An overview
of
Real and Complex Microlocal Analysis.

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Abstract

I will discuss some links between sheaves on a real manifold and D-modules on a complex manifold, from a microlocal point of view.

Contents

1 Introduction

The history began in the years 1959-60, when Mikio Sato (see [Sch07]) introduced his theory of hyperfunctions [Sat59, Sat60]. This theory gave a radically new approach to analysis and has entirely modified the mathematical landscape in this area. Ten years later in [Sat70], Sato introduced the theory of microfunctions, opening the way to what is now called "microlocal analysis".

At the same time, in the 70's appeared the theory of D-modules, with Kashiwara's thesis [Kas70] and Bernstein's paper [Ber71]. This theory is now extremely popular and plays a central role in many fields of mathematics.

Since the characteristic variety of a coherent D-module on a complex manifold X lives in the cotangent bundle T^*X, it is natural to try to localize the sheaf of rings $\mathscr{D}_X$ and one gets the sheaf $\mathscr{E}_{T^*X}$ of microdifferential operators of [SKK73].

On the other-hand, there is no analogue to the sheaf $\mathscr{D}_X$ on a real manifold M (the sheaf $\mathscr{D}_M$ of real differential operators is not a good candidate) but there exists a "microlocal theory of sheaves" introduced and developed in [KS82, KS85, KS90] which plays a similar role.

The aim of these Notes is to present both aspects of microlocalization, the real and the complex one, and to stress their relations.

Remark 1.1. For the readers who are not familiar with homological algebra and sheaf theory, we suggest the reading of [Sch02, Sch06].

2 Sheaves

References for this section are made to [KS90].

2.1 Micro-support

We assume the reader familiar with (derived) sheaf theory and the six operations.

Let M be a real C^∞-manifold. One denotes by $\pi_M : T^*M \to M$ its cotangent bundle and by $a_M : M \to \mathrm{pt}$ the unique map from M to one point.

For a commutative unital ring with finite global dimension $\mathbf{k}$, we denote by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ the bounded derived category of sheaves of $\mathbf{k}$-modules on M.

For a locally closed subset Z of M, we denote by $\mathbf{k}_Z$ the constant sheaf with stalk $\mathbf{k}$ on Z, extended by 0 on $M \setminus Z$.

The dualizing complex ω_M is defined as $\omega_M = a_M^!(\mathbf{k})$. One has the isomorphism

$$\omega_M \simeq \mathrm{or}_M\,[\dim M] \tag{2.1}$$

where or_M is the orientation sheaf on M.

The duality functors D'_M and D_M are given by

$$\mathrm{D}'_M F = \mathrm{R}\mathscr{H}om\,(F, \mathbf{k}_M),$$
$$\mathrm{D}_M F = \mathrm{R}\mathscr{H}om\,(F, \omega_M).$$

One also sets $\omega_M^{\otimes -1} = \mathrm{D}'_M \omega_M$.

The singular support, also called the micro-support, of a sheaf is a closed $\mathbb{R}^+$-conic subset of the cotangent bundle T^*M.

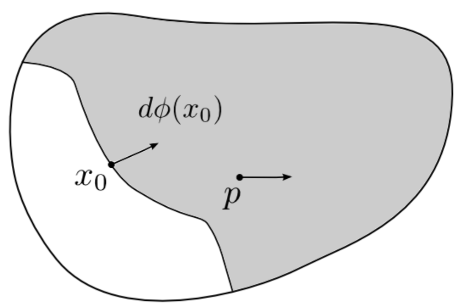
Definition 2.1. Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. The singular support $\mathrm{SS}(F)$ of F is the closed conic subset of T^*M defined as follows. An open subset W of T^*M does not intersect $\mathrm{SS}(F)$ if for any C^1-function $\varphi\colon M \to \mathbb{R}$ and any $x_0 \in M$ such that $(x_0; d\varphi(x_0)) \in W$

$$(\mathrm{R}\Gamma_{\{x;\varphi(x)\geq\varphi(x_0)\}}F)_{x_0} = 0. \tag{2.2}$$

Setting $U = \{x; \varphi(x) < \varphi(x_0)\}$, (2.2) is equivalent to:

$$\varinjlim_{V \ni x_0} H^j(U \cup V; F) \simeq H^j(U; F) \text{ for all } j \in \mathbb{Z}. \tag{2.3}$$

Therefore, if $(x_0; d\varphi(x_0)) \notin \mathrm{SS}(F)$, then any cohomology class defined on an open subset U as above extends through the boundary in a neighborhood of x_0.



- The microsupport is closed and is $\mathbb{R}^+$-conic, that is, invariant by the action of $\mathbb{R}^+$ on T^*M.

- $\mathrm{SS}(F) \cap T^*_M M = \pi_M(\mathrm{SS}(F)) = \mathrm{Supp}(F)$, where $\pi_M\colon T^*M \to M$ is the projection.

- The microsupport satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.

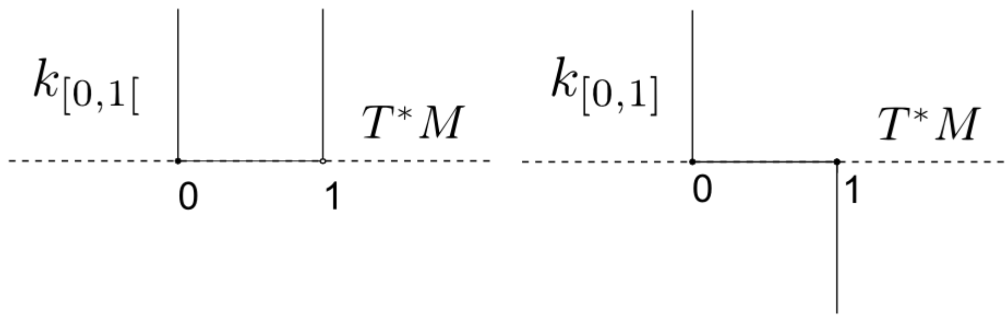
- The microsupport is involutive (i.e., co-isotropic). (A precise definition will come later.)

Example 2.2. (i) If F is a non-zero local system on M and M is connected, then $\mathrm{SS}(F) = T^*_M M$, the zero-section.

(ii) If N is a closed submanifold of M and $F = \mathbf{k}_N$, then $\mathrm{SS}(F) = T^*_N M$, the conormal bundle to N in M.

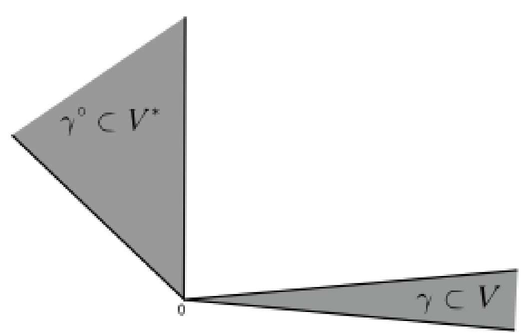
(iii) Let φ be a C^1-function such that $d\varphi(x) \neq 0$ whenever $\varphi(x) = 0$. Let $U = \{x \in M; \varphi(x) > 0\}$ and let $Z = \{x \in M; \varphi(x) \geq 0\}$. Then

$$\mathrm{SS}(\mathbf{k}_U) = U \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \leq 0\},$$
$$\mathrm{SS}(\mathbf{k}_Z) = Z \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \geq 0\}.$$



(iv) Assume $M = V$ is a vector space and let γ be a closed proper convex cone with vertex at 0. Then $\mathrm{SS}(\mathbf{k}_\gamma) \cap \pi_M^{-1}(\{0\}) = \gamma^\circ$ where $\gamma^\circ \subset V^*$ is the polar cone given by

$$\gamma^\circ = \{\theta \in V^*; \langle \theta, v \rangle \geq 0 \text{ for all } v \in \gamma\}.$$



(v) Let $(X, \mathscr{O}_X)$ be a complex manifold and let $\mathscr{M}$ be a coherent module over the ring $\mathscr{D}_X$ of holomorphic differential operators. (Hence, $\mathscr{M}$ represents a system of linear partial differential equations on X.) Denote by $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ the complex of holomorphic solutions of $\mathscr{M}$. Then $\mathrm{SS}(F) = \mathrm{char}(\mathscr{M})$, the characteristic variety of $\mathscr{M}$.

Involutivity

For a closed submanifold N of M, the Whitney normal cone $C_N(S) \subset T_N M$ is given in a local coordinate system $(x) = (x', x'')$ on M with $N = \{x' = 0\}$ by

$$(2.4)\qquad \begin{cases} (x''_0; v_0) \in C_N(S) \subset T_N M \text{ if and only if there exists a sequence} \\ \{(x_n, c_n)\}_n \subset S \times \mathbb{R}^+ \text{ with } x_n = (x'_n, x''_n) \text{ such that } x'_n \xrightarrow{n} 0, \\ x''_n \xrightarrow{n} x''_0 \text{ and } c_n(x'_n) \xrightarrow{n} v_0. \end{cases}$$

The Whitney's normal cone $C(S_1, S_2)$ is given in a local coordinate system (x) on M by:

$$(2.5)\qquad \begin{cases} (x_0; v_0) \in C(S_1, S_2) \subset TM \text{ if and only if there exists a sequence} \\ \{(x_n, y_n, c_n)\}_n \subset S_1 \times S_2 \times \mathbb{R}^+ \text{ such that } x_n \xrightarrow{n} x_0,\ y_n \xrightarrow{n} x_0 \text{ and} \\ c_n(x_n - y_n) \xrightarrow{n} v_0. \end{cases}$$

Definition 2.3 (KS82)**.** A locally closed subset S of a symplectic manifold X is co-isotropic (or involutive) at $p \in X$ if

$$C_p(S, S)^\perp \subset C_p(S).$$

Here, $\perp$ is defined through the Hamiltonian isomorphism $TX \xrightarrow{\sim} T^*X$.

Example 2.4. Let $M = \mathbb{R}$, let $(t; \tau)$ denote the coordinates on T^*M and let $p = (0; 0)$.

$$A = \{(t;\tau); \tau = 0, t \geq 0\}, C_p(A) = A, C_p(A, A)^\perp = \{(t;\tau); \tau = 0\}, A \text{ is not involutive.}$$
$$B = \{(t;\tau); t \cdot \tau = 0, t \geq 0, \tau \geq 0\}, C_p(B) = B, C_p(B, B)^\perp = \{0\}, B \text{ is involutive.}$$

Theorem 2.5 (KS82)**.** *Let $F \in \mathrm{D^b}(\mathbf{k}_M)$. Then $\mathrm{SS}(F)$ is co-isotropic in T^*M.*

2.2 Operations

Let $f\colon M \to N$ be a morphism of real manifolds. To f are associated the diagrams

$$(2.6)\qquad \begin{tikzcd} TM \arrow[r, "f'"] \arrow[d, "\tau_M"] & M \times_N TN \arrow[r, "f_\tau"] \arrow[d] & TN \arrow[d, "\tau_N"] \\ M \arrow[r, equal] & M \arrow[r, "f"] & N, \end{tikzcd} \qquad \begin{tikzcd} T^*M \arrow[d, "\pi_M"] & M \times_N T^*N \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d] & T^*N \arrow[d, "\pi_N"] \\ M \arrow[r, equal] & M \arrow[r, "f"] & N. \end{tikzcd}$$

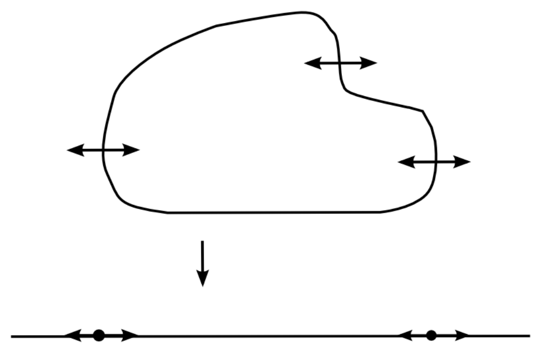
Let $\Lambda_M \subset T^*M$ be a closed $\mathbb{R}^+$-conic subset. Then f_π is proper on $f_d^{-1}\Lambda_M$ if and only if f is proper on $\Lambda_M \cap T^*_M M$.

Let $\Lambda_N \subset T^*N$ be a closed $\mathbb{R}^+$-conic subset. Then f_d is proper on $f_\pi^{-1}\Lambda_N$ if and only if $f_\pi^{-1}\Lambda_N \cap f_d^{-1} T^*_M M \subset M \times_N T^*_N N$. In this case, one says that f is non-characteristic for Λ_N.

