Quantization of complex manifolds: a survey

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Abstract

These Notes are the written version of an informal zoom seminar with a few people interested in the subject. I will recall classical results on quantization of complex contact and symplectic (or more generally Poisson) manifolds and their Lagrangian submanifolds. These Notes contain no new or original results.

Contents

General notations

Geometry

All manifolds and morphisms of manifolds are either real (C^∞ or real analytic) or complex analytic. In § 1 to 7 all manifolds are complex analytic.

- For a space X we denote by Δ_X or simply Δ the diagonal of $X \times X$ and δ_X the diagonal embedding. For a topological space X and a subset A of X, we denote by $\overline{A}$ its closure and $\mathrm{Int}(A)$ its interior.

- For a real manifold M, we denote by $\dim M$ its dimension and by or_M the orientation sheaf.

- For a complex manifold $(M, \mathscr{O}_M)$ we denote by d_M its complex dimension, by $\Omega_M^{(j)}$ the sheaf of holomorphic forms of degree j and we set $\Omega_M := \Omega_M^{(d_M)}$. On a product $M \times N$, the notation $\Omega_{M\times N}^{(i,j)}$ is clear.

- Let M be a real or complex manifold. We denote by $\tau_M \colon TM \to M$ its tangent bundle and by $\pi_M \colon T^*M \to M$ its cotangent bundle. (We shall often write π instead of π_M.) For $N \subset M$ a smooth submanifold, we denote by T_NM the normal bundle to N in M and by T^*_NM the conormal bundle to N in M. In particular, T^*_MM denotes the zero-section. We shall identify T^*M to $T^*_\Delta(M\times M)$ by the first projection $T^*(M\times M) \to T^*M$ and similarly we identify $T_\Delta(M\times M)$ with TM. We set $\dot{T}^*M := T^*M \setminus T^*_MM$.

- For a real manifold M and two subsets $A, B \subset M$, we denote by $\mathrm{C}(B, A) \subset TM$ the Whitney normal cone. If N is a smooth closed submanifold, we denote by $\mathrm{C}_N(A)$ the image of $\mathrm{C}(N, A)$ in T_NM. (See the footnote in § 9.)

- We denote by $a \colon T^*M \to T^*M$ the antipodal map. In a local coordinate system, $a : (x;\xi) \mapsto (x;-\xi)$. For a subset $S \subset T^*M$, we write S^a instead of $a(S)$.

- Note that $X := T^*M$ is a symplectic manifold, the symplectic 2-form ω_X being the differential of the Liouville 1-form α_X and there is an $\mathbb{R}^+$-action in the real case, a $\mathbb{C}^\times$-action in the complex case, on X. The manifold $\dot{T}^*M$ is an example of a homogeneous symplectic manifold (see § 6 for more details). In a local coordinate system x on M, with associated coordinates $(x;\xi)$ on T^*M, the Liouville form is

$$\alpha_X = \sum_j \xi_j \, d\, x_j$$

- In the complex case, we denote by $Y := P^*M = \dot{T}^*M/\mathbb{C}^\times$ the projective cotangent bundle, a complex contact manifold and by γ the projection:

$$\gamma \colon \dot{T}^*M \to P^*M.$$

Sheaves

We shall mainly follow the notations of [KS90] for homological algebra and sheaf theory. Here, all rings are unital and associative.

A topological space X is *good* if it is Hausdorff, locally compact, countable at infinity and of finite flabby dimension.

Let X be a good topological space and $\mathbf{k}$ a commutative ring of finite global dimension. We denote by $\mathrm{Mod}(\mathbf{k}_X)$ the abelian category of sheaves of $\mathbf{k}$-modules on X, by $\mathrm{D}(\mathbf{k}_X)$ its derived category and by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ the bounded derived category. More generally, for a sheaf of rings $\mathscr{R}$, we denote by $\mathrm{D}(\mathscr{R})$ the derived category of sheaves of left $\mathscr{R}$-modules and by $\mathrm{D}^{\mathrm{b}}(\mathscr{R})$ the bounded derived category.

For a locally closed subset Z of X, we denote by $\mathbf{k}_Z$ the constant sheaf on Z with stalk $\mathbf{k}$, extended by 0 on $X \setminus Z$. We denote by ω_X the dualizing complex on X.

On a real manifold M, we denote by $\mathrm{SS}(F)$ the micro-support of a sheaf $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$, a closed $\mathbb{R}^+$-conic subset of T^*M. The set $\mathrm{SS}(F)$ is co-isotropic (in some sense that we do not recall here).

For a commutative Noetherian ring $\mathbf{k}$ and a real analytic manifold M we denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of $\mathbb{R}$-constructible sheaves. When M is a complex manifold, we denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathbb{C}$-constructible sheaves.

A sheaf of rings $\mathscr{R}$ on X is (left) Noetherian if it is (left) coherent, the stalks $\mathscr{R}_x$ are (left) Noetherian for all $x \in X$ and on any open subset, an increasing family of coherent (left) ideals is locally staionnary.

1 Microdifferential operators on a cotangent bundle

References are made to [SKK73]. See also [Kas86, Sch85] for an exposition[1].

Let M be a complex manifold. The $\mathbb{C}^\times$-conic sheaf of $\mathbb{C}$-algebras $\mathscr{E}_X$ of microdifferential operators on $X = T^*M$ has been constructed *functorially* in [SKK73] as a sub-sheaf of the $\mathbb{R}^+$-conic sheaf of $\mathbb{C}$-algebras of microlocal operators $\mathscr{E}^{\mathbb{R}}_X$

$$\mathscr{E}^{\mathbb{R}}_X := \mu_\Delta(\Omega^{(0,d_M)}_{M\times M})\,[d_M]$$

where μ_Δ is the Sato's microlocalization functor along the diagonal of $M \times M$. By identifying $T^*_\Delta(M\times M)$ with T^*M by the first projection, we consider $\mathscr{E}^{\mathbb{R}}_X$ as a sheaf on T^*M. We shall not describe here this ring which is extremely complicated (it contains as a sub-ring that of microdifferential operators of infinite order) and remains quite mysterious.

The sheaf $\mathscr{E}_X$ is constant on the fibers of the projection $\gamma\colon \dot{T}^*X \to P^*X$, and we denote its direct images on $Y = P^*M$ by $\mathscr{E}_Y$.

$$\mathscr{E}_Y := \gamma_*\mathscr{E}_X.$$

(It is proved that the derived direct image $\mathrm{R}\gamma_*\mathscr{E}_X$ is concentrated in degree 0.)

Most of the results below hold either for $\mathscr{E}_X$-modules or for $\mathscr{E}_Y$-modules.

The sheaf $\mathscr{E}_X$ is filtered over $\mathbb{Z}$, and one denotes by $\mathscr{E}_X(m)$ the sheaf of operators of order less than or equal to m. We denote by

$$\sigma_m(\cdot)\colon \mathscr{E}_X(m) \to \mathscr{E}_X(m)/\mathscr{E}_X(m-1) \simeq \mathscr{O}_{T^*X}(m)$$

[1] See https://webusers.imj-prg.fr/~pierre.schapira/books/ at the bottom of the page for a few errata

the symbol map, where $\mathscr{O}_{T^*X}(m)$ is the sheaf of holomorphic functions homogeneous of degree m in the fiber variable. Hence

$$gr(\mathscr{E}_X) \simeq \bigoplus_{j\in\mathbb{Z}} \mathscr{O}_{T^*M}(j).$$

If P is a section of $\mathscr{E}_X(m)$ and not of $\mathscr{E}_X(m-1)$ (on a connected open subset of X), $\sigma_m(P)$ is called the principal symbol of P.

