Grothendieck topologies for the analysts

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Sheaf theory was created by Jean Leray (an analyst!) when he was a war prisoner in the 40's (see the historical notes in [2]).

In spirit, sheaves are local objects, and the cohomology of sheaves calculates the obstruction to go from local to global. But what means local? It depends on the topology of the space and of the notion of coverings, and Grothendieck, by introducing what is now called Grothendieck topologies, has shown that these classical notions can easily be generalized without changing one word to sheaf theory, opening new horizons in algebraic geometry, and, perhaps, in analysis.

Here, I will endow real analytic manifolds with what we call the subanalytic topology and try to convince the audience of the beauty, naturallity and usefulness of this topology.

With M. Kashiwara in 2001, we have defined the subanalytic topology and constructed in this way various sheaves which would have no meaning in the classical setting, such as sheaves of C^∞-functions with polynomial growth, the sheaf of Whitney C^∞-functions and the sheaf of temperate distributions.

With Guillermou, in 2012, we refine this topology and introduce the linear subanalytic topology. This allows us to define for example sheaves of C^∞-functions with a precise growth, or sheaves of C^∞-functions with Gevrey growth. Using these tools, Gilles Lebeau recently constructed the Sobolev sheaves (which, in fact, are objects of the derived category).

References

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- M. Kashiwara and P. Schapira, *Ind-sheaves,* Astérisque Soc. Math. France **271** (2001).
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Subanalytic sets

In all this lecture, M will denote a real analytic manifold. The subanalytic sets were introduced by Gabrielov and Hironaka after the pioneering work of Lojasiewicz.

On M, a semi-analytic subset S is a locally closed subset locally defined by a finite number of analytic inequalities

$$S = \{x \in M; f_i(x) < 0, g_j(x) \leq 0, i = 1, \dots, p, j = 1, \dots, q\}$$

Unfortunately this family is not stable by proper direct images. The smallest family containing the semi-analytic sets and stable by proper direct images is that of subanalytic sets.

To each real analytic manifold M one can associate a family $\mathrm{SbA}(M)$ of subsets of M, called the subanalytic subsets of M with the following properties:

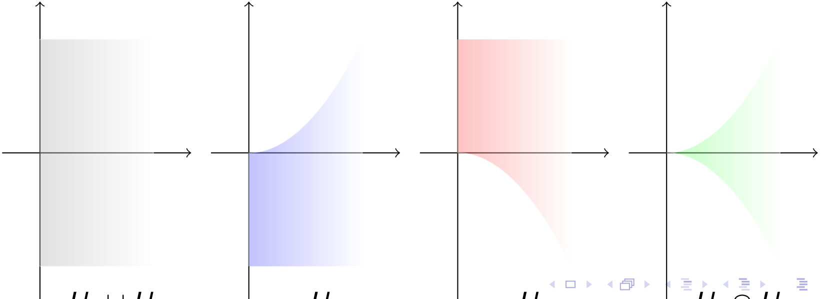
- a semi-analytic set is subanalytic and conversely if $\dim M \leq 2$,
- the property of being subanalytic is local on M,
- the family $\mathrm{SbA}(M)$ is stable by finite intersection, finite union, $\varnothing, M \in \mathrm{SbA}(M)$ and if $S \in \mathrm{SbA}(M)$ then $M \setminus S, \mathrm{Int}(S), \overline{S} \in \mathrm{SbA}(M)$,
- for $f\colon M \to N$ a morphism of real analytic manifolds and $S \in \mathrm{SbA}(M)$, $f(S) \in \mathrm{SbA}(N)$ if f is proper on $\overline{S}$,
- for $Z \in \mathrm{SbA}(N)$, $f^{-1}(Z) \in \mathrm{SbA}(M)$,
- a compact subanalytic set is topologically isomorphic to a CW-complex and in particular, it has finite many connected components.

Lojasiewicz inequalities

When necessary, we choose a distance d_M on M locally equivalent in a local chart to the Euclidean distance on $\mathbb{R}^n$. We adopt the convention $d(x, \varnothing) = D_M + 1$, where D_M is the diameter of M.

Let $U = \bigcup_{j \in J} U_j$ be a finite covering of open relatively compact subanalytic sets. Then there exist a constant $C > 0$ and a positive integer N such that

$$d(x, M \setminus U)^N \leq C \cdot (\max_{j \in J} d(x, M \setminus U_j)).$$



Sheaves

In all this lecture, $\mathbf{k}$ denotes a field (for example, $\mathbf{k} = \mathbb{C}$).

