St Aloysius College, Edathua, Kerala, India December 29, 2020.

Webinar Series by Department of Mathematics, SAC.

Srinivasa Ramanujan
His life and his work

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Abstract

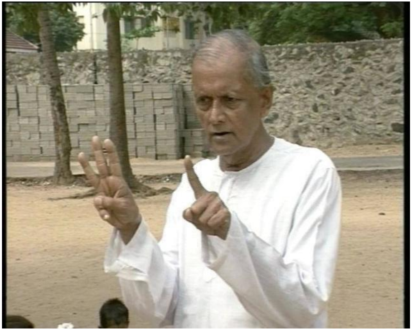
This lecture includes a few biographical informations about Srinivasan Ramanujan. Among the topics which we discuss are Euler constant, nested roots, divergent series, Ramanujan – Nagell equation, partitions, Ramanujan tau function, Hardy Littlewood and the circle method, highly composite numbers and transcendence theory, the number π, and the lost notebook.

Srinivasa Ramanujan

Erode December 22, 1887 —

Chetput, (Madras), April 26, 1920

P.K. Srinivasan (November 4, 1924-June 20, 2005)



PKS was the first biographer of Srinivas Ramanujan.

The Hindu, November 1, 2009
Passion for numbers by Soudhamini

http://beta.thehindu.com/education/article41732.ece

Biography of Srinivasa Ramanujan

(December 22, 1887 — April 26, 1920)

1887: born in Erode (near Tanjore)

1894-1903: school in Kumbakonam

In 1900 he began to work on his own on mathematics summing geometric and arithmetic series.

Gopuram Sarangapani Kumbakonam

Sarangapani Sannidhi Street Kumbakonam

Ramanujan House Kumbakonam

Ramanujan House in Kumbakonam

Ramanujan House Kumbakonam

Town High School Kumbakonam

Town High School Kumbakonam

1903: G.S.Carr - *A synopsis of elementary results — a book on pure mathematics* (1886) $5\,000$ formulae

$$\sqrt{x} + y = 7, \qquad x + \sqrt{y} = 11$$

$$x = 9, \qquad y = 4.$$

Biography *(continued)*

1903 (December): exam at Madras University

1904 (January): enters Government Arts College, Kumbakonam

Sri K. Ranganatha Rao Prize

Subrahmanyam scholarship

MacTutor History of Mathematics

http://www-history.mcs.st-andrews.ac.uk/

By 1904 Ramanujan had begun to undertake deep research. He investigated the series

$$\sum_n \frac{1}{n}$$

and calculated Euler's constant to 15 decimal places.

He began to study the Bernoulli numbers, although this was entirely his own independent discovery.

Euler constant

$$S_N = \sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N}$$

$$\int_1^N \frac{dx}{x+1} < S_N < 1 + \int_1^N \frac{dx}{x}$$

$$\gamma = \lim_{N \to \infty} (S_N - \log N).$$

Reference

Jeffrey C. Lagarias
Euler's constant: Euler's work and modern developments
Bulletin Amer. Math. Soc.
50 (2013), No. 4, 527–628.

arXiv:1303.1856 [math.NT]
Bibliography: 314 references.

M.W. *Is the Euler constant a rational number, an algebraic irrational number or else a transcendental number?*

http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/EulerConstantVI.pdf

Euler archives and Eneström index

http://eulerarchive.maa.org/

Gustaf Eneström (1852–1923)
Die Schriften Euler's chronologisch nach den Jahren geordnet, in denen sie verfasst worden sind
Jahresbericht der Deutschen Mathematiker–Vereinigung, 1913.


Gustaf Eneström.
Efter fotografi.

http://www.math.dartmouth.edu/~euler/index/enestrom.html http://www.eulerarchive.org/

(Reference [86] of the text by Lagarias)

http://www.eulerarchive.org/

(Reference [86] of the text by Lagarias)

Harmonic numbers

$$H_1 = 1, \quad H_2 = 1 + \frac{1}{2} = \frac{3}{2}, \quad H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6},$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{j=1}^{n} \frac{1}{j}.$$

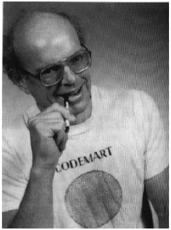
Sequence:

$$1, \quad \frac{3}{2}, \quad \frac{11}{6}, \quad \frac{25}{12}, \quad \frac{137}{60}, \quad \frac{49}{20}, \quad \frac{363}{140}, \quad \frac{761}{280}, \quad \frac{7129}{2520}, \ldots$$

The online encyclopaedia of integer sequences

https://oeis.org/

Neil J. A. Sloane



Numerators and denominators

Numerators: https://oeis.org/A001008

$$1, 3, 11, 25, 137, 49, 363, 761, 7129, 7381, 83711, 86021, 1145993,$$

$$1171733, 1195757, 2436559, 42142223, 14274301, 275295799,$$

$$55835135, 18858053, 19093197, 444316699, 1347822955, \ldots$$

Denominators: https://oeis.org/A002805

$$1, 2, 6, 12, 60, 20, 140, 280, 2520, 2520, 27720, 27720, 360360,$$

$$360360, 360360, 720720, 12252240, 4084080, 77597520,$$

$$15519504, 5173168, 5173168, 118982864, 356948592, \ldots$$

Euler (1731)

De progressionibus harmonicis observationes

The sequence

$$H_n - \log n$$

has a limit $\gamma = 0,577\,21\underline{8}\ldots$
when n tends to infinity.

