An introduction to Algebra and Topology

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Introduction

These Notes are an elementary introduction to the language of categories and sheaves.

In **Chapter 1** we recall some basic notions of linear algebra over a ring, putting the emphasis on the operations: kernels and cokernels, products and direct sums, Hom and tens, projective and inductive limits. We also study with some details the Koszul complexes in this framework.

Chapter 2 is a very sketchy introduction to the language of categories and functors, including the notion of derived functors. Many examples are treated, in particular in relation with the categories **Set** of sets and $\mathrm{Mod}(A)$ of A-modules.

In **Chapter 3**, we study abelian sheaves on topological spaces. We define the cohomology of sheaves by using the derived functors of the functor of global sections and show how to calculate this cohomology in some situations, in particular on real or complex manifolds with the help of the De Rham and Dolbeault complexes.

Chapter 1

Linear algebra over a ring

We start by recalling some basic notions on sets and on modules over a (non necessarily commutative) ring.

1.1 Sets and maps

The aim of this section is to fix some notations and to recall some elementary constructions on sets.

If $f\colon X \to Y$ is a map from a set X to a set Y, we shall often say that f is a morphism from X to Y. If f is bijective we shall say that f is an isomorphism and write $f\colon X \xrightarrow{\sim} Y$. If there exists an isomorphism $f\colon X \xrightarrow{\sim} Y$, we say that X and Y are isomorphic and write $X \simeq Y$.

We shall denote by $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$, or simply $\mathrm{Hom}\,(X, Y)$, the set of all maps from X to Y. If $g\colon Y \to Z$ is another map, we can define the composition $g \circ f\colon X \to Z$. Hence, we get two maps:

$$\begin{aligned} g\circ &\colon \mathrm{Hom}\,(X, Y) \to \mathrm{Hom}\,(X, Z), \\ \circ f &\colon \mathrm{Hom}\,(Y, Z) \to \mathrm{Hom}\,(X, Z). \end{aligned}$$

Notice that if $X = \{x\}$ and $Y = \{y\}$ are two sets with one element each, then there exists a unique isomorphism $X \xrightarrow{\sim} Y$. Of course, if X and Y are finite sets with the same cardinal $\pi > 1$, X and Y are still isomorphic, but the isomorphism is no more unique.

In the sequel we shall denote by $\emptyset$ the empty set and by $\{\mathrm{pt}\}$ a set with one element. Note that for any set X, there is a unique map $\emptyset \to X$ and a unique map $X \to \{\mathrm{pt}\}$.

Let $\{X_i\}_{i\in I}$ be a family of sets indexed by a set I. The product of the

X_i's, denoted $\prod_{i\in I} X_i$, or simply $\prod_i X_i$, is the defined as

$$\prod_i X_i = \{\{x_i\}_{i\in I}; x_i \in X_i \text{ for all } i \in I\}. \tag{1.1}$$

If $I = \{1, 2\}$ one uses the notation $X_1 \times X_2$. If $X_i = X$ for all $i \in I$, one uses the notation X^I. Note that

$$\mathrm{Hom}\,(I, X) \simeq X^I. \tag{1.2}$$

For any set Y, there is a natural isomorphism

$$\mathrm{Hom}\,(Y, \prod_i X_i) \xrightarrow{\sim} \prod_i \mathrm{Hom}\,(Y, X_i). \tag{1.3}$$

For three sets I, X, Y, there are natural isomorphisms

$$\begin{aligned} \mathrm{Hom}\,(I \times X, Y) &\simeq \mathrm{Hom}\,(I, \mathrm{Hom}\,(X, Y)) \\ &\simeq \mathrm{Hom}\,(X, Y)^I. \end{aligned} \tag{1.4}$$

If $\{X_i\}_{i\in I}$ is a family of sets indexed by a set I, one may also consider their disjoint union, also called their coproduct. The coproduct of the X_i's is denoted $\coprod_{i\in I} X_i$ or $\bigsqcup_{i\in I} X_i$ or simply $\bigsqcup_i X_i$. If $I = \{1, 2\}$ one uses the notation $X_1 \sqcup X_2$. If $X_i = X$ for all $i \in I$, one uses the notation $X^{(I)}$. Note that

$$X \times I \simeq X^{(I)}. \tag{1.5}$$

For any set Y, there is a natural isomorphism

$$\mathrm{Hom}\,(\coprod_i X_i, Y) \xrightarrow{\sim} \prod_i \mathrm{Hom}\,(X_i, Y). \tag{1.6}$$

Consider two sets X and Y and two maps f, g from X to Y. We write for short $f, g \colon X \rightrightarrows Y$. The kernel (or equalizer) of (f, g), denoted $\mathrm{Ker}(f, g)$, is defined as

$$\mathrm{Ker}(f, g) = \{x \in X; f(x) = g(x)\}. \tag{1.7}$$

Note that for a set Z, one has

$$\mathrm{Hom}\,(Z, \mathrm{Ker}(f, g)) \simeq \mathrm{Ker}(\mathrm{Hom}\,(Z, X) \rightrightarrows \mathrm{Hom}\,(Z, Y)). \tag{1.8}$$

Let us recall a few elementary definitions.