A presheaf F on a topological space M associates a vector space $F(U)$ to each open sets U of M, and a linear map (the restriction) $F(V) \to F(U)$ to open inclusion $U \subset V$, with the natural compatibility conditions, the restriction of the identity $U \subset U$ is the identity and the restriction of the composition $U \subset V \subset W$ is the composition of the restrictions.

A sheaf F is a presheaf which satisfies two properties

- if a section $s \in F(U)$ is locally zero then it is zero, that is, if one has an open covering $U = \bigcup_i U_i$ and $u|_{U_i} = 0$ for all i, then $u = 0$
- a family of locally defined sections which coincide two by two define a global section, that is, if one has an open covering $U = \bigcup_i U_i$ and $s_i \in F(U_i)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists $s \in F(U)$ with $s|_{U_i} = s_i$.

One sees that to define sheaves one needs two things: the notion of open sets and that of a covering. There is no necessity to take all open sets, and no necessity to take all coverings.

Example

Given a presheaf F one "functiorially" associates a sheaf F^a to it.

(i) Let $X = \mathbb{C}$ and consider the presheaf $F\colon U \mapsto \mathcal{O}_X(U)/\partial_z\mathcal{O}_X(U)$, that is, the cockernel of $\frac{\partial}{\partial z}$ acting on $\mathcal{O}_X(U)$. Then F is locally 0 but $F(\mathbb{C}\setminus\{0\}) \neq 0$. The associated sheaf if the sheaf 0.

(ii) Let F be the presheaf on $M = \mathbb{R}^n$ which, to U open in $\mathbb{R}^n$ associates the space $L^1(U;dx)$ of Lebesgue integrable functions on U. Then clearly, F is not a sheaf (for the usual topology). The associated sheaf is the sheaf L^1_{loc}.

Sheaves 2

Let $f: M \to N$ be a continuous map (more generally, a morphism of sites). One defines the functors of direct and inverse images for sheaves:

$$\mathrm{Mod}(\mathbf{k}_M) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathrm{Mod}(\mathbf{k}_N).$$

The direct image f_*F of a sheaf F on M is the sheaf on N given by

$$f_*F(V) = F(f^{-1}V).$$

The inverse image $f^{-1}G$ of a sheaf G on N is the sheaf associated with the presheaf

$$U \mapsto \varinjlim_{f(U) \subset V} G(V).$$

$\mathbb{R}$-constructible sheaves

A sheaf F on the real analytic manifold M is $\mathbb{R}$-constructible if there exists a stratification $M = \bigsqcup_\alpha M_\alpha$ of M by locally closed subanalytic subsets M_α such that $F|_{M_\alpha}$ is locally constant of finite rank.

One denotes by $\mathbb{R}\text{-}\mathrm{C}(\mathbf{k}_M)$ the abelian category of $\mathbb{R}$-constructible sheaves on M.

Example

(i) If $S \subset M$ is locally closed and subanalytic, the constant sheaf $\mathbf{k}_S$ (extended by 0 on $M \setminus S$) is $\mathbb{R}$-constructible.

(ii) Let $M = N = \mathbb{C} \setminus \{0\}$ and $f : M \to N$ the map $z \mapsto z^2$. Then $f_*\mathbf{k}_M$ is locally isomorphic to $\mathbf{k}_N^2$, but not globally. It is called a local system of rank 2.

The subanalytic topology

The subanalytic sheaves on M were introduced by Kashiwara-S in 2001 as a byproduct of the theory of indsheaves.

Denote by $\mathrm{Op}_{M_{\mathrm{sa}}}$ the category consisting of subanalytic and relatively compact open subsets (the morphisms are the inclusions). Let us say that a family $\{U_i\}_{i\in I}$ of subobjects of $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ is a subanalytic covering of U if there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j = U$.

The family of subanalytic coverings satisfies the axioms of Grothendieck topologies, namely:

(1) U is a covering of U,
(2) if $\{U_i\}_i$ and $\{V_j\}_j$ are families of open subsets of U, $\{U_i\}_i$ is a covering of U and is a refinement of $\{V_j\}_j$ then $\{V_j\}_j$ is a covering of U,
(3) if $\{U_i\}_i$ is a covering of U and V is open, then $\{U_i \cap V\}_i$ is a covering of $U \cap V$,
(4) if $\{U_i\}_i$ and $\{V_j\}_j$ are families of open subsets of U, $\{U_i\}_i$ is a covering of U and $\{U_i \cap V_j\}_j$ is a covering of U_i for all i, then $\{V_j\}_j$ is a covering of U.