Theorem 2.6. *Let $F \in \mathrm{D^b}(\mathbf{k}_M)$ and let $G \in \mathrm{D^b}(\mathbf{k}_N)$.*

(i) *Assume that f is proper on* $\mathrm{Supp}(F)$*. Then* $\mathrm{SS}(\mathrm{R}f_*F) \subset f_\pi f_d^{-1}\,\mathrm{SS}(F)$.

(ii) *Assume that f is non-characteristic for* $\mathrm{SS}(G)$*. Then* $\mathrm{SS}(f^{-1}G) \subset f_d f_\pi^{-1}\,\mathrm{SS}(G)$.



Note that Theorem (i) is a kind of stationary phase lemma for sheaves.

Corollary 2.7 (The Morse lemma for sheaves)**.** *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$, let $\psi\colon M \to \mathbb{R}$ be a function of class C^1 and assume that ψ is proper on* $\mathrm{Supp}(F)$*. For $t \in \mathbb{R}$, set $M_t = \psi^{-1}(]-\infty, t[)$. Let $a < b$ in $\mathbb{R}$ and assume that $d\varphi(x) \notin \mathrm{SS}(F)$ for $a \leq \psi(x) < b$. Then the restriction morphism $\mathrm{R}\Gamma(M_b; F) \to \mathrm{R}\Gamma(M_a; F)$ is an isomorphism.*

Proof. Consider $G = \mathrm{R}\psi_* F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}})$. Then $\mathrm{SS}(G) \cap \{(t; dt); t \in [a, b[\} = \emptyset$ and it follows that $\mathrm{R}\Gamma(]-\infty, b[; G) \to \mathrm{R}\Gamma(]-\infty, a[; G)$ is an isomorphism by the definition of the micro-support. Q.E.D.

2.3 $\mathbb{R}$- and $\mathbb{C}$-constructible sheaves

In this subsection, we assume that $\mathbf{k}$ is a field.

First, we assume that M is real analytic. Recall that on M there is a family of subsets called subanalytic subsets. This family contains the family of semi-analytic subsets (those defined by inequalities of real analytic functions) and is stable by all natural operations, finite unions and intersections, closure and interior, complements, inverse and direct image by a real analytic morphism (for direct images, one needs the hypothesis that the map is proper on the closure of the set). If $S \subset M$ is a locally closed subanalytic subset, there exists an open dense subset $S_{\mathrm{reg}} \subset S$ such that S_{reg} is a smooth locally closed real analytic submanifold of M.

A sheaf F is weakly $\mathbb{R}$-constructible on M if there exists a subanalytic stratification $M = \bigsqcup_\alpha M_\alpha$ such that for all α, $F|_{M_\alpha}$ is locally constant. The sheaf F is constructible if moreover F_x is finite dimensional for all $x \in M$. One denotes by $\mathbb{R}$-$\mathrm{C}(\mathbf{k}_M)$ the abelian category of $\mathbb{R}$-constructible sheaves.

An object $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ is weakly constructible (resp. constructible) if all H^jF are. One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ (resp. $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$) the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of weakly constructible (resp. constructible) sheaves.

Let $\Lambda \subset T^*M$ be a locally closed subanalytic subset. One says that Λ is Lagrangian if so is Λ_{reg}.

Theorem 2.8. *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. Then the conditions below are equivalent.*

(a) $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$,

(b) $\mathrm{SS}(F)$ *is a Lagrangian subanalytic subset of* T^*M,

(c) $\mathrm{SS}(F)$ *is contained in a Lagrangian subanalytic subset of* T^*M.

Assume now that X is a complex manifold. One defines similarly the notion of (weakly) $\mathbb{C}$-constructible sheaves, replacing the subanalytic stratifications by $\mathbb{C}$-analytic stratifications. One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{C}\mathrm{c}}(\mathbf{k}_X)$ (resp. $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbf{k}_X)$) the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ consisting of weakly $\mathbb{C}$-constructible (resp. $\mathbb{C}$-constructible) sheaves.

Let $\Lambda \subset T^*M$ be a locally closed subset. One says that Λ is $\mathbb{C}$-Lagrangian if Λ is a complex analytic subvariety of T^*X and Λ_{reg} is Lagrangian.

We shall say that a locally closed subset Z is locally $\mathbb{C}^\times$-conic if Z is closed in an open subset U such that $U \cap (\mathbb{C}^\times \cdot Z) = Z$.

Theorem 2.9. *Let $\Lambda \subset T^*M$ be a locally closed subanalytic Lagrangian subset. Then Λ is $\mathbb{C}$-Lagrangian if and only if Λ is locally $\mathbb{C}^\times$-conic.*

Theorem 2.10. *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. Then $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{C}\mathrm{c}}(\mathbf{k}_X)$ if and only if $\mathrm{SS}(F)$ is a complex analytic Lagrangian subvariety.*

3 D-modules

References for this section are made to [SKK73, Kas03]. In this section, $(X, \mathscr{O}_X)$ denotes a complex manifold of dimension d_X.

3.1 The ring $\mathscr{D}_X$

One denotes by Ω_X^p the sheaf of holomorphic p-forms, by Ω_X the sheaf $\Omega_X^{d_X}$ and by Θ_X the sheaf of holomorphic vector fields.

Definition 3.1. One denotes by $\mathscr{D}_X$ the subalgebra of $\mathscr{E}nd_{\mathbb{C}}(\mathscr{O}_X)$ generated by $\mathscr{O}_X$ and Θ_X.

If $(x_1, \dots, x_n)$ is a local coordinate system on a local chart U of X, then a section P of $\mathscr{D}_X$ on U may be uniquely written as a polynomial

$$P = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha \tag{3.1}$$

where $a_\alpha \in \mathscr{O}_X$, $\partial_i = \partial_{x_i} = \frac{\partial}{\partial_{x_i}}$ and we use the classical notations for multi-indices:

$$\begin{cases} \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \\ |\alpha| = \alpha_1 + \cdots + \alpha_n, \\ \text{if } y = (y_1, \dots, y_n), \text{ then } y^\alpha = y^{\alpha_1} \dots y^{\alpha_n}. \end{cases}$$

Assume that X is affine, that is, X is open in a finite dimensional complex vector space $\mathbb{V}$. One defines the total symbol, of P

$$\sigma_{\mathrm{tot}}(P)(x;\xi) := \exp\langle -x,\xi\rangle P(\exp\langle x,\xi\rangle) \;=\; \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha. \tag{3.2}$$

Using (3.1), one gets that $\sigma_{\mathrm{tot}}(P)$ is a function on $X\times\mathbb{V}^*$, polynomial with respect to $\xi\in\mathbb{V}^*$. One defines the order of P, $\operatorname{ord}(P)$ as the order of the polynomial $\sigma_{\mathrm{tot}}(P)$.

The function $\sigma_{\mathrm{tot}}(P)$ highly depends on the affine structure, but its order (a locally constant function on X) does not. It is called the order of P and denoted $\operatorname{ord}(P)$.

If Q is another differential operator with total symbol $\sigma_{\mathrm{tot}}(Q)$, it follows easily from the Leibniz formula that the total symbol $\sigma_{\mathrm{tot}}(R)$ of $R=P\cdot Q$ is given by:

$$\sigma_{\mathrm{tot}}(R) = \sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\partial_\xi^\alpha(\sigma_{\mathrm{tot}}(P))\partial_x^\alpha(\sigma_{\mathrm{tot}}(Q)). \tag{3.3}$$

By this formula, one gets that

$$\begin{aligned}\operatorname{ord}(P\cdot Q) &= \operatorname{ord}(P)+\operatorname{ord}(Q),\\ \operatorname{ord}([P,Q]) &\le \operatorname{ord}(P)+\operatorname{ord}(Q)-1.\end{aligned}$$

The ring $\mathscr{D}_X$ is filtered by the order, $F_m\mathscr{D}_X=\{P;\operatorname{ord}(P)\le m\}$. The associated graded ring $\operatorname{Gr}\mathscr{D}_X$ is isomorphic to the commutative ring $\pi_*\mathscr{O}_{[T^*X]}$ where $\mathscr{O}_{[T^*X]}$ is the subsheaf of $\mathscr{O}_{T^*X}$ consisting of functions polynomial in the fibers of $\pi\colon T^*X\to X$. One denotes by $\sigma\colon \operatorname{Fl}\mathscr{D}_X\to\operatorname{Gr}\mathscr{D}_X$ the "principal symbol map" and by $\sigma_m\colon \operatorname{Fl}_m\mathscr{D}_X\to\operatorname{Gr}_m\mathscr{D}_X$ the map "symbol of order m".

Let $\mathscr{R}$ be a sheaf of $\mathbb{C}$-algebras on X. Recall that an $\mathscr{R}$-module $\mathscr{M}$ is Noetherian (see [Kas03, Def. A.7]) if it is coherent, $\mathscr{M}_x$ is a Noetherian $\mathscr{R}_x$-module for any $x\in X$, and for any open subset $U\subset X$, any small filtrant family of coherent submodules of $\mathscr{M}|_U$ is locally stationary.

Using the commutative graded ring $\operatorname{Gr}\mathscr{D}_X$, one proves

Theorem 3.2. *The sheaf of rings $\mathscr{D}_X$ is right and left Noetherian.*

One denotes by $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ the bounded derived category of left $\mathscr{D}_X$-modules and by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ consisting of objects with coherent cohomologies.

A left coherent $\mathscr{D}_X$-module $\mathscr{M}$ may be locally represented as the cokernel of a matrix $\cdot P_0$ of differential operators acting on the right:

$$\mathscr{M}\simeq\mathscr{D}_X^{N_0}/\mathscr{D}_X^{N_1}\cdot P_0.$$

By classical arguments, $\mathscr{M}$ is locally isomorphic to the cohomology of a bounded complex

$$\mathscr{M}^\bullet := 0\to\mathscr{D}_X^{N_r}\to\cdots\to\mathscr{D}_X^{N_1}\xrightarrow{\cdot P_0}\mathscr{D}_X^{N_0}\to 0. \tag{3.4}$$

A filtration $\mathrm{F}\mathscr{M}$ on the coherent module $\mathscr{M}$ is the data of a sequence of subsheaves $\mathrm{F}_k\mathscr{M} \subset \mathscr{M}$, $k \in \mathbb{Z}$, satisfying $\mathscr{D}_X(m) \cdot \mathrm{F}_k\mathscr{M} \subset \mathrm{F}_{k+m}\mathscr{M}$ and $\bigcup_k \mathrm{F}_k\mathscr{M} = \mathscr{M}$. The filtration is coherent if moreover,

$$
\begin{cases}
\text{locally on } X,\ \mathrm{F}_j\mathscr{M} = 0 \text{ for } j \ll 0, \\
\mathrm{F}_j\mathscr{M} \text{ is } \mathscr{O}_X\text{-coherent}, \\
\text{locally on } X,\ \mathscr{D}_X(m) \cdot \mathrm{F}_k\mathscr{M} = \mathrm{F}_{k+m}\mathscr{M} \text{ for } k \gg 0 \text{ and all} \\
m \geq 0.
\end{cases}
\tag{3.5}
$$

If $\mathscr{M}_0 \subset \mathscr{M}$ is a coherent $\mathscr{O}_X$-submodule which generates $\mathscr{M}$, setting $\mathrm{F}_k\mathscr{M} = \mathscr{D}_X(k) \cdot \mathscr{M}_0$, the sequence $\{\mathrm{F}_k\mathscr{M}\}_k$ is a coherent filtration.

One says that a coherent $\mathscr{D}_X$-module $\mathscr{M}$ is good if for any relatively compact open subset $U \subset\subset X$, there exists a coherent $\mathscr{O}_X$-submodule $\mathscr{M}_0 \subset \mathscr{M}$ which generates $\mathscr{M}$.

One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of objects with good cohomologies.

There is an equivalence of categories between left and right $\mathscr{D}_X$-modules

$$
\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X) \arrow[r, shift left, "\bullet \otimes_{\mathscr{O}_X} \Omega_X"] & \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}}). \arrow[l, shift left, "\Omega_X^{\otimes -1} \otimes_{\mathscr{O}_X} \bullet"]
\end{tikzcd}
$$

3.2 Characteristic variety

Locally on X, one may endow $\mathscr{M}$ with a good filtration. The support of the associated graded module $\mathrm{Gr}\,\mathscr{M}$ over $\mathrm{Gr}\,\mathscr{D}_X$ does not depend on the choice of the good filtration. One denotes this set by $\mathrm{char}(\mathscr{M})$ and calls it the characteristic variety of $\mathscr{M}$. If $\mathscr{M} = \mathscr{D}_X/\mathscr{I}$ for a locally finitely generated left ideal of $\mathscr{D}_X$, then

$$
\mathrm{char}(\mathscr{M}) = \{(z;\zeta) \in T^*X; \sigma(P)(z;\zeta) = 0 \text{ for all } P \in \mathscr{I}\}.
$$

Theorem 3.3. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then* $\mathrm{char}(\mathscr{M})$ *is a closed conic complex analytic* involutive (i.e., *co-isotropic*) *subset of T^*X.*

The involutivity result was first proved by Sato-Kashiwara-Kawai [SKK73] using differential operators of infinite order. Then Gabber [Gab81] gave a purely algebraic proof.

