Khon Kaen University November 8, 2016

Srinivasa Ramanujan
His life and his work

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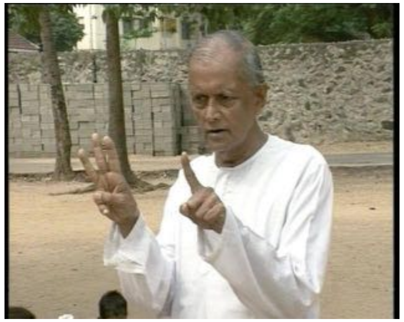
Abstract

This lecture includes a few biographical informations about Srinivasan Ramanujan. Among the topics which we discuss are Euler constant, nested roots, divergent series, Ramanujan – Nagell equation, partitions, Ramanujan tau function, Hardy Littlewood and the circle method, highly composite numbers and transcendence theory, the number π, and the lost notebook.

Srinivasa Ramanujan

Erode December 22, 1887 —
Chetput, (Madras), April 26, 1920

P.K. Srinivasan (November 4, 1924-June 20, 2005)



PKS was the first biographer of Srinivas Ramanujan.

The Hindu, November 1, 2009
Passion for numbers by Soudhamini

http://beta.thehindu.com/education/article41732.ece

Biography of Srinivasa Ramanujan

(December 22, 1887 — April 26, 1920)

1887 : born in Erode (near Tanjore)

1894-1903 : school in Kumbakonam

In 1900 he began to work on his own on mathematics summing geometric and arithmetic series.

Gopuram Sarangapani Kumbakonam

Sarangapani Sannidhi Street Kumbakonam

Ramanujan House Kumbakonam

Ramanujan House in Kumbakonam

Ramanujan House Kumbakonam

Town High School Kumbakonam

Town High School Kumbakonam

1903 : G.S.Carr - *A synopsis of elementary results — a book on pure mathematics* (1886) $5\,000$ formulae

$$\sqrt{x} + y = 7, \qquad x + \sqrt{y} = 11$$

$$x = 9, \qquad y = 4.$$

Biography *(continued)*

1903 (December) : exam at Madras University

1904 (January) : enters Government Arts College, Kumbakonam

Sri K. Ranganatha Rao Prize

Subrahmanyam scholarship

MacTutor History of Mathematics

http://www-history.mcs.st-andrews.ac.uk/

By 1904 Ramanujan had begun to undertake deep research. He investigated the series

$$\sum_n \frac{1}{n}$$

and calculated Euler's constant to 15 decimal places.

He began to study the Bernoulli numbers, although this was entirely his own independent discovery.

Euler constant

$$S_N = \sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N}$$

$$\int_1^N \frac{dx}{x+1} < S_N < 1 + \int_1^N \frac{dx}{x}$$

$$\gamma = \lim_{N\to\infty} (S_N - \log N).$$

Reference

Jeffrey C. Lagarias
Euler's constant : Euler's work and modern developments
Bulletin Amer. Math. Soc. **50** (2013), No. 4, 527–628.

arXiv:1303.1856 [math.NT]
Bibliography : 314 references.

Euler archives and Eneström index

http://eulerarchive.maa.org/

Gustaf Eneström (1852–1923)
Die Schriften Euler's chronologisch nach den Jahren geordnet, in denen sie verfasst worden sind
Jahresbericht der Deutschen Mathematiker–Vereinigung, 1913.

Gustaf Eneström.
Efter fotografi.

http://www.math.dartmouth.edu/~euler/index/enestrom.html

http://www.eulerarchive.org/

(Référence [86] of the text by Lagarias)

Harmonic numbers

$$H_1 = 1, \quad H_2 = 1 + \frac{1}{2} = \frac{3}{2}, \quad H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6},$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{j=1}^{n} \frac{1}{j}.$$

Sequence :

$$1, \quad \frac{3}{2}, \quad \frac{11}{6}, \quad \frac{25}{12}, \quad \frac{137}{60}, \quad \frac{49}{20}, \quad \frac{363}{140}, \quad \frac{761}{280}, \quad \frac{7129}{2520}, \dots$$