On denotes by M_{sa} the site so-defined and by $\rho_{\mathrm{sa}} \colon M \to M_{\mathrm{sa}}$ the natural morphism of sites.

We have a pair of adjoint functors $(\rho_{\mathrm{sa}}^{-1}, \rho_{\mathrm{sa}*})$:

$$\mathrm{Mod}(\mathbf{k}_M) \underset{\rho_{\mathrm{sa}}^{-1}}{\overset{\rho_{\mathrm{sa}*}}{\rightleftarrows}} \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}).$$

Subanalytic sheaves 1

From now on, $\mathbf{k} = \mathbb{C}$. As usual, we denote by $\mathcal{C}^\infty_M$ the sheaf of complex valued functions of class C^∞ and by $\mathcal{D}b_M$ (resp. $\mathcal{B}_M$) the sheaf of Schwartz's distributions (resp. Sato's hyperfunctions). One denotes by $\mathcal{D}_M$ the sheaf of analytic finite-order differential operators.

Definition

Let $f \in \mathcal{C}^\infty_M(U)$. One says that f has *polynomial growth* at $p \in M$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} \big(\mathrm{dist}(x, K \setminus U)\big)^N |f(x)| < \infty \,.$$

We say that f is temperate at p if all its derivatives have polynomial growth at p. We say that f is temperate if it is temperate at any point.

Subanalytic sheaves 2

For an open subanalytic set U in M, denote by $\mathcal{C}_M^{\infty,\mathrm{tp}}(U)$ the subspace of $\mathcal{C}_M^{\infty}(U)$ consisting of tempered functions.
Denote by $\mathcal{D}b_M^{\mathrm{tp}}(U)$ the space of tempered distributions on U (i.e., distributions on U which can be extended to M).

Using Lojasiewicz's inequalities, one easily proves that

- the presheaf $U \mapsto \mathcal{C}^{\infty,\mathrm{tp}}(U)$ is a sheaf on M_{sa},
- the presheaf $U \mapsto \mathcal{D}b_M^{\mathrm{tp}}(U)$ is a sheaf on M_{sa}.

One denotes by $\mathcal{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{tp}}$ the first one and calls it the sheaf of temperate C^∞-functions. One denotes by $\mathcal{D}b_{M_{\mathrm{sa}}}^{\mathrm{tp}}$ the second one and calls it the sheaf of temperate distributions.

Application: integral of distributions

We shall only treat an example.
Let $M = N \times \mathbb{R}^d$, let f be the projection and let $t = (t_1, \dots, t_d)$ be the coordinates on $\mathbb{R}^d$.
Let $Z \subset M$ be a closed subanalytic subset of M such that f is proper on Z. Consider the complex

$$0 \to \Gamma_Z(M; \mathcal{D}b_M^{\mathrm{tp}(0)}) \xrightarrow{d_t} \cdots \xrightarrow{d_t} \Gamma_Z(M; \mathcal{D}b_M^{\mathrm{tp}(d)}) \to 0$$

in which $\mathcal{D}b_M^{\mathrm{tp}(j)}$ is the space of differential forms in the variables t with coefficients in $\mathcal{D}b_M^{\mathrm{tp}}$ and $\mathcal{D}b_M^{\mathrm{tp}(d)}$ is in degree 0.
Then the integration $\int_{\mathbb{R}^d} \bullet \, dt$ in the fibers of f induces a quasi-isomorphism between this complex and the complex

$$\mathrm{RHom}\,(\mathrm{R}f_* \mathbb{C}_Z, \mathcal{D}b_N^{\mathrm{tp}}).$$

In other words, the characterization of the space $\int_f \mathcal{D}b_M^{\mathrm{tp}}$ is reduced to a topological problem, namely the calculation of $\mathrm{R}f_* \mathbb{C}_Z$.

Holomorphic functions with temperate growth

Now let X be a *complex* manifold of complex dimension n. We still denote by X the real underlying manifold. One defines the sheaf of temperate holomorphic functions $\mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ as the Dolbeault complex with coefficients in $\mathcal{C}^{\infty,\mathrm{tp}}_{X_{\mathrm{sa}}}$. In other words

$$\mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} \quad = \quad 0 \to \mathcal{C}^{\infty,\mathrm{tp}(0,0)}_{X_{\mathrm{sa}}} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{C}^{\infty,\mathrm{tp}(0,n)}_{X_{\mathrm{sa}}} \to 0.$$

When replacing in this complex the sheaves $\mathcal{C}^{\infty,\mathrm{tp}(0,p)}_{X_{\mathrm{sa}}}$ with the sheaves $\mathcal{D}b^{\mathrm{tp}(0,p)}_{X_{\mathrm{sa}}}$, one gets the same result.

