Microlocal Euler classes and Hochschild homology

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Introduction

This is a joint work with Masaki Kashiwara.

On a complex manifold $(X, \mathscr{O}_X)$, the Hochschild homology is a powerful tool to construct characteristic classes of coherent modules and to get index theorems. Here, I will show how to adapt this formalism to a wide class of sheaves on a real manifold M by using the functor μhom of microlocalization. This construction applies in particular to constructible sheaves on real manifolds and $\mathscr{D}$-modules on complex manifolds, or more generally to elliptic pairs.

Hochschild homology

Consider a complex manifold $(X, \mathscr{O}_X)$ of complex dimension d_X. We shall use the following notations:

- $\Omega_X = \Omega_X^{d_X}$, $\omega_X^{\mathrm{hol}} \simeq \Omega_X[d_X]$, the dualizing complex,
- $\mathrm{D}_{\mathscr{O}}(\,\bullet\,) = \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\,\bullet\,, \omega_X^{\mathrm{hol}})$ the duality functor and $\mathrm{D}'_{\mathscr{O}}(\,\bullet\,) = \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\,\bullet\,, \mathscr{O}_X)$
- $\delta\colon X \hookrightarrow X \times X$ the diagonal embedding and $\Delta = \delta(X)$. We set $\mathscr{O}_\Delta := \delta_*\mathscr{O}_X$, $\omega_\Delta^{\mathrm{hol}} := \delta_*\omega_X^{\mathrm{hol}}$, etc.

The Hochschild homology of $\mathscr{O}_X$ is defined by

$$\begin{aligned}
\mathscr{H}\mathscr{H}(\mathscr{O}_X) &= \delta^{-1}(\mathscr{O}_\Delta \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}} \mathscr{O}_\Delta) \\
&\simeq \delta^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{O}_{X\times X}}(\omega_\Delta^{\mathrm{hol},\otimes -1}, \mathscr{O}_\Delta) \simeq \delta^*\delta_*\mathscr{O}_X \\
&\simeq \delta^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{O}_{X\times X}}(\mathscr{O}_\Delta, \omega_\Delta^{\mathrm{hol}}) \simeq \delta^!\delta_!\omega_X.
\end{aligned}$$

Hochschild classes

Let $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$. The morphisms $\mathrm{D}'_{\mathscr{O}}\mathscr{F} \otimes \mathscr{F} \to \mathscr{O}_X$ and $\mathrm{D}_{\mathscr{O}}\mathscr{F} \otimes \mathscr{F} \to \omega_X^{\mathrm{hol}}$ give by adjunction the morphisms

$$\mathrm{D}'_{\mathscr{O}}\mathscr{F} \boxtimes \mathscr{F} \to \mathscr{O}_\Delta, \quad \mathrm{D}_{\mathscr{O}}\mathscr{F} \boxtimes \mathscr{F} \to \omega_\Delta^{\mathrm{hol}}$$

and then by duality the morphisms

$$\omega_\Delta^{\mathrm{hol},\otimes -1} \to \mathrm{D}'_{\mathscr{O}}\mathscr{F} \boxtimes \mathscr{F} \to \mathscr{O}_\Delta, \quad \mathscr{O}_\Delta \to \mathrm{D}_{\mathscr{O}}\mathscr{F} \boxtimes \mathscr{F} \to \omega_\Delta^{\mathrm{hol}}$$

and the composition defines the Hochschild classes of $\mathscr{F}$:

$$\mathrm{hh}_{\mathscr{O}}(\mathscr{F}) \in H^0_{\mathrm{supp}(\mathscr{F})}(X; \delta^{-1}\delta_*\mathscr{O}_X), \quad \widetilde{\mathrm{hh}}_{\mathscr{O}}(\mathscr{F}) \in H^0_{\mathrm{supp}(\mathscr{F})}(X; \delta^!\delta_!\omega_X).$$

Functoriality of Hochschild classes

Let X_i $(i=1,2,3)$ be complex manifolds. Set $X_{ij} = X_i \times X_j$, etc. Denote by $q_{ij}\colon X_{123} \to X_{ij}$ the projections. For $K_{ij} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X_{ij}})$ $(ij = 12, 23, 13)$, we set

$$K_{12} \underset{2}{\circ} K_{23} := \mathrm{R}q_{13!}(q_{12}^* K_{12} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{123}} q_{23}^* K_{23}).$$