The online encyclopaedia of integer sequences

https://oeis.org/

Neil J. A. Sloane

Numerators and denominators

Numerators : https://oeis.org/A001008

$1, 3, 11, 25, 137, 49, 363, 761, 7129, 7381, 83711, 86021, 1145993,$

$1171733, 1195757, 2436559, 42142223, 14274301, 275295799,$

$55835135, 18858053, 19093197, 444316699, 1347822955, \ldots$

Denominators : https://oeis.org/A002805

$1, 2, 6, 12, 60, 20, 140, 280, 2520, 2520, 27720, 27720, 360360,$

$360360, 360360, 720720, 12252240, 4084080, 77597520,$

$15519504, 5173168, 5173168, 118982864, 356948592, \ldots$

Euler (1731)

De progressionibus harmonicis observationes

The sequence

$$H_n - \log n$$

has a limit $\gamma = 0,577\,21\underline{8}\ldots$
when n tends to infinity.

Leonhard Euler
(1707–1783)

Moreover,

$$\gamma = \sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m}.$$

Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

Euler : $s \in \mathbb{R}$.

Riemann : $s \in \mathbb{C}$.

Numerical value of Euler's constant

The online encyclopaedia of integer sequences
https://oeis.org/A001620
Decimal expansion of Euler's constant
(or Euler–Mascheroni constant) gamma.

Yee (2010) computed $29\,844\,489\,545$ decimal digits of gamma.

$$\gamma = 0,577\,215\,664\,901\,532\,860\,606\,512\,090\,082\,402\,431\,042\,\ldots$$

Euler constant

Euler–Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) = 0.577\,215\,664\,9\ldots$$

Neil J. A. Sloane's encyclopaedia
http://www.research.att.com/∼njas/sequences/A001620

Bernoulli numbers

$$B_0 = 1, \qquad \sum_{k=0}^{n-1}\binom{n}{k}B_k = 0 \quad \text{for } n > 1.$$

Jacob Bernoulli (1654 – 1705)

$$B_0 + 2B_1 = 0 \qquad B_1 = -\frac{1}{2}$$

$$B_0 + 3B_1 + 3B_2 = 0 \qquad B_2 = \frac{1}{6}$$

$$B_0 + 4B_1 + 6B_2 + 4B_3 = 0 \qquad B_3 = 0$$

$$B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 = 0 \qquad B_4 = -\frac{1}{30}$$

$$\vdots$$

Sloane A027642 and A000367

Kumbakonam

1905 : Fails final exam

1906 : Enters Pachaiyappa's College, Madras

Ill, goes back to Kumbakonam

1907 (December) : Fails final exam.

1908 : continued fractions and divergent series

1909 (April) : underwent an operation

1909 (July 14) : marriage with S Janaki Ammal (1900—1994)

S Janaki Ammal

Madras

1910 : meets Ramaswami Aiyar

1911 : first mathematical paper

1912 : clerk office, Madras Port Trust — Sir Francis Spring and Sir Gilbert Walker get a scholarship for him from the University of Madras starting May 1913 for 2 years.

1912 Questions in the Journal of the Indian Mathematical Society

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}} = ?$$

$$\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\cdots}}}} = ?$$

Answers from Ramanujan

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}} = 3$$

$$\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\cdots}}}} = 4$$

"Proofs" $n(n+2)$

$$(n+2)^2 = 1 + (n+1)(n+3)$$

$$n(n+2) = n\sqrt{1+(n+1)(n+3)}$$

$$f(n) = n(n+2)$$

$$f(n) = n\sqrt{1+f(n+1)}$$

$$f(n) = n\sqrt{1+(n+1)\sqrt{1+f(n+2)}}$$

$$= n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\cdots}}}$$

$$f(1) = 3$$

“Proofs” $n(n+3)$

$$(n+3)^2 = n+5+(n+1)(n+4)$$

$$n(n+3) = n\sqrt{n+5+(n+1)(n+4)}$$

$$g(n) = n(n+3)$$

$$g(n) = n\sqrt{n+5+g(n+1)}$$

$$g(n) = n\sqrt{n+5+(n+1)\sqrt{n+6+g(n+2)}}$$

$$= n\sqrt{n+5+(n+1)\sqrt{n+6+(n+2)\sqrt{n+7+\cdots}}}$$

$$g(1) = 4$$

Letter of S. Ramanujan to M.J.M. Hill in 1912

$$1+2+3+\cdots+\infty = -\frac{1}{12}$$

$$1^2+2^2+3^2+\cdots+\infty^2 = 0$$

$$1^3+2^3+3^3+\cdots+\infty^3 = \frac{1}{120}$$

Answer of M.J.M. Hill in 1912

$$1+2+3+\cdots+n = \frac{1}{2}n(n+1)$$

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Renormalisation of divergent series