- The sheaf $\mathscr{E}_X$ is Noetherian and the support of a coherent module is a complex analytic variety, *co-isotropic* with respect to the homogeneous symplectic (or contact) structure. Note that this fundamental result of [SKK73] was obtained much before (and certainly is at the origin of) the famous Gabber theorem [Gab81]. (One shall be aware that the sheaf $\mathscr{E}_X(0)$ is coherent on $\dot{T}^*M$ but not on T^*M, although $\mathscr{E}_X(0)|_M \simeq \mathscr{O}_M$.)

- A coherent module whose support is Lagrangian is called holonomic.

- A coherent $\mathscr{E}_X$-module $\mathscr{L}$ is simple along a ($\mathbb{C}^\times$-conic) Lagrangian submanifold Λ if it is locally generated by a coherent $\mathscr{E}_X(0)$-module $\mathscr{L}_0$ such that setting $\mathscr{L}_{-1} = \mathscr{E}_X(-1)\mathscr{L}_0$, one has $\mathscr{L}_0/\mathscr{L}_{-1} \simeq \mathscr{O}_\Lambda$. This definition extends to smooth co-isotropic submanifolds.

- Given two open subsets $U \subset X$ and $V \subset X$ (of course, one can also choose $U \subset T^*M$ and $V \subset T^*N$ with $d_M = d_N$), there locally exists a homogeneous symplectic isomorphism $\chi\colon U \xrightarrow{\sim} V$ which can be quantized as an isomorphism of $\mathbb{C}$-algebras $\mathscr{E}_X|_U \xrightarrow{\sim} \mathscr{E}_X|_V$. The graph of χ is a Lagrangian submanifold $\Lambda \subset U \times V^a$ and the quantization is associated with the choice of a simple holonomic module $\mathscr{L}$ and a "non degenerate" generator of this module. Such a quantization is called a QCT (quantized contact transform) in [SKK73].

One can also define the sheaf of $\mathbb{C}$-algebras $\widehat{\mathscr{E}}_X$ of formal microdifferential operators by

$$\widehat{\mathscr{E}}_X(m) = \varprojlim_p \mathscr{E}_X(m)/\mathscr{E}_X(m-p).$$

Then, $\mathscr{E}_X \subset \widehat{\mathscr{E}}_X$, as a subring.

In a local coordinate system x on M, with associated homogeneous symplectic coordinates $(x;\xi)$ on T^*M, a formal microdifferential operator P of order m (*i.e.,* a section of $\widehat{\mathscr{E}}_X(m)$) defined on an open subset V of T^*M may be identified with its total symbol $\sigma_{\rm tot}(P)$:

$$\sigma_{\rm tot}(P) = \sum_{j=-\infty}^{m} p_j(x,\xi), \;\; p_j \in \mathscr{O}_{T^*M}(j)(V). \tag{1.1}$$

The product structure on $\widehat{\mathscr{E}}_X$ (hence, on $\mathscr{E}_X$) is then given by the Leibniz formula. If Q is a formal microdifferential operator of total symbol $\sigma_{\mathrm{tot}}(Q)$, then

$$\sigma_{\mathrm{tot}}(P \circ Q) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{\mathrm{tot}}(P) \partial_x^\alpha \sigma_{\mathrm{tot}}(Q). \tag{1.2}$$

The ring $\mathscr{E}_X$ is the subring of $\widehat{\mathscr{E}}_X$ of operators whose total symbol (1.1) satisfies the estimates

$$\begin{cases} \text{for any compact subset } K \text{ of } V \text{ there exists a constant} \\ C_K > 0 \text{ such that for all } j < 0,\ \sup_K |p_j| \le C_K^{-j}(-j)!. \end{cases} \tag{1.3}$$

In particular, a section P in $\widehat{\mathscr{E}}_X$ or in $\mathscr{E}_X$ is invertible on an open subset V of T^*X if and only if its principal symbol is nowhere vanishing on V. For example, on $T^*\mathbb{C}^n$, the Laplace operator $\sum_{j=1}^n (\frac{\partial}{\partial x_j})^2$ is invertible on the open set $\{(x;\xi) \in T^*\mathbb{C}^n; \sum_{j=1}^n \xi_j^2 \neq 0\}$.

Let $\mathscr{D}_M$ denote the sheaf of $\mathbb{C}$-algebras of (finite order) differential operators. Then there is a natural embedding

$$\pi^{-1}\mathscr{D}_M \hookrightarrow \mathscr{E}_X,$$

and an isomorphism $\mathscr{E}_X|_{T^*_M M} \simeq \mathscr{D}_M$.

Remark 1.1. The formal properties of the rings $\mathscr{E}_X$ and $\widehat{\mathscr{E}}_X$ and their modules are very similar. However, microdifferential operators act on holomorphic cohomology classes, which is not the case in the formal case. More precisely, (this is for specialists) if F is a sheaf on the real manifold underlying M (an $\mathbb{R}$-constructible sheaf in practice), then the cohomology classes $H^j(\mu hom(F, \mathscr{O}_M))$ are $\mathscr{E}_X$-modules. There is a stronger (and difficult) result: $\mu hom(F, \mathscr{O}_M)$ is well-defined in the derived category $\mathrm{D^b}(\mathscr{E}_X)$ (see Corollary 9.3).

Problem 1.2. To extend the construction of $\mathscr{E}_Y$ to complex contact manifolds. The answer is given in [Kas96] and explained in § 5.

2 The WKB-algebra on a cotangent bundle

In all this paper, we set for short

$$\mathbb{C}^\hbar := \mathbb{C}[[\hbar]], \quad \mathbb{C}^{\hbar,\mathrm{loc}} := \mathbb{C}((\hbar)). \tag{2.1}$$

The sheaf $\widehat{\mathscr{E}}_X$ is associated with the homogeneous symplectic structure on T^*M. As usual, one can "kill" this homogeneity by adding a variable. Recall that $\dot{T}^*M = T^*M \setminus T^*_M M$ is the cotangent bundle with the zero-section removed and P^*M the projective cotangent bundle.

Let t denote a coordinate on $\mathbb{C}$ and $(t;\tau)$ the associated coordinates on $T^*\mathbb{C}$. Set $Z = X \times \dot{T}^*\mathbb{C}$, $Y = Z/\mathbb{C}^\times \subset P^*(M\times\mathbb{C})$ and consider the maps

$$(2.2)\qquad \begin{array}{ccccc} Z = T^*M \times T^*_{\tau\neq 0}\mathbb{C} & \xrightarrow[\gamma]{} & Y = P^*_{\tau\neq 0}(M\times\mathbb{C}) & \xrightarrow[\rho]{} & X = T^*M \\ (x,\xi),(t;\tau) & \longrightarrow & (x,t;[\xi,\tau]) & \longrightarrow & (x;\xi/\tau). \end{array}$$

We shall say that Y is a contactification of X and Z a homogenization of Y.

Recall that $\widehat{\mathscr{E}}_Y(0)$ denotes the sheaf of microdifferential operators of order ≤ 0 on Y. One denotes by $\widehat{\mathscr{E}}_{Y,\hat{t}}(0)$ the sub-ring consisting of operators not depending on t, that is, commutating with ∂_t in $\widehat{\mathscr{E}}_Y$. Then one sets

$$(2.3)\qquad \hbar := \partial_t^{-1}$$

and define the so-called WKB-algebra (a particular case of a DQ-algebra, see below)

$$\widehat{\mathscr{W}_X}(0) := \rho_*\widehat{\mathscr{E}}_{Y,\hat{t}}(0).$$

Since ∂_t is central in $\widehat{\mathscr{E}}_{Y,\hat{t}}(0)$, $\mathscr{W}_X(0)$ is a $\mathbb{C}^\hbar$-algebra.