Similarly, for closed subsets $A_{ij} \subset X_{ij}$ we set

$$A_{12} \underset{2}{\circ} A_{23} = q_{13}(A_{12} \times_{X_2} A_{23}).$$

Theorem

(a) *There is a natural morphism*

$$\mathscr{HH}(\mathscr{O}_{12}) \underset{2}{\circ} \mathscr{HH}(\mathscr{O}_{23}) \to \mathscr{HH}(\mathscr{O}_{13}),$$

(b) *let $K_{ij} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X_{ij}})$ with $\mathrm{supp}(K_{ij}) \subset A_{ij}$ and assume that q_{13} is proper on $A_{12} \times_{X_2} A_{23}$.*

$$\mathrm{hh}_{X_{13}}(K_{12} \underset{2}{\circ} K_{23}) = \mathrm{hh}_{X_{12}}(K_{12}) \underset{2}{\circ} \mathrm{hh}_{X_{23}}(K_{23}),$$

$$\widetilde{\mathrm{hh}}_{X_{13}}((K_{12} \otimes \omega_2^{\mathrm{hol}\otimes -1}) \underset{2}{\circ} K_{23}) = \widetilde{\mathrm{hh}}_{X_{12}}(K_{12}) \underset{2}{\circ} \widetilde{\mathrm{hh}}_{X_{23}}(K_{23}),$$

in $H^0_{A_{13}}(X_{13}; \mathscr{HH}(\mathscr{O}_{13}))$.

Hochschild-Kostant-Rosenberg isomorphism

There is a commutative diagram constructed by Kashiwara in 1991 in which α_X is the HKR (Hochschild-Kostant-Rosenberg) isomorphism and β_X is a kind of dual HKR isomorphism:

$$\begin{array}{ccc} & \mathscr{HH}(\mathscr{O}_X) & \\ {}^{\alpha_X^1}\swarrow\sim & & \sim\searrow^{\beta_X^1} \\ \delta^*\delta_*\mathscr{O}_X & \xrightarrow[\text{td}]{\sim} & \delta^!\delta_!\omega_X^{\text{hol}} \\ \alpha_X \downarrow \wr & & \wr \downarrow \beta_X \\ \bigoplus_{i=0}^{d_X}\Omega_X^i[i] & \xrightarrow[\tau]{\sim} & \bigoplus_{i=0}^{d_X}\Omega_X^i[i]. \end{array}$$

Hochschild-Kostant-Rosenberg isomorphism

For $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, one sets $\mathrm{ch}(\mathscr{F}) = \alpha_X \circ \alpha^1_X(\mathrm{hh}_{\mathscr{O}}(\mathscr{F}))$, the Chern character of $\mathscr{F}$ and $\mathrm{eu}(\mathscr{F}) = \beta_X \circ \beta^1_X(\mathrm{hh}_{\mathscr{O}}(\mathscr{F}))$, the Euler class of $\mathscr{F}$. Then ch commutes with inverse images and eu commutes with proper direct images.

Kashiwara made in 1991 the conjecture that the arrow τ making the diagram commutative is given by the cup product by the Todd class of TX. This conjecture has recently been proved by Ramadoss (2008) (after previous work by Markarian) in the algebraic case and Grivaux (2009) in the analytic case (and with a very simple proof).
This gives a new and functorial approach to the Riemann-Roch-Hirzebruch-Grothendieck theorem.

Let M be a real manifold, $\pi\colon T^*M \to M$ its cotangent bundle.

- $\mathbf{k}$ a commutative unital ring with finite global dimension,
- $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ the derived category of sheaves of $\mathbf{k}$-modules on M.
- $\omega_M \simeq \mathrm{or}_M\,[\dim M]$ the dualizing complex,
- $\mathrm{D}'_M = \mathrm{R}\mathscr{H}om\,(\bullet\,,\mathbf{k}_M)$ and $\mathrm{D}_M = \mathrm{R}\mathscr{H}om\,(\bullet\,,\omega_M)$ the duality functors.

Microsupport

For $F \in \mathrm{D^b}(\mathbf{k}_M)$ one defines its micro-support, or singular support, $\mathrm{SS}(F)$, a closed $\mathbb{R}^+$-conic subset of T^*M.