Definition 3.4. A coherent $\mathscr{D}_X$-module $\mathscr{M}$ is holonomic if $\mathrm{char}(\mathscr{M})$ is Lagrangian.

One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of objects with holonomic cohomologies.

3.3 Operations

Duality

One defines the duality functor $\mathbb{D}$ for left (resp. right) D-modules

$$
\begin{aligned}
\mathbb{D}_X\mathscr{M} &= \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})[d_X] \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X) \quad \text{for } \mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X), \\
\mathbb{D}_X\mathscr{N} &= \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\mathrm{op}}}(\mathscr{N}, \Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X)[d_X] \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}}) \quad \text{for } \mathscr{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}}).
\end{aligned}
$$

Of course, if $\mathscr{M}$ is coherent, then $\mathbb{D}_X\mathbb{D}_X\mathscr{M} \simeq \mathscr{M}$. in this case, one has

$$\mathrm{char}(\mathscr{M}) = \mathrm{char}(\mathbb{D}_X\mathscr{M}).$$

The de Rham and Sol functors

One defines the functors

$$\begin{aligned}
\mathcal{S}ol_X \quad &: \quad \mathrm{D^b}(\mathscr{D}_X)^{\mathrm{op}} \to \mathrm{D^b}(\mathbb{C}_X), \quad \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X),\\
\mathcal{DR}_X \quad &: \quad \mathrm{D^b}(\mathscr{D}_X) \to \mathrm{D^b}(\mathbb{C}_X), \quad \mathscr{M} \mapsto \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}.
\end{aligned}$$

For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, by representing it by the complex (3.4), one gets

$$\mathcal{S}ol_X(\mathscr{M}) \quad \simeq \quad 0 \to \mathscr{O}_X^{N_0} \xrightarrow{P_0 \cdot} \mathscr{O}_X^{N_1} \to \cdots \to \mathscr{O}_X^{N_r} \to 0. \tag{3.6}$$

Note that

$$\mathcal{S}ol_X(\mathscr{M}) \simeq \mathcal{DR}_X(\mathbb{D}\mathscr{M})\,[-d_X]. \tag{3.7}$$

Theorem 3.5 ([KS90])**.** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then*

$$\mathrm{char}(\mathscr{M}) = \mathrm{SS}(\mathcal{S}ol_X(\mathscr{M})).$$

The proof uses essentially the Cauchy-Kowalewsky theorem.

This formula, together with Theorem 2.5, gives another totally different proof of the involutivity of $\mathrm{char}(\mathscr{M})$.

External product

Let X and Y be two manifolds. For a $\mathscr{D}_X$-module $\mathscr{M}$ and a $\mathscr{D}_Y$-module $\mathscr{N}$, we define their external product, denoted $\mathscr{M} \underline{\boxtimes} \mathscr{N}$, by

$$\mathscr{M} \underline{\boxtimes} \mathscr{N} := \mathscr{D}_{X\times Y} \otimes_{\mathscr{D}_X \boxtimes \mathscr{D}_Y} (\mathscr{M} \boxtimes \mathscr{N}).$$

Note that the functor $\mathscr{M} \mapsto \mathscr{M} \underline{\boxtimes} \mathscr{N}$ is exact.

If $\mathscr{M}$ and $\mathscr{N}$ are coherent, then so is $\mathscr{M} \underline{\boxtimes} \mathscr{N}$ and $\mathrm{char}(\mathscr{M} \underline{\boxtimes} \mathscr{N}) = \mathrm{char}(\mathscr{M}) \times \mathrm{char}(\mathscr{N})$.

Transfert bimodule

Let $f\colon X \to Y$ be a morphism of complex manifolds. The sheaf

$$\mathscr{D}_{X\to Y} := \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y \tag{3.8}$$

is naturally endowed with a structure of an $(\mathscr{O}_X, f^{-1}\mathscr{D}_Y)$-bimodule. We shall endow it of a structure of a left $\mathscr{D}_X$-module by first defining the action Θ_X. Let $v \in \Theta_X$. Then $f'_* v \in \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\Theta_Y$. Hence

$$f'_* v = \sum_j a_j \otimes w_j,$$

with $a_j \in \mathscr{O}_X$ and $w_j \in f^{-1}\Theta_Y$. Define the action of v on $a \otimes P \in \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$ by setting

$$v(a \otimes P) = v(a) \otimes P + \sum_j a a_j \otimes w_j \circ P. \tag{3.9}$$

Then one checks that the action of Θ_X extends as a left action of $\mathscr{D}_X$. If one chooses a local coordinate system $(y_1, \dots, y_m)$ on Y and writes $f = (f_1, \dots, f_m)$, then

$$f'_* v = \sum_{j=1}^{m} v(f_j) \otimes \partial_{y_j}.$$

Inverse images

The inverse image functor $\mathrm{D}f^*$ for $\mathscr{D}$-modules is given by

$$\mathrm{D}f^*\mathscr{N} = \mathscr{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{N}, \quad \mathscr{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y).$$

One says that f is non characteristic for $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$ if f_d is proper (hence, finite) on $f_\pi^{-1}\,\mathrm{char}(\mathscr{N})$.

Theorem 3.6 (Cauchy-Kowalevsky-Kashiwara theorem). *Let $\mathscr{N}$ be a coherent $\mathscr{D}_Y$-module and assume that f is non-characteristic for $\mathscr{N}$. Then $\mathrm{D}f^*\mathscr{N}$ is concentrated in degree 0, is a coherent $\mathscr{D}_Y$-module, $\mathrm{char}(\mathrm{D}f^*\mathscr{N}) = f_d f_\pi^{-1}\,\mathrm{char}(\mathscr{N})$ and there is a natural ismorphism*

$$f^{-1}\mathrm{Sol}_Y\mathscr{N} \xrightarrow{\sim} \mathrm{Sol}_X(\mathrm{D}f^*\mathscr{N}).$$

Product

The product $\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{L}$ of two left $\mathscr{D}_X$-modules is naturally endowed with a structure of a left $\mathscr{D}_X$-module. We denote it by $\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L}$. One has

$$\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L} \simeq \mathrm{D}\delta^*(\mathscr{M} \underline{\boxtimes} \mathscr{L})$$

where $\delta \colon X \hookrightarrow X \times X$ is the diagonal embedding. Even when $\mathscr{M}$ and $\mathscr{N}$ are coherent $\mathscr{D}_X$-modules, the product is not coherent in general, but it follows from Theorem 3.6 that it is coherent as soon as $\mathrm{char}(\mathscr{M}) \cap \mathrm{char}(\mathscr{N}) \subset T^*_X X$.

Direct images

The proper direct image functor $\mathrm{D}f_!$ for (right) $\mathscr{D}$-modules is given by

$$\mathrm{D}f_!\mathscr{M} := \mathrm{R}f_!(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}), \quad \mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}}).$$

One defines the proper direct image functor $\mathrm{D}f_!$ for left $\mathscr{D}$-modules by using the line bundles Ω_X and Ω_Y (or their inverse) which interwine the left and right structures.

Example 3.7. Let Y be a complex curve and let $f\colon X \to Y$ be a proper map. Then $\mathrm{D}f_*\mathscr{O}_X$ is called a Gauss-Manin connection on Y.

Theorem 3.8 ([Kas03, Sch86]). *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$ and assume that f is proper on the support of $\mathscr{M}$. Then $\mathrm{D}f_!\mathscr{M}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_Y)$. Moreover, there is a natural isomorphism*

$$\mathrm{R}f_!\mathcal{S}ol_X(\mathscr{M})\,[d_X] \simeq \mathcal{S}ol_Y(\mathrm{D}f_!\mathscr{M})\,[d_Y], \tag{3.10}$$

$$\mathrm{R}f_!(\mathcal{DR}_X(\mathscr{M})) \simeq \mathcal{DR}_Y(\mathrm{D}f_!\mathscr{M}). \tag{3.11}$$

The coherency of direct images easily follows from the Grauert's direct images theorem.

3.4 Holonomic modules

Theorem 3.9 ([Kas75]). *The functor $\mathcal{S}ol_X\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$ induces a functor*

$$\mathcal{S}ol_X\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X)$$

Sketch of proof. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ and set $F = \mathcal{S}ol_X(\mathscr{M})$. It follows from Theorem 3.5 that $\mathrm{SS}(F)$ is a Lagrangian $\mathbb{C}$-analytic subset of T^*X and therefore F is weakly $\mathbb{C}$-constructible. It remains to show that F_x is finite dimensional for all $x \in X$.

We may assume that X is open in $\mathbb{C}^n$. Denote by $B(x;\varepsilon)$ the open ball of radius $\varepsilon > 0$ centered at x. The object $\mathrm{R}\Gamma(B(x;\varepsilon);F)$ may be represented by a complex

$$F_\varepsilon \;:=\; 0 \to \Gamma(B(x;\varepsilon);\mathscr{O}_X^{N_0}) \xrightarrow{P_0\cdot} \cdots \Gamma(B(x;\varepsilon);\mathscr{O}_X^{N_n}) \to 0.$$

This is a complex of topological vector spaces of type FS (Fréchet-Schwartz). Since F is weakly constructible, the restriction morphisms $F_\varepsilon \to F_{\varepsilon'}$ are quasi-isomorphisms for $0 < \varepsilon \leq \varepsilon' \ll 1$. It follows from the Montel theorem that the restriction maps $\Gamma(B(x;\varepsilon);\mathscr{O}_X) \to \Gamma(B(x;\varepsilon');\mathscr{O}_X)$ are compact for $0 < \varepsilon' < \varepsilon$. Then the cohomology of the complex is finite dimensional by a classical theorem of Riesz . Q.E.D.

The functor $\mathrm{Sol}_X\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X)$ is not faithful. For example, if $X = \mathbb{A}^1(\mathbb{C})$, the complex line with coordinate t, $P = t^2\partial_t - 1$ and $Q = t^2\partial_t + t$, then the two holonomic $\mathscr{D}_X$-modules $\mathscr{D}_X/\mathscr{D}_XP$ and $\mathscr{D}_X/\mathscr{D}_XQ$ have the same sheaves of solutions.

Hence, a natural question is to look for a full triangulated category of $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ on which Sol_X is fully faithful and induces an equivalence with $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X)$. This is the subject of § 6.

4 Microlocalization on complex manifolds

4.1 Microdifferential modules

References for this subsection are made to [SKK73] (see also [Sch85] for an exposition).

For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, its characteristic variety $\mathrm{char}(\mathscr{M})$ lies in T^*X. It is thus natural to look for a localization of $\mathscr{D}_X$ on T^*X. One such a localization is given by the sheaf $\widehat{\mathscr{E}}_X$ of formal microdifferential operators.

Assume X is open in a finite dimensional complex vector space E. In this case, the total symbol of a differential operator P is a section of $\pi_*\mathscr{O}_{T^*X}$, polynomial in the fiber variable. If order to invert $\sigma_{\mathrm{tot}}(P)(x;\xi)$ on an open set on which its principal symbol does not vanish, one is lead to consider the new ring

$$\widehat{\mathscr{S}}_{T^*X} = \varinjlim_{m\in\mathbb{Z}} \prod_{-\infty<j\leq m} \mathscr{O}_{T^*X}(j)$$

where $\mathscr{O}_{T^*X}(j)$ denotes the subsheaf of $\mathscr{O}_{T^*X}$ consisting of section homogeneous of degree j in the fiber variable. Hence a section f of $\widehat{\mathscr{S}}_{T^*X}$ on an open subset U of T^*X is a formal series

$$f(x;\xi) = \sum_{-\infty<j\leq m} f_j(x;\xi)$$

where $f_j \in \Gamma(U;\mathscr{O}_{T^*X})$ is homogeneous of degree j in ξ.

If f_m is not identically zero, we say f is of order m and call f_m the principal symbol of f. We set

$$\sigma(f) = f_m.$$

One extends the product (3.3) to this new sheaf. Hence, we define the algebra $(\widehat{\mathscr{S}}_{T^*X}, \star)$ by setting for for $f, g \in \widehat{\mathscr{S}}_{T^*X}$

$$(f \star g)(x,\xi) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial_\xi^\alpha(f(x;\xi))\partial_x^\alpha(g(x;\xi)). \tag{4.1}$$

We call this product $\star$, the Leibniz product. The next result is obvious.