In a local coordinate system x on M, with associated symplectic coordinates $(x;u)$ on T^*M, a section of $\widehat{\mathscr{W}_X}(0)$ has a total symbol $\sigma_{\rm tot}(P)$:

$$(2.4)\qquad \sigma_{\rm tot}(P) = \sum_{j=0}^{+\infty} f_j(x,u)\hbar^j, \quad f_j \in \mathscr{O}_X.$$

This corresponds to (1.1) by setting for $j \leq 0$:

$$p_j(x;\xi) = p_j(x;\xi/\tau,1)\tau^j = f_j(x,u)\hbar^{-j}.$$

The product structure on $\widehat{\mathscr{W}_X}(0)$ is given by the Leibniz formula. If Q is another section of total symbol $\sigma_{\rm tot}(Q)$, then

$$(2.5)\qquad \sigma_{\rm tot}(P\circ Q) = \sum_{\alpha\in\mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!}\partial_u^\alpha\sigma_{\rm tot}(P)\partial_x^\alpha\sigma_{\rm tot}(Q).$$

Replacing $\widehat{\mathscr{E}}_Y(0)$ with $\widehat{\mathscr{E}}_Y$, one defines the $\mathbb{C}^{\hbar,\rm loc}$-algebra $\widehat{\mathscr{W}_X}$. Of course

$$\widehat{\mathscr{W}_X} = \mathbb{C}_X^{\hbar,\rm loc} \otimes_{\mathbb{C}_X^\hbar} \widehat{\mathscr{W}_X}(0).$$

Consider the subring $\mathbb{C}^\hbar_w$ of $\mathbb{C}^\hbar$ defined as the set of series $\sum_{j=0}^{+\infty} c_j\hbar^j$ satisfying (similarly as (1.3)) $\sum_{j=0}^{+\infty} c_j\hbar^j/j! < \infty$ and its localization $\mathbb{C}_w^{\hbar,\rm loc} = \mathbb{C}[\hbar,\hbar^{-1}]\otimes_{\mathbb{C}[\hbar]}\mathbb{C}^\hbar_w$.

By replacing the sheaf $\widehat{\mathscr{E}}_Y$ with the sheaf $\mathscr{E}_Y$, one defines similarly the sheaf $\mathscr{W}_X$, a sheaf of $\mathbb{C}_w^{\hbar,\rm loc}$-algebras.

The sheaves $\widehat{\mathscr{W}_X}$ and $\mathscr{W}_X$ inherit of most of the properties of the sheaves $\widehat{\mathscr{E}}_X$ and $\mathscr{E}_X$. In particular, they are Noetherian and the support of a coherent $\widehat{\mathscr{W}_X}$ or $\mathscr{W}_X$-module (which is no more $\mathbb{C}^\times$-conic) is co-isotropic. On the other-hand, any closed complex analytic subvariety of X is the support of a coherent $\mathscr{W}_X(0)$-module since $\mathscr{O}_X \simeq \mathscr{W}_X(0)/\hbar\mathscr{W}_X(0)$.

However, one shall be aware that there exist holonomic $\mathscr{W}_X$-modules which are not, even locally, the image by ρ of any holonomic $\mathscr{E}_Y$-module on $Y = P^*_{\tau\neq 0}(M\times\mathbb{C})$.

3 Star-products

The product on the ring $\widehat{\mathscr{W}}_X(0)$ is a particular case of a so called star-product, which can be defined on any complex manifold.

Let $(X, \mathscr{O}_X)$ be a complex manifold. An associative multiplication law $\star$ on $\mathscr{O}_X[[\hbar]]$ is a star-product if it is $\hbar$-bilinear and satisfies

$$f \star g = fg + \sum_{i \geq 1} P_i(f,g)\hbar^i \text{ for } f, g \in \mathscr{O}_X,$$

where the P_i's are bi-differential operators such that $P_i(f,1) = P_i(1,f) = 0$ for all $f \in \mathscr{O}_X$ and $i > 0$. We call $(\mathscr{O}_X[[\hbar]], \star)$ a star-algebra. (See Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer and also Berezin [BFF+78, Ber74].)

- A star-product defines a Poisson structure on $(X, \mathscr{O}_X)$:

$$\{f, g\} = \hbar^{-1}(f \star g - g \star f) \bmod \hbar\mathscr{O}_X[[\hbar]],$$

- any morphism of $\mathbb{C}^\hbar$-algebras $(\mathscr{O}_X[[\hbar]], \star) \to (\mathscr{O}_X[[\hbar]], \star')$ is an isomorphism and is given by a sequence of differential operators,

Example 3.1. When the Poisson bracket is symplectic, the star product is locally isomorphic to the Leibniz product given in'(2.5). With symplectic coordinates $(x; u)$:

$$f \star g = \sum_{\alpha \in \mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!}(\partial_u^\alpha f)(\partial_x^\alpha g). \tag{3.1}$$

Definition 3.2. A DQ-algebra $\mathscr{A}$ on X is a $\mathbb{C}^\hbar$-algebra locally isomorphic to a star-algebra $(\mathscr{O}_X[[\hbar]], \star)$.

Recall that any $\mathbb{C}$-algebra endomorphism of $\mathscr{O}_X$ is the identity. It follows that the $\mathbb{C}$-algebra isomorphism $\mathscr{A}/\hbar\mathscr{A} \xrightarrow{\sim} \mathscr{O}_X$ is unique. Clearly a DQ-algebra $\mathscr{A}$ satisfies the conditions:

$$\begin{cases} \text{(i)} \ \hbar\colon \mathscr{A} \to \mathscr{A} \text{ is injective } (\mathscr{A} \text{ has no } \hbar\text{-torsion}), \\ \text{(ii)} \ \mathscr{A} \to \varprojlim_n \mathscr{A}/\hbar^n\mathscr{A} \text{ is an isomorphism } (\mathscr{A} \text{ is } \hbar\text{-adic complete}), \\ \text{(iii)} \ \mathscr{A}/\hbar\mathscr{A} \text{ is isomorphic to } \mathscr{O}_X \text{ as a } \mathbb{C}\text{-algebra.} \end{cases}$$

We denote by σ_0 the composition

$$\sigma_0\colon \mathscr{A} \to \mathscr{A}/\hbar\mathscr{A} \xrightarrow{\sim} \mathscr{O}_X.$$

One proves that $\mathscr{A}$ is right and left Noetherian (in particular, coherent).

Problem 3.3. Given a complex contact manifold X, can we construct a DQ-algebra $\mathscr{A}_X$ such that the Poisson bracket associated with the star-product is the given Poisson bracket of X? This extremely difficult question was solved by Kontsevich in the real case, locally in the complex case and globally in the complex case after replacing the notion of sheaves of algebras by that of algebroid stacks (see [Kon01] for the complex case).

4 Algebroid stacks

Roughly speaking, an algebroid stack is locally a sheaf of algebras but the usual glueing condition (vanishing of a 2-cocycle) does not hold and is replaced by the vanishing of a 3-cocycle.

Let X be a topological space (or a site admitting fiber products) and let $\mathbf{k}$ be a field. Let $\{U_i\}_i$ be a family of open subsets. We set as usual $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$, etc.

We shall not give the definition of an algebroid stack but simply say that, in our situation of the analytic topology, it can be represented by the following data:

- an open covering $X = \bigcup_i U_i$,
- a sheaf of $\mathbf{k}$-algebras $\mathscr{A}_i$ on U_i,
- isomorphisms of $\mathbf{k}$-algebras $f_{ji} \colon \mathscr{A}_j \xrightarrow{\sim} \mathscr{A}_i$ on U_{ij},
- $a_{ijk} \in \mathscr{A}_i^\times(U_{ijk})$ ($\mathscr{A}_i^\times$ is the sheaf of invertible sections of $\mathscr{A}_i$),

these data satisfing the following conditions:

(i) $f_{ij} \circ f_{jk} = \mathrm{ad}(a_{ijk}) f_{ik}$ on U_{ijk} as isomorphisms $\mathscr{A}_k \xrightarrow{\sim} \mathscr{A}_i$,

(ii) $a_{ijk} a_{ikl} = f_{ij}(a_{jkl}) a_{ijk}$.