Leonhard Euler

(1707 – 1783)

Introductio in analysin infinitorum

(1748)

Euler

Values of Riemann zeta function at negative integers :

$$\zeta(-k) = -\frac{B_{k+1}}{k+1} \qquad (n \geq 1)$$

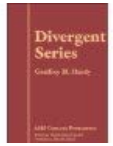
$$\zeta(-2n) = 1^{2n} + 2^{2n} + 3^{2n} + 4^{2n} + \cdots = 0 \qquad (n \geq 1)$$

$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

$$\zeta(-3) = 1^3 + 2^3 + 3^3 + 4^3 + \cdots = \frac{1}{120}$$

$$\zeta(-5) = 1^5 + 2^5 + 3^5 + 4^5 + \cdots = -\frac{1}{252}$$

G.H. Hardy : Divergent Series (1949)



Niels Henrik Abel
(1802 – 1829)

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.

Letters to H.F. Baker and E.W. Hobson in 1912

No answer to his letters to H.F. Baker and E.W. Hobson in 1912. . .

Letter of Ramanujan to Hardy (January 16, 1913)

I have had no university education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at mathematics. I have not trodden through the conventional regular course which is followed in a university course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as "startling".

Godfrey Harold Hardy (1877 – 1947)

John Edensor Littlewood (1885 – 1977)

Hardy and Littlewood

Letter from Ramanujan to Hardy (January 16, 1913)

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$$

$$1 - 1! + 2! - 3! + \cdots = .596\cdots$$

Answer from Hardy (February 8, 1913)

I was exceedingly interested by your letter and by the theorems which you state. You will however understand that, before I can judge properly of the value of what you have done, it is essential that I should see proofs of some of your assertions. Your results seem to me to fall into roughly three classes :

(1) there are a number of results that are already known, or easily deducible from known theorems ;

(2) there are results which, so far as I know, are new and interesting, but interesting rather from their curiosity and apparent difficulty than their importance ;

(3) there are results which appear to be new and important...

1913–1920

1913, February 27 : New letter from Ramanujan to Hardy

1913 : Visit of Neville to India

1914, March 17 to April 14 : travel to Cambridge.

1918 : (May) Fellow of the Royal Society
(November) Fellow of Trinity College, Cambridge.

1919, February 27 to March 13 : travel back to India.

Ramanujan – Taxi Cab Number 1729

Hardy's obituary of Ramanujan :

> *I had ridden in taxi-cab No* 1729*, and remarked that the number (*$7 \cdot 13 \cdot 19$*) seemed to me a rather dull one...*

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

$$12^3 = 1728, \qquad 9^3 = 729$$

Narendra Jadhav — Taxi Cab Number 1729

Narendra Jadhav (born 1953) is a noted Indian bureaucrat, economist, social scientist, writer and educationist. He is a member of Planning Commission of India as well as a member of National Advisory Council (NAC), since 31 May 2010. Prior to this, he had worked with International Monetary Fund (IMF) and headed economic research at Reserve Bank of India (RBI).

He was Vice-Chancellor (from 24 August 2006 to 15 June 2009) of University of Pune
Author of *Outcaste – A Memoir, Life and Triumphs of an Untouchable Family In India* (2003).

http://www.drnarendrajadhav.info

Ramanujan – Taxi Cab Number 1729

$$12^3 = 1728, \qquad 9^3 = 729$$

$$50 = 7^2 + 1^2 = 5^2 + 5^2$$

$$\begin{aligned} 4\,104 &= 2^3 + 16^3 = 9^3 + 15^3 \\ 13\,832 &= 2^3 + 24^3 = 18^3 + 20^3 \\ 40\,033 &= 9^3 + 34^3 = 16^3 + 33^3 \\ &\vdots \end{aligned}$$