Proposition 4.1. *The sheaf $\widehat{\mathscr{S}}_{T^*X}$ endowed with the Leibniz product given by (4.1) is a sheaf of filtered unitary $\mathbb{C}$-algebras.*

One denotes by $\widehat{\mathscr{S}}_{T^*X}(m)$ the subring of $\widehat{\mathscr{S}}_{T^*X}$ consisting of sections of order $\leq m$.

Proposition 4.2. *Let U be an open subset of T^*X and let $f \in \Gamma(U;\widehat{\mathscr{S}}_{T^*X})$. Assume that $\sigma(f)(x;\xi) \neq 0$ for any $(x;\xi) \in U$. Then f is invertible in $\Gamma(U;\widehat{\mathscr{S}}_{T^*X})$.*

Proof. We may assume U is connected and f is of order $m \in \mathbb{Z}$. The operator g with total symbol f_m^{-1} is well defined. By Proposition 4.1, the operator $g \circ f$ has order 0 and principal symbol 1. Hence

$$g \circ f = 1 - r$$

where $r \in \widehat{\mathscr{S}}_{T^*X}(-1)$. Therefore the series $\sum_{j=0}^{\infty} r^j$ is well defined in $\widehat{\mathscr{S}}_{T^*X}(0)$ and is a right and left inverse of $g \circ f$, which shows that f admits a left inverse. One proves similarly that f admits a right inverse. Q.E.D.

Theorem 4.3. (Sato-Kashiwara-Kawai) *For each complex manifold X, there exists a filtered sheaf $\widehat{\mathscr{E}}_{T^*X}$ of unitary $\mathbb{C}$-algebras together with*

(i) *a natural isomorphism* $\mathrm{Gr}\,\widehat{\mathscr{E}}_{T^*X} \simeq \bigoplus_{m\in\mathbb{Z}} \mathscr{O}_{T^*X}(m)$,

(ii) *a monomorphism of algebras* $\pi^{-1}\mathscr{D}_X \hookrightarrow \widehat{\mathscr{E}}_{T^*X}$,

(iii) *and, when X is affine, a canonical isomorphism of filtered rings*

$$\sigma_{\mathrm{tot}} \colon \widehat{\mathscr{E}}_{T^*X} \xrightarrow{\sim} (\widehat{\mathscr{S}_{T^*X}}, \star),$$

such that (i) *and* (ii) *are induced by* (iii).

In practice, when X is affine, one identifies these two sheaves.

Proposition 4.4. *Let X be a complex manifold. The sheaf of rings $\widehat{\mathscr{E}}_{T^*X}$ is*

(a) $\mathbb{C}^\times$*-conic,*

(b) *right and left Noetherian,*

(c) *flat over $\pi^{-1}\mathscr{D}_X$ and over $\widehat{\mathscr{E}}_{T^*X}(0)$.*

Proof. These properties are local and we may assume that X is affine. Then they follows from the corresponding properties on $\mathrm{Gr}\,\widehat{\mathscr{E}}_{T^*X}$ and general results on filtered and graded rings. Q.E.D.

One defines the notion of a good filtration on coherent $\widehat{\mathscr{E}}_{T^*X}$-modules similarly as for $\mathscr{D}_X$-modules. For a coherent $\widehat{\mathscr{E}}_{T^*X}$-module endowed with a good filtration, we set

$$\widetilde{\mathrm{Gr}(\mathscr{M})} := \mathscr{O}_{T^*X} \otimes_{\mathrm{Gr}\,\widehat{\mathscr{E}}_{T^*X}} \mathrm{Gr}(\mathscr{M}).$$

Outside of the zero section T^*_XX, to give a good filtration on a coherent $\widehat{\mathscr{E}}_{T^*X}$-module $\mathscr{M}$ is equivalent to give a coherent $\widehat{\mathscr{E}}_{T^*X}(0)$-module $\mathscr{M}_0 \subset \mathscr{M}$ which generates $\mathscr{M}$.

Theorem 4.5. *Let $\mathscr{M}$ be a coherent $\widehat{\mathscr{E}}_{T^*X}$-module defined on an open subset U of T^*X. Assume that $\mathscr{M}$ is endowed with a good filtration.*

(i) *One has* $\mathrm{supp}(\mathscr{M}) = \mathrm{supp}(\widetilde{\mathrm{Gr}(\mathscr{M})})$.

(ii) $\mathrm{supp}(\mathscr{M})$ *is a closed $\mathbb{C}^\times$-conic complex analytic subset of $U \subset T^*X$.*

(iii) $\mathrm{supp}(\mathscr{M})$ *is an involutive subset of $U \subset T^*X$.*

Corollary 4.6. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then*

$$\mathrm{char}(\mathscr{M}) = \mathrm{supp}(\widehat{\mathscr{E}}_{T^*X} \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{M}).$$

Since $\widehat{\mathscr{E}}_{T^*X}$ is a $\mathbb{C}^\times$-conic sheaf, it is natural to take its direct image on the projective cotangent bundle P^*X. Denote by

$$\rho\colon T^*X \setminus X \to P^*X = (T^*X \setminus X)/\mathbb{C}^\times$$

the canonical projection on the cotangent bundle. One sets

$$\widehat{\mathscr{E}}_{P^*X} = \rho_*\widehat{\mathscr{E}}_{T^*X}.$$

The manifold P^*X is a complex contact manifold and any such a manifold Y is locally isomorphic to an open subset of P^*X. We shall not recall here what is an algebroid stack (a notion introduced in [Kon03]), but let us simply say that it is a kind of sheaf of categories locally equivalent to a sheaf of algebras.

Theorem 4.7 ([Kas96])**.** *Let Y be a complex contact manifold. There is a "canonical" algebroid stack $\widehat{\mathscr{E}}_Y$ locally isomorphic to $\widehat{\mathscr{E}}_{P^*X}$.*

Remark 4.8. The sheaf of algebras $\widehat{\mathscr{E}}_{T^*X}$ is the sheaf of *formal* microdifferential operators. It contains the sheaf of algebras $\mathscr{E}_{T^*X}$ of microdifferentail operators defined as follows. In a local chart U, a section P of $\widehat{\mathscr{E}}_{T^*X}$ is a section of $\mathscr{E}_{T^*X}$ if its total symbol is Borel summable on any compact subset of U. Recall that a formal series $\sum_{j\in\mathbb{N}} a_j$ is Borel summable if $\sum_j |a_j|/j! < \infty$.

The sheaf $\mathscr{E}_{T^*X}$ satisfies all properties of $\widehat{\mathscr{E}}_{T^*X}$ mentioned above and is in fact more natural and more important that its formal version. It has been constructed functorially in [SKK73].

4.2 $\widehat{\mathscr{W}}_{T^*X}$-modules

As already mention, the sheaf of microdifferential operators on T^*X is $\mathbb{C}^\times$-conic. As usual, it is possible to pass from a conic situation to a non-conic one by adding a variable.

Notation 4.9. We consider the ring $\mathbb{C}[[\hbar]]$ of formal power series in an indeterminate $\hbar$ and its fraction field, the field $\mathbb{C}((\hbar))$ of Laurent series in $\hbar$.

We set

$$\mathscr{O}_X^\hbar := \varprojlim_n \mathscr{O}_X \otimes (\mathbb{C}[[\hbar]]/\hbar^n\mathbb{C}[[\hbar]]) \simeq \prod_{n\geq 0} \mathscr{O}_X\hbar^n,$$
$$\mathscr{O}_X((\hbar)) := \mathbb{C}((\hbar)) \otimes_{\mathbb{C}[[\hbar]]} \mathscr{O}_X^\hbar.$$

Recall that a section f of $\widehat{\mathscr{S}}_{T^*X}$ on an open subset U of T^*X is a formal series

$$f(x;\xi) = \sum_{-\infty<j\leq m} f_j(x;\xi)$$

where $f_j \in \Gamma(U;\mathscr{O}_{T^*X})$ is homogeneous of degree j in ξ. Let us introduce a formal parameter $\hbar$ and set

$$u_i = \hbar\xi_i, (i = 1,\dots,n).$$

By this change of variable, we may embed the sheaf $\widehat{\mathscr{S}}_{T^*X}$ into the sheaf $\mathscr{O}_{T^*X}((\hbar))$. The Leibniz product on $\widehat{\mathscr{S}}_{T^*X}$ defines the star product on $\mathscr{O}_{T^*X}((\hbar))$:

$$f \star g \;=\; \sum_{\alpha\in\mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!}(\partial_u^\alpha f)(\partial_x^\alpha g). \tag{4.2}$$

Theorem 4.10. *For each complex manifold X, there exists canonically a filtered sheaf $\widehat{\mathscr{W}}_{T^*X}$ of unitary $\mathbb{C}((\hbar))$-algebras together with*

(i) *a natural isomorphism* $\mathrm{Gr}\,\widehat{\mathscr{W}}_{T^*X} \simeq \mathscr{O}_{T^*X}[\hbar^{-1},\hbar]$,

(ii) *monomorphisms of $\mathbb{C}$-algebras*

$$\pi^{-1}\mathscr{D}_X \hookrightarrow \widehat{\mathscr{E}}_{T^*X} \hookrightarrow \widehat{\mathscr{W}}_{T^*X}, \tag{4.3}$$

(iii) *and, when X is affine, a canonical isomorphism of $\mathbb{C}((\hbar))$-algebras*

$$\sigma_{\mathrm{tot}}\colon \widehat{\mathscr{W}}_{T^*X} \xrightarrow{\sim} (\mathscr{O}_{T^*X}((\hbar)),\star),$$

such that (i) *and* (ii) *are induced by* (iii) *and the product in $\widehat{\mathscr{W}}_{T^*X}$ induces the Leibniz product* (4.2).

One may construct intrinsically $\widehat{\mathscr{W}}_{T^*X}$ as follows (see [PS04]). Consider the complex line $\mathbb{C}$ endowed with the coordinate t and the subsheaf $\widehat{\mathscr{E}}_{T^*(\mathbb{C}\times X),\hat{t}}$ of $\widehat{\mathscr{E}}_{T^*(\mathbb{C}\times X)}$ consisting of sections which commute with ∂_t. Denote by $\rho\colon T^*_{\tau\neq 0}(\mathbb{C}\times X) \to T^*X$ the map $(t,x;\tau,\xi)\mapsto(x,\xi/\tau)$. Then $\widehat{\mathscr{W}}_{T^*X} = \rho_*\widehat{\mathscr{E}}_{T^*(\mathbb{C}\times X),\hat{t}}$.

By adapting the method of [Kas96], one constructs an algebroid stack $\widehat{\mathscr{W}}_Y$ on any complex symplectic manifold Y, but this is a particular case of a theorem of [Kon03] (see below).

4.3 DQ-modules

References for this subsection are made to [KS01].

From now on, we denote by Y a complex manifold, but in fact, Y will be endowed with a Poisson structure. When this Poisson structure is symplectic, Y will locally play the role of the cotangent bundle encountered in the previous sections.

Let us recall a classical definition which generalizes the product in (4.2).

Definition 4.11 (see [BFF+78])**.** A star-product on $\mathscr{O}_Y[[\hbar]]$ is a $\mathbb{C}[[\hbar]]$-bilinear associative multiplication law $\star$ which satisfies

$$f\star g = \sum_{i\geq 0} P_i(f,g)\hbar^i \text{ for } f,g\in\mathscr{O}_Y, \tag{4.4}$$

where the P_i's are bidifferential operators such that $P_0(f,g) = fg$ and $P_i(f,1) = P_i(1,f) = 0$ for all $f\in\mathscr{O}_Y$ and $i>0$. We call $(\mathscr{O}_Y[[\hbar]],\star)$ a star-algebra.

Note that $1 \in \mathscr{O}_Y \subset \mathscr{O}_Y[[\hbar]]$ is a unit with respect to $\star$. Note also that we have

$$(\sum_{i\geq 0} f_i\hbar^i) \star (\sum_{j\geq 0} g_j\hbar^i) = \sum_{n\geq 0}\big(\sum_{i+j+k=n} P_k(f_i, g_j)\big)\hbar^n.$$

A star-product defines a Poisson structure on $(Y, \mathscr{O}_Y)$, by setting for $f, g \in \mathscr{O}_Y$:

$$\{f, g\} = P_1(f, g) - P_1(g, f) = \hbar^{-1}(f \star g - g \star f) \bmod \hbar\mathscr{O}_Y[[\hbar]]. \tag{4.5}$$

Example 4.12. (i) If the star product is commutative, then it is isomorphic to the usual product on $\mathscr{O}_X^{\hbar}$.
(ii) If the Poisson structure is symplectic, then the star product is locally isomorphic to that given by the Leibniz product (4.2).