Leonhard Euler
(1707–1783)

Moreover,

$$\gamma = \sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m}.$$

Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

Euler: $s \in \mathbb{R}$.

Riemann: $s \in \mathbb{C}$.

Numerical value of Euler's constant

The online encyclopaedia of integer sequences
https://oeis.org/A001620
Decimal expansion of Euler's constant
(or Euler–Mascheroni constant) gamma.

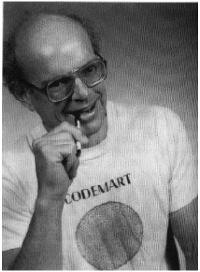
Yee (2010) computed $29\,844\,489\,545$ decimal digits of gamma.

$$\gamma = 0,577\,215\,664\,901\,532\,860\,606\,512\,090\,082\,402\,431\,042\ \dots$$

Euler constant

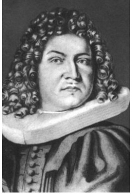
Euler–Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) = 0.577\,215\,664\,9\ldots$$



Neil J. A. Sloane's encyclopaedia
http://www.research.att.com/∼njas/sequences/A001620

Bernoulli numbers



$$B_0 = 1, \qquad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad \text{for } n > 1.$$

Jacob Bernoulli (1654 – 1705)

$$B_0 + 2B_1 = 0 \qquad B_1 = -\frac{1}{2}$$

$$B_0 + 3B_1 + 3B_2 = 0 \qquad B_2 = \frac{1}{6}$$

$$B_0 + 4B_1 + 6B_2 + 4B_3 = 0 \qquad B_3 = 0$$

$$B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 = 0 \qquad B_4 = -\frac{1}{30}$$

$$\vdots$$

Sloane A027642 and A000367

Kumbakonam

1905: Fails final exam

1906: Enters Pachaiyappa's College, Madras

Ill, goes back to Kumbakonam

1907 (December): Fails final exam.

1908: continued fractions and divergent series

1909 (April): underwent an operation

1909 (July 14): marriage with S Janaki Ammal (1900—1994)

S Janaki Ammal

Madras

1910: meets Ramaswami Aiyar

1911: first mathematical paper

1912: clerk office, Madras Port Trust — Sir Francis Spring and Sir Gilbert Walker get a scholarship for him from the University of Madras starting May 1913 for 2 years.

Nested roots

Journal of the Indian Mathematical Society (1912) – problems solved by Ramanujan

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}} = ?$$

$$\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\cdots}}}} = ?$$

Answers from Ramanujan

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}} = 3$$

$$\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\cdots}}}} = 4$$

“Proofs” $n(n+2)$

$$(n+2)^2 = 1 + (n+1)(n+3)$$

$$n(n+2) = n\sqrt{1 + (n+1)(n+3)}$$

$$f(n) = n(n+2)$$

$$f(n) = n\sqrt{1 + f(n+1)}$$

$$f(n) = n\sqrt{1 + (n+1)\sqrt{1 + f(n+2)}}$$

"Proofs" $n(n+3)$

$$(n+3)^2 = n + 5 + (n+1)(n+4)$$

$$n(n+3) = n\sqrt{n + 5 + (n+1)(n+4)}$$

$$g(n) = n(n+3)$$

$$g(n) = n\sqrt{n + 5 + g(n+1)}$$

$$g(n) = n\sqrt{n + 5 + (n+1)\sqrt{n + 6 + g(n+2)}}$$

Letter of S. Ramanujan to M.J.M. Hill in 1912

$$1 + 2 + 3 + \cdots + \infty = -\frac{1}{12}$$

$$1^2 + 2^2 + 3^2 + \cdots + \infty^2 = 0$$

$$1^3 + 2^3 + 3^3 + \cdots + \infty^3 = \frac{1}{120}$$

Answer of M.J.M. Hill in 1912

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Renormalisation of divergent series

Leonhard Euler

(1707 – 1783)

Introductio in analysin infinitorum

(1748)

Euler

Values of Riemann zeta function at negative integers :

$$\zeta(-k) = -\frac{B_{k+1}}{k+1} \qquad (n \geq 1)$$

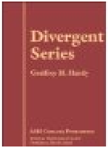
$$\zeta(-2n) = 1^{2n} + 2^{2n} + 3^{2n} + 4^{2n} + \cdots = 0 \qquad (n \geq 1)$$

$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

$$\zeta(-3) = 1^3 + 2^3 + 3^3 + 4^3 + \cdots = \frac{1}{120}$$

$$\zeta(-5) = 1^5 + 2^5 + 3^5 + 4^5 + \cdots = -\frac{1}{252}$$

G.H. Hardy: Divergent Series (1949)



Niels Henrik Abel
(1802 – 1829)

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.