Example

(i) Let Z be a closed complex analytic subset of the complex manifold X. We have the isomorphism

$$\rho_{\mathrm{sa}}^{-1}\mathrm{R}\mathcal{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\mathbb{C}_Z, \mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}) \simeq \mathrm{R}\Gamma_{[Z]}(\mathcal{O}_X) \text{ (algebraic cohomology).}$$

For example, if Z is a complex hypersurface, one finds the sheaf of meromorphic functions with poles in Z.

(ii) Let M be a real analytic manifold and X a complexification of M. We have the isomorphism

$$\rho_{\mathrm{sa}}^{-1}\mathrm{R}\mathcal{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\mathrm{D}'_X\mathbb{C}_M, \mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}) \simeq \mathcal{D}b_M.$$

Notice that with this approach, the sheaf $\mathcal{D}b_M$ of Schwartz's distributions is constructed similarly as the sheaf $\mathcal{B}_M$ of Sato's hyperfunctions. In particular, functional analysis is not used in the construction.

Linear coverings

Definition

Let $\{U_i\}_{i\in I}$ be a finite family in $\mathrm{Op}_{M_{\mathrm{sa}}}$. We say that this family is 1-regularly situated if there is a constant C such that for any $x \in M$

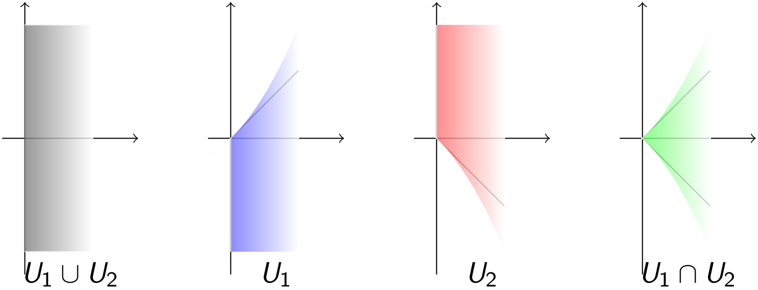
$$d(x, M \setminus \bigcup_{i\in I} U_i) \leq C \cdot \max_{i\in I} d(x, M \setminus U_i).$$

Definition

A linear covering of U is a family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sa}}}$ such that $U_i \subset U$ for all $i \in I$ and

$$\begin{cases} \text{there exists a finite subset } I_0 \subset I \text{ such that the family } \{U_i\}_{i\in I_0} \\ \text{is 1-regularly situated and } \bigcup_{i\in I_0} U_i = U. \end{cases}$$

One proves that the family of linear coverings defines a Grothendieck topology.



Sheaves on the linear subanalytic topology

Definition

The linear subanalytic site M_{sal} is the presite M_{sa} endowed with the Grothendieck topology for which the coverings are the linear coverings.
We denote by $\rho_{\mathrm{sal}} \colon M_{\mathrm{sa}} \to M_{\mathrm{sal}}$ the natural morphisms of sites.

A presheaf F on M_{sa} is a sheaf on M_{sal} if and only if, for any $U_1, U_2 \in \mathrm{Op}_{M_{\mathrm{sa}}}$ such that $\{U_1, U_2\}$ is a linear covering of $U := U_1 \cup U_2$, the sequence

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2)$$

is exact.

If moreover the sequence

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2) \to 0$$

is exact, then one proves that $\mathrm{R}\Gamma(U; F)$ is concentrated in degree 0 for all $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. In this case, one says that F is Γ-acyclic on M_{sal}.

The linear subanalytic topology 2

The functor $\rho_{\mathrm{sal}*}$ is left exact and its left adjoint ρ_{sal}^{-1} is exact. Hence, we have the pairs of adjoint functors

$$\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \underset{\rho_{\mathrm{sal}}^{-1}}{\overset{\rho_{\mathrm{sal}*}}{\rightleftarrows}} \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sal}}}), \qquad \mathrm{D}(\mathbf{k}_{M_{\mathrm{sa}}}) \underset{\rho_{\mathrm{sal}}^{-1}}{\overset{\mathrm{R}\rho_{\mathrm{sal}*}}{\rightleftarrows}} \mathrm{D}(\mathbf{k}_{M_{\mathrm{sal}}})$$

One proves that the functor $\mathrm{R}\rho_{\mathrm{sal}*}$ commutes with small direct sums. Using the Brown representability theorem, one deduces:

Theorem

The functor $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}})$ *admits a right adjoint* $\rho_{\mathrm{sal}}^{!}\colon \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}}) \to \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}})$.