Definition 4.13. A DQ-algebra $\mathscr{A}$ on Y is a $\mathbb{C}[[\hbar]]$-algebra locally isomorphic to a star-algebra $(\mathscr{O}_Y[[\hbar]], \star)$ as a $\mathbb{C}[[\hbar]]$-algebra.

Locally, (globally in the real case), any Poisson manifold $(Y, \mathscr{O}_Y)$ may be endowed with a DQ-algebra to which the Poisson structure is associated. This is a famous theorem of Kontsevich [Kon01]. There is a similar global result on a complex Poisson manifold after replacing the notion of a sheaf of algebras by that of an algebroid stack [Kon03].

Clearly a DQ-algebra $\mathscr{A}$ satisfies the conditions:

$$\begin{cases} \text{(i)} \ \hbar\colon \mathscr{A} \to \mathscr{A} \text{ is injective,} \\ \text{(ii)} \ \mathscr{A} \to \varprojlim_n \mathscr{A}/\hbar^n\mathscr{A} \text{ is an isomorphism,} \\ \text{(iii)} \ \mathscr{A}/\hbar\mathscr{A} \text{ is isomorphic to } \mathscr{O}_Y \text{ as a } \mathbb{C}\text{-algebra.} \end{cases} \tag{4.6}$$

For a $\mathbb{C}[[\hbar]]$-algebra $\mathscr{A}$ satisfying (4.6), the $\mathbb{C}$-algebra isomorphism $\mathscr{A}/\hbar\mathscr{A} \xrightarrow{\sim} \mathscr{O}_Y$ in (4.6) (iii) is unique. This follows from the fact that any $\mathbb{C}$-algebra endomorphism of $\mathscr{O}_Y$ is equal to the identity.

We denote by

$$\sigma_0\colon \mathscr{A} \to \mathscr{O}_Y \tag{4.7}$$

the $\mathbb{C}[[\hbar]]$-algebra morphism $\mathscr{A} \to \mathscr{A}/\hbar\mathscr{A} \xrightarrow{\sim} \mathscr{O}_Y$.

One proves that a DQ-algebra is right and left Noetherian. As usual, one denotes by $\mathrm{D}^{\mathrm{b}}(\mathscr{A})$ the bounded derived category of left $\mathscr{A}$-modules and by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A})$ the full triangulated subcategory consisting of objects with coherent cohomology.

Definition 4.14. Let Λ be a smooth submanifold of Y and let $\mathscr{L}$ be a coherent $\mathscr{A}$-module supported by Λ. One says that $\mathscr{L}$ is simple along Λ if $\mathrm{gr}_\hbar(\mathscr{L})$ is concentrated in degree 0 and $H^0(\mathrm{gr}_\hbar(\mathscr{L}))$ is an invertible $\mathscr{O}_\Lambda \otimes_{\mathscr{O}_X} \mathrm{gr}_\hbar(\mathscr{A})$-module. (In particular, $\mathscr{L}$ has no $\hbar$-torsion.)

$\hbar$-localization

To a DQ-algebra $\mathscr{A}$ we associate its *$\hbar$-localization*, the $\mathbb{C}((\hbar))$-algebra

$$\mathscr{A}^{\mathrm{loc}} = \mathbb{C}((\hbar)) \otimes_{\mathbb{C}[[\hbar]]} \mathscr{A}. \tag{4.8}$$

There exists a pair of adjoint exact functors $(\bullet \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar)), \mathit{for})$:

$$\mathrm{Mod}(\mathscr{A}^{\mathrm{loc}}) \underset{\bullet\otimes_{\mathbb{C}[[\hbar]]}\mathbb{C}((\hbar))}{\overset{\mathit{for}}{\rightleftarrows}} \mathrm{Mod}(\mathscr{A}). \tag{4.9}$$

The algebra $\mathscr{A}^{\mathrm{loc}}$ is Noetherian.

If $\mathscr{M}$ is an $\mathscr{A}^{\mathrm{loc}}$-module, $\mathscr{M}_0$ an $\mathscr{A}$-submodule and $\mathscr{M}_0 \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar)) \xrightarrow{\sim} \mathscr{M}$, we shall say that $\mathscr{M}_0$ generates $\mathscr{M}$.

A coherent $\mathscr{A}^{\mathrm{loc}}$-module $\mathscr{M}$ is *good* if, for any open relatively compact subset U of X, there exists a coherent $(\mathscr{A}|_U)$-module which generates $\mathscr{M}|_U$. One denotes by $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}})$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}^{\mathrm{loc}})$ consisting of good modules. Similarly as in [Kas03, Prop. 4.23], one proves that $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}})$ is a thick subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}^{\mathrm{loc}})$.

We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}^{\mathrm{loc}})$ (resp. $\mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{A}^{\mathrm{loc}})$) the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{A}^{\mathrm{loc}})$ consisting of objects $\mathscr{M}$ such that $H^j(\mathscr{M})$ is coherent (resp. good) for all $j \in \mathbb{Z}$. The notion of good $\mathscr{A}_X$-module is similar to that of good $\mathscr{D}$-module of loc. cit.

A complex analytic subset S of a complex Poisson manifold Y is involutive if for any $f, g \in \mathscr{O}_Y$ such that $f|_S = 0$, $g|_S = 0$, one has $\{f, g\}|_S = 0$.

Proposition 4.15. *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}^{\mathrm{loc}})$. Then* $\mathrm{supp}(\mathscr{M})$ *is a closed $\mathbb{C}$-analytic involutive subset of Y.*

The involutivity result easily follows from Gabber's theorem [Gab81].

Holonomic DQ-modules

We assume now that the Poisson structure associated with the DQ-algebra $\mathscr{A}$ is symplectic.

Definition 4.16. (a) An $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is holonomic if it is coherent and its support is a Lagrangian subvariety of X.

(b) An $\mathscr{A}$-module $\mathscr{N}$ is holonomic if it is coherent, without $\hbar$-torsion and $\mathscr{N}^{\mathrm{loc}}$ is a holonomic $\mathscr{A}_X^{\mathrm{loc}}$-module.

(c) Let Λ be a smooth Lagrangian submanifold of X. We say that an $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is simple holonomic along Λ if there exists locally an $\mathscr{A}$-module $\mathscr{M}_0$ simple along Λ such that $\mathscr{M} \simeq \mathscr{M}_0^{\mathrm{loc}}$.

Theorem 4.17. *Let X be a complex symplectic manifold and let $\mathscr{M}$ and $\mathscr{L}$ be two holonomic $\mathscr{A}_X^{\mathrm{loc}}$-modules. Then*

(i) *the object* $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L})$ *belongs to* $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}c}(\mathbf{k}_X)$,

(ii) *There is a canonical isomorphism:*

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L}) \xrightarrow{\sim} \big(\mathrm{D}'_X \mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L},\mathscr{M})\big)\,[d_X]. \tag{4.10}$$

(iii) *the object* $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L})[d_X/2]$ *is perverse.*

Let Λ be a smooth Lagrangian submanifold of Y. Denote by $\mathrm{DLoc}_\Lambda(\mathbb{C}((\hbar)))$ the full subcategory of the derived category of sheaves of $\mathbb{C}((\hbar))$-modules consisting of objects whose cohomology are local systems. Denote by $\mathrm{D}^{\mathrm{b}}_\Lambda(\mathscr{A}^{\mathrm{loc}})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}^{\mathrm{loc}})$ consisting of holonomic modules supported by Λ. If $\mathscr{M}$ and $\mathscr{L}$ belong to $\mathrm{D}^{\mathrm{b}}_\Lambda(\mathscr{A}^{\mathrm{loc}})$, one easily proves that $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L},\mathscr{M})$ belongs to $\mathrm{DLoc}_\Lambda(\mathbb{C}((\hbar))$.

Assume now that there exists a simple $\mathscr{A}^{\mathrm{loc}}$-module $\mathscr{L}$ along Λ. We get a functor

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}^{\mathrm{loc}}}(\mathscr{L},\bullet)\colon \mathrm{D}^{\mathrm{b}}_\Lambda(\mathscr{A}^{\mathrm{loc}}) \to \mathrm{DLoc}_\Lambda(\mathbb{C}((\hbar))) \tag{4.11}$$

and one easily proves that this functor is an equivalence of categories.

The existence of a simple $\mathscr{A}^{\mathrm{loc}}$-module along smooth Lagrangian submanifolds is discussed in [DS07] (see also [BGKP14]).

Remark 4.18. In case of $\mathscr{D}_X$-modules, there is a holonomic module which plays a particular role, namely the $\mathscr{D}_X$-module $\mathscr{O}_X$. This is related the fact that on T^*X there is a Lagrangian submanifold which plays a particular role, namely the zero-section. On a symplectic manifold, there are no zero-section and all smooth Lagrangian submanifolds play the same role. Then one can ask if there exists an analogue of the Riemann-Hilbert correspondence in this case.

5 Microlocalization on real manifolds

References are made to [KS90, [§ 3.7, Ch. 4, §6.2 §7.2].

5.1 Fourier-Sato transform

The classical Fourier transform interchanges (generalized) functions on a vector space $\mathbb{V}$ and (generalized) functions on the dual vector space $\mathbb{V}^*$. The idea of extending this formalism to sheaves, hence of replacing an isomorphism of spaces with an equivalence of categories, seems to have appeared first in Mikio Sato's construction of microfunctions in [Sat70].

Let $\tau : E \to M$ be a finite dimensional real vector bundle over a real manifold M with fiber dimension n and let $\pi : E^* \to M$ be the dual vector bundle. Denote by p_1 and p_2 the first and second projection defined on $E \times_M E^*$, and define:

$$P = \{(x,y) \in E \times_M E^*; \langle x,y\rangle \geq 0\},$$
$$P' = \{(x,y) \in E \times_M E^*; \langle x,y\rangle \leq 0\}.$$

Consider the diagram:

$$\begin{tikzcd} & E \times_M E^* \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ E \arrow[dr, "\tau"'] & & E^* \arrow[dl, "\pi"] \\ & M. & \end{tikzcd}$$

Denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_E)$ consisting of conic sheaves, that is, objects with locally constant cohomology on the orbits of the action of $\mathbb{R}^+$.

Definition 5.1. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$, $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}^*_E)$. One sets:

$$\begin{aligned} F^\wedge &:= \mathrm{R}p_{2!}(p_1^{-1}F)_{P'} \simeq \mathrm{R}p_{2*}(\mathrm{R}\Gamma_P p_1^{-1}F), \\ G^\vee &:= \mathrm{R}p_{1*}(\mathrm{R}\Gamma_{P'} p_2^! G) \simeq \mathrm{R}p_{1!}(p_2^! G)_P. \end{aligned}$$

The main result of the theory is the following.

Theorem 5.2. *The two functors $(\cdot)^\wedge$ and $(\cdot)^\vee$ are inverse to each other, hence define an equivalence of categories $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E) \simeq \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{E^*})$ and for $F_1, F_2 \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$, one has the isomorphism*

$$\mathrm{RHom}\,(F_1^\wedge, F_2^\wedge) \simeq \mathrm{RHom}\,(F_1, F_2). \tag{5.1}$$

Example 5.3. (i) Let γ be a closed proper convex cone in E with $M \subset \gamma$. Then:

$$(\mathbf{k}_\gamma)^\wedge \simeq \mathbf{k}_{\mathrm{Int}(\gamma^\circ)}.$$

Here γ° is the polar cone to γ, a closed convex cone in E^* and $\mathrm{Int}\gamma^\circ$ denotes its interior.
(ii) Let γ be an open convex cone in E. Then:

$$(\mathbf{k}_\gamma)^\wedge \simeq \mathbf{k}_{\gamma^{\circ a}} \otimes \mathrm{or}_{E^*/M}\,[-n].$$

Here $\lambda^a = -\lambda$, the image of λ by the antipodal map.
(iii) Let $(x) = (x', x'')$ be coordinates on $\mathbb{R}^n$ with $(x') = (x_1, \ldots, x_p)$ and $(x'') = (x_{p+1}, \ldots, x_n)$. Denote by $(y) = (y', y'')$ the dual coordinates on $(\mathbb{R}^n)^*$. Set

$$\gamma = \{x; x'^2 - x''^2 \geq 0\}, \quad \lambda = \{y; y'^2 - y''^2 \leq 0\}.$$

Then $(\mathbf{k}_\gamma)^\wedge \simeq \mathbf{k}_\lambda[-p]$.

5.2 Specialization

Let $\iota\colon N \hookrightarrow M$ be the embedding of a closed submanifold N of M. Denote by $\tau_M\colon T_N M \to M$ the normal bundle to N.