- A relation $\mathcal{R}$ on a set X is a subset of $X \times X$. One writes $x\mathcal{R}y$ if $(x, y) \in \mathcal{R}$.
- The opposite relation $\mathcal{R}^{\mathrm{op}}$ is defined by $x\mathcal{R}^{\mathrm{op}}y$ if and only if $y\mathcal{R}x$.
- A relation $\mathcal{R}$ is reflexive if it contains the diagonal, that is, $x\mathcal{R}x$ for all $x \in X$.
- A relation $\mathcal{R}$ is symmetric if $x\mathcal{R}y$ implies $y\mathcal{R}x$.
- A relation $\mathcal{R}$ is anti-symmetric if $x\mathcal{R}y$ and $y\mathcal{R}x$ implies $x = y$.
- A relation $\mathcal{R}$ is transitive if $x\mathcal{R}y$ and $y\mathcal{R}z$ implies $x\mathcal{R}z$.
- A relation $\mathcal{R}$ is an equivalence relation if it is reflexive, symmetric and transitive.
- A relation $\mathcal{R}$ is a pre-order if it is reflexive and transitive. If moreover it is anti-symmetric, then one says that $\mathcal{R}$ is an order on X. A pre-order is often denoted $\leq$. A set endowed with a pre-order is called a poset.
- Let $(I, \leq)$ be a poset. One says that $(I, \leq)$ is filtrant (one also says "directed") if I is non empty and for any $i, j \in I$ there exists k with $i \leq k$ and $j \leq k$.
- Assume $(I, \leq)$ is a filtrant poset and let $J \subset I$ be a subset. One says that J is cofinal to I if for any $i \in I$ there exists $j \in J$ with $i \leq j$.

If $\mathcal{R}$ is a relation on a set X, there is a smaller equivalence relation which contains $\mathcal{R}$. (Take the intersection of all subsets of $X \times X$ which contain $\mathcal{R}$ and which are equivalence relations.)

Let $\mathcal{R}$ be an equivalence relation on a set X. A subset S of X is saturated if $x \in S$ and $x\mathcal{R}y$ implies $y \in S$. A subset S of X is an equivalence class of $\mathcal{R}$ if it is saturated, non empty, and $x, y \in S$ implies $x\mathcal{R}y$. One then defines a new set $X/\mathcal{R}$ and a canonical map $f\colon X \to X/\mathcal{R}$ as follows: the elements of $X/\mathcal{R}$ are the equivalence classes of $\mathcal{R}$ and the map f associates to $x \in X$ the unique equivalence class S such that $x \in S$.

1.2 Modules and linear maps

All along these Notes, a ring A means an associative and unital ring, but A is not necessarily commutative.

We will demote by $\mathbf{k}$ a commutative ring. Recall that a $\mathbf{k}$-algebra A is a ring endowed with a morphism of rings $\varphi\colon \mathbf{k} \to A$ such that the image of $\mathbf{k}$ is contained in the center of A (*i.e.*, $\varphi(x)a = a\varphi(x)$ for any $x \in \mathbf{k}$ and $a \in A$). Notice that a ring A is always a $\mathbb{Z}$-algebra. If A is commutative, then A is an A-algebra.

Since we do not assume A is commutative, we have to distinguish between left and right structures. Unless otherwise specified, a module M over A means a left A-module.

Let $a \in A$. We denote by $a\cdot$ the left action of a on A and by $\cdot a$ the right action.

Recall that an A-module M is an additive group (whose operations and zero element are denoted $+, 0$) endowed with an external law $A \times M \to M$ (denoted $(a, m) \mapsto a \cdot m$ or simply $(a, m) \mapsto am$) satisfying:

$$\begin{cases} (ab)m = a(bm) \\ (a+b)m = am + bm \\ a(m+m') = am + am' \\ 1 \cdot m = m \end{cases}$$

where $a, b \in A$ and $m, m' \in M$.

Note that M inherits a structure of a $\mathbf{k}$-module via φ. In the sequel, if there is no risk of confusion, we shall not write φ.