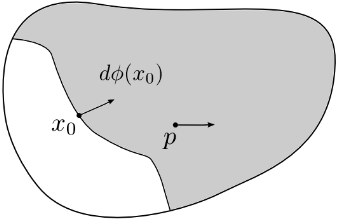
Definition

An open subset W of T^*M does not intersect $\mathrm{SS}(F)$ if for any C^1-function $\varphi\colon M \to \mathbb{R}$ and any $x_0 \in M$ such that $(x_0; d\varphi(x_0)) \in W$, setting $U = \{x; \varphi(x) < \varphi(x_0)\}$, one has for all $j \in \mathbb{Z}$

$$\varinjlim_{V \ni x_0} H^j(U \cup V; F) \simeq H^j(U; F).$$

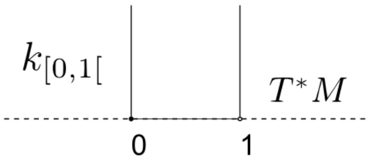
Equivalently, $\mathrm{R}\Gamma_{\{x;\varphi(x)\geq 0\}}(F)_{x_0} \simeq 0$.

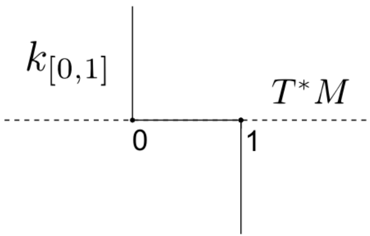
Roughly speaking, F "propagates" in the codirections which do not belong to $\mathrm{SS}(F)$.



Properties and examples of the microsupport

- $\mathrm{SS}(F)$ is co-isotropic,
- if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a d.t. then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for $j \neq k$,
- Let N be a closed submanifold of M. Then $\mathrm{SS}(\mathbf{k}_N) = T^*_N M$,
- Let X be a complex manifold, $\mathscr{M}$ a coherent $\mathscr{D}_X$-module. Set $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$. Then $\mathrm{SS}(F) = \mathrm{char}(\mathscr{M})$.
- Here, $M = \mathbb{R}$ and $T^*M = \mathbb{R}^2$





The functor μhom

Let $N \subset M$ be a closed submanifold. denote by
$\tau\colon T_N M \to M$ the normal bundle,
$\pi\colon T^*_N M \to M$ the conormal bundle.
Recall the functor ν_N of specialization, μ_N of Sato's microlocalization and its variant, μhom:

$$\nu_N\colon \mathrm{D^b}(\mathbf{k}_M) \to \mathrm{D^b}(\mathbf{k}_{T_N M}) \text{ similar to the deformation to the normal cone,}$$
$$\mu_N\colon \mathrm{D^b}(\mathbf{k}_M) \to \mathrm{D^b}(\mathbf{k}_{T^*_N M}) \text{ the Fourier-Sato transform of } \nu_M,$$
$$\mu hom(F,G) = \mu_\Delta \mathrm{R}\mathscr{H}om\,(q_2^{-1}F, q_1^! G)$$
$$\colon \mathrm{D^b}(\mathbf{k}_M)^{\mathrm{op}} \times \mathrm{D^b}(\mathbf{k}_M) \to \mathrm{D^b}(\mathbf{k}_{T^*M}).$$

Properties of the functor μhom

We can "microlocalize" the category of sheaves.
For $A \subset T^*M$ one denotes by $\mathrm{D^b}(\mathbf{k}_M; A)$ the localization of $\mathrm{D^b}(\mathbf{k}_M)$ by the full triangulated subcategory consisting of sheaves F with $\mathrm{SS}(F) \cap A = \emptyset$. Then

$$\text{for } p \in T^*M,\ H^0\mu hom(F, G)_p \simeq \mathrm{Hom}_{\mathrm{D^b}(\mathbf{k}_M;\{p\})}(F, G).$$

Moreover

$$\mathrm{R}\pi_{M*}\mu hom \simeq \mathrm{R}\mathscr{H}om\,,$$
$$\mathrm{supp}\,\mu hom(F, G) \subset \mathrm{SS}(F) \cap \mathrm{SS}(G).$$

Assume M is real analytic and $\mathbf{k}$ is a field (for simplicity). Let $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ denote the category of $\mathbb{R}$-constructible sheaves on M. This category does not admit a Serre functor. However, we have for $F, G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$

$$\mathrm{D}_{T^*M}\mu hom(F, G) \simeq \mu hom(G, F) \otimes \pi_M^{-1}\omega_M.$$

Notations

As above, M is a real manifold. One sets:

- $\delta\colon M \hookrightarrow M \times M$ the diagonal embedding, $\Delta = \delta(M)$, $\mathbf{k}_\Delta = \delta_*\mathbf{k}_M$, $\omega_\Delta = \delta_*\omega_M$, etc.
- $\delta^a\colon T^*M \hookrightarrow T^*(M \times M)$, $\delta^a((x;\xi)) = (x, x; \xi, -\xi)$.
- Let M_i $(i = 1, 2, 3)$ be manifolds. For short, we write $M_{ij} := M_i \times M_j$ $(1 \leq i, j \leq 3)$, $M_{123} = M_1 \times M_2 \times M_3$, etc.
- $q_{ij}\colon M_{123} \to M_{ij}$ the projections,
  $p_{ij}\colon T^*M_{123} \to T^*M_{ij}$ the projections,
  p_{ij^a}, the composition of p_{ij} and the antipodal map on T^*M_j.

For $K_{ij} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{ij}})$ and for $L_{ij} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*_{M_{ij}}})$ we set

$$K_{12} \underset{2}{\circ} K_{23} := \mathrm{R}q_{13!}(q_{12}^{-1}K_{12} \otimes q_{23}^{-1}K_{23}),$$

$$L_{12} \overset{a}{\underset{2}{\circ}} L_{23} := \mathrm{R}p_{13^a!}(p_{12^a}^{-1}L_{12} \otimes p_{23^a}^{-1}L_{23}).$$

We also define the corresponding operations for subsets of cotangent bundles. Let $A \subset T^*M_{12}$ and $B \subset T^*M_{23}$. We set

$$A \overset{a}{\underset{2}{\circ}} B = p_{13^a}(A \overset{a}{\underset{2}{\times}} B) \text{ where } A \overset{a}{\underset{2}{\times}} B = p_{12^a}^{-1}(A) \cap p_{23^a}^{-1}(B).$$

Microlocal homology 1

Let Λ be a closed conic subset of T^*M. We set

$$\begin{aligned} \mathscr{MH}(\mathbf{k}_M) &:= (\delta^a)^{-1}\mu hom(\mathbf{k}_\Delta, \omega_\Delta) \\ \mathrm{MH}^0_\Lambda(\mathbf{k}_M) &:= H^0_\Lambda(T^*M; \mathscr{MH}(\mathbf{k}_M)). \end{aligned}$$

We call $\mathscr{MH}(\mathbf{k}_M)$ the microlocal homology of M. Of course, we have an isomorphism which plays a role similar to that of the HKR isomorphism

$$\mathscr{MH}(\mathbf{k}_M) \simeq \pi_M^{-1}\omega_M.$$

Let $ij = 12, 23, 13$ and let Λ_{ij} be a closed conic subset of T^*M_{ij}. Assume that $\Lambda_{12} \overset{a}{\underset{2}{\times}} \Lambda_{23}$ is proper over T^*M_{13} and set $\Lambda_{13} = \Lambda_{12} \overset{a}{\underset{2}{\circ}} \Lambda_{23}$. There is a natural morphism

$$\mathscr{MH}(\mathbf{k}_{M_{12}}) \underset{2}{\circ} \mathscr{MH}(\mathbf{k}_{M_{23}}) \to \mathscr{MH}(\mathbf{k}_{M_{13}}).$$

and this morphism induces a map

$$\underset{2}{\circ} : \mathrm{MH}^0_{\Lambda_{12}}(\mathbf{k}_{M_{12}}) \otimes \mathrm{MH}^0_{\Lambda_{23}}(\mathbf{k}_{X_{23}}) \to \mathrm{MH}^0_{\Lambda_{13}}(\mathbf{k}_{X_{13}}).$$

Microlocal homology 2

The construction of the morphism above uses the composition of μhom, which makes the computations not easy. Fortunately, we have the following result, a kind of HKR isomorphism for sheaves.
We have a commutative diagram

$$\begin{array}{ccc} \mathscr{MH}(\mathbf{k}_{12}) \overset{a}{\underset{2}{\circ}} \mathscr{MH}(\mathbf{k}_{23}) & \longrightarrow & \mathscr{MH}(\mathbf{k}_{13}) \\ \downarrow \wr & & \downarrow \wr \\ \pi_{12}^{-1}\omega_{12} \overset{a}{\underset{2}{\circ}} \pi_{23}^{-1}\omega_{23} & \longrightarrow & \pi_{13}^{-1}\omega_{13}. \end{array}$$