Leonhard Euler (1707 – 1783)

$$59^4 + 158^4 = 133^4 + 134^4 = 635\,318\,657$$

Diophantine equations

$$x^3 + y^3 + z^3 = w^3$$

$$(x, y, z, w) = (3, 4, 5, 6)$$

$$3^3 + 4^3 + 5^3 = 27 + 64 + 125 = 216 = 6^3$$

Parametric solution :

$$\begin{aligned} x &= 3a^2 + 5ab - 5b^2 & y &= 4a^2 - 4ab + 6b^2 \\ z &= 5a^2 - 5ab - 3b^2 & w &= 6a^2 - 4ab + 4b^2 \end{aligned}$$

Ramanujan – Nagell Equation

Trygve Nagell (1895 – 1988)

$$x^2 + 7 = 2^n$$

$$\begin{aligned} 1^2 + 7 &= 2^3 = 8 \\ 3^2 + 7 &= 2^4 = 16 \\ 5^2 + 7 &= 2^5 = 32 \\ 11^2 + 7 &= 2^7 = 128 \\ 181^2 + 7 &= 2^{15} = 32\,768 \end{aligned}$$

$x^2 + D = 2^n$

Nagell (1948) : for $D = 7$, no further solution

R. Apéry (1960) : for $D > 0$, $D \neq 7$, the equation $x^2 + D = 2^n$ has at most 2 solutions.

Examples with 2 solutions :

$$D = 23 : \qquad 3^2 + 23 = 32, \quad 45^2 + 23 = 2^{11} = 2\,048$$

$$D = 2^{\ell+1} - 1,\ \ell \geq 3 : \qquad (2^\ell - 1)^2 + 2^{\ell+1} - 1 = 2^{2\ell}$$

$x^2 + D = 2^n$

F. Beukers (1980) : at most one solution otherwise.

M. Bennett (1995) : considers the case $D < 0$.

Partitions

$$\begin{array}{lcll} 1 & & & p(1) = 1 \\ 2 & = & 1+1 & p(2) = 2 \\ 3 & = & 2+1 = 1+1+1 & p(3) = 3 \\ 4 & = & 3+1 = 2+2 = 2+1+1 & \\ & = & 1+1+1+1 & p(4) = 5 \end{array}$$

$$p(5) = 7, \quad p(6) = 11, \quad p(7) = 15, \ldots$$

MacMahon : table of the first 200 values

Neil J. A. Sloane's encyclopaedia
http://www.research.att.com/∼njas/sequences/A000041

Ramanujan

$p(5n+4)$ is a multiple of 5

$p(7n+5)$ is a multiple of 7

$p(11n+6)$ is a multiple of 11

$p(25n+24)$ is a multiple of 25

$p(49n+47)$ is a multiple of 49

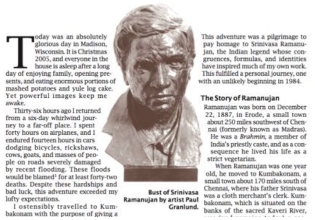
$p(121n+116)$ is a multiple of 121

Partitions - Ken Ono

Manjul Bhargava, left, and Ken Ono in front of Ramanujan's house.

Honoring a Gift from Kumbakonam

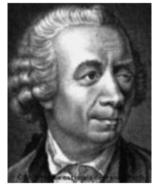
Ken Ono



Notices of the AMS, **53** N°6 (July 2006), 640–651

`http://www.ams.org/notices/200606/fea-ono.pdf`

Leonhard Euler (1707 – 1783)



$$1 + p(1)x + p(2)x^2 + \cdots + p(n)x^n + \cdots$$

$$= \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^n)\cdots}$$

$$1 + \sum_{n=1}^{\infty} p(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-1}$$

Eulerian products

Riemann zeta function

For $s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1-p^{-s})^{-1}$$

Georg Friedrich Bernhard Riemann (1826 - 1866)

Ramanujan tau function

$$x(1-x)^{-1} = \sum_{n=1}^{\infty} x^n$$

$$x \prod_{n=1}^{\infty}(1-x^n)^{24} = \sum_{n=1}^{\infty} \tau(n)x^n.$$

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p (1 - \tau(p)p^{-s} + p^{11-2s})^{-1}$$

