d+\ell$. *Let* $x_1, \dots, x_d$ *be* $\mathbf{Q}$*–linearly independent complex numbers and* $y_1, \dots, y_\ell$ *be also* $\mathbf{Q}$*–linearly independent complex numbers. Then one at least of the* $d\ell$ *numbers*

$$e^{x_iy_j}, \quad (1 \le i \le d,\ 1 \le j \le \ell)$$

is transcendental.

Six exponentials Theorem

Theorem (Siegel, Lang, Ramachandra). *Let*

$$\begin{pmatrix} \log\alpha_1 & \log\alpha_2 & \log\alpha_3 \\ \log\beta_1 & \log\beta_2 & \log\beta_3 \end{pmatrix}$$

be a 2 *by* 3 *matrix whose entries are logarithms of algebraic numbers. Assume the three columns are linearly independent over* $\mathbf{Q}$ *and the two rows are also linearly independent over* $\mathbf{Q}$. *Then the matrix has rank* 2.

Density of an additive subgroup of $\mathbf{R}$

Kronecker : The additive group

$$\mathbf{Z} + \mathbf{Z}\theta = \left\{a + b\theta\ ;\ (a,b) \in \mathbf{Z}^2\right\}$$

is dense in $\mathbf{R}$ if and only if θ is irrational (means : 1 and θ are $\mathbf{Q}$ linearly independent).

Example : $\mathbf{Z} + \mathbf{Z}e$ and $\mathbf{Z} + \mathbf{Z}\pi$ are dense in $\mathbf{R}$.

Also $\mathbf{Z} + \mathbf{Z}e\pi + \mathbf{Z}(e+\pi)$ is dense in $\mathbf{R}$. Hence there exists a subgroup of rank 2 which is also dense. But no one knows how to produce one explicitly.

Density of an additive subgroup of $\mathbf{R}^n$

Kronecker : Let $\theta_1, \dots, \theta_n$ be real numbers. *Then the subgroup*

$$\mathbf{Z}^n + \mathbf{Z}(\theta_1, \dots, \theta_n)$$

of $\mathbf{R}^n$ *is dense if and only if the numbers* $1, \theta_1, \dots, \theta_n$ *are linearly independent over* $\mathbf{Q}$.

Footnote : According to his own taste, the reader will find a reference either in
N. Bourbaki, *Eléments de Mathématique*, Topologie Générale, Herman 1974, Chap. VII, § 1, N°1, Prop. 2 ;
or else in
G.H. Hardy and A.M. Wright, *An Introduction to the Theory of Numbers*, Oxford Sci. Publl., 1938, Chap. XXIII.

Additive subgroups of $\mathbf{R}^n$

$$\mathbf{Z}^n + \mathbf{Z}(\theta_1, \dots, \theta_n) \subset \mathbf{R}^n$$

is the set of tuples

$$(a_1 + a_0\theta_1, \dots, a_n + a_0\theta^n)$$

with $(a_0, a_1, \dots, a_n) \in \mathbf{Z}^{n+1}$.

Multiplicative groups of $(\mathbf{R}_+^\times)^n$

Multiplicative analog : for positive real numbers α_{ij}, $(1 \le i \le n,\ 1 \le j \le n+1)$ consider the set of tuples

$$\left(\alpha_{i,1}^{a_1} \cdots \alpha_{i,n+1}^{a_{n+1}}\right)_{1 \le i \le n} \in (\mathbf{R}_+^\times)^n$$

with $(a_1, a_2, \dots, a_{n+1}) \in \mathbf{Z}^{n+1}$.

Additive vs multiplicative groups :
Take exponential or logarithm and change basis.

Density of a multiplicative subgroup of $\mathbf{R}_+^\times$

The multiplicative subgroup of rank 1 of $\mathbf{R}_+^\times$ generated by 2 :

$$\left\{\dots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1,\ 2,\ 4,\ 8,\ 16, \dots\right\}$$

is not dense in $\mathbf{R}_+^\times$,

but the subgroup of rank 2 generated by $2, 3$, namely

$$\left\{2^a 3^b \ ;\ (a,b) \in \mathbf{Z}^2\right\}$$

is dense in $\mathbf{R}_+^\times$.