Letters to H.F. Baker and E.W. Hobson in 1912

No answer to his letters to H.F. Baker and E.W. Hobson in 1912...

Letter of Ramanujan to Hardy (January 16, 1913)

I have had no university education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at mathematics. I have not trodden through the conventional regular course which is followed in a university course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as "startling".

Godfrey Harold Hardy (1877 – 1947)

John Edensor Littlewood (1885 – 1977)

Hardy and Littlewood

Letter from Ramanujan to Hardy (January 16, 1913)

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$$

$$1 - 1! + 2! - 3! + \cdots = .596\cdots$$

Answer from Hardy (February 8, 1913)

I was exceedingly interested by your letter and by the theorems which you state. You will however understand that, before I can judge properly of the value of what you have done, it is essential that I should see proofs of some of your assertions. Your results seem to me to fall into roughly three classes:

(1) there are a number of results that are already known, or easily deducible from known theorems;

(2) there are results which, so far as I know, are new and interesting, but interesting rather from their curiosity and apparent difficulty than their importance;

(3) there are results which appear to be new and important. . .

1913–1920

1913, February 27: New letter from Ramanujan to Hardy

1913: Visit of Neville to India

1914, March 17 to April 14: travel to Cambridge.

1918: (May) Fellow of the Royal Society
(November) Fellow of Trinity College, Cambridge.

1919, February 27 to March 13: travel back to India.

An interesting street number

The puzzle itself was about a street in the town of Louvain in Belgium, where houses are numbered consecutively. One of the house numbers had the peculiar property that the total of the numbers lower than it was exactly equal to the total of the numbers above it. Furthermore, the mysterious house number was greater than 50 but less than 500.

Prasanta Chandra Mahalanobis

1893 – 1972

Srinivasa Ramanujan

1887 – 1920

http://mathshistory.st-andrews.ac.uk/Biographies/Mahalanobis.html

https://www.math.auckland.ac.nz/~butcher/miniature/miniature2.pdf

Balancing numbers

Answer: house number 204 in a street with 288 houses.
Sequence of balancing numbers (number of the house)
https://oeis.org/A001109

$0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, 7997214\dots$

This is a linear recurrence sequence

$$u_{n+1} = 6u_n - u_{n-1}$$

with the initial conditions $u_0 = 0$, $u_1 = 1$.
The number of houses is https://oeis.org/A001108

$$0, 1, 8, 49, 288, 1681, 9800, 57121, 332928, 1940449, \dots$$

M.W. *The square root of 2, the Golden ratio and the Fibonacci sequence.* http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/sqrt2.pdf
YouTube: https://www.youtube.com/watch?v=CIQ0w3SRetE&t=247s

Ramanujan – Taxi Cab Number 1729

Hardy's obituary of Ramanujan:

> *I had ridden in taxi-cab No* 1729*, and remarked that the number (*$7 \cdot 13 \cdot 19$*) seemed to me a rather dull one...*

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

$$12^3 = 1728, \qquad 9^3 = 729$$

Narendra Jadhav — Taxi Cab Number 1729

Narendra Jadhav (born 1953), Member of Rajya Sabha, the upper house of Indian Parliament.

Former Vice-Chancellor (from 24 August 2006 to 15 June 2009) of Savitribai Phule Pune University.
Author of *Outcaste – A Memoir, Life and Triumphs of an Untouchable Family In India* (2003).

http://www.drnarendrajadhav.info

Ramanujan – Taxi Cab Number 1729

$$12^3 = 1728, \qquad 9^3 = 729$$

$$50 = 7^2 + 1^2 = 5^2 + 5^2$$

$$\begin{aligned} 4\,104 &= 2^3 + 16^3 = 9^3 + 15^3 \\ 13\,832 &= 2^3 + 24^3 = 18^3 + 20^3 \\ 40\,033 &= 9^3 + 34^3 = 16^3 + 33^3 \\ &\vdots \end{aligned}$$

Leonhard Euler (1707 – 1783)

$$59^4 + 158^4 = 133^4 + 134^4 = 635\,318\,657$$

Diophantine equations

$$x^3 + y^3 + z^3 = w^3$$

$$(x, y, z, w) = (3, 4, 5, 6)$$

$$3^3 + 4^3 + 5^3 = 27 + 64 + 125 = 216 = 6^3$$

Parametric solution:

$$\begin{aligned} x &= 3a^2 + 5ab - 5b^2 & \qquad y &= 4a^2 - 4ab + 6b^2 \\ z &= 5a^2 - 5ab - 3b^2 & \qquad w &= 6a^2 - 4ab + 4b^2 \end{aligned}$$

Ramanujan – Nagell Equation

Trygve Nagell (1895 – 1988)

$$x^2 + 7 = 2^n$$

$$\begin{aligned} 1^2 + 7 &= 2^3 &= 8 \\ 3^2 + 7 &= 2^4 &= 16 \\ 5^2 + 7 &= 2^5 &= 32 \\ 11^2 + 7 &= 2^7 &= 128 \\ 181^2 + 7 &= 2^{15} &= 32\,768 \end{aligned}$$