Ramanujan's Congruences

$\tau(pn)$ is divisible by p for $p = 2,\ 3,\ 5,\ 7,\ 23.$

also : congruences modulo 691
(numerator of Bernoulli number B_{12})

Pierre Deligne

Ramanujan's Conjecture,
proved by Deligne in 1974

$$|\tau(p)| < 2p^{11/2}$$

Hardy–Ramanujan

For almost all integers n, the number of prime factors of n is $\log\log n$.

$$A_\epsilon(x) = \bigl\{n \le x\ ;\ (1-\epsilon)\log\log n < \omega(n) < (1+\epsilon)\log\log n\bigr\}.$$

$$\frac{1}{x}A_\epsilon(x) \to 1 \quad \text{when} \quad x \to \infty.$$

Highly composite numbers

(Proc. London Math. Soc. 1915)

$$\begin{array}{rccccccccc} n = & 2 & 4 & 6 & 12 & 24 & 36 & 48 & 60 & 120\dots \\ d(n) = & 2 & 3 & 4 & 6 & 8 & 9 & 10 & 12 & 16\dots \end{array}$$

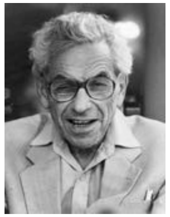
Question : *For $t \in \mathbf{R}$, do the conditions $2^t \in \mathbf{Z}$ and $3^t \in \mathbf{Z}$ imply $t \in \mathbf{Z}$?*

Example : For $t = \dfrac{\log 1729}{\log 2}$, we have $2^t = 1729 \in \mathbf{Z}$, but

$$3^t = \exp((\log 3)(\log 1729)/\log 2) = 135451.447153\dots$$

is not an integer.

Pàl Erdős Carl Ludwig Siegel



Alaoglu and Erdős : *On highly composite and similar numbers*, 1944.

C.L. Siegel : *For* $t \in \mathbf{R}$, *the conditions* $2^t \in \mathbf{Z}$, $3^t \in \mathbf{Z}$ *and* $5^t \in \mathbf{Z}$ *imply* $t \in \mathbf{Z}$.

Serge Lang, K. Ramachandra : *six exponentials theorem, four exponentials conjecture.*

Five exponentials Theorem and generalizations

1985 : Five exponentials Theorem.

1993, Damien Roy, Matrices whose coefficients are linear forms in logarithms.
J. Number Theory **41** (1992), no. 1, 22–47.

Approximation for π due to Ramanujan

$$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right) = 3.141592653\ 80568820189839000630\dots$$

$$\pi = 3.141592653\ 58979323846264338328\dots$$

Another formula due to Ramanujan for π

$$\pi = \frac{9\,801}{\sqrt{8}}\left(\sum_{n=0}^{\infty}\frac{(4n)!(1\,103+26\,390n)}{(n!)^4 396^{4n}}\right)^{-1}$$

$n=0$: 6 exact digits for $3.141592\dots$

$n \rightarrow n+1$: 8 more digits

Ramanujan's formula for $1/\pi$

$$\frac{1}{\pi} = \sum_{m=0}^{\infty}\binom{2m}{m}\frac{42m+5}{2^{12m+4}}.$$

Decimals of π

Ramanujan's formulae were used in 1985 : $1.7 \cdot 10^7$ digits for π (1.7 crores)

In 1999 : $2 \cdot 10^{10}$ digits ($2\ 000$ crores)

18 Aug 2009 : Pi Calculation Record Destroyed : 2.5 Trillion Decimals ($2.5 \cdot 10^{12}$).

$2,576,980,377,524$ decimal places in 73 hours 36 minutes

Massive parallel computer called : T2K Tsukuba System.

Team leader professor Daisuke Takahashi.