Explanation : *the number* $(\log 2)/(\log 3)$ *is irrational.*

Density of a multiplicative subgroup of $(\mathbf{R}_+^\times)^2$

Let α_i and β_i be positive real numbers. The multiplicative subgroup generated by (α_1, β_1), (α_2, β_2), (α_3, β_3) in $(\mathbf{R}_+^\times)^2$, namely the set of

$$\left(\alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3},\ \beta_1^{a_1}\beta_2^{a_2}\beta_3^{a_3}\right)$$

for $(a_1, a_2, a_3) \in \mathbf{Z}^2$, is dense in $(\mathbf{R}_+^\times)^2$ if and only if, for any $(s_1, s_2, s_3) \in \mathbf{Z}^3 \setminus \{0\}$, the 3×3 matrix

$$\begin{pmatrix} \log\alpha_1 & \log\alpha_2 & \log\alpha_3 \\ \log\beta_1 & \log\beta_2 & \log\beta_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

has maximal rank 3.

A question from transcendental number theory

Equivalent :
the matrix

$$\begin{pmatrix} s_3 \log \alpha_1 - s_1 \log \alpha_3 & s_3 \log \alpha_2 - s_2 \log \alpha_3 \\ s_3 \log \beta_1 - s_1 \log \beta_3 & s_3 \log \beta_2 - s_2 \log \beta_3 \end{pmatrix}$$

has maximal rank 2.

A fundamental problem is to study the rank of matrices with entries which are logarithms of algebraic numbers : Ramachandra initiated such investigation by means of transcendence methods.

Example of an open question

For $\alpha = a + b\sqrt{2} \in \mathbf{Q}(\sqrt{2})$, write $\overline{\alpha} = a - b\sqrt{2}$.

Define

$$\alpha_1 := 2\sqrt{2} - 1, \quad \alpha_2 := 3\sqrt{2} - 1, \quad \alpha_2 := 4\sqrt{2} - 1,$$

and let Γ be the set of elements in $(\mathbf{R}^\times)^2$ of the form

$$\left(\alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3},\ \overline{\alpha}_1^{a_1}\overline{\alpha}_2^{a_2}\overline{\alpha}_3^{a_3}\right)$$

with $(a_1, a_2, a_3) \in \mathbf{Z}^3$.

Question : *Is Γ dense in $(\mathbf{R}^\times)^2$?*

Algebraic independence of logarithms of algebraic numbers

One of the main problems in transcendental number theory is to prove that $\mathbf{Q}$ *–linearly independent logarithms of algebraic numbers are algebraically independent*. Such a result would solve the question of the rank of matrices having entries in the space of logarithms of algebraic numbers.

Baker's results provide a satisfactory answer for the *linear independence of such numbers over the field of algebraic numbers*. But he says nothing about algebraic independence.

Schanuel's Conjecture

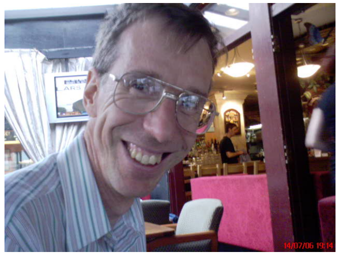
Let $x_1, \dots, x_n$ be $\mathbf{Q}$-linearly independent complex numbers. Then at least n of the $2n$ numbers $x_1, \dots, x_n,\ e^{x_1}, \dots, e^{x_n}$ are algebraically independent.

In other terms, the conclusion is

$$\operatorname{tr\,deg}_{\mathbf{Q}} \mathbf{Q}\left(x_1, \dots, x_n,\ e^{x_1}, \dots, e^{x_n}\right) \geq n.$$

Dale Brownawell and Stephen Schanuel

Roy's Theorem (1999)



The conjecture on algebraic independence of logarithms is equivalent to the question of the rank of matrices with determinants logarithms of algebraic numbers.