$x^2 + D = 2^n$

Nagell (1948): for $D = 7$, no further solution

R. Apéry (1960): for $D > 0$, $D \neq 7$, the equation $x^2 + D = 2^n$ has at most 2 solutions.

Examples with 2 solutions:

$$D = 23: \qquad 3^2 + 23 = 32, \quad 45^2 + 23 = 2^{11} = 2\,048$$

$$D = 2^{\ell+1} - 1,\ \ell \geq 3: \qquad (2^\ell - 1)^2 + 2^{\ell+1} - 1 = 2^{2\ell}$$

$x^2 + D = 2^n$

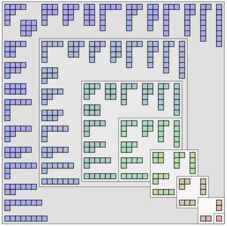
F. Beukers (1980): at most one solution otherwise.

M. Bennett (1995): considers the case $D < 0$.

Partitions

$$\begin{aligned} 1 & \\ 2 &= 1+1 \\ 3 &= 2+1 = 1+1+1 \\ 4 &= 3+1 = 2+2 = 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

$p(1) = 1$
$p(2) = 2$
$p(3) = 3$
$p(4) = 5$



$$p(5) = 7, \quad p(6) = 11, \quad p(7) = 15, \ldots$$

MacMahon: table of the first 200 values

Neil J. A. Sloane's encyclopaedia
http://www.research.att.com/~njas/sequences/A000041

Ramanujan

$p(5n+4)$ is a multiple of 5

$p(7n+5)$ is a multiple of 7

$p(11n+6)$ is a multiple of 11

$p(25n+24)$ is a multiple of 25

$p(49n+47)$ is a multiple of 49

$p(121n+116)$ is a multiple of 121

Ramanujan conjecture

If $m = 5^a 7^b 11^c$ and $24\ell - 1$ is a multiple of m, then $p(nm + \ell)$ is a multiple of m for all n.

Hansraj Gupta : $p(243) = 133\,978\,259\,344\,888$ is not a multiple of 7^3.
S. Chowla : As $24 \cdot 243 - 1$ is divisible by 7^3, the above conjecture is false.

S. Chowla, . "Congruence Properties of Partitions." J. London Math. Soc. **9**, 247, 1934.

https://doi.org/10.1112/jlms/s1-9.4.247a

Sarvadaman Chowla
(1907–1995)

CONGRUENCE PROPERTIES OF PARTITIONS

S. CHOWLA*.

Let $p(n)$ denote the number of unrestricted partitions of n. Ramanujan conjectured† that

If $\delta = 5^a 7^b 11^c$ *and* $24\lambda \equiv 1 \pmod{\delta}$, *then*

$$p(\lambda),\ p(\lambda+\delta),\ p(\lambda+2\delta),\ \ldots \equiv 0 \quad (\bmod\ \delta).$$

This result has been proved in a number of special cases, *e.g.* when‡

$$\delta = 5,\ 7,\ 11,\ 5^2,\ 7^2,\ 11^2,\ 5^3.$$

A table of partitions recently calculated by H. Gupta§ shows that *the above conjecture is false for* $\delta = 7^3$. In fact,

$$24 . 243 = 24(343-100) \equiv -2400 = -7^4+1 \equiv 1(7^3),$$

but
$$p(243) = 133978259344888,$$

and it is easily verified that $p(243) \equiv 0(7^2)$ but $p(243) \not\equiv 0(7^3)$.

Partitions - Ken Ono

Manjul Bhargava, left, and Ken Ono in front of Ramanujan's house.

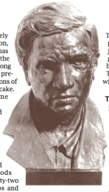
Honoring a Gift from Kumbakonam

Ken Ono

Today was an absolutely glorious day in Madison, Wisconsin. It is Christmas 2005, and everyone in the house is asleep after a long day of enjoying family, opening presents, and eating enormous portions of mashed potatoes and yule log cake. Yet powerful images keep me awake.

Thirty-six hours ago I returned from a six-day whirlwind journey to a far-off place. I spent forty hours on airplanes, and I endured fourteen hours in cars dodging bicycles, rickshaws, cows, goats, and masses of people on roads severely damaged by recent flooding. These floods would be blamed[1] for at least forty-two deaths. Despite these hardships and bad luck, this adventure exceeded my lofty expectations.

I ostensibly travelled to Kumbakonam with the purpose of giving a



Bust of Srinivasa Ramanujan by artist Paul Granlund.

This adventure was a pilgrimage to pay homage to Srinivasa Ramanujan, the Indian legend whose congruences, formulas, and identities have inspired much of my own work. This fulfilled a personal journey, one with an unlikely beginning in 1984.

The Story of Ramanujan

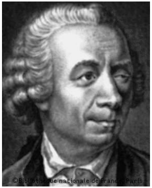
Ramanujan was born on December 22, 1887, in Erode, a small town about 250 miles southwest of Chennai (formerly known as Madras). He was a *Brahmin*, a member of India's priestly caste, and as a consequence he lived his life as a strict vegetarian.