If F is a sheaf on M, its restriction to N, denoted $F|_N$, may be viewed as a global object, namely the direct image by τ_M of a sheaf $\nu_N F$ on $T_N M$, called the specialization of F along N. Intuitively, $T_N M$ is the set of light rays issued from N in M and the germ of $\nu_N F$ at a normal vector $(x;v) \in T_N M$ is the germ at x of the restriction of F along the light ray v.

One constructs a new manifold $\widetilde{M}_N$, called the normal deformation of M along N, together with the maps

$$\begin{tikzcd} T_N M \arrow[r, "s"] \arrow[d, "\tau_M"'] & \widetilde{M}_N \arrow[d, "p"] & \Omega \arrow[l, "j"'] \arrow[dl, "\widetilde{p}"] \\ N \arrow[r, "\iota"'] & M & \end{tikzcd} \quad t\colon \widetilde{M}_N \to \mathbb{R},\ \Omega = \{t^{-1}(\mathbb{R}_{>0})\} \tag{5.2}$$

with the following properties. Locally, after choosing a local coordinate system (x', x'') on M such that $N = \{x' = 0\}$, we have $\widetilde{M}_N = M \times \mathbb{R}$, $t\colon \widetilde{M}_N \to \mathbb{R}$ is the projection, $\Omega = \{(x;t) \in M \times \mathbb{R}; t > 0\}$, $p(x', x'', t) = (tx', x'')$, $T_N M = \{t = 0\}$.

Definition 5.4. (a) Let $S \subset M$ be a locally closed subset. The Whitney normal cone $C_N(S)$ is a closed conic subset of $T_N M$ given by

$$C_N(S) = \overline{\widetilde{p}^{-1}(S)} \cap T_N M.$$

(b) For two subsets $S_1, S_2 \subset M$, their Whitney's normal cone is given by

$$C(S_1, S_2) = C_\Delta(S_1 \times S_2) \tag{5.3}$$

where Δ is the diagonal of $M \times M$ and TM is identified to $T_\Delta(M \times M)$ by the first projection $T(M \times M) \to TM$.

One defines the specialization functor

$$\nu_N \colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T_N M})$$

by a formula mimicking Definition 5.4, namely:

$$\nu_N F := s^{-1} \mathrm{R} j_* \widetilde{p}^{-1} F.$$

Clearly, $\nu_N F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T_N M})$, that is, $\nu_N F$ is a conic sheaf for the $\mathbb{R}^+$-action on $T_N M$. Moreover,

$$\mathrm{R}\tau_{M*} \nu_N F \simeq \nu_N F|_N \simeq F|_N.$$

For an open cone $V \subset T_N M$, one has

$$H^j(V; \nu_N F) \simeq \varinjlim_U H^j(U; F)$$

where U ranges through the family of open subsets of M such that

$$C_N(M \setminus U) \cap V = \emptyset.$$

5.3 Microlocalization

Denote by $\pi_M\colon T^*_N M \to M$ the conormal bundle to N in M, that is, the dual bundle to $\tau_M\colon T_N M \to M$.

If F is a sheaf on M, the sheaf of sections of F supported by N, denoted $\mathrm{R}\Gamma_N F$, may be viewed as a global object, namely the direct image by π_M of a sheaf $\mu_M F$ on $T^*_N M$. Intuitively, $T^*_N M$ is the set of "walls" (half-spaces) in M containing N in their boundary and the germ of $\mu_N F$ at a conormal vector $(x;\xi) \in T^*_N M$ is the germ at x of the sheaf of sections of F supported by closed tubes with edge N and which are almost the half-space associated with ξ.

More precisely, the microlocalization of F along N, denoted $\mu_N F$, is the Fourier-Sato transform of $\nu_N F$, hence is an object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T^*_N M})$. It satisfies:

$$\mathrm{R}\pi_{M*}\mu_N F \simeq \mu_N F|_N \simeq \mathrm{R}\Gamma_N F.$$

For a convex open cone $V \subset T^*_N M$, one has

$$H^j(V;\mu_N F) \simeq \varinjlim_{U,Z} H^j_{U\cap Z}(U;F),$$

where U ranges over the family of open subsets of M such that $U \cap N = \pi_M(V)$ and Z ranges over the family of closed subsets of M such that $C_M(Z) \subset V^\circ$ where V° is the polar cone to V.

If $H \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T^*M})$ is a conic sheaf on T^*M, then $\mathrm{R}\pi_{M!}H \simeq \mathrm{R}\Gamma_M H$ and one gets Sato's distinguished triangle

$$\mathrm{R}\pi_{M!}H \to \mathrm{R}\pi_{M*}H \to R\dot{\pi}_{M*}H \xrightarrow{+1} . \tag{5.4}$$

Applying this result to the conic sheaf $\mu_N F$, one gets the distinguished triangle

$$F|_N \otimes \omega_{N/M} \to \mathrm{R}\Gamma_N F|_N \to R\dot{\pi}_{M*}\mu_N F \xrightarrow{+1} . \tag{5.5}$$

5.4 The functor μhom

Let us briefly recall the main properties of the functor μhom, a variant of Sato's microlocalization functor.

Recall that Δ denotes the diagonal of $M \times M$. We shall denote by $\widetilde{\delta}$ the isomorphism

$$\widetilde{\delta}\colon T^*M \xrightarrow{\sim} T^*_M(M\times M), \quad (x;\xi) \mapsto (x,x;\xi,-\xi).$$

Definition 5.5. One defines the functor $\mu hom\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*M})$ by

$$\mu hom(F,G) = \widetilde{\delta}^{-1}\mu_\Delta \mathrm{R}\mathscr{H}om\,(q_2^{-1}F, q_1^! G)$$

where q_i $(i=1,2)$ denotes the i-th projection on $M \times M$.

Note that

- $\mathrm{R}\pi_{M*}\mu hom(F,G) \simeq \mathrm{R}\mathscr{H}om\,(F,G)$,

- $\mu hom(\mathbf{k}_N, F) \simeq \mu_N(F)$ for N a closed submanifold of M,
- $\operatorname{supp} \mu hom(F,G) \subset \mathrm{SS}(G) \cap \mathrm{SS}(F)$,
- assuming that F is $\mathbb{R}$-constructible, there is a distinguished triangle $\mathrm{D}'F \otimes G \to \mathrm{R}\mathscr{H}om\,(F,G) \to R\dot{\pi}_{M*}\mu hom(F,G) \xrightarrow{+1}$, where $R\dot{\pi}_{M*}$ is the projection $\dot{T}^*M \to M$, $\dot{T}^*M$ denoting $T^*M \setminus M$, the cotangent bundle with the zero-section removed.

Microlocal Serre functor

Assume that $\mathbf{k}$ is a field.

Consider a $\mathbf{k}$-triangulated category $\mathscr{T}$ such that for any pair of objects A, B,

$$\dim \sum_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{T}}(A, B\,[j]) < \infty \tag{5.6}$$

Following [BK89], one says that a triangulated functor $S\colon \mathscr{T} \to \mathscr{T}$ is a Serre functor if there is an isomorphism, functorial in A, B:

$$eq:serrefct2 \qquad \mathrm{Hom}_{\mathscr{T}}(A,B)^* \simeq \mathrm{Hom}_{\mathscr{T}}(B, S(A)) \tag{5.7}$$

where * is the duality functor for $\mathbf{k}$-vector spaces.

Now assume that M is real analytic. If $\mathscr{T}$ denotes the full triangualted category of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ consisting of objects with compact support, then (5.6) is satisfied in $\mathscr{T}$ but it is well-known that there are no Serre functors in this category as soons as $\dim M > 0$. However, there is an interesting phenomena which holds with μhom and not with $\mathrm{R}\mathscr{H}om$. Indeed, the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ admits a kind of microlocal Serre functor, as shown by the isomorphism, functorial in G and F (see [KS90, Prop. 8.4.14]):

$$\mathrm{D}_{T^*M}\mu hom(F,G) \simeq \mu hom(G,F) \otimes \pi_M^{-1}\omega_M.$$

This confirms the fact that to fully understand $\mathbb{R}$-constructible sheaves, it is natural to look at them microlocally, that is, in T^*M. This is also in accordance with the "philosophy" of Mirror Symmetry which interchanges the category of coherent $\mathscr{O}_X$-modules on a complex manifold X with the Fukaya category on a symplectic manifold Y. In case of $Y = T^*M$, the Fukaya category is equivalent (in some sense that we do not discuss here) to the category of $\mathbb{R}$-constructible sheaves on M, according to Nadler-Zaslow [Nad09, NZ09].

Microlocal Fourier-Sato transform

The Fourier-Sato transfom is by no means local: it interchanges sheaves on a vector bundle E and sheaves on E^*. However, this transformation is *microlocal* in the following sense.

Let $E \to Z$ be vector bundle over a manifold Z. There is a natural isomorphism $T^*E \simeq T^*E^*$ given in local coordinates

$$T^*E \ni (z, x; \zeta, \xi) \mapsto (z, \xi; \zeta, -x) \in T^*E^*. \tag{5.8}$$

Theorem 5.6 ([KS90, Exe. VII.2]). *Let $F, G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$. After identifying T^*E and T^*E^* as above, there is a natural isomorphism*

$$\mu hom(F, G) \simeq \mu hom(F^\wedge, G^\wedge). \tag{5.9}$$

5.5 Localization

Let A be a subset of T^*M and let $Z = T^*M \setminus A$. The full subcategory $\mathrm{D}^{\mathrm{b}}_Z(\mathbf{k}_M)$ of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of sheaves F such that $\mathrm{SS}(F) \subset Z$ is a triangulated subcategory. One sets

$$\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; A) := \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)/\mathrm{D}^{\mathrm{b}}_Z(\mathbf{k}_M),$$

the localization of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ by $\mathrm{D}^{\mathrm{b}}_Z(\mathbf{k}_M)$. Hence, the objects of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; A)$ are those of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ but a morphism $u\colon F_1 \to F_2$ in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ becomes an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; A)$ if, after embedding this morphism in a distinguished triangle $F_1 \to F_2 \to F_3 \xrightarrow{+1}$, one has $\mathrm{SS}(F_3) \cap A = \emptyset$. When $A = \{p\}$ for some $p \in T^*M$, one simply writes $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; p)$ instead of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; \{p\})$.

The functor μhom describes in some sense the microlocal morphisms of the category $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. More precisely, for U open in T^*M, it follows from (**??**) that μhom induces a bifunctor:

$$\mu hom\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; U)^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; U) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_U).$$

Moreover, the sequence of morphisms

$$\begin{aligned} \mathrm{RHom}\,(G, F) &\simeq \mathrm{R}\Gamma(M; \mathrm{R}\mathscr{H}om\,(G, F)) \\ &\simeq \mathrm{R}\Gamma(T^*M; \mu hom(G, F)) \\ &\to \mathrm{R}\Gamma(U; \mu hom(G, F)) \end{aligned}$$

define the morphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; U)}(G, F) \to H^0\mathrm{R}\Gamma(U; \mu hom(G, F)). \tag{5.10}$$

The morphism (5.10) is not an isomorphism, but it induces an isomorphism at each $p \in T^*M$:

Theorem 5.7 (See [KS90, Th. 6.1.2]). *Let $p \in T^*M$. Then*

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; p)}(G, F) \simeq H^0(\mu hom(G, F)_p).$$

Pure and simple sheaves

Let S be a smooth submanifold of M and let $\Lambda = T^*_S M$. Let $p \in \Lambda, p \notin T^*_M M$ and let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; p)$. Let us say that F is pure at p if $F \simeq V\,[d]$ for some $\mathbf{k}$-module V and some shift d and let us say that F is simple if moreover V is free of rank one. A natural question is to generalize this definition to the case where Λ is a smooth Lagrangian submanifold of $\dot{T}^*M$ but is no more necessarily a conormal bundle. Another natural question would be to calculate the shift d. This last point makes use of the Maslov index and we refer to [KS90, §. 7.5].

Notation 5.8. Let Λ be a smooth $\mathbb{R}^+$-conic Lagrangian locally closed submanifold of $\dot{T}^*M$, closed in an open conic neighborhood W of Λ.

(i) We denote by $\mathrm{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of objects F such that there exists an open neighborhood W of Λ (containing Λ as a closed subset) in T^*M such that $\mathrm{SS}(F) \cap W \subset \Lambda$.

(ii) One denotes by $\mathrm{DLoc}(\mathbf{k}_\Lambda)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_\Lambda)$ consisting of objects F such that for each $j \in \mathbb{Z}$, $H^j(F)$ is a local system on Λ. Equivalently, $\mathrm{DLoc}(\mathbf{k}_\Lambda)$ is the subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_\Lambda)$ consisting of sheaves with microsupport contained in the zero-section $T^*_\Lambda\Lambda$.