An important remark of [Kas96] is the following:

Lemma 4.1. *If the sections a_{ijk} belong to some subsheaf of $\mathscr{A}_i$ for which the condition $\mathrm{ad}(a) = \mathrm{ad}(b)$ implies $a = b$, then condition* (ii) *above follows from* (i)*.*

5 Quantization of complex contact manifolds

References are made to [Kas96].

First, consider a complex manifold M, its cotangent bundle $X = T^*M$ and its projective cotangent bundle $Y = P^*M$.

Since all results of this section about formal/non formal modules are almost the same, we shall not make any difference, writing for short $\mathscr{E}_X$ both for $\mathscr{E}_X$ or $\widehat{\mathscr{E}}_X$.

A volume form on M defines an anti-automorphism

$$* \colon \mathscr{E}_X \to a_* \mathscr{E}_X \tag{5.1}$$

(recall that a is the antipodal map). This anti-isomorphism of algebras is called the transposition (with respect to the volume form).

This leads to consider the sheaf of rings:

$$\mathscr{E}_X^{\sqrt{v}} := \pi^{-1}\Omega_M^{\otimes 1/2} \otimes_{\pi^{-1}\mathscr{O}_M} \mathscr{E}_X \otimes_{\pi^{-1}\mathscr{O}_M} \pi^{-1}\Omega_M^{\otimes -1/2}. \tag{5.2}$$

Note that $\Omega_M^{\otimes 1/2}$ and $\Omega_M^{\otimes -1/2}$ are not globally defined as sheaves but are globally defined as twisted sheaves. On the other-hand $\mathscr{E}_X^{\sqrt{v}}$ is a well-defined sheaf of rings on $X = T^*M$, locally isomorphic to $\mathscr{E}_X$. The sheaf $\mathscr{E}_Y^{\sqrt{v}} = \gamma_* \mathscr{E}_X^{\sqrt{v}}$ on $Y = P^*M$ is thus endowed with an anti-automorphism $*$ such that $** = \mathrm{id}$.

Now introduce the subring $\mathscr{E}_{Y,*}^{\sqrt{v}}(0)$ of $\mathscr{E}_Y^{\sqrt{v}}(0)$ consisting of section P satisfying

$$\sigma_0(P) = 1, \quad P \cdot P^* = 1.$$

One shows that

- given two open subsets U and V of a contact manifold Y, there locally exists a contact isomorphism $\chi \colon U \xrightarrow{\sim} V$ which can be quantized as a $*$-preserving isomorphism of $\mathbb{C}$-algebras $\mathscr{E}_U^{\sqrt{v}} \xrightarrow{\sim} \mathscr{E}_V^{\sqrt{v}}$,
- any $*$-preserving automorphism of $\mathscr{E}_Y^{\sqrt{v}}$ is of the form $\mathrm{ad}(P)$ with $P \in \mathscr{E}_{Y,*}^{\sqrt{v}}(0)$,
- for P and Q in $\mathscr{E}_{Y,*}^{\sqrt{v}}(0)$, $\mathrm{ad}(P) = \mathrm{ad}(Q)$ implies $P = Q$.

Applying Lemma 4.1, one gets:

Theorem 5.1 (Kashiwara 96)**.** *Let Y be a complex contact manifold. There exists* canonically *an algebroid stack $\mathscr{E}_Y^{\sqrt{v}}$ locally isomorphic to $\mathscr{E}_{P^*M}^{\sqrt{v}}$ for a complex manifold M.*

In the same paper, Kashiwara announces without proof that one can quantize smooth Lagrangian (=Legendrian) submanifolds after a twist. Let $\Lambda \subset Y$ be such a manifold.

Recall that on a topological space X, given a class $c \in H^2(X; \mathbf{k}_X)$, one can associate a twisted sheaf $\mathbf{k}_{X,c}$ locally isomorphic to $\mathbf{k}_X$.

On a complex manifold X, one has an exact sequence

$$1 \to \mathbb{C}_X^\times \to \mathscr{O}_X^\times \xrightarrow{d\log} d\mathscr{O}_X \to 0$$

giving rise to the long exact sequence

$$\cdots \to H^1(X; \mathscr{O}_X^\times) \xrightarrow{\beta} H^1(X; d\mathscr{O}_X) \xrightarrow{\gamma} H^2(X; \mathbb{C}_X^\times) \to \cdots.$$

We shall denote by $\mathbb{C}_{\sqrt{\Omega}}$ the twisted sheaf corresponding to $\gamma(\frac{1}{2}\beta([\Omega_X]))$.

Theorem 5.2 (Kashiwara 96)**.** *Let $\Lambda \subset Y$ be a smooth Lagrangrian submanifold of the complex contact manifold Y. There exists $\mathscr{L}$ an $\mathscr{E}_Y^{\sqrt{v}} \otimes \mathbb{C}_{\sqrt{\Omega_\Lambda}}$-module, simple along Λ.*

The proof of this theorem is written in [DS07].

6 Quantization of complex symplectic manifolds

In this section, X denotes a complex symplectic manifold. When $X = T^*M$, we have constructed in § 2 the $\mathbb{C}^{\hbar,\mathrm{loc}}$-algebra $\mathscr{W}_X$. In this situation, similarly as in (5.2) we set

$$\mathscr{W}_{T^*M}^{\sqrt{v}} := \pi^{-1}\Omega_M^{\otimes 1/2} \otimes_{\pi^{-1}\mathscr{O}_M} \mathscr{W}_{T^*M} \otimes_{\pi^{-1}\mathscr{O}_M} \pi^{-1}\Omega_M^{\otimes -1/2}. \tag{6.1}$$

Theorem 6.1 ([PS04]). *Let X be a complex symplectic manifold. There exists* canonically *a $\mathbb{C}^\hbar$-algebroid stack $\mathscr{W}_X^{\sqrt{v}}$ locally isomorphic to $\mathscr{W}_{T^*M}^{\sqrt{v}}$ for a complex manifold M.*

The proof goes along the same lines as for Theorem 5.1 with a new difficulty, namely that Lemma 4.1 does not apply directly. Indeed, the sections appearing when glueing the local models are of the form $\mathrm{ad}(P)\delta_a$ with P the same as in the contact case and δ_a denoting a translation with respect to t (see Diagram 2.2).

Theorem 6.2 ([DS07]). *Let $\Lambda \subset X$ be a smooth Lagrangian submanifold of the complex symplectic manifold X. There exists $\mathscr{L}$ a $\mathscr{W}_X^{\sqrt{v}} \otimes \mathbb{C}_{\sqrt{\Omega_\Lambda}}$-module, simple along Λ.*

The proof follows immediately from Theorem 5.2 and Proposition 6.4 below.

Let us first recall some well-known facts on contact and symplectic geometry. All manifolds are complex analytic. However, most of the facts and results below hold for real manifolds after replacing the multiplicative group $\mathbb{C}^\times$ with $\mathbb{R}^+$.

If v is a vector field on a manifold Z, we denote by $\mathrm{L}_v(\cdot)$ the Lie derivative and $\iota_v(\cdot)$ the interior derivative. They are linked by Cartan's formula $\mathrm{L}_v = d \circ \iota_v + \iota_v \circ d$ where d is the differential.

Definition 6.3. (a) A homogeneous symplectic manifold (Z, ω_Z, v) is a symplectic manifold (Z, ω_Z) endowed with a never vanishing vector field v satisfying $\mathrm{L_v}(\omega_\mathrm{Z}) = \omega_\mathrm{Z}$. In this situation one sets $\alpha_Z = \iota_v(\omega_Z)$, so that $d\alpha_Z = \omega_Z$ and $\iota_v(\alpha_Z) = 0$.
A subvariety Λ of Z is conic if it is the union of integral curves of v.