When Ramanujan was one year old, he moved to Kumbakonam, a small town about 170 miles south of Chennai, where his father Srinivasa was a cloth merchant's clerk. Kumbakonam, which is situated on the banks of the sacred Kaveri River,

Notices of the AMS, **53** N°6 (July 2006), 640–651

http://www.ams.org/notices/200606/fea-ono.pdf

Leonhard Euler (1707 – 1783)



$$1 + p(1)x + p(2)x^2 + \cdots + p(n)x^n + \cdots$$

$$= \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^n)\cdots}$$

$$1 + \sum_{n=1}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} (1-x^n)^{-1}$$

Eulerian products

Riemann zeta function

For $s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - p^{-s}\right)^{-1}$$

Georg Friedrich Bernhard Riemann (1826 - 1866)

Circle method

Srinivasa Ramanujan
(1887 – 1920)

G.H. Hardy
(1877 – 1947)

J.E. Littlewood
(1885 – 1977)

Hardy, ICM Stockholm, 1916
Hardy and Ramanujan (1918): partitions
Hardy and Littlewood (1920 – 1928):
Some problems in Partitio Numerorum

Circle method

This method was further developed by Hardy, Littlewood, Rademacher, Vinogradov, Davenport,. . .
This gave rise to:

- Large Sieve
- Ternary Goldbach Conjecture
- Progress on binary Goldbach Conjecture
- Waring's problem

All these problems looked beyond reach before the birth of Circle method.

Ramanujan tau function

$$x(1-x)^{-1} = \sum_{n=1}^{\infty} x^n$$

$$x \prod_{n=1}^{\infty} (1-x^n)^{24} = \sum_{n=1}^{\infty} \tau(n) x^n.$$

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_{p} \left(1 - \tau(p) p^{-s} + p^{11-2s}\right)^{-1}$$

Ramanujan's Congruences

$\tau(pn)$ is divisible by p for $p = 2,\ 3,\ 5,\ 7,\ 23.$

also: congruences modulo 691
(numerator of Bernoulli number B_{12})

Pierre Deligne

Ramanujan's Conjecture,
proved by Deligne in 1974

$$|\tau(p)| < 2p^{11/2}$$

Hardy–Ramanujan

For almost all integers n, the number of prime factors of n is $\log\log n$.

$$A_\epsilon(x) = \{n \le x\ ;\ (1-\epsilon)\log\log n < \omega(n) < (1+\epsilon)\log\log n\}.$$

$$\frac{1}{x} A_\epsilon(x) \to 1 \quad \text{when} \quad x \to \infty.$$

Highly composite numbers

(Proc. London Math. Soc. 1915)

$$\begin{array}{rccccccccc} n = & 2 & 4 & 6 & 12 & 24 & 36 & 48 & 60 & 120\ldots \\ d(n) = & 2 & 3 & 4 & 6 & 8 & 9 & 10 & 12 & 16\ldots \end{array}$$

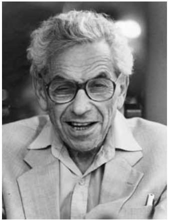
Question: *For $t \in \mathbf{R}$, do the conditions $2^t \in \mathbf{Z}$ and $3^t \in \mathbf{Z}$ imply $t \in \mathbf{Z}$?*

Example: For $t = \dfrac{\log 1729}{\log 2}$, we have $2^t = 1729 \in \mathbf{Z}$, but

$$3^t = \exp((\log 3)(\log 1729)/\log 2) = 135451.447153\ldots$$

is not an integer.

Pàl Erdős Carl Ludwig Siegel



Alaoglu and Erdős: *On highly composite and similar numbers*, 1944.

C.L. Siegel: *For* $t \in \mathbf{R}$, *the conditions* $2^t \in \mathbf{Z}$, $3^t \in \mathbf{Z}$ *and* $5^t \in \mathbf{Z}$ *imply* $t \in \mathbf{Z}$.

Serge Lang, K. Ramachandra: *six exponentials theorem, four exponentials conjecture.*

Five exponentials Theorem and generalizations

1985: Five exponentials Theorem.

1993, Damien Roy, Matrices whose coefficients are linear forms in logarithms.
J. Number Theory **41** (1992), no. 1, 22–47.

Approximation for π due to Ramanujan

$$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right) = 3.141592653\ 805688201898390006 30\ldots$$

$$\pi = 3.141592653\ 58979323846264338328\ldots$$

Another formula due to Ramanujan for π

$$\pi = \frac{9\,801}{\sqrt{8}} \left(\sum_{n=0}^{\infty} \frac{(4n)!(1\,103 + 26\,390n)}{(n!)^4 396^{4n}} \right)^{-1}$$

$n = 0$: 6 exact digits for $3.141592\ldots$

$n \to n+1$: 8 more digits

Ramanujan's formula for $1/\pi$

$$\frac{1}{\pi} = \sum_{m=0}^{\infty} \binom{2m}{m} \frac{42m+5}{2^{12m+4}}.$$

Decimals of π

Ramanujan's formulae were used in 1985: $1.7 \cdot 10^7$ digits for π (1.7 crores)

In 1999: $2 \cdot 10^{10}$ digits ($2\ 000$ crores)

18 Aug 2009: Pi Calculation Record Destroyed: 2.5 Trillion Decimals ($2.5 \cdot 10^{12}$).

$2,576,980,377,524$ decimal places in 73 hours 36 minutes

Massive parallel computer called : T2K Tsukuba System.