For $F, G \in \mathrm{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$, $\operatorname{supp} \mu hom(F, G) \subset T^*_\Lambda\Lambda$. Therefore, the functor μhom induces a functor

$$\mu hom \colon \mathrm{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M) \to \mathrm{DLoc}(\mathbf{k}_\Lambda).$$

Lemma 5.9. *Let $L \in \mathrm{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M; W)$. There is a natural morphism $\mathbf{k}_\Lambda \to \mu hom(L, L)$.*

Proof. Represent $L \in \mathrm{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$ by $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. The morphism $\mathbf{k}_M \to \mathrm{R}\mathscr{H}om\,(F, F) \simeq \mathrm{R}\pi_*\mu hom(F, F)$ defines the morphism $\mathbf{k}_{T^*M} \to \mu hom(F, F)$. Since $\mu hom(L, L)$ is supported by Λ in a neighborhoodd of Λ, this last morphism factorizes through $\mathbf{k}_\Lambda$. Q.E.D.

For simplicity, we assume that $\mathbf{k}$ is a field. Then, there is an easy definition of purity and simplicity.

Definition 5.10. Assume that $\mathbf{k}$ is a field and let $F \in \mathrm{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$.

(a) One says that F is pure on Λ if $\mu hom(F, F)|_\Lambda$ is concentrated in degree 0. One denotes by $\mathrm{Pure}(\Lambda, \mathbf{k})$ the subcategory of $\mathrm{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$ consisting of pure sheaves.

(b) One says that F simple on Λ if $\mathbf{k}_\Lambda \xrightarrow{\sim} \mu hom(L, L)|_\Lambda$. One denotes by $\mathrm{Simple}(\Lambda, \mathbf{k})$ the subcategory of $\mathrm{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$ consisting of simple sheaves.

Let $L \in \mathrm{Simple}(\Lambda, \mathbf{k})$. Then we get the functor

$$\mu hom(L, \bullet) \colon \mathrm{Pure}(\Lambda, \mathbf{k}) \to \mathrm{DLoc}(\mathbf{k}_\Lambda). \tag{5.11}$$

Proposition 5.11 (see [KS90, Cor.7.5.4])**.** *Let $F \in \mathrm{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$. Then the set of $p \in \Lambda$ in a neighborhood of which F is pure (resp. simple) is open and closed in Λ.*

The category $\mathrm{Simple}(\Lambda, \mathbf{k})$ is an important invariant associated with Λ. One shall be aware that it may be empty.

Remark 5.12. Pure sheaves are intensively (and implicitely) used in [?STZ14] in their study of Legendrain knots.

6 The regular Riemann-Hilbert correspondence

Definition 6.1. Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module, Λ its characteristic variety in T^*X and $\mathscr{I}_\Lambda$ the ideal of $\mathrm{Gr}(\mathscr{D}_X)$ of functions vanishing on Λ. We say that $\mathscr{M}$ is *regular* if there exists locally a good filtration on $\mathscr{M}$ such that $\mathscr{I}_\Lambda \cdot \mathrm{Gr}(\mathscr{M}) = 0$.

Theorem 6.2 ([KK81]). *The full subcategory of* $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ *consisting of regular holonomic* $\mathscr{D}_X$*-modules is a thick abelian subcategory.*

Denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ whose objects have regular holonomic cohomologies. It follows from [KK81, Cor. 5.1.11] that $\mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathscr{D}_X)$ is contained in $\mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$.

For holonomic D-modules, the property of being regular holonomic is stable by duality, external product, inverse image and projective direct images (again, see [KK81]).

6.1 Subanalytic topology

In order to prove the Riemann-Hilbert correspondence, Kashiwara introduced the functor $Thom$ of temperate cohomology in[Kas80]. This functor was studied systematically in [KS96], as well as its dual, the functor of Whitney cohomology (that we shall not use here). These two functors appeared later as particular case of a more general construction, namely the subanalytic topology.

References for this subsection are made to [KS01] .

Let M be a real analytic manifold. Denote by Op_M the category of its open subsets, the morphisms being the inclusion and by $\mathrm{Op}_{M_{\mathrm{sa}}}$ the full subcategory of Op_M consisting of subanalytic relatively compact open subsets. The site M_{sa} is obtained by deciding that a family $\{U_i\}_{i\in I}$ of subobjects of $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ is a covering of U if there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j = U$. One calls M_{sa} the subanalytic site associated to M.

Note that

$$\begin{cases} \text{a presheaf } F \text{ on } M_{\mathrm{sa}} \text{ is a sheaf if and only if } F(\emptyset)=0 \text{ and for any} \\ \text{pair } (U_1,U_2) \text{ in } \mathrm{Op}_{M_{\mathrm{sa}}}, \text{ the sequence below is exact:} \\ \qquad 0 \to F(U_1\cup U_2) \to F(U_1)\oplus F(U_2) \to F(U_1\cap U_2). \end{cases} \tag{6.1}$$

Let us denote by

$$\rho_M \colon M \to M_{\mathrm{sa}} \tag{6.2}$$

the natural morphism of sites and, as usual, by $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ the Grothendieck category of sheaves of $\mathbf{k}$-modules on M_{sa}. Hence, (ρ_M^{-1}, ρ_{M*}) is a pair of adjoint functors.

The functor ρ_M^{-1} also admits a left adjoint, denoted by $\rho_{M!}$. For $F \in \mathrm{Mod}(\mathbf{k}_M)$, $\rho_{M!}F$ is the sheaf associated to the presheaf $U \mapsto F(\overline{U})$, $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Hence we have the two pairs of adjoint functors (ρ_M^{-1}, ρ_{M*}) and $(\rho_{M!}, \rho_M^{-1})$

$$\mathrm{Mod}(\mathbf{k}_M) \underset{\rho_{M!}}{\overset{\rho_{M*}}{\rightrightarrows}} \!\!\!\xleftarrow{\ \rho_M^{-1}\ } \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}).$$

The functor ρ_{M*} is fully faithful.

Proposition 6.3. *The restriction of the functor ρ_{M*} to the subcategory $\mathbb{R}\text{-}\mathrm{C}(\mathbf{k}_M)$ is exact and fully faithful.*

Hence, in the sequel, we consider an object of $\mathbb{R}\text{-}\mathrm{C}(\mathbf{k}_M)$ as an object of $\mathrm{Mod}(\mathbf{k}_M)$ as well as an object of $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$.

One denotes by $\mathrm{Op}_{M_{\mathrm{sb}}}$ the category of open subanalytic subsets of M and says that a family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{M_{\mathrm{sb}}}$ is a covering of $U \in \mathrm{Op}_{M_{\mathrm{sb}}}$ if $U_i \subset U$ for all $i \in I$ and, for each compact subset K of M, there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j \cap K = U \cap K$. We denote by M_{sb} the site so-defined. The next result is obvious.

Proposition 6.4. *The morphism of sites $M_{\mathrm{sb}} \to M_{\mathrm{sa}}$ induces an equivalence of categories $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sb}}}) \simeq \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$.*

Now let $f\colon M \to N$ be a morphism of real analytic manifolds. In general, it does not induces a morphism of sites from M_{sa} to N_{sa} but it induces a morphism of sites $M_{\mathrm{sb}} \to N_{\mathrm{sb}}$ and one can define the pair of adjoint functors

$$\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} \mathrm{Mod}(\mathbf{k}_{N_{\mathrm{sa}}}).$$

which induce

$$\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sa}}}) \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{N_{\mathrm{sa}}}).$$

One also define the proper direct image functor $\mathrm{R}f_{!!}$ and its right adjoint $f^!$.

6.2 The sheaf of temperate holomorphic functions

References for this subsection are made to [KS01].

Definition 6.5. Let U be an open subset of M and $f \in \mathscr{C}^\infty(U)$. One says that f has *polynomial growth* at $p \in M$ if f satisfies the following condition: for a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x\in K\cap U} \big(\mathrm{dist}(x, K \setminus U)\big)^N |f(x)| < \infty\,. \tag{6.3}$$

Here, $\mathrm{dist}(x, K \setminus U) := \inf\{|y - x|\,;\, y \in K \setminus U\}$, and we understand that the left-hand side of (6.3) is 0 if $K \cap U = \emptyset$ or $K \setminus U = \emptyset$. Hence f has polynomial growth at any point of U. We say that f is tempered at p if all its derivatives have polynomial growth at p. We say that f is *tempered* if it is tempered at any point of M.

Lemma 6.6. *Let U and V be two relatively compact open subanalytic subsets of $\mathbb{R}^n$. There exist a positive integer N and $C > 0$ such that*

$$\mathrm{dist}\big(x, \mathbb{R}^n \setminus (U \cup V)\big)^N \le C\big(\mathrm{dist}(x, \mathbb{R}^n \setminus U) + \mathrm{dist}(x, \mathbb{R}^n \setminus V)\big).$$

For an open subanalytic subset U in M, denote by $\mathcal{C}^{\infty,\mathrm{t}}_M(U)$ the subspace of $\mathscr{C}^\infty(U)$ consisting of tempered C^∞-functions.

Denote by $\mathcal{D}b^{\mathrm{t}}_M(U)$ the image of the restriction map $\Gamma(M;\mathrm{D}^{\mathrm{b}}_M)\to\Gamma(U;\mathrm{D}^{\mathrm{b}}_M)$, and call it the space of *tempered distributions* on U. Using Lemma 6.6 and (6.1) one proves:

- the presheaf $U\mapsto\mathcal{C}^{\infty,\mathrm{t}}_M(U)$ is a sheaf on M_{sa},
- the presheaf $U\mapsto\mathcal{D}b^{\mathrm{t}}_M(U)$ is a sheaf on M_{sa}.

One denotes these sheaves by $\mathcal{C}^{\infty,\mathrm{t}}_{M_{\mathrm{sa}}}$ and $\mathcal{D}b^{\mathrm{t}}_{M_{\mathrm{sa}}}$, respectively.

Now let X be complex manifold. We denote by X^c the complex conjugate manifold to X and by $X^{\mathbb{R}}$ the underlying real analytic manifold. If there is no risk of confusion, we write X instead of $X^{\mathbb{R}}$. We set

$$\mathscr{D}_{X_{\mathrm{sa}}}=\rho_{X!}\mathscr{D}_X$$

and define similarly $\mathscr{D}_{X_{\mathrm{sa}}\to Y_{\mathrm{sa}}}$.

One defines $\mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}}$ as the Dolbeault complex of $\mathcal{D}b^{\mathrm{t}}_{X^{\mathbb{R}}_{\mathrm{sa}}}$. More precisely, one sets

$$\begin{aligned}\mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}} &:= \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X^c_{\mathrm{sa}}}}(\rho_{X!}\mathscr{O}_{X^c_{\mathrm{sa}}},\mathcal{D}b^{\mathrm{t}}_{X^{\mathbb{R}}_{\mathrm{sa}}})\in\mathrm{D}^{\mathrm{b}}(\mathscr{D}_{X_{\mathrm{sa}}})\\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X^c_{\mathrm{sa}}}}(\rho_{X!}\mathscr{O}_{X^c_{\mathrm{sa}}},\mathcal{C}^{\infty,\mathrm{t}}_{X^{\mathbb{R}}_{\mathrm{sa}}})\text{ in }\mathrm{D}^{\mathrm{b}}(\mathscr{D}_{X_{\mathrm{sa}}}).\end{aligned}$$

We also introduce

$$\Omega^{\mathrm{t}}_{X_{\mathrm{sa}}} := \rho_{X!}\Omega_X\otimes_{\rho_{X!}\mathscr{O}_X}\mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}}.$$

We define the tempered de Rham and solution functors by

$$\begin{aligned}\mathcal{DR}^{\mathrm{t}}_X &: \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)\to\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}}),\mathscr{M}\mapsto\Omega^{\mathrm{t}}_{X_{\mathrm{sa}}}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_{\mathrm{sa}}}}\rho_{X!}\mathscr{M},\\ \mathcal{S}ol^{\mathrm{t}}_X &: \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}}\to\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}}),\mathscr{M}\mapsto\mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\rho_{X!}\mathscr{M},\mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}}).\end{aligned}$$

One has

$$\mathcal{S}ol_X\simeq\rho_X^{-1}\mathcal{S}ol^{\mathrm{t}}_X,\quad \mathcal{DR}_X\simeq\rho_X^{-1}\mathcal{DR}^{\mathrm{t}}_X.$$

Definition 6.7.

$$\mathcal{TH}_X\colon\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbb{C}_X)\to\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X),\quad F\mapsto\rho_X^{-1}\mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(F,\mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}}).\tag{6.4}$$

This functor was already defined in [Kas80] directly, without the use of the subanalytic topology, and denoted *Thom*.