Here the bottom horizontal arrow is induced by

$$p_{12^a}^{-1}\pi_{12}^{-1}\omega_{12} \otimes p_{23^a}^{-1}\pi_{23^a}^{-1}\omega_{23} \simeq \pi_1^{-1}\omega_1 \boxtimes \omega_{T^*M_2} \boxtimes \pi_3^{-1}\omega_3 \text{ and}$$
$$\mathrm{R}p_{13^a\,!}\left(\pi_1^{-1}\omega_1 \boxtimes \omega_{T^*M_2} \boxtimes \pi_{M_3}^{-1}\omega_3\right) \longrightarrow \pi_1^{-1}\omega_1 \boxtimes \pi_3^{-1}\omega_3.$$

As a particular case, we get canonical isomorphisms

$$\mathscr{MH}(\mathbf{k}_M) \otimes \mathscr{MH}(\mathbf{k}_M) \simeq \pi^{-1}\omega_M \otimes \pi^{-1}\omega_M \simeq \omega_{T^*M}.$$

Trace kernels

A trace kernel (K, u, v) on M is the data of $K \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M\times M})$ together with morphisms (u, v)

$$\mathbf{k}_\Delta \xrightarrow{u} K \xrightarrow{v} \omega_\Delta.$$

Setting $\mathrm{SS}_\Delta(K) := \mathrm{SS}(K) \cap T^*_\Delta(M \times M)$, the morphism u gives an element of $H^0_{\mathrm{SS}_\Delta(K)}(T^*M; \mu hom(\mathbf{k}_\Delta, K))$ whose image by v is the microlocal Euler class of K

$$\mu\mathrm{eu}_M(K) \in \mathbb{MH}^0_{\mathrm{SS}_\Delta(K)}(\mathbf{k}_M)) \simeq H^0_{\mathrm{SS}_\Delta(K)}(T^*M; \pi^{-1}\omega_M).$$

If $M = \mathrm{pt}$, a trace kernel K is nothing but an object of $\mathrm{D}^{\mathrm{b}}(\mathbf{k})$ together with linear maps $\mathbf{k} \to K \to \mathbf{k}$. The composition gives the element $\mu\mathrm{eu}(K)$ of $\mathbf{k}$. If $\mathbf{k}$ is a field of characteristic zero and $K = L \otimes L^*$ where $L \in \mathrm{D}^{\mathrm{b}}_f(\mathbf{k})$, then $\mu\mathrm{eu}(K) = \chi(L)$.

Functoriality of trace kernels

Theorem

Let K_{ij} be a trace kernel on M_{ij} and assume for simplicity that $\mathrm{SS}(K_{ij}) \subset \Lambda_{ij} \times \Lambda_{ij}^a$ $(ij = 12, 23)$.
We make the hypothesis:

$$\Lambda_{12} \overset{a}{\underset{2}{\times}} \Lambda_{23} \text{ is proper over } T^*M_{13}.$$

Set $\widetilde{K}_{12} = K_{12} \otimes q_{22}^{-1}(\mathbf{k}_2 \boxtimes \omega_2^{\otimes -1})$. Then $K_{13} := \widetilde{K}_{12} \underset{22}{\circ} K_{23}$ is a trace kernel on M_{13} and

$$\mu\mathrm{eu}_{M_{13}}(K_{13}) = \mu\mathrm{eu}_{M_{12}}(K_{12}) \overset{a}{\underset{2}{\circ}} \mu\mathrm{eu}_{M_{23}}(K_{23})$$

as elements of $\mathbb{MH}^0_{\Lambda_{13}}(\mathbf{k}_{13})$, where $\Lambda_{13} = \Lambda_{12} \overset{a}{\underset{2}{\circ}} \Lambda_{23}$

As an application, one can perform the external product, the proper direct image and the non characteristic inverse image of trace kernels and compute their microlocal Euler classes. In particular, we get:

Corollary

Let K_i be a trace kernel with $\mathrm{SS}(K_i) \subset \Lambda_i \times \Lambda_i^a$ $(i = 1, 2)$ and set $\widetilde{K}_1 = K_1 \otimes (\mathbf{k}_M \boxtimes \omega_M^{\otimes -1})$.