We denote by A^{op} the ring A with the opposite structure. Hence the product ab in A^{op} is the product ba in A and an A^{op}-module is a right A-module.

Note that if the ring A is a field (here, a field is always commutative), then an A-module is nothing but a vector space.

Examples 1.2.1. (i) The first example of a ring is $\mathbb{Z}$, the ring of integers. Since a field is a ring, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are rings. If A is a commutative ring, then $A[x_1, \dots, x_n]$, the ring of polynomials in n variables with coefficients in A, is also a commutative ring. It is a sub-ring of $A[[x_1, \dots, x_n]]$, the ring of formal powers series with coefficients in A.
(ii) Let $\mathbf{k}$ be a field. Then for $n > 1$, the ring $M_n(\mathbf{k})$ of square matrices of rank n with entries in $\mathbf{k}$ is non commutative.
(iii) Let $\mathbf{k}$ be a field. The *Weyl algebra* in n variables, denoted $W_n(\mathbf{k})$, is the non commutative ring of polynomials in the variables x_i, ∂_j $(1 \leq i, j \leq n)$ with coefficients in $\mathbf{k}$ and relations :

$$[x_i, x_j] = 0,\ [\partial_i, \partial_j] = 0,\ [\partial_j, x_i] = \delta^i_j$$

where $[p, q] = pq - qp$ and δ^i_j is the Kronecker symbol.

The Weyl algebra $W_n(\mathbf{k})$ may be regarded as the ring of differential operators with coefficients in $\mathbf{k}[x_1,\dots,x_n]$, and $\mathbf{k}[x_1,\dots,x_n]$ becomes a left $W_n(\mathbf{k})$-module: x_i acts by multiplication and ∂_i is the derivation with respect to x_i. Indeed, an element $P(x,\partial)$ of $W_n(\mathbf{k})$ may be written uniquely as a polynomial in $\partial_1,\cdots,\partial_n$ with coefficients in $\mathbf{k}[x_1,\dots,x_n]$:

$$P(x,\partial)=\sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha.$$

Here $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $a_\alpha(x)\in\mathbf{k}[x_1,\dots,x_n]$ and $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$.

A morphism $f\colon M\to N$ of A-modules is an A-linear map, *i.e.,* f satisfies:

$$\begin{cases} f(m+m')=f(m)+f(m') & m,m'\in M\\ f(am)=af(m) & m\in M, a\in A.\end{cases}$$

A morphism f is an isomorphism if there exists a morphism $g:N\to M$ with $f\circ g=\mathrm{id}_N, g\circ f=\mathrm{id}_M$.

If f is bijective, it is easily checked that the inverse map $f^{-1}:N\to M$ is itself A-linear. Hence f is an isomorphism if and only if f is A-linear and bijective.

A submodule N of M is a subset N of M such that $n,n'\in N$ implies $n+n'\in N$ and $n\in N, a\in A$ implies $an\in N$. A submodule of the A-module A is called an ideal of A. Note that if A is a field, it has no non trivial ideal, *i.e.,* its only ideals are $\{0\}$ and A. If $A=\mathbb{C}[x]$, then $I=\{P\in\mathbb{C}[x]; P(0)=0\}$ is a non trivial ideal.

If N is a submodule of M, it defines an equivalence relation $m\mathcal{R}m'$ if and only if $m-m'\in N$. One easily checks that the quotient set $M/\mathcal{R}$ is naturally endowed with a structure of a left A-module. This module is called the quotient module and is denoted M/N.

Let $f\colon M\to N$ be a morphism of A-modules. One sets:

$$\begin{aligned}\operatorname{Ker} f &= \{m\in M;\quad f(m)=0\}\\ \operatorname{Im} f &= \{n\in N;\quad \text{there exists } m\in M,\quad f(m)=n\}.\end{aligned}$$

These are submodules of M and N respectively, called the kernel and the image of f, respectively. One also introduces the cokernel and the coimage of f:

$$\operatorname{Coker} f = N/\operatorname{Im} f,\quad \operatorname{Coim} f = M/\operatorname{Ker} f.$$

Note that the natural morphism $\operatorname{Coim} f\to\operatorname{Im} f$ is an isomorphism.

A family of elements $\{m_i\}_{i\in I}$ of an A-module M is a system of generators of M if any $m \in M$ may be written as a *finite* sum $m = \sum_{i\in I} a_i m_i$ with $a_i \in A$. One says that M is of finite type, or is finitely generated, if it admits a finite system of generators. This is equivalent to saying that there exists s surjective linear map $A^{N_0} \twoheadrightarrow M$, for some $N_0 \in \mathbb{N}$. If a system of generators consists of a single element $\{m\}$, then one says that m is a generator of M.