A result of Damien Roy

Let k be a field and $P \in k[X_1, \ldots, X_n]$ a polynomial in n variables. Then there exists a square matrix M, whose coefficients are linear polynomials in $1, X_1, \ldots, X_n$, such that P is the determinant of M.

The Strong Six Exponentials Theorem

Denote by $\mathcal{L}$ the $\mathbf{Q}$-vector subspace of $\mathbf{C}$ of logarithms of algebraic numbers : it consists of the complex numbers λ for which e^λ is algebraic. Further denote by $\widetilde{\mathcal{L}}$ the $\overline{\mathbf{Q}}$-vector space spanned by 1 and $\mathcal{L}$: hence $\widetilde{\mathcal{L}}$ is the set of linear combinations with algebraic coefficients of logarithms of algebraic numbers :

$$\widetilde{\mathcal{L}} = \{\beta_0 + \beta_1\lambda_1 + \cdots + \beta_n\lambda_n \ ; \ n \geq 0, \beta_i \in \overline{\mathbf{Q}}, \ \lambda_i \in \mathcal{L}\}.$$

Theorem (D.Roy). *If x_1, x_2 are $\overline{\mathbf{Q}}$–linearly independent complex numbers and y_1, y_2, y_3 are $\overline{\mathbf{Q}}$–linearly independent complex numbers, then one at least of the six numbers*

$$x_1y_1, \ x_1y_2, \ x_1y_3, \ \ x_2y_1, \ x_2y_2, \ x_2y_3$$

is not in $\widetilde{\mathcal{L}}$.

The Strong Four Exponentials Conjecture

Conjecture. *If* x_1, x_2 *are* $\overline{\mathbf{Q}}$*–linearly independent complex numbers and* y_1, y_2 *are* $\overline{\mathbf{Q}}$*–linearly independent complex numbers, then one at least of the four numbers*

$$x_1y_1, \; x_1y_2, \; x_2y_1, \; x_2y_2$$

is not in $\widetilde{\mathcal{L}}$.

Lower bound for the rank of matrices

- Rank of matrices. An alternate form of the strong Six Exponentials Theorem (resp. the strong Four Exponentials Conjecture) is the fact that *a* 2×3 *(resp.* 2×2*) matrix with entries in* $\widetilde{\mathcal{L}}$

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix} \qquad \textit{(resp.} \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \quad \textit{)},$$

the rows of which are linearly independent over $\overline{\mathbf{Q}}$ *and the columns of which are also linearly independent over* $\overline{\mathbf{Q}}$*, has maximal rank* 2.

The strong Six Exponentials Theorem

References :

D. Roy – « Matrices whose coefficients are linear forms in logarithms », *J. Number Theory* **41** (1992), no. 1, p. 22–47.

M. Waldschmidt – *Diophantine approximation on linear algebraic groups*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. **326**, Springer-Verlag, Berlin, 2000.

Further results by Ramachandra in his Acta Arithmetica paper of 1968

Elliptic functions : Weierstrass $\wp$–functions ζ–functions, σ–functions.

Further open problems in Ramachandra's paper.

Development of Ramachandra's contributions

Generalizations : abelian varieties, extensions of abelian varieties by the additive or multiplicative group, and more generally by commutative linear groups.

Density of rational points on varieties ; Mazur's problems

Periods of $K3$ surfaces (2003 with H. Shiga)

A special case of a conjecture by Ramachandra : problem of Mahler and Manin, solved by K. Barré–Siriex, G. Diaz, F. Gramain and G. Philibert (1996).

References on density questions related to the four exponentials Conjecture and the six exponentials Theorem

- C. Bertolin – « Périodes de 1-motifs et transcendance », *J. Number Theory* **97** (2002), no. 2, p. 204–221.
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Ramachandra's contributions to transcendental number theory

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