The next result is a reformulation of a theorem of [Kas84] (see also [KS96, Th. 5.7], [KS01, Th. 7.4.1])

Theorem 6.8. *Let $f\colon X\to Y$ be a morphism of complex manifolds. There is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{D}^{\mathrm{op}}_{X_{\mathrm{sa}}})$:*

$$\Omega^{\mathrm{t}}_{X_{\mathrm{sa}}}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_{\mathrm{sa}}}}\mathscr{D}_{X_{\mathrm{sa}}\to Y_{\mathrm{sa}}}[d_X]\xrightarrow{\sim}f^!\Omega^{\mathrm{t}}_{Y_{\mathrm{sa}}}[d_Y].\tag{6.5}$$

Corollary 6.9. *Let $f\colon X \to Y$ be a morphism of complex manifolds, let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbb{C}_X)$ and assume that f is proper on* $\operatorname{supp} F$. *Then*

$$\mathrm{D}f_!\mathrm{R}\mathscr{H}om\,(F, \mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}})\,[d_X] \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(\mathrm{R}f_!F, \mathscr{O}^{\mathrm{t}}_{Y_{\mathrm{sa}}})\,[d_Y].$$

Theorem 6.10 (Tempered Grauert theorem [KS96, Th. 7.3])**.** *Let $f\colon X \to Y$ be a morphism of complex manifolds, let $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ and assume that f is proper on* $\mathrm{Supp}(\mathscr{F})$. *Then there is a natural isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{O}_{Y_{\mathrm{sa}}})$:*

$$\mathrm{R}f_{!!}(\mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}}\overset{\mathrm{L}}{\otimes}_{\rho_{X!}\mathscr{O}_X}\rho_{X!}\mathscr{F}) \simeq \mathscr{O}^{\mathrm{t}}_{Y}\overset{\mathrm{L}}{\otimes}_{\rho_{Y!}\mathscr{O}_Y}\rho_{Y!}\mathrm{R}f_!\mathscr{F}. \tag{6.6}$$

Corollary 6.11 ([KS01, Th. 7.4.6])**.** *Let $f\colon X \to Y$ be a morphism of complex manifolds. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$ and assume that f is proper on* $\mathrm{Supp}(\mathscr{M})$. *Then there is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{Y_{\mathrm{sa}}})$*

$$\mathcal{DR}^{\mathrm{t}}_Y(\mathrm{D}f_*\mathscr{M}) \ \xrightarrow{\sim}\ \mathrm{R}f_*\mathcal{DR}^{\mathrm{t}}_X(\mathscr{M}). \tag{6.7}$$

6.3 The regular Riemann-Hilbert correspondence

References for this subsection are made to [Kas80, Kas84, KK81]. Our presentation essentially follows [KS16].

A precise formulation of the Riemann-Hilbert correspondence was formulated in 1977 by Masaki Kashiwara and a detailed sketch of proof of the theorem appeared in [Kas80] where the functor *Thom* was introduced, a detailed proof appearing in [Kas84]. Note that a different proof, in which the inverse of the functor $\mathcal{S}ol_X$ is not constructed, appeared in [Meb84].

Definition 6.12. Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module, Λ its characteristic variety in T^*X and $\mathscr{I}_\Lambda$ the ideal of $\mathrm{Gr}(\mathscr{D}_X)$ of functions vanishing on Λ. We say that $\mathscr{M}$ is *regular* if there exists locally a good filtration on $\mathscr{M}$ such that $\mathscr{I}_\Lambda \cdot \mathrm{Gr}(\mathscr{M}) = 0$.

Theorem 6.13 ([KK81, Prop. 1.1.17])**.** *The full subcategory of* $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ *consisting of regular holonomic $\mathscr{D}_X$-modules is a thick abelian subcategory.*

Denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ whose objects have regular holonomic cohomologies. It follows from [KK81, Cor. 5.1.11] that $\mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathscr{D}_X)$ is contained in $\mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$.

For holonomic D-modules, the property of being regular holonomic is stable by duality, external product, inverse image and projective direct images.

We shall prove:

Theorem 6.14 (Regular Riemann-Hilbert correspondence [Kas80])**.** *The functor $\mathcal{S}ol_X$ induces an equivalence of categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathscr{D}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X)$.*

Note that the functor is well-defined thanks to Theorem 3.9.

Before entering into the proof, we need some preliminary results.

Definition 6.15. Let X be a complex manifold and $D \subset X$ a normal crossing divisor. We say that a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ has *regular normal form* along D if locally on D, for a local coordinate system $(z_1, \dots, z_n)$ on X such that $D = \{z_1 \cdots z_r = 0\}$, $\mathscr{M} \simeq \mathscr{D}_X/\mathscr{I}_\lambda$ for $\lambda = (\lambda_1, \dots, \lambda_r) \in (\mathbb{C} \setminus \mathbb{Z}_{\geq 0})^r$. Here, $\mathscr{I}_\lambda$ is the left ideal generated by the operators $(z_i\partial_i - \lambda_i)$ and ∂_j for $i \in \{1, \dots, r\}$, $j \in \{r+1, \dots, n\}$.

Clearly, a holonomic module with regular normal form is regular holonomic.

The next lemma is an essential tool in the study of regular holonomic D-modules. Its proof uses Hironaka's desingularization theorem and the results of [Del70, KK81].

Lemma 6.16. *Let $P_X(\mathscr{M})$ be a statement concerning a complex manifold X and a regular holonomic object $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathscr{D}_X)$. Consider the following conditions.*

(a) *Let $X = \bigcup_{i \in I} U_i$ be an open covering. Then $P_X(\mathscr{M})$ is true if and only if $P_{U_i}(\mathscr{M}|_{U_i})$ is true for any $i \in I$.*

(b) *If $P_X(\mathscr{M})$ is true, then $P_X(\mathscr{M}[n])$ is true for any $n \in \mathbb{Z}$.*

(c) *Let $\mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \xrightarrow{+1}$ be a distinguished triangle in $\mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathscr{D}_X)$. If $P_X(\mathscr{M}')$ and $P_X(\mathscr{M}'')$ are true, then $P_X(\mathscr{M})$ is true.*

(d) *Let $\mathscr{M}$ and $\mathscr{M}'$ be regular holonomic $\mathscr{D}_X$-modules. If $P_X(\mathscr{M} \oplus \mathscr{M}')$ is true, then $P_X(\mathscr{M})$ is true.*

(e) *Let $f\colon X \to Y$ be a projective morphism and let $\mathscr{M}$ be a good regular holonomic $\mathscr{D}_X$-module. If $P_X(\mathscr{M})$ is true, then $P_Y(\mathrm{D}f_*\mathscr{M})$ is true.*

(f) *If $\mathscr{M}$ is a regular holonomic $\mathscr{D}_X$-module with a regular normal form along a normal crossing divisor of X, then $P_X(\mathscr{M})$ is true.*

If conditions (a)–(f) *are satisfied, then $P_X(\mathscr{M})$ is true for any complex manifold X and any $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathscr{D}_X)$.*

Theorem 6.17. *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathscr{D}_X)$. Then there are isomorphisms:*

$$\mathcal{DR}^{\mathrm{t}}_X(\mathscr{M}) \xrightarrow{\sim} \mathcal{DR}_X(\mathscr{M}) \text{ in } \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}}), \tag{6.8}$$

$$\mathcal{S}ol^{\mathrm{t}}_X(\mathscr{M}) \xrightarrow{\sim} \mathcal{S}ol_X(\mathscr{M}) \text{ in } \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}}). \tag{6.9}$$

Sketch of proof. (i) By duality, it is enough to prove the first isomorphism.

(ii) We shall apply Lemma 6.16. Denote by $P_X(\mathscr{M})$ the statement which asserts that the morphism in (6.8) is an isomorphism.

(a)–(d) of this lemma are clearly satisfied.

(e) follows from isomorphism (6.7) in Corollary 6.11 and its non-tempered version, isomorphism (3.11) in Theorem 3.8.

(f) Let us check property (f). Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module with regular normal form along a normal crossing divisor D. After a blow up of the divisor, one reduces to the case where $\mathscr{M} = \mathscr{O}_X$. Q.E.D.

By using a similar method one also proves (see [KS16, Th. 4.3.2]):

Theorem 6.18. *There is an isomorphism functorial in* $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathscr{D}_X)$

$$\mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}} \overset{\mathrm{D}}{\otimes} \mathscr{M} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\mathcal{S}ol_X(\mathscr{M}), \mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}}) \text{ in } \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{X_{\mathrm{sa}}}). \tag{6.10}$$

There is an analogue of Lemma 6.16 for constructible sheaves, whose proof simply uses Hironaka's desingularization theorem, namely:

Lemma 6.19. *Let* $P_X(F)$ *be a statement concerning a complex manifold* X *and an object* $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X)$. *Consider the following conditions.*

(a) *Let* $X = \bigcup_{i\in I} U_i$ *be an open covering. Then* $P_X(F)$ *is true if and only if* $P_{U_i}(F|_{U_i})$ *is true for any* $i \in I$.

(b) *If* $P_X(F)$ *is true, then* $P_X(F[n])$ *is true for any* $n \in \mathbb{Z}$.

(c) *Let* $F' \to F \to F'' \xrightarrow{+1}$ *be a distinguished triangle in* $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X)$. *If* $P_X(F')$ *and* $P_X(F'')$ *are true, then* $P_X(F)$ *is true.*

(d) *Let* F *and* F' *be objects of* $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X)$. *If* $P_X(F \oplus F')$ *is true, then* $P_X(F)$ *is true.*

(e) *Let* $f\colon X \to Y$ *be a projective morphism and let* $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X)$. *If* $P_X(F)$ *is true, then* $P_Y(\mathrm{R}f_*F)$ *is true.*

(f) *If* F *is a local system in the complement of a normal crossing divisor of* X, *then* $P_X(F)$ *is true.*

If conditions (a)–(f) *are satisfied, then* $P_X(F)$ *is true for any complex manifold* X *and any* $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X)$.

Theorem 6.20. *The functor* $\mathcal{TH}_X$ *in Definition* (6.7) *induces a functor*

$$\mathcal{RH}_X\colon \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathscr{D}_X). \tag{6.11}$$

Proof. We shall apply Lemma 6.19.
(i) Properties (a)–(d) of this lemma are clearly satisfied.
(ii) Property (e) follows from Corollary 6.9.
(iii) Finally, property (f) is proved as follows. First assume that Y is a normal crossing divisor in X and $F = \mathbb{C}_{X\setminus Y}$. The problem being local, we choose a local coordinate system $x = (x_1, \dots, x_n)$ such that $Y = \bigcup_{1\leq j\leq n} Y_j$ and $Y_j = \{x_j = 0\}$. In this case, one proves by a direct calculation that $\mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\mathbb{C}_{X\setminus Y}, \mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}}) \simeq \mathscr{O}_X(*Y)$, the sheaf of meromorphic functions with poles on Y. To treat the general case, denote by $\exp(2\pi\sqrt{-1}A_j$ the monodromy around Y_j for $(N \times N)$-matrices such that $[A_j, A_k] = 0$ for $1 \leq k \leq j$. Then $x^A\mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(F, \mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}})$ is isomorphic to $\mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\mathbb{C}_{X\setminus Y}, \mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}})$. (See [Kas84, § 7] for details.) Q.E.D.

Sketch of proof of Theorem 6.14. (i) Applying the functor ρ_X^{-1} to the isomorphism (6.10), we get for $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathscr{D}_X)$

$$\mathcal{RH}_X \circ \mathcal{S}ol_X(\mathscr{M}) \simeq \mathscr{M}. \tag{6.12}$$

Hence, the functor $\mathcal{S}ol_X$ is essentially surjective.
(ii) It remains to prove that the functor $\mathcal{S}ol_X$ is fully faithful. Let $\mathscr{M}, \mathscr{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$. One has

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(\mathcal{S}ol_X(\mathscr{N}), \mathrm{Sol}_X(\mathscr{M})) &\simeq \mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(\mathcal{S}ol_X(\mathscr{N}), \mathrm{R}\mathscr{H}om_{\rho_{X!}\mathscr{D}_X}(\rho_{X!}\mathscr{M}, \mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}})) \\
&\simeq \mathrm{R}\mathscr{H}om_{\rho_{X!}\mathscr{D}_X}(\rho_{X!}\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\mathcal{S}ol_X(\mathscr{N}), \mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}}) \\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \rho_X^{-1}\mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\mathcal{S}ol_X(\mathscr{N}), \mathscr{O}^{\mathrm{t}}_{X_{\mathrm{sa}}}) \\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}).
\end{aligned}$$

Q.E.D.

Remark 6.21. We have not only proved that the functor $\mathcal{S}ol_X$ is an equivalence, we also have constructed an inverse to this functor, namely the functor $\mathcal{RH}_X$.

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