Hence, for $F \in \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}})$ and $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we have

$$\mathrm{R}\Gamma(U; \rho_{\mathrm{sal}}^{!} F) \simeq \mathrm{RHom}\,(\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_U; F).$$

Open sets with Lipschitz boundary

Definition

We say that $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ has Lipschitz boundary or simply that U is Lipschitz (in M) if, for any $x \in \partial U$, there exist an open neighborhood V of x and a bi-Lipschitz subanalytic homeomorphism $\psi \colon V \xrightarrow{\sim} W$ with W an open subset of $\mathbb{R}^n$ such that $\psi(V \cap U) = W \cap \{x_n > 0\}$.

Example

Assume that M is open in some real vector space V and let γ be a proper closed convex cone with nonempty interior. If $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ satisfies

$$U = (U + \gamma) \cap M$$

then U is Lipschitz in M.

Open sets with Lipschitz boundary 2

Recall the functor $\rho_{\mathrm{sal}}^{!}\colon \mathrm{D}^{+}(\mathbf{k}_{M_{\mathrm{sal}}}) \to \mathrm{D}^{+}(\mathbf{k}_{M_{\mathrm{sa}}})$ and recall that for $F \in \mathrm{D}^{+}(\mathbf{k}_{M_{\mathrm{sal}}})$ and $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we have

$$\mathrm{R}\Gamma(U;\rho_{\mathrm{sal}}^{!}F) \simeq \mathrm{RHom}\,(\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_U;F).$$

Theorem

Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and assume that U is Lipschitz. Then $\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_U \simeq \mathbf{k}_U$. (In particular, it is concentrated in degree zero.)

The proof uses a result of Parusinski.

Corollary

For $F \in$ mod $(\mathbf{k}_{M_{\mathrm{sal}}})$, one has $\mathrm{R}\Gamma(U;\rho_{\mathrm{sal}}^{!}F) \simeq \mathrm{R}\Gamma(U;F)$.
If moreover F is Γ-acyclic, then $\mathrm{R}\Gamma(U;\rho_{\mathrm{sal}}^{!}F) \simeq F(U)$.

In other words, for a Γ-acyclic sheaf F on M_{sal}, the object $\rho_{\mathrm{sal}}^{!}F$ of $\mathrm{D}^{+}(\mathbf{k}_{M_{\mathrm{sa}}})$ "coincides" with F on the open sets U which are Lipschitz.

Open sets with Lipschitz boundary 3

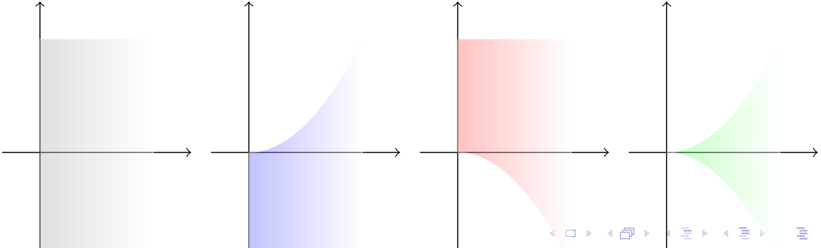
Application: let U_1 and U_2 be two Lipschitz open sets such that $U_1 \cup U_2$ is also Lipschitz but $U_1 \cap U_2)$ is not Lipschitz and $\{U_1, U_2\}$ is not a linear covering of $U_1 \cap U_2$. Let F be a sheaf on M_{sal} and assume that F is Γ-acyclic. Then the sequence

$$0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2)$$

is not exact in general, and the complex

$$F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2)$$

in which $F(U_1) \oplus F(U_2)$ is in degree 0 is a natural substitute to the space $F(U_1 \cap U_2)$.



Some sheaves on the linear subanalytic topology

Definition

Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, let $f \in \mathcal{C}^{\infty}_{M}(U)$ and let $s \in \mathbb{R}_{\geq 0}$. We say that f has *growth of order* $\leq s$ at $p \in M$ if it satisfies the following condition. For a local coordinate system $(x_1, \ldots, x_n)$ around p, there exists a sufficiently small compact neighborhood K of p such that

$$\sup_{x \in K \cap U} \big(\mathrm{dist}(x, K \setminus U)\big)^{s} |f(x)| < \infty \, .$$

We say that f is temperate of order s at p if, for each $m \in \mathbb{N}$, all its derivatives of order $\leq m$ have polynomial growth of order $\leq s + m$ at p. We say that f is temperate of order s if it is temperate of order s at any point.