Team leader professor Daisuke Takahashi.

Ramanujan Notebooks

Written from 1903 to 1914

First: 16 chapters, 134 pages
Second: 21 chapters, 252 pages
Third: 33 pages

B.M. Wilson, G.N. Watson
Edited in 1957 in Bombay

The lost notebook

George Andrews, 1976

Bruce Berndt, 1985–87 (5 volumes)

Last work of Ramanujan

Mock theta functions

S. Zwegers – "Mock ϑ-functions and real analytic modular forms.", in *Berndt, Bruce C. (ed.) et al., q-series with applications to combinatorics, number theory, and physics. Proceedings of a conference, University of Illinois, Urbana-Champaign, IL, USA, October 26-28, 2000. Providence, RI: American Mathematical Society (AMS). Contemp. Math. 291, 269-277*, 2001.

SASTRA Ramanujan Prize

The SASTRA Ramanujan Prize, founded by Shanmugha Arts, Science, Technology & Research Academy (SASTRA) located near Kumbakonam, India, Srinivasa Ramanujan's hometown, is awarded every year to a young mathematician judged to have done outstanding work in Ramanujan's fields of interest. The age limit for the prize has been set at 32 (the age at which Ramanujan died), and the current award is $10,000.

https://sas.sastra.edu/ramanujan/Ramanujan-Awards.php

SASTRA Ramanujan Prize

Year	Name	Institution
2005	Manjul Bhargava Kannan Soundararajan	Princeton University University of Michigan
2006	Terence Tao	University of California at Los Angeles
2007	Ben Green	Cambridge University
2008	Akshay Venkatesh	Stanford University
2009	Kathrin Bringmann	University of Cologne, Germany, and University of Minnesota, USA
2010	Wei Zhang	Harvard University
2011	Roman Holowinsky[1]	Ohio State University
2012	Zhiwei Yun[2]	Stanford University
2013	Peter Scholze[3]	University of Bonn
2014	James Maynard [4]	Oxford University, England, and University of Montreal, Canada
2015	Jacob Tsimerman	University of Toronto, Canada
2016	Kaisa Matomäki Maksym Radziwill[5]	University of Turku, Finland; McGill University, Canada, and Rutgers University, USA
2017	Maryna Viazovska[6]	École Polytechnique Fédérale de Lausanne, Switzerland
2018	Yifeng Liu Jack Thorne[7]	Yale University, USA; Cambridge University, United Kingdom
2019	Adam Harper[8]	University of Warwick, England (UK)
2020	Shai Evra [9]	Princeton University, USA, and Hebrew University of Jerusalem, Israel

https://en.wikipedia.org/wiki/SASTRA_Ramanujan_Prize

ICTP Ramanujan Prize

The Ramanujan Prize

For young mathematicians from developing countries

The Ramanujan Prize for young mathematicians from developing countries has been awarded annually since 2005. It was originally instituted by ICTP, the Niels Henrik Abel Memorial Fund, and the International Mathematical Union (IMU). The participation of the Abel Fund ended in 2012; the Department of Science and Technology of the Government of India (DST) has now agreed to fund the Prize for a 5 year period, starting with the 2014 Prize.

The Prize is awarded annually to a researcher from a developing country who is less than 45 years of age on 31 December of the year of the award, and who has conducted outstanding research in a developing country. Researchers working in any branch of the mathematical sciences are eligible. The Prize carries a $15,000 cash award. The winner will be invited to ICTP to receive the Prize and deliver a lecture. The Prize is usually awarded to one person, but may be shared equally among recipients who have contributed to the same body of work.

The Selection Committee takes into account not only the scientific quality of the research, but also the background of the candidate and the environment in which the work was carried out. The Committee in particular favours candidates who have overcome adversity to achieve distinction in mathematics.

The Committee consists of eminent mathematicians appointed in consultation between ICTP, the IMU, and the DST.

https://www.ictp.it/about-ictp/prizes-awards/the-ramanujan-prize/the-ramanujan-prize-winners.aspx

ICTP Ramanujan Prize

The Ramanujan Prize Winners

2020	Carolina Araujo
2019	Hoàng Hiệp Phạm
2018	Ritabrata Munshi (India)
2017	Eduardo Teixeira (Brazil)
2016	Chenyang Xu (People's Republic of China)
2015	Amalendu Krishna (India)
2014	Miguel Walsh (Argentina)
2013	Ye Tian (People's Republic of China)
2012	Fernando Codá Marques (Brazil)
2011	Philibert Nang (Gabon)
2010	Yuguang Shi (People's Republic of China)
2009	Ernesto Lupercio (Mexico)
2008	Enrique R. Pujals (Brazil/Argentina)
2007	Jorge Lauret (Argentina)
2006	Sujatha Ramdorai (India)
2005	Marcelo Viana (Brazil)