(b) A contact manifold (Y, Z, α_Z, γ) is a manifold Y of dimension $2n+1$ endowed with a principal $\mathbb{C}^\times$-bundle $\gamma\colon Z \to Y$ and a 1-form $\alpha_Z \in \Gamma(Z; \Omega_Z^1)$ such that denoting by v the vector field associated with the infinitesimal action of $\mathbb{C}^\times$ on Z, $(Z, d\alpha_Z, v)$ is a homogeneous symplectic manifold, $\iota_v(\alpha_Z) = 0$ and $L_v(\alpha_Z) = \alpha_Z$.
A subvariety Λ of Y is Lagrangian $\gamma^{-1}(\Lambda)$ is Lagrangian in Z.

(c) A contactification of a symplectic manifold (X, ω_X) is the data of a contact manifold (Y, Z, α_Z, γ) together with a section $\tau\colon Y \to Z$ of γ and a morphism of complex manifolds $\rho\colon Y \to X$ such that there exist an open covering $X = \bigcup_{i \in I} U_i$, holomorphic functions t_i on $\rho^{-1}(U_i)$ and primitives $\alpha_i \in \Omega_{U_i}^1$ of ω_{U_i} such that $\chi_i := (\rho, t_i)$ gives an isomorphism $\chi_i\colon \rho^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}$, and $dt_i + \rho^*(\alpha_X|_{U_i}) = (\alpha_Z/\tau)|_{\rho^{-1}(U_i)}$.

Note that if (Y, Z, α_Z, γ) is a contact manifold, one may look at the 1-form α_Z as a section of $\Omega_Y^1 \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(1)$ where $\mathscr{O}_Y(1)$ denotes the sheaf of section of the bundle Z homogeneous of degree 1.

Proposition 6.4 ([DS07, Lem. 8.3, 8.4]). *Let (X, ω_X) be a symplectic manifold.*

(a) *The manifold X admits a contactification if and only if the de Rham cohomology class $[\omega_X] \in H^2(X; \mathbb{C}_X)$ vanishes.*

(b) *Let Λ be a smooth Lagrangian submanifold of X. Then there exists an open neighborhood U of Λ such that $(U, \omega_X|_U)$ admits a contactification $\rho\colon Y \to U$ and there exists a Lagrangian submanifold $\widetilde{\Lambda} \subset Y$ such that ρ induces an isomorphism $\widetilde{\Lambda} \xrightarrow{\sim} \Lambda$.*

Remark 6.5. One can ask the problem of quantizing line or vector bundles over a smooth Lagrangian submanifold Λ. One cas also want to replace Lagrangian manifolds with co-isotropic manifolds as suggested in [KO01, KR09]. These questions are treated in [BGKP14, BC17]. Note that one can *locally* reduce the case of a co-isotropic manifold V to that of a Lagrangian manifold by sending V into the symplectic manifold $X \times (V/\sim)$ where $V/\sim$ is the symplectic manifold, the quotient of V by the bicharacteristic relation. In particular the diagonal embedding identifies X with a Lagrangian submanifold of $X \times X^a$, where X^a denotes the symplectic manifold $(X, -\omega_X)$.

7 DQ-modules

References are made to [KS12].

Recall notations (2.1) in which we set $\mathbb{C}^\hbar := \mathbb{C}[[\hbar]]$ and $\mathbb{C}^{\hbar,\mathrm{loc}} := \mathbb{C}((\hbar))$.

On a complex manifold X, we have already defined in § 3 what is a DQ-algebra. A DQ-algebroid is a $\mathbb{C}^\hbar$-algebroid stack $\mathscr{A}_X$ on X locally equivalent to a DQ-algebra.

- We set $\mathscr{A}_{X^a} = \mathscr{A}_X^{\mathrm{op}}$.
- Let X and Y be complex manifolds endowed with DQ-algebroids $\mathscr{A}_X$ and $\mathscr{A}_Y$ respectively. There is a canonical DQ-algebroid $\mathscr{A}_{X\times Y} := \mathscr{A}_X \underline{\boxtimes} \mathscr{A}_Y$ on $X \times Y$. For an $\mathscr{A}_X$-module $\mathscr{M}$ and an $\mathscr{A}_Y$-module $\mathscr{N}$, one defines their exterior product
$$\mathscr{M} \underline{\boxtimes} \mathscr{N} := \mathscr{A}_{X\times Y} \otimes_{\mathscr{A}_X \boxtimes \mathscr{A}_Y} (\mathscr{M} \boxtimes \mathscr{N}).$$
- Let $\mathscr{A}_X$ be a DQ-algebroid on X and let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$. Its dual $\mathrm{D}'_{\mathscr{A}_X}\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X^a})$ is given by
$$\mathrm{D}'_{\mathscr{A}_X}\mathscr{M} := \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M}, \mathscr{A}_X).$$
- Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$. Then, locally, $\mathscr{M}$ admits a resolution by free modules of finite rank of length $\leq d_X + 1$.
- We denote by $\delta\colon X \hookrightarrow X \times X^a$ the diagonal embedding. Then $\delta_*\mathscr{A}_X$ is a coherent $\mathscr{A}_{X\times X^a}$-module.

We set

$$\begin{aligned}&\mathrm{gr}_\hbar(\mathscr{A}_X) := \mathscr{A}_X/\hbar\mathscr{A}_X, \text{ the associated graded ring,}\\&\mathscr{A}_X^{\mathrm{loc}} := \mathbb{C}_X^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}_X^\hbar} \mathscr{A}_X, \text{ the localization.}\end{aligned}$$

Note that $\mathrm{gr}_\hbar(\mathscr{A}_X)$ is locally isomorphic to $\mathscr{O}_X$ (globally in the algebraic case or when $\mathscr{A}_X$ is a sheaf of algebras).

We introduce the functors:

$$\begin{aligned}&\mathrm{gr}_\hbar\colon \mathrm{D^b}(\mathscr{A}_X) \to \mathrm{D^b}(\mathrm{gr}_\hbar(\mathscr{A}_X)), \quad \mathscr{M} \mapsto \mathbb{C}_X \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} \mathscr{M}\\&\mathrm{loc}\colon \mathrm{D^b_{coh}}(\mathscr{A}_X) \to \mathrm{D^b_{coh}}(\mathscr{A}_X^{\mathrm{loc}}), \quad \mathscr{M} \mapsto \mathbb{C}_X^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}_X^\hbar} \mathscr{M}.\end{aligned}$$

Note that

- the functor $\mathrm{gr}_\hbar$ is conservative on coherent modules,
- the support of a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module is co-isotropic (Gabber's theorem),
- any coherent $\mathscr{O}_X$-module may be locally regarded as an $\mathscr{A}_X$-module which implies that any closed analytic subset of X is locally the support of a coherent $\mathscr{A}_X$-module,
- for $X = T^*M$ a cotangent bundle, $\pi\colon X \to M$ the projection, there is a canonical DQ-algebra $\mathscr{A}_X$, namely $\mathscr{W}_X(0)$, and $\pi^{-1}\mathscr{D}_M$ is naturally a sub-algebra of $\mathscr{A}_X^{\mathrm{loc}}$.

Hence, in some sense, the theory of $\mathscr{A}$-modules contains that of $\mathscr{O}$-modules (when the Poisson bracket is zero) and the theory of $\mathscr{A}^{\mathrm{loc}}$-modules contains (in the symplectic case) that of $\mathscr{D}$-modules.

An important tool in the study of (analytic) DQ-modules is the following.

Cohomologically completeness

On a topological space X, let $\mathscr{R}$ be a sheaf of $\mathbb{Z}[\hbar]$-algebras with no $\hbar$-torsion. For an $\mathscr{R}$-module $\mathscr{M}$, set $\mathscr{M}^{\mathrm{loc}} = \mathbb{Z}_X[\hbar,\hbar^{-1}] \otimes_{\mathbb{Z}_X[\hbar]} \mathscr{R}$.