Ramanujan Notebooks

Written from 1903 to 1914

First : 16 chapters, 134 pages
Second : 21 chapters, 252 pages
Third : 33 pages

B.M. Wilson, G.N. Watson
Edited in 1957 in Bombay

The lost notebook

George Andrews, 1976

Bruce Berndt, 1985–87 (5 volumes)

Last work of Ramanujan

Mock theta functions

S. Zwegers – « Mock ϑ-functions and real analytic modular forms. », in *Berndt, Bruce C. (ed.) et al., q-series with applications to combinatorics, number theory, and physics. Proceedings of a conference, University of Illinois, Urbana-Champaign, IL, USA, October 26-28, 2000. Providence, RI : American Mathematical Society (AMS). Contemp. Math. 291, 269-277*, 2001.

SASTRA Ramanujan Prize

SASTRA Ramanujan Prize 2009 : Kathrin Bringmann.

International Conference in

Number Theory & Mock Theta Function

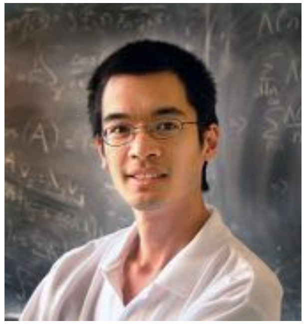
Srinivasa Ramanujan Center, Sastra University, Kumbakonam, Dec. 22, 2009.

www.math.ufl.edu/sastra-prize/2009.html

Previous SASTRA Ramanujan Prize winners

Manjul Bhargava and Kannan Soundararajan (2005)

Terence Tao (2006) and Ben Green (2007)



Akshay Venkatesh (2008)

Sastra Ramanujan Prize

Year	Name	University
2005	Manjul Bhargava	Princeton University
	Kannan Soundararajan	University of Michigan
2006	Terence Tao	University of California at Los Angeles
2007	Ben Green	Cambridge University
2008	Akshay Venkatesh	Stanford University
2009	Kathrin Bringmann	University of Cologne, University of Minnesota
2010	Wei Zhang	Harvard University
2011	Roman Holowinsky [1]	Ohio State University
2012	Zhiwei Yun [2]	Stanford University
2013	Peter Scholze [3]	University of Bonn
2014	James Maynard [4]	Oxford University, England, and University of Montreal, Canada
2015	Jacob Tsimerman [5]	University of Toronto, Canada

https://en.wikipedia.org/wiki/SASTRA_Ramanujan_Prize

ICTP Ramanujan Prize

- 2005 Marcelo Viana, Brazil[3]
- 2006 Ramdorai Sujatha, India[4]
- 2007 Jorge Lauret, Argentina[5]
- 2008 Enrique Pujals, Argentina/Brazil[6]
- 2009 Ernesto Lupercio, Mexico[7]
- 2010 Shi Yuguang, China[8]
- 2011 Philibert Nang, Gabon[9]
- 2012 Fernando Codá Marques, Brazil[10]
- 2013 Tian Ye, China[11]
- 2014 Miguel Walsh, Argentina[12]
- 2015 Amalendu Krishna, India[13]

https://en.wikipedia.org/wiki/ICTP_Ramanujan_Prize

ICTP Ramanujan Prize

Call for Nominations, 2016 Prize

Nomination deadline: 1 March 2016

Call for Nominations, 2016 Prize

The Ramanujan Prize for young mathematicians from developing countries has been awarded annually since 2005. The Prize is now funded by the Department of Science and Technology of the Government of India (DST), and will be administered jointly by ICTP, the International Mathematical Union (IMU), and the DST.

The Prize winner must be less than 45 years of age on 31 December of the year of the award, and have conducted outstanding research in a developing country. Researchers working in any branch of the mathematical sciences are eligible. The Prize carries a $15,000 cash award. The winner will be invited to ICTP to receive the Prize and deliver a lecture. The Prize is usually awarded to one person, but may be shared equally among recipients who have contributed to the same body of work.