References (continued)

Don Zagier (March 16, 2005, BNF/SMF) :

"Ramanujan to Hardy, from the first to the last letter..."
http://smf.emath.fr/VieSociete/Rencontres/BNF/2005/

MacTutor History of Mathematics
http://www-groups.dcs.st-and.ac.uk/~history/

Eric Weisstein worlds of mathematics, Wolfram Research
http://scienceworld.wolfram.com/

Wikipedia, the free encyclopedia.
http://en.wikipedia.org/wiki/Ramanujan

Ramanujan according to Wikipedia

Erode December 22, 1887 — Chetput, (Madras), April 26, 1920

Landau–Ramanujan constant
Mock theta functions
Ramanujan prime
Ramanujan–Soldner constant
Ramanujan theta function
Ramanujan's sum
Rogers–Ramanujan identities

http://en.wikipedia.org/wiki/Srinivasa_Ramanujan

Landau–Ramanujan constant

In mathematics, the Landau–Ramanujan constant occurs in a number theory result stating that the number of positive integers less than x which are the sum of two square numbers, for large x, varies as

$$x/\sqrt{\ln(x)}.$$

The constant of proportionality is the Landau–Ramanujan constant, which was discovered independently by Edmund Landau and Srinivasa Ramanujan.
More formally, if $N(x)$ is the number of positive integers less than x which are the sum of two squares, then

$$\lim_{x\to\infty} \frac{N(x)\sqrt{\ln(x)}}{x} \approx 0.76422365358922066299069873125.$$

Landau–Ramanujan constant

M.W. *Representation of integers by cyclotomic binary forms*.
Number Theory Web Seminar, Tuesday, May 12, 2020.
Link to recording and slides

https://sites.google.com/view/ntwebseminar/previous-talks

Étienne Fouvry

Claude Levesque

EF+CL+MW, *Representation of integers by cyclotomic binary forms.* Acta Arithmetica, **184**.1 (2018), 67 – 86.

EF+MW, *Sur la représentation des entiers par des formes cyclotomiques de grand degré.* Bull. Soc. Math. France, **148** (2020), 253–282.

International Conference on Number theory and Discrete Mathematics. Cochin

Web conference using Google Meet December 11–14, 2020.

Organized by the Ramanujan Mathematical Society (RMS)

and hosted by the Rajagiri School of Engineering & Technology (RSET), Kakkanad.

https://www.rajagiritech.ac.in/icntdm/index.asp

M.W. Lecture on *On some families of binary forms and the integers they represent.*
Monday, December 14, 12:00 IST = 7:30 am CST.

http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/FamiliesBinaryForms.pdf

Ramanujan primes

In mathematics, a Ramanujan prime is a prime number that satisfies a result proven by Srinivasa Ramanujan relating to the prime-counting function.

$$\pi(x) - \pi(x/2) \geq 1, 2, 3, 4, 5, \ldots \quad \text{for all} \quad x \geq 2, 11, 17, 29, 41, \ldots$$

respectively, where $\pi(x)$ is the prime-counting function, that is, the number of primes less than or equal to x.

Ramanujan primes

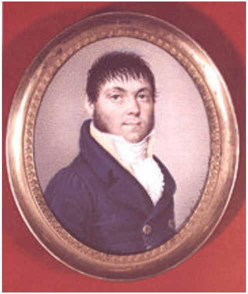
$$2,\ 11,\ 17,\ 29,\ 41,\ 47,\ 59,\ 67,\ 71,\ 97,\ 101,\ 107,\ 127,\ 149,$$
$$151,\ 167,\ 179,\ 181,\ 227,\ 229,\ 233,\ 239,\ 241,\ 263,\ 269,\ \ldots$$

$a(n)$ is the smallest number such that if $x \geq a(n)$, then $\pi(x) - \pi(x/2) \geq n$, where $\pi(x)$ is the number of primes $\leq x$.

$\pi(2) - \pi(1) = 1$, $a(1) = 2$ (Bertrand's Postulate).
$\pi(10) - \pi(5) = 1$, a single prime (namely 7) in $(5, 10]$;
$\pi(n) - \pi(n/2) \geq 2$ for $n \geq 11$, hence $a(2) = 11$.

Neil J. A. Sloane's encyclopaedia
http://www.research.att.com/~njas/sequences/A104272

Ramanujan–Soldner constant



In mathematics, the Ramanujan–Soldner constant is a mathematical constant defined as the unique positive zero of the logarithmic integral function. It is named after Srinivasa Ramanujan and Johann Georg von Soldner (16 July 1776 - 13 May 1833).

Its value is approximately

$$1.451369234883381050283968485892027449493\ldots$$

Ramanujan's sum (1918)

In number theory, a branch of mathematics, Ramanujan's sum, usually denoted $c_q(n)$, is a function of two positive integer variables q and n defined by the formula

$$c_q(n) = \sum_{\substack{1 \le a \le q \\ (a,q)=1}} e^{2\pi i \frac{a}{q} n}.$$

Ramanujan's sums are used in the proof of Vinogradov's theorem that every sufficiently-large odd number is the sum of three primes.