Definition 7.1. One says that an object $\mathscr{M}$ of $\mathrm{D}(\mathscr{R})$ is cohomologically complete (cc for short) if $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}},\mathscr{M}) \simeq \mathrm{R}\mathscr{H}om_{\mathbb{Z}_X[\hbar]}(\mathbb{Z}_X[\hbar,\hbar^{-1}],\mathscr{M}) \simeq 0$.

One denote by $\mathrm{D}_{cc}(\mathscr{R})$ the full triangulated subcategory of $\mathrm{D}(\mathscr{R})$ consisting of cc-modules.

- If $f\colon X \to Y$ is continuous and $\mathscr{M} \in \mathrm{D}(\mathbb{Z}_X[\hbar])$ is cc, then $\mathrm{R}f_*\mathscr{M}$ is cc.

Let us come back to the case of $\mathscr{A}_X$-modules.

- If $\mathscr{M} \in \mathrm{D^b_{coh}}(\mathscr{A}_X)$, then $\mathscr{M}$ is cc.
- Let $\mathscr{M} \in \mathrm{D^b}(\mathscr{A}_X)$ and assume that $\mathscr{M}$ is cc and $\mathrm{gr}_\hbar(\mathscr{M}) \in \mathrm{D^b_{coh}}(\mathrm{gr}_\hbar(\mathscr{A}_X))$. Then $\mathscr{M} \in \mathrm{D^b_{coh}}(\mathscr{A}_X)$.

As an application we get easily a kind of Grauert's theorem for DQ-modules (see [KS12, Ch 3]).

Holonomic modules

Here, we assume X is symplectic. Recall that the support of a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is a complex subvariety of X and is co-isotropic. One says that $\mathscr{M}$ is holonomic if its support is Lagrangian.

Theorem 7.2. *Let X be a complex symplectic manifold and let $\mathscr{L}_0$ and $\mathscr{L}_1$ be two holonomic $\mathscr{A}_X^{\mathrm{loc}}$-modules. Then the object $F := \mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L}_0, \mathscr{L}_1)$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X^{\hbar,\mathrm{loc}})$ and is perverse.*

In general, one does not know how to estimate the microsupport $\mathrm{SS}(F)$ of F, except when $\mathscr{L}_0$ and $\mathscr{L}_1$ are simple along smooth Lagrangian submanifolds Λ_0 and Λ_1 of X in which case one proves that

$$\mathrm{SS}\big(\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L}_0, \mathscr{L}_1)\big) \subset \mathrm{C}(\Lambda_0, \Lambda_1),$$

where $\mathrm{C}(\Lambda_0, \Lambda_1)$ is the Whitney normal cone, a closed subset of T^*X, after identifying T^*X and TX by the Hamiltonian isomorphism.

Two Lagrangians

Let us say after D. Joyce that a smooth Lagrangian submanifold Λ of the complex symplectic manifold X is *orientable* is there exists a line bundle $\mathscr{L}$ on Λ such that $\mathscr{L}^{\otimes 2} \simeq \Omega_\Lambda$.

Let Λ_0 and Λ_1 be two smooth orientable Lagrangian submanifolds of X. By the results of § 6, there exist holonomic DQ-modules $\mathscr{L}_i$ simple along Λ_i ($i = 0, 1$). Applying Theorem 7.2, we get a perverse sheaf F along $\Lambda_0 \cap \Lambda_1$. This perverse sheaf is defined over the field $\mathbb{C}^{\hbar,\mathrm{loc}}$.

Problem 7.3. Given a field $\mathbf{k}$ and two smooth Lagrangian orientable submanifolds Λ_0 and Λ_1 of X, construct naturally a perverse sheaf over $\mathbf{k}$ supported by $\Lambda_0 \cap \Lambda_1$.

This question is raised in [JS12] and is motivated by the theory of Donaldson-Thomas invariants (see [Beh09]). It is solved in [BBD⁺12, Bus14]. This leads to another question.

Problem 7.4. Given a field $\mathbf{k}$ and a smooth Lagrangian submanifold Λ of X, can we "quantize" Λ with something analogue to the simple holonomic modules in case $\mathbf{k} = \mathbb{C}^{\hbar,\mathrm{loc}}$ and get an alternative proof to Problem 7.3?

A possible candidate would be based on the notion of microlocal perverse sheaf which, until now, exists only on cotangent bundles. In order to defines such "sheaves", one needs to introduce first the notion of ind-sheaves, second to construct the microlocalization functor μ_X.

8 Ind-sheaves

References are made to [KS01]

Recall that for a category $\mathscr{C}$ (in a given universe $\mathscr{U}$), the category $\mathscr{C}^\wedge$ denotes the category of functors from $\mathscr{C}^{\mathrm{op}}$ to **Set**[2]. This category admits colimits, denoted here "$\varinjlim$" , but if the category $\mathscr{C}$ admits itself colimits, denoted here $\varinjlim$, the Yoneda functor $\mathscr{C} \hookrightarrow \mathscr{C}^\wedge$ does not commute with colimits in general and this is the reason for the different notations.

The category $\mathrm{Ind}(\mathscr{C})$ is the full subcategory of $\mathscr{C}^\wedge$ consisting of objects which can be written as small and filtrant colimits of objects of $\mathscr{C}$.

Now assume that $\mathscr{C}$ is an abelian category. Then $\mathrm{Ind}(\mathscr{C})$ is also abelian and the natural fully faithful functor $\iota\colon \mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ is exact and one can consider $\mathscr{C}$ as a full abelian subcategory of $\mathrm{Ind}(\mathscr{C})$.

Example 8.1. Let $\mathbf{k}$ be a field, $\mathrm{Mod}(\mathbf{k})$ the category of $\mathbf{k}$-vector spaces and $\mathrm{Mod}^{\mathrm{f}}(\mathbf{k})$ the full subcategory of finite dimensional vector spaces. Let V and W be an infinite dimensional vector spaces and consider

$$V \simeq \varinjlim_i V_i, \quad \widetilde{V} := \text{“}\varinjlim_i\text{”}\, V_i,$$

where V_i ranges over the family of finite dimensional vector subspaces of V. Then, writing $\mathrm{I}\mathbf{k}$ for $\mathrm{Ind}(\mathrm{Mod}(\mathbf{k}))$,

$$\mathrm{Hom}_{\mathrm{I}\mathbf{k}}(W, \widetilde{V}) \simeq \varinjlim \mathrm{Hom}_{\mathbf{k}}(W, V_i) \neq \mathrm{Hom}_{\mathbf{k}}(W, V).$$

Moreover, the kernel of the natural epimorphism $\alpha\colon \widetilde{V} \to V$ is an object $Z \in \mathrm{Ind}(\mathrm{Mod}(\mathbf{k}))$ satisfying $Z \neq 0$ and $\mathrm{Hom}_{\mathrm{I}\mathbf{k}}(\mathbf{k}, Z) \simeq 0$.

It is proved in [KS06, Prop. 15.1.2] that the category $\mathrm{Ind}(\mathrm{Mod}(\mathbf{k}))$ does not admit enough injectives.

Now, let X be a good topological space, $\mathbf{k}$ a field, and denote by $\mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_X)$ the abelian sub-category of $\mathrm{Mod}(\mathbf{k}_X)$ consisting of sheaves with compact support. We set

$$\mathrm{I}\mathbf{k}_{\mathrm{X}} := \mathrm{Ind}(\mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_{\mathrm{X}}))$$

and call an object of $\mathrm{I}\mathbf{k}_{\mathrm{X}}$ an ind-sheaf on X. Here are some properties of this category.