The Selection Committee will take into account not only the scientific quality of the research, but also the background of the candidate and the environment in which the work was carried out.

https://www.ictp.it/about-ictp/prizes-awards/the-ramanujan-prize/call-for-nominations.aspx

References (continued)

[2] Don Zagier (March 16, 2005, BNF/SMF) :

"Ramanujan to Hardy, from the first to the last letter..."
http://smf.emath.fr/VieSociete/Rencontres/BNF/2005/

[3] MacTutor History of Mathematics
http://www-groups.dcs.st-and.ac.uk/∼history/

[4] Eric Weisstein worlds of mathematics, Wolfram Research
http://scienceworld.wolfram.com/

[5] Wikipedia, the free encyclopedia.
http://en.wikipedia.org/wiki/Ramanujan

Ramanujan according to Wikipedia

Erode December 22, 1887 — Chetput, (Madras), April 26, 1920

Landau–Ramanujan constant
Mock theta functions
Ramanujan prime
Ramanujan–Soldner constant
Ramanujan theta function
Ramanujan's sum
Rogers–Ramanujan identities

http://en.wikipedia.org/wiki/Srinivasa_Ramanujan

Landau–Ramanujan constant

In mathematics, the Landau–Ramanujan constant occurs in a number theory result stating that the number of positive integers less than x which are the sum of two square numbers, for large x, varies as

$$x/\sqrt{\ln(x)}.$$

The constant of proportionality is the Landau–Ramanujan constant, which was discovered independently by Edmund Landau and Srinivasa Ramanujan.
More formally, if $N(x)$ is the number of positive integers less than x which are the sum of two squares, then

$$\lim_{x\to\infty} \frac{N(x)\sqrt{\ln(x)}}{x} \approx 0.76422365358922066299069873125.$$

Ramanujan primes

In mathematics, a Ramanujan prime is a prime number that satisfies a result proven by Srinivasa Ramanujan relating to the prime-counting function.

$\pi(x) - \pi(x/2) \geq 1, 2, 3, 4, 5, \ldots$ for all $x \geq 2, 11, 17, 29, 41, \ldots$

respectively, where $\pi(x)$ is the prime-counting function, that is, the number of primes less than or equal to x.

Ramanujan primes

$2,\ 11,\ 17,\ 29,\ 41,\ 47,\ 59,\ 67,\ 71,\ 97,\ 101,\ 107,\ 127,\ 149,$

$151,\ 167,\ 179,\ 181,\ 227,\ 229,\ 233,\ 239,\ 241,\ 263,\ 269,\ 281,$

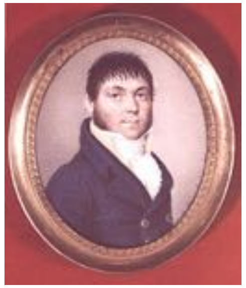
$307,\ 311,\ 347,\ 349,\ 367,\ 373,\ 401,\ 409,\ 419,\ 431,\ 433,\ 439,$

$461,\ 487,\ 491,\ 503,\ 569,\ 571,\ 587,\ 593,\ 599,\ 601,\ 607,\ 641, \ldots$

$a(n)$ is the smallest number such that if $x \geq a(n)$, then $\pi(x) - \pi(x/2) \geq n$, where $\pi(x)$ is the number of primes $\leq x$.

Neil J. A. Sloane's encyclopaedia
http://www.research.att.com/~njas/sequences/A104272

Ramanujan–Soldner constant



In mathematics, the Ramanujan–Soldner constant is a mathematical constant defined as the unique positive zero of the logarithmic integral function. It is named after Srinivasa Ramanujan and Johann Georg von Soldner (16 July 1776 - 13 May 1833).

Its value is approximately

$$1.451369234883381050283968485892027449493\ldots$$

Ramanujan's sum (1918)

In number theory, a branch of mathematics, Ramanujan's sum, usually denoted $c_q(n)$, is a function of two positive integer variables q and n defined by the formula

$$c_q(n) = \sum_{\substack{1\le a\le q\\(a,q)=1}} e^{2\pi i \frac{a}{q} n}.$$

Ramanujan's sums are used in the proof of Vinogradov's theorem that every sufficiently-large odd number is the sum of three primes.

Ramanujan theta function

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2}$$

Carl Gustav Jacob Jacobi (1804–1851)

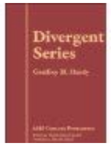
In mathematics, the Ramanujan theta function generalizes the form of the Jacobi theta functions, while capturing their general properties. In particular, the Jacobi triple product takes on a particularly elegant form when written in terms of the Ramanujan theta.

Rogers–Ramanujan identities

Leonard James Rogers
1862 - 1933

In mathematics, the Rogers–Ramanujan identities are a set of identities related to basic hypergeometric series. They were discovered by Leonard James Rogers (1894) and subsequently rediscovered by Srinivasa Ramanujan (1913) as well as by Issai Schur (1917).