(a) *Assume $\Lambda_1 \cap \Lambda_2^a \subset T_M^* M$. Then $\widetilde{K}_1 \otimes K_2$ is a trace kernel on M and*

$$\mu\mathrm{eu}_M(\widetilde{K}_1 \otimes K_2) = \mu\mathrm{eu}_M(K_1) \star \mu\mathrm{eu}_M(K_2).$$

(b) *Assume moreover that* $\operatorname{supp} K_1 \cap \operatorname{supp} K_2$ *is compact. Then*

$$\mu\mathrm{eu}(\mathrm{R}\Gamma(M \times M; \widetilde{K}_1 \otimes K_2)) = \int_{T^*M} \mu\mathrm{eu}(K_1) \cup \mu\mathrm{eu}(K_2).$$

Constructible sheaves

We assume now that M is real analytic and $\mathbf{k}$ is a field. Let $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ be an $\mathbb{R}$-constructible sheaf.

The evaluation morphism $G \overset{\mathrm{L}}{\otimes} \mathrm{D}G \to \omega_M$ gives by adjunction and duality:

$$\mathbf{k}_\Delta \to G \overset{\mathrm{L}}{\boxtimes} \mathrm{D}G \to \omega_\Delta.$$

Denote by $\mathrm{TK}(G)$ the trace kernel so constructed. Then $\mu\mathrm{eu}(\mathrm{TK}(G))$ is nothing but the Lagrangian cycle of G constructed by Kashiwara in 1985 and one recovers the classical functorial properties of Lagrangian cycles. Let $f\colon M \to N$ be a morphism of manifolds. To f one associates the maps

$$T^*M \xleftarrow{f_d} M \times_N T^*N \xrightarrow{f_\pi} T^*N$$

There are natural morphsim

$$f_\mu\colon f_{\pi\,!} f_d^{-1} \pi_M^{-1} \omega_M \to \pi_N^{-1} \omega_N,$$
$$f^\mu\colon f_{d\,!} f_\pi^{-1} \pi_N^{-1} \omega_N \to \pi_M^{-1} \omega_M.$$

- Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ and assume f is proper on $\mathrm{supp}(F)$, or equivalently, f_π is proper on $f_d^{-1}\mathrm{SS}(F)$. Then $\mu\mathrm{eu}(\mathrm{R}f_* F) = f_\mu \mu\mathrm{eu}(F)$,
- Let $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_N)$ and assume that f is non characteristic for G, that is, f_d is proper on $f_\pi^{-1}\mathrm{SS}(G)$. Then $\mu\mathrm{eu}(f^{-1}G) = f^\mu \mu\mathrm{eu}(D)$.

$\mathscr{D}$-modules

- X a complex manifold of complex dimension d_X, $\Delta \hookrightarrow X \times X$ the diagonal
- $\mathrm{D}_{\mathscr{D}}\mathscr{M} := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}[d_X]$, (duality for left $\mathscr{D}$-modules),
- $\mathscr{M} \underline{\boxtimes} \mathscr{N} := \mathscr{D}_{X\times X} \otimes_{\mathscr{D}_X \boxtimes \mathscr{D}_X} (\mathscr{M} \boxtimes \mathscr{N})$ (external product),
- $\mathscr{B}_\Delta := H^{d_X}_{[\Delta]}(\mathscr{O}_{X\times X})$ and $\mathscr{B}_\Delta^\vee := \mathscr{B}_\Delta[2d_X]$.

Note that $\mathrm{D}_{\mathscr{D}}\mathscr{B}_\Delta \simeq \mathscr{B}_\Delta$. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, we have the isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X\times X}}(\mathscr{B}_\Delta, \mathscr{M} \underline{\boxtimes} \mathrm{D}_{\mathscr{D}}\mathscr{M})[d_X].$$

We deduce the morphisms (the second one by duality from the first one):

$$\mathscr{B}_\Delta \to \mathscr{M} \underline{\boxtimes} \mathrm{D}_{\mathscr{D}}\mathscr{M}[d_X] \to \mathscr{B}_\Delta^\vee.$$

$\mathscr{E}$-modules

Denote by $\mathscr{E}_{T^*X}$ the sheaf on T^*X of microdifferential operators. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$ set

$$\mathscr{M}^E := \mathscr{E}_{T^*X} \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{M}.$$