Ramanujan theta function

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2}$$

Carl Gustav Jacob Jacobi (1804–1851)

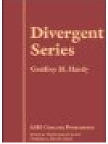
In mathematics, the Ramanujan theta function generalizes the form of the Jacobi theta functions, while capturing their general properties. In particular, the Jacobi triple product takes on a particularly elegant form when written in terms of the Ramanujan theta.

Rogers–Ramanujan identities

Leonard James Rogers
1862 - 1933

In mathematics, the Rogers–Ramanujan identities are a set of identities related to basic hypergeometric series. They were discovered by Leonard James Rogers (1894) and subsequently rediscovered by Srinivasa Ramanujan (1913) as well as by Issai Schur (1917).

G.H. Hardy: Divergent Series



In

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

set $z = -1$, as Euler does:

$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2}.$$

Similarly, from the derivative of the previous series

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

deduce

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}.$$

$s = 1 - 2 + 3 - 4 + \cdots$

There are further reasons to attribute the value $1/4$ to s. For instance

$$s = 1 - (1 - 1 + 1 - 1 + \cdots) - (1 - 2 + 3 - 4 + \cdots) = 1 - \frac{1}{2} - s$$

gives $2s = 1/2$, hence $s = 1/4$.

Also computing the square by expanding the product

$$(1 - 1 + 1 - 1 + \cdots)^2 = (1 - 1 + 1 - 1 + \cdots)(1 - 1 + 1 - 1 + \cdots)$$

yields to

$$1 - 2 + 3 - 4 + \cdots = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Cesaro convergence

For a series

$$a_0 + a_1 + \cdots + a_n + \cdots = s$$

converging (in the sense of Cauchy), the partial sums

$$s_n = a_0 + a_1 + \cdots + a_n$$

have a mean value

$$\frac{s_0 + \cdots + s_n}{n+1}$$

which is a sequence which converges (in the sense of Cauchy) to s.

For the diverging series

$$1 - 1 + 1 - 1 + \cdots$$

the limit exists and is $1/2$.

Cesaro convergence

Ernesto Cesàro
(1859 – 1906)

For the series

$$1 + 0 - 1 + 1 + 0 - 1 + \cdots$$

the Cesaro limit

$$\lim_{n\to\infty} \frac{s_0 + \cdots + s_n}{n+1}$$

exists and is $2/3$.

Rules for summing divergent series

$a_0 + a_1 + a_2 + \cdots = s$ implies $ka_0 + ka_1 + ka_2 + \cdots = ks.$

$a_0 + a_1 + a_2 + \cdots = s$ and $b_0 + b_1 + b_2 + \cdots = t$ implies
$a_0 + b_0 + a_1 + b_1 + a_2 + b_2 + \cdots = s + t.$

$a_0 + a_1 + a_2 + \cdots = s$ if and only if $a_1 + a_2 + \cdots = s - a_0.$

$1^2 - 2^2 + 3^2 - 4^2 + \cdots$

Recall

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

Take one more derivative, you find also

$$1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - 4 \cdot 5 + \cdots = \frac{1}{4}$$

from which you deduce

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots = 0.$$

Further examples of divergent Series

$$1 + 1 + 1 + \cdots = 0$$

$$1 - 2 + 4 - 8 + \cdots = \frac{1}{3}$$

$$1 + 2 + 4 + 8 + \cdots = -1$$

$$1^{2k} + 2^{2k} + 3^{2k} + 4^{2k} + 5^{2k} + \cdots = 0 \quad \text{for} \quad k \geq 1.$$

Euler:

$$1 - 1! + 2! - 3! + 4! + \cdots = -e(\gamma - 1 + \frac{1}{2 \cdot 2!} - \frac{1}{3 \cdot 3!} + \cdots)$$

gives the value $0.5963\ldots$ also found by Ramanujan.

Ramanujan's method (following Joseph Oesterlé)

Here is Ramanujan's method for computing the value of divergent series and for accelerating the convergence of series.

The series

$$a_0 - a_1 + a_2 - a_3 + \cdots$$

can be written

$$\frac{1}{2}\left(a_0 + (b_0 - b_1 + b_3 - b_4 + \cdots)\right)$$

where $b_n = a_n - a_{n+1}$.

Acceleration of convergence

For instance in the case $a_n = 1/n^s$ we have $b_n \sim s/n^{s+1}$. Repeating the process yields the analytic continuation of the Riemann zeta function.

For $s = -k$ where k is a positive integer, Ramanujan's method yields the Bernoulli numbers.

In the case of convergent series like

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2,$$

or for Euler constant, Ramanujan's method gives an efficient way of accelerating the convergence.

St Aloysius College, Edathua, Kerala, India December 29, 2020.

Webinar Series by Department of Mathematics, SAC.

Srinivasa Ramanujan
His life and his work

Michel Waldschmidt

Sorbonne University
Institut de Mathématiques de Jussieu
http://www.imj-prg.fr/~michel.waldschmidt/