For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, we denote by $\mathcal{C}^{\infty,s}_M(U)$ the subspace of $\mathcal{C}^{\infty}_M(U)$ consisting of functions tempered of order s and we denote by $\mathcal{C}^{\infty,s}_{M_{\mathrm{sal}}}$ the presheaf on M_{sal} so obtained.

(i) The presheaves $\mathcal{C}^{\infty,s}_{M_{\mathrm{sal}}}$ are sheaves on M_{sal},

(ii) the sheaf $\mathcal{C}^{\infty,0}_{M_{\mathrm{sal}}}$ is a sheaf of rings,

(iii) for $s \geq 0$, $\mathcal{C}^{\infty,s}_{M_{\mathrm{sal}}}$ is a $\mathcal{C}^{\infty,0}_{M_{\mathrm{sal}}}$-module,

(iv) The sheaves $\mathcal{C}^{\infty,s}_{M_{\mathrm{sal}}}$ are Γ-acyclic.

Sheaves on the subanalytic topology 2

The fact that these sheaves are Γ-acyclic follows from the existence of partitions of unity in $\mathcal{C}^{\infty,0}_{M_{\mathrm{sal}}}$, a result which is implicitly proved in Hörmander's book.
Then, for $0 \leq s \leq s'$, there are natural monomorphisms of sheaves on M_{sal} :

$$\mathcal{C}^{\infty,0}_{M_{\mathrm{sal}}} \hookrightarrow \mathcal{C}^{\infty,s}_{M_{\mathrm{sal}}} \hookrightarrow \mathcal{C}^{\infty,s'}_{M_{\mathrm{sal}}} \hookrightarrow \mathcal{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sal}}}.$$

One sets

$$\mathcal{C}^{\infty,s}_{M_{\mathrm{sa}}} := \rho^{!}_{\mathrm{sal}} \mathcal{C}^{\infty,s}_{M_{\mathrm{sal}}} \text{ an object of } \mathrm{D}^{+}(\mathbb{C}_{M_{\mathrm{sa}}}).$$

If U is Lipschitz, then $\mathrm{R}\Gamma(U; \mathcal{C}^{\infty,s}_{M_{\mathrm{sa}}})$ is concentrated in degree 0 and coincides with $\mathcal{C}^{\infty,s}_{M}(U)$.

Sobolev sheaves, after Lebeau

As usual, $H^s(\mathbb{R}^n)$ is the space of tempered distributions f such that the Fourier transform $\widehat{f}$ belongs to $L^2_{loc}(\mathbb{R}^{n*})$ and satisfies

$$\|f\|^2_{H^s} = \int (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi < \infty.$$

For U open in $\mathbb{R}^n$, one defines

$$H^s(U) = \{f \in \mathcal{D}b(U);\ \text{there exists } g \in H^s(\mathbb{R}^n), g|_U = f\}.$$

One easily extends this definition to the case where U open in a real manifold.

Theorem (Gilles Lebeau, 2014)
Let M be a real analytic manifold and let $s \leq 0$. There exists a Γ-acyclic sheaf $\mathcal{H}^s_{M_{\mathrm{sal}}}$ on the site M_{sal} such that if $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ is Lipschitz, then $\mathcal{H}^s_{M_{\mathrm{sal}}}(U) = H^s(U)$.
In particular, there exists an object $\mathcal{H}^s_{M_{\mathrm{sa}}} \in \mathrm{D}^+(\mathbb{C}_{M_{\mathrm{sa}}})$ such that if $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ is Lipschitz, then $\mathrm{R}\Gamma(U; \mathcal{H}^s_{M_{\mathrm{sa}}})$ is concentrated in degree 0 and coincides with the space $H^s(U)$.

As an application, consider the case of two Lipschitz open sets U_1 and U_2 in $\mathrm{Op}_{M_{\mathrm{sa}}}$ and assume that $U_1 \cup U_2$ is Lipschitz. This suggests that it would be natural to replace the space $H^s(U_1 \cap U_2)$ with the complex

$$H^s(U_1 \cup U_2) \to H^s(U_1) \oplus H^s(U_2)$$

in which $H^s(U_1) \oplus H^s(U_2)$ is in degree 0.