We define the duality functor $\mathrm{D}_{\mathscr{E}}$ for $\mathscr{E}$-modules and the external product similarly as for $\mathscr{D}$-modules.
Recall that $\mathrm{char}(\mathscr{M}) = \mathrm{supp}(\mathscr{M}^E)$ where $\mathrm{char}(\mathscr{M})$ is the characteristic variety of $\mathscr{M}$. Set

$$\mathscr{C}_\Delta := \mathscr{B}_\Delta^E, \quad \mathscr{C}_\Delta^\vee := (\mathscr{B}_\Delta^\vee)^E.$$

We have the morphisms

$$\mathscr{C}_\Delta \to \mathscr{M}^E \underline{\boxtimes} \mathrm{D}_{\mathscr{E}} \mathscr{M}^E [d_X] \to \mathscr{C}_\Delta^\vee.$$

Setting

$$\mathcal{HH}(\mathscr{E}_{T^*X}) = (\delta^a)^{-1} \mathrm{R}\mathscr{H}om_{\mathscr{E}_{X\times X}}(\mathscr{C}_\Delta, \mathscr{C}_\Delta^\vee),$$

we get the Hochschild class of $\mathscr{M}$:

$$\mathrm{hh}_{\mathscr{E}}(\mathscr{M}) \in H^0_{\mathrm{char}(\mathscr{M})}(T^*X; \mathcal{HH}(\mathscr{E}_{T^*X})).$$

Hochschild class and microlocal Euler class 1

We have the natural morphism in $\mathrm{D}^{\mathrm{b}}(\pi^{-1}\mathscr{D}_X \otimes \pi^{-1}\mathscr{D}_X^{\mathrm{op}})$

$$\mathscr{E}_X \to \mu hom(\Omega_X, \Omega_X).$$

We deduce the morphism for $\mathscr{N}_1$ and $\mathscr{N}_2$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$;

$$\mathrm{R}\mathscr{H}om_{\mathscr{E}}(\mathscr{N}_1^E, \mathscr{N}_2^E) \to \mu hom(\Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{N}_1, \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{N}_2).$$

We have

$$\Omega_{X\times X}[-d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \mathscr{B}_\Delta \simeq \mathbb{C}_\Delta, \quad \Omega_{X\times X}[-d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \mathscr{B}_\Delta^\vee \simeq \omega_\Delta.$$

One deduces the morphism and isomorphism

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathscr{E}_{X\times X}}(\mathscr{C}_\Delta, \mathscr{C}_\Delta^\vee) &\xrightarrow{\sim} \mu hom(\Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \mathscr{B}_\Delta, \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \mathscr{B}_\Delta^\vee) \\ &\simeq \mu hom(\mathbb{C}_\Delta, \omega_\Delta). \end{aligned}$$

An easy calculation shows that the first arrow is also an isomorphism. Therefore, we get (a result of Brylinski-Getzler 1987)

$$\mathcal{HH}(\mathscr{E}_X) \xrightarrow{\sim} \mathscr{MH}(\mathbb{C}_X).$$

Hochschild class and microlocal Euler class 2

Recall the morphisms

$$\mathscr{B}_\Delta \to \mathscr{M} \underline{\boxtimes} \mathrm{D}_{\mathscr{D}} \mathscr{M}\,[d_X] \to \mathscr{B}_\Delta^\vee.$$

Applying $\Omega_{X\times X}[-d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \bullet$ to these morphisms, we get the morphisms

$$\mathbb{C}_\Delta \to \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} (\mathscr{M} \underline{\boxtimes} \mathrm{D}_{\mathscr{D}} \mathscr{M}) \to \omega_\Delta.$$

For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, we set

$$\mathrm{TK}(\mathscr{M}) = \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} (\mathscr{M} \underline{\boxtimes} \mathrm{D}_{\mathscr{D}} \mathscr{M}).$$

Then

$$\mathrm{hh}_{\mathscr{E}}(\mathscr{M}) = \mu\mathrm{eu}_X(\mathrm{TK}(\mathscr{M})) \text{ in } H^0_{\mathrm{char}(\mathscr{M})}(T^*X; \pi^{-1}\omega_X).$$

Elliptic pairs 1

Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ and $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. Recall that $(\mathscr{M}, G)$ is an elliptic pair (S-Schneiders 1994) if

$$\mathrm{char}(\mathscr{M}) \cap \mathrm{SS}(G) \subset T^*_X X.$$

We shall assume now that $(\mathscr{M}, G)$ is an elliptic pair and we set

$$\mathrm{TK}(\mathscr{M}, G) \quad := \quad \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} (\mathscr{M} \underline{\boxtimes} \mathrm{D}_{\mathscr{D}} \mathscr{M}) \otimes G \boxtimes \mathrm{D}' G.$$