- The presheaf of categories $U \mapsto \mathrm{I}\mathbf{k}_{\mathrm{U}}$ is a stack.
- The category $\mathrm{I}\mathbf{k}_{\mathrm{X}}$ is an abelian category which admits small limits and colimits and filtrant colimits are exact. It is NOT a Grothendieck category (since it does not admit enough injectives).
- The natural functor $\iota_X\colon \mathrm{Mod}(\mathbf{k}_X) \hookrightarrow \mathrm{I}\mathbf{k}_{\mathrm{X}}$ is exact, fully faithful and admits a left adjoint $\alpha_X\colon \mathrm{I}\mathbf{k}_{\mathrm{X}} \to \mathrm{Mod}(\mathbf{k}_{\mathrm{X}})$. In particular $\alpha_X \circ \iota_X \simeq \mathrm{id}_{\mathrm{Mod}(\mathbf{k}_X)}$. For a small and filtrant category I, $\alpha_X(\text{“}\varinjlim\text{”}\, F_i) \simeq \varinjlim F_i$, $i \in I$, the F_i's being usual sheaves.

[2]Ind-objects were defined in SGA4. We do not give precise references for such a classical subject.

- The functor α_X also admits a left adjoint $\beta_X\colon \mathrm{Mod}(\mathbf{k}_X) \hookrightarrow \mathrm{I}\mathbf{k}_\mathrm{X}$ and this last functor is exact and fully faithful. In particular $\alpha_X \circ \beta_X \simeq \mathrm{id}_{\mathrm{Mod}(\mathbf{k}_X)}$.

- There are natural bifunctors $\otimes\colon \mathrm{I}\mathbf{k}_\mathrm{X} \times \mathrm{I}\mathbf{k}_\mathrm{X} \to \mathrm{I}\mathbf{k}_\mathrm{X}$ and $\mathscr{I}hom : (\mathrm{I}\mathbf{k}_\mathrm{X})^{\mathrm{op}} \times \mathrm{I}\mathbf{k}_\mathrm{X} \to \mathrm{I}\mathbf{k}_\mathrm{X}$. Moreover, denoting by $\mathscr{H}om$ the internal hom of the stack of ind-sheaves, one has $\mathscr{H}om = \alpha_X \circ \mathscr{I}hom$.

- Let $f\colon X \to Y$ be a morphism of good topological spaces. One constructs naturally the exact functor $f^{-1}\colon \mathrm{I}\mathbf{k}_\mathrm{Y} \to \mathrm{I}\mathbf{k}_\mathrm{X}$ and its right adjoint, the functor $f_*\colon \mathrm{I}\mathbf{k}_\mathrm{X} \to \mathrm{I}\mathbf{k}_\mathrm{Y}$. These functors commute with ι and that is why we keep the same notations. One defines the proper direct image functor $f_{!!}\colon \mathrm{I}\mathbf{k}_\mathrm{X} \to \mathrm{I}\mathbf{k}_\mathrm{Y}$ by setting for $F \in \mathrm{I}\mathbf{k}_\mathrm{X}$,

$$f_{!!}F := \text{“}\varinjlim_{U}\text{”}\, f_*F_U$$

where U ranges over the family of relatively compact open subsets of X and $F_U = F \otimes \mathbf{k}_U$.

Example 8.2. Let U be an open subset of X and S a closed subset. Then

$$\beta_X(\mathbf{k}_U) \simeq \text{“}\varinjlim_{V \subset\subset U}\text{”}\, \mathbf{k}_V, \quad \beta_X(\mathbf{k}_S) \simeq \text{“}\varinjlim_{S \subset W}\text{”}\, \mathbf{k}_W,\ V \text{ and } W \text{ open.}$$

Remark 8.3. Note that if we would have defined $\mathrm{I}\mathbf{k}_\mathrm{X}$ as the category $\mathrm{Ind}(\mathrm{Mod}(\mathbf{k}_X))$, then we would not have obtained a stack. For example, choose $X = \mathbb{R}$ and consider $F = \text{“}\varinjlim\text{”}\, \mathbf{k}_{[a,+\infty[}$ with $a \in \mathbb{R}_{\geq 0}$. Then $F \simeq 0$ in $\mathrm{I}\mathbf{k}_\mathrm{X}$ but $F \neq 0$ in the category $\mathrm{Ind}(\mathrm{Mod}(\mathbf{k}_X))$.

This is similar to the fact that the space of distributions on a real manifold M is not the dual of the space of C^∞-functions on M but the dual of compactly supported such functions and it follows that the presheaf of distributions is a sheaf.

One shall be aware that there exist objects $F \in \mathrm{I}\mathbf{k}_\mathrm{X}$ such that $F \neq 0$ and $\Gamma(U; F) \simeq 0$ for all U open in X. When $X = \mathrm{pt}$, choose $F = Z_X$ with Z given by Example 8.1.

Denote by $\mathrm{D^b}(\mathrm{I}\mathbf{k}_\mathrm{X})$ the bounded derived category of ind-sheaves. Although the category $\mathrm{I}\mathbf{k}_\mathrm{X}$ does not admit enough injectives (even when $X = \mathrm{pt}$), one can construct the six Grothendieck operations for ind-sheaves.

Remark 8.4. The category $\mathrm{I}\mathbf{k}_\mathrm{X}$ is not easy to manipulate since it has not enough injective. When X is a real analytic manifold, it contains the full subcategory of subanalytic ind-sheaves that we briefly describe now. Denote by $\mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ the abelian category of $\mathbb{R}$-constructible sheaves with compact support on X and set $\mathrm{I}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X) = \mathrm{Ind}(\mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X))$. Then $\mathrm{I}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ is a full abelian subcategory of $\mathrm{I}\mathbf{k}_\mathrm{X}$ and one proves the equivalence

$$\mathrm{I}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X) \simeq \mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}}), \tag{8.1}$$

where X_{sa} is the subanalytic site associated with X. The open sets of X_{sa} are the relatively compact subanalytic open subsets of X and the coverings are the finite coverings (see [KS01, Ch 7]). Denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{IR}\text{-c}}(\mathrm{I}\mathbf{k}_{\mathrm{X}})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{X}})$ consisting of objects whose cohomologies belong (up to isomorphism) to $\mathrm{I}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$. There is an equivalence of triangulated categories

$$\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X_{\mathrm{sa}}}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathbb{IR}\text{-c}}(\mathrm{I}\mathbf{k}_{\mathrm{X}}). \tag{8.2}$$

The category $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X_{\mathrm{sa}}})$ is much easier to manipulate than the category $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{X}})$.

9 The microlocalization functor

The functor μ_X has been constructed in [KSIW06] (to which all references of this section are made) along the lines of a manuscript of Kashiwara. See also [KS99].

In this section, all manifolds are real and $\mathbf{k}$ is a field.

let us first recall the construction of the normal deformation, following the exposition of [KS90, Ch III].

Let $\iota\colon N \hookrightarrow M$ be the embedding of a closed submanifold N of M and recall that $\tau_M\colon T_NM \to N$ denotes the normal bundle to N. One constructs a new manifold $\widetilde{M}_N$, called the normal deformation of M along N, together with the maps

$$\begin{array}{ccccc}
\{0\} & \hookrightarrow & \mathbb{R} & \hookleftarrow & \mathbb{R}_{>0}\,, \\
\uparrow & & \uparrow{\scriptstyle t} & & \uparrow \\
T_NM & \overset{s}{\hookrightarrow} & \widetilde{M}_N & \overset{j}{\hookleftarrow} & \Omega \\
{\scriptstyle \tau_M}\downarrow & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle \widetilde{p}} & \\
N & \underset{\iota}{\hookrightarrow} & M & &
\end{array} \tag{9.1}$$

where $\Omega = t^{-1}(\mathbb{R}_{>0})$ and $T_NM = t^{-1}(0)$. Locally, after choosing a local coordinate system (x', x'') on M such that $N = \{x' = 0\}$, we have $\widetilde{M}_N = M \times \mathbb{R}$, $t\colon \widetilde{M}_N \to \mathbb{R}$ is the projection, $p(x', x'', t) = (tx', x'')$. [3]

We shall use the normal deformation of $M \times M$ along Δ. Hence, in Diagram 9.1, replace M with $M \times M$, N with Δ, ι with δ_M and identify TM with $T_\Delta(M \times M)$.