G.H. Hardy : Divergent Series



In
$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$
set $z = -1$, as Euler does :
$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2} \cdots$$

Similarly, from the derivative of the previous series
$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$$
deduce
$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}.$$

$s = 1 - 2 + 3 - 4 + \cdots$

There are further reasons to attribute the value $1/4$ to s. For instance
$$s = 1 - (1 - 1 + 1 - 1 + \cdots) - (1 - 2 + 3 - 4 + \cdots) = 1 - \frac{1}{2} - s$$
gives $2s = 1/2$, hence $s = 1/4$.

Also computing the square by expanding the product
$$(1 - 1 + 1 - 1 + \cdots)^2 = (1 - 1 + 1 - 1 + \cdots)(1 - 1 + 1 - 1 + \cdots)$$
yields to
$$1 - 2 + 3 - 4 + \cdots = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Cesaro convergence

For a series
$$a_0 + a_1 + \cdots + a_n + \cdots = s$$
converging (in the sense of Cauchy), the partial sums
$$s_n = a_0 + a_1 + \cdots + a_n$$
have a mean value
$$\frac{s_0 + \cdots + s_n}{n+1}$$
which is a sequence which converges (in the sense of Cauchy) to s.
For the diverging series
$$1 - 1 + 1 - 1 + \cdots$$
the limit exists and is $1/2$.

Cesaro convergence

Ernesto Cesàro
(1859 – 1906)

For the series
$$1 + 0 - 1 + 1 + 0 - 1 + \cdots$$
the Cesaro limit
$$\lim_{n\to\infty} \frac{s_0 + \cdots + s_n}{n+1}$$
exists and is $2/3$.

Rules for summing divergent series

$a_0 + a_1 + a_2 + \cdots = s$ implies $ka_0 + ka_1 + ka_2 + \cdots = ks.$

$a_0 + a_1 + a_2 + \cdots = s$ and $b_0 + b_1 + b_2 + \cdots = t$ implies

$$a_0 + b_0 + a_1 + b_1 + a_2 + b_2 + \cdots = s + t.$$

$a_0 + a_1 + a_2 + \cdots = s$ if and only if $a_1 + a_2 + \cdots = s - a_0.$

$1^2 - 2^2 + 3^2 - 4^2 + \cdots$

Recall

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

Take one more derivative, you find also

$$1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - 4 \cdot 5 + \cdots = \frac{1}{4}$$

from which you deduce

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots = 0.$$

Further examples of divergent Series

$$1 + 1 + 1 + \cdots = 0$$
$$1 - 2 + 4 - 8 + \cdots = \frac{1}{3}$$
$$1 + 2 + 4 + 8 + \cdots = -1$$
$$1^{2k} + 2^{2k} + 3^{2k} + 4^{2k} + 5^{2k} + \cdots = 0 \quad \text{for} \quad k \geq 1.$$

Euler :

$$1 - 1! + 2! - 3! + 4! + \cdots = -e(\gamma - 1 + \frac{1}{2 \cdot 2!} - \frac{1}{3 \cdot 3!} + \cdots)$$

gives the value $0.5963\ldots$ also found by Ramanujan.

Ramanujan's method (following Joseph Oesterlé)

Here is Ramanujan's method for computing the value of divergent series and for accelerating the convergence of series.

The series

$$a_0 - a_1 + a_2 - a_3 + \cdots$$

can be written

$$\frac{1}{2}\left(a_0 + (b_0 - b_1 + b_3 - b_4 + \cdots)\right)$$

where $b_n = a_n - a_{n+1}$.

Acceleration of convergence

For instance in the case $a_n = 1/n^s$ we have $b_n \sim s/n^{s+1}$.
Repeating the process yields the analytic continuation of the Riemann zeta function.

For $s = -k$ where k is a positive integer, Ramanujan's method yields the Bernoulli numbers.

In the case of convergent series like

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2,$$

or for Euler constant, Ramanujan's method gives an efficient way of accelerating the convergence.

Srinivasa Ramanujan
His life and his work

Michel Waldschmidt

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