It follows from the functoriality of trace kernels that $\mathrm{TK}(\mathscr{M}, G)$ is a trace kernel and moreover:

$$\mu\mathrm{eu}_X(\mathrm{TK}(\mathscr{M}, G)) = \mu\mathrm{eu}_X(\mathscr{M}) \star \mu\mathrm{eu}_X(G).$$

Elliptic pairs 2

We have the natural isomorphism (a Petrovsky's theorem for sheaves)

$$\mathrm{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{D}'G \otimes \mathscr{O}_X) \xrightarrow{\sim} \mathrm{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M} \otimes G, \mathscr{O}_X).$$

Example Assume M is a real analytic manifold and X is a complexification of M. Choose $G = \mathrm{D}'_X \mathbb{C}_M$. Then $(\mathscr{M}, G)$ is an elliptic pair off $\mathscr{M}$ is elliptic in the usual sense and we get

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M} \otimes G, \mathscr{O}_X) &\simeq \mathrm{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M) \\ &\simeq \mathrm{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_M). \end{aligned}$$

Assuming that $\mathrm{supp}(\mathscr{M}) \cap \mathrm{supp}(G)$ is compact, it follows that the complex

$$\mathrm{Sol}(\mathscr{M} \otimes G) := \mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M} \otimes G, \mathscr{O}_X)$$

may be represented both by a complex of topological vector spaces of type FN and a complex of type DFN. Therefore its cohomology is finite dimensional. Moreover

$$\mathrm{R}\Gamma(X \times X; \mathrm{TK}(\mathscr{M}, G)) \simeq \mathrm{Sol}(\mathscr{M} \otimes G) \otimes \mathrm{Sol}(\mathscr{M} \otimes G)^*.$$

Elliptic pairs 3

Applying the general results on trace kernels , we get

Theorem

Let $(\mathscr{M}, G)$ *be an elliptic pair and assume that* $\operatorname{supp}\mathscr{M} \cap \operatorname{supp} G$ *is compact. Then*

$$\chi(\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M} \otimes G, \mathscr{O}_X)) = \int_{T^*X} (\mathrm{hh}_{\mathscr{E}}(\mathscr{M}) \cup \mu\mathrm{eu}_X(G)).$$

This formula has many applications, as far as one is able to calculate $\mathrm{hh}_{\mathscr{E}}(\mathscr{M})$.

Elliptic pairs 3

Assume that $\mathscr{M}$ is endowed with a good filtration and $\mathrm{char}(\mathscr{M}) \subset \Lambda$. Set

$$\widetilde{\mathrm{gr}}\mathscr{M} := \mathscr{O}_{T^*X} \otimes_{\pi^{-1}\mathrm{gr}\mathscr{D}_X} \pi^{-1}\mathrm{gr}\mathscr{M}$$

$$\sigma_\Lambda(\mathscr{M}) = \mathrm{ch}_\Lambda(\widetilde{\mathrm{gr}}\mathscr{M}) \in \bigoplus_j H^{2j}_\Lambda(T^*X; \mathbb{C}_{T^*X}),$$

$$\mu\mathrm{ch}_\Lambda(\mathscr{M}) = \sigma_\Lambda(\mathscr{M}) \cup \pi^*\mathrm{Td}_X(T^*X) \text{ for a left } \mathscr{D}\text{-module}$$

$$\mu\mathrm{ch}_\Lambda(\mathscr{M}) = \sigma_\Lambda(\mathscr{M}) \cup \pi^*\mathrm{Td}_X(TX) \text{ for a right } \mathscr{D}\text{-module.}$$

Note that $\mu\mathrm{ch}$ commutes with proper direct images (Laumon's version of the RR theorem for $\mathscr{D}$-modules) and non characteristic inverse images.
The formula

$$\mu\mathrm{eu}_\Lambda(\mathscr{M}) = [\mu\mathrm{ch}_\Lambda(\mathscr{M})]^{2d_X}$$

was conjectured by S-Schneiders in 1994 and proved by Bressler-Nest-Tsygan in 2002.
If M is a compact real analytic manifold and X is a complexification of M, one recovers the Atiyah-Singer theorem by choosing $G = \mathrm{D}'\mathbb{C}_M$.

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