Assume to be given a section $\sigma\colon M \to T^*M$ of π_M. One then constructs an object $L_\sigma \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M\times M}})$ as follows. Set

$$P = \{(x, v_x) \in TM; \langle \sigma(x), v_x\rangle \geq 0\}.$$

Then set

$$L_\sigma = \mathrm{R}p_{!!}(\mathbf{k}_{\overline{\Omega}} \otimes \beta_{\widetilde{M\times M}_M}(\mathbf{k}_P)) \otimes \beta_{M\times M}(\delta_{M*}(\omega_M)) \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M\times M}}).$$

The ind-sheaf L_σ defines a functor $L_\sigma\circ\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M}})$ by setting (for q_1 and q_2 the two projections defined on $M \times M$):

$$L_\sigma \circ G = \mathrm{R}q_{1!!}(L_\sigma \otimes q_2^{-1}G).$$

[3] For $S \subset M$, the Whitney normal cone $C_N(S)$ is the closed conic subset of T_NM given by $\mathrm{C}_N(S) = \overline{\widetilde{p}^{-1}(S)} \cap T_NM$. For two subsets A, B of M, one sets $\mathrm{C}(A, B) = \mathrm{C}_\Delta(A \times B)$.

This construction applies in particular to a homogeneous symplectic manifold (X, α_X), choosing for σ the 1-form α_X. One sets $K_X = L_{\alpha_X} \in \mathrm{D^b}(\mathrm{I}\mathbf{k}_{\mathrm{X}\times\mathrm{X}})$ which defines a functor

$$K_X \circ \colon \mathrm{D^b}(\mathrm{I}\mathbf{k}_{\mathrm{X}}) \to \mathrm{D^b}(\mathrm{I}\mathbf{k}_{\mathrm{X}}).$$

Now assume that $X = T^*M$ and α_X is the Liouville 1-form. The microlocalization functor is defined as the composition

$$\mu_X \colon \mathrm{D^b}(\mathbf{k}_M) \hookrightarrow \mathrm{D^b}(\mathrm{I}\mathbf{k}_{\mathrm{M}}) \xrightarrow{\pi_{\mathrm{M}}^{-1}} \mathrm{D^b}(\mathrm{I}\mathbf{k}_{\mathrm{X}}) \xrightarrow{\mathrm{K_X}\,\circ} \mathrm{D^b}(\mathrm{I}\mathbf{k}_{\mathrm{X}}).$$

The functor μ_X has the following properties.

Theorem 9.1. *For $F, G \in \mathrm{D^b}(\mathbf{k}_M)$, one has*

$$\begin{aligned} &\mathrm{SS}(F) = \mathrm{supp}(\mu_X(F)), \\ &\mu hom(F,G) \simeq \mathrm{R}\mathscr{H}om\,(\mu_X(F), \mu_X(G)) \simeq \mathrm{R}\mathscr{H}om\,(\pi_M^{-1}(F), \mu_X(G)). \end{aligned} \tag{9.2}$$

Now choose $\mathbf{k} = \mathbb{C}$ and assume that M is a complex manifold.

Theorem 9.2. *The object $\mu_X(\mathscr{O}_M)\,[d_M]$ is concentrated in degree 0 on $\dot{T}^*M$ and is well-defined in $\mathrm{D^b}(\mathscr{E}_X|_{\dot{T}^*M})$.*

Corollary 9.3. *Let $F \in \mathrm{D^b}(\mathbb{C}_M)$. Then $\mu hom(F, \mathscr{O}_M)$ is well-defined as an object of $\mathrm{D^b}(\mathscr{E}_X|_{\dot{T}^*M})$.*

10 Microlocal perverse sheaves

References are made to [Was04, Was05]. Note that the study of microlocal perverse sheaves was initiated by Emmanuel Andronikov in [And94].

In this section M is a complex manifold and $\mathbf{k}$ is a field. Consider the maps

$$X = T^*M \supset Z = \dot{T}^*M \xrightarrow{\gamma} Y = P^*M \xrightarrow{\pi} M.$$

We denote by $\mathrm{D^b_{perv}}(\mathbf{k}_M)$ the abelian subcategory of $\mathrm{D^b_{\mathbb{C}c}}(\mathbf{k}_M)$ consisting of perverse sheaves.

Let U be a $\mathbb{C}^\times$-conic open subset of $\dot{T}^*M$ and let Λ be a $\mathbb{C}^\times$-conic Lagrangian complex analytic subvariety, closed in U. Denote by:

- $\mathrm{D^b_{\mathbb{R}c}}(\mathbf{k}_M; U)$ the localization of $\mathrm{D^b_{\mathbb{R}c}}(\mathbf{k}_M)$ by the subcategory of sheaves F satisfying $\mathrm{SS}(F) \cap U = \varnothing$ (see [KS90, § 6.1]),
- $\mathrm{D^b_{\mathbb{C}c,\Lambda}}(\mathbf{k}_M; U)$ the full subcategory of $\mathrm{D^b_{\mathbb{R}c}}(\mathbf{k}_M; U)$ consisting of sheaves microlocally supported by Λ on U,
- $\mathrm{D^b_{perv,\Lambda}}(\mathbf{k}_M; U)$ the full subcategory of $\mathrm{D^b_{\mathbb{C}c,\Lambda}}(\mathbf{k}_M; U)$ consisting of sheaves of shift 0 on Λ_{reg} (see [KS90, § 7.5]),

- $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(\mathbf{k}_M;U)$ the union in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}c}(\mathbf{k}_M;U)$ of the categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(\mathbf{k}_M;U)$.

Definition 10.1. Let V be an open subset of Y. The category $\mu\mathrm{Perv}(\mathbf{k}_{\mathrm{Y}})(\mathrm{V})$ is the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\gamma^{-1}\mathrm{V}})$ consisting of objects F such that, for each $p \in V$, there exist an open neighborhood W of p and $F_p \in \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(\mathbf{k}_{\pi(W)})$ such that $F|_W \simeq \mu_X(F_p)$.

Denote by $\mu\mathfrak{P}\mathrm{erv}(\mathbf{k}_{\mathrm{Y}})$ the prestack $V \mapsto \mu\mathrm{Perv}(\mathbf{k}_{\mathrm{Y}})(\mathrm{V})$, V open in Y.

Theorem 10.2. *The prestack $\mu\mathfrak{P}\mathrm{erv}(\mathbf{k}_{\mathrm{Y}})$ is a $\mathbf{k}$-abelian stack on Y. Moreover, this stack is equivalent to the stack associated with the prestack $V \mapsto \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(\mathbf{k}_M;\gamma^{-1}(V))$.*

Assume that $\mathbf{k} = \mathbb{C}$ and denote by $\mathfrak{Mod}_{\mathrm{reghol}}(\mathscr{E}_Y)$ the abelian stack of regular holonomic $\mathscr{E}_Y$-modules on Y. One gets the microlocal Riemann-Hilbert corrspondance:

Theorem 10.3. *The functor*

$$\mathrm{R}\mathscr{H}om_{\mathscr{E}_Y}(\bullet,\mu_X(\mathscr{O}_M))\colon\ \mathfrak{Mod}_{\mathrm{reghol}}(\mathscr{E}_Y)^{\mathrm{op}} \to \mu\mathrm{Perv}(\mathbb{C}_{\mathrm{Y}})$$

defines an equivalence of abelian stacks on Y.

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