Example 1.2.2. Let $W_n(\mathbf{k})$ denote as above the Weyl algebra. Consider the left $W_n(\mathbf{k})$-linear map $W_n(\mathbf{k}) \to \mathbf{k}[x_1, \dots, x_n]$, $W_n(\mathbf{k}) \ni P \mapsto P(1) \in \mathbf{k}[x_1, \dots, x_n]$. This map is clearly surjective and its kernel is the left ideal generated by $(\partial_1, \cdots, \partial_n)$. Hence, one has the isomorphism of left $W_n(\mathbf{k})$-modules:

$$W_n(\mathbf{k})/\sum_j W_n(\mathbf{k})\partial_j \xrightarrow{\sim} \mathbf{k}[x_1, \dots, x_n]. \tag{1.9}$$

Of course, the polynomial 1 is a generator of the $W_n(\mathbf{k})$-module $\mathbf{k}[x_1, \dots, x_n]$, but one easily check that if $\mathbf{k}$ has characteristic 0, then any non-zero polynomial $P(x)$ is a generator of $\mathbf{k}[x_1, \dots, x_n]$.

1.3 Operations on modules

Linear maps

Let M and N be two A-modules. Recall that an A-linear map $f\colon M \to N$ is also called a morphism of A-modules. One denotes by $\operatorname{Hom}_A(M, N)$ the set of A-linear maps $f\colon M \to N$. This is clearly a $\mathbf{k}$-module. In fact one defines the action of $\mathbf{k}$ on $\operatorname{Hom}_A(M, N)$ by setting: $(\lambda f)(m) = \lambda(f(m))$. Hence $(\lambda f)(am) = \lambda f(am) = \lambda a f(m) = a\lambda f(m) = a(\lambda f(m))$, and $\lambda f \in \operatorname{Hom}_A(M, N)$.

There is a natural isomorphism $\operatorname{Hom}_A(A, M) \simeq M$: to $u \in \operatorname{Hom}_A(A, M)$ one associates $u(1)$ and to $m \in M$ one associates the linear map $A \to M, a \mapsto am$. More generally, if I is an ideal of A then $\operatorname{Hom}_A(A/I, M) \simeq \{m \in M; Im = 0\}$.

Note that if A is a $\mathbf{k}$-algebra and $L \in \operatorname{Mod}(\mathbf{k})$, $M \in \operatorname{Mod}(A)$, the $\mathbf{k}$-module $\operatorname{Hom}_{\mathbf{k}}(L, M)$ is naturally endowed with a structure of a left A-module. If N is a right A-module, then $\operatorname{Hom}_{\mathbf{k}}(N, L)$ becomes a left A-module.

Products and direct sums

Let I be a set, and let $\{M_i\}_{i\in I}$ be a family of A-modules indexed by I. The set $\prod_i M_i$ is naturally endowed with a structure of a left A-module by setting

$$\{m_i\}_i + \{m'_i\}_i = \{m_i + m'_i\}_i,$$
$$a \cdot \{m_i\}_i = \{a \cdot m_i\}_i.$$

For each $j \in I$ there is a natural linear map $\pi_j \colon \prod_i M_i \to M_j$, called the j-th projection. It is given by $\{m_i\}_{i\in I} \mapsto m_j$.

The direct sum $\bigoplus_i M_i$ is the submodule of $\prod_i M_i$ whose elements are the $\{m_i\}_i$'s such that $m_i = 0$ for all but a finite number of $i \in I$. In particular, if the set I is finite, the natural injection $\bigoplus_i M_i \to \prod_i M_i$ is an isomorphism. For each $j \in I$ there is a natural linear map $\sigma_j \colon M_j \to \bigoplus_i M_i$. It is given by $m_j \mapsto \{m_i\}_{i\in I}$, where $m_i = 0$ for $i \neq j$.

Tensor product

Consider a right A-module N, a left A-module M and a $\mathbf{k}$-module L. Let us say that a map $f \colon N \times M \to L$ is $(A, \mathbf{k})$-bilinear if f is additive with respect to each of its arguments and satisfies $f(na, m) = f(n, am)$ and $f(n\lambda, m) = \lambda(f(n, m))$ for all $(n, m) \in N \times M$ and $a \in A, \lambda \in \mathbf{k}$.

Let us identify a set I to a subset of $\mathbf{k}^{(I)}$ as follows: to $i \in I$, we associate $\{l_j\}_{j\in I} \in \mathbf{k}^{(I)}$ given by

$$l_j = \begin{cases} 1 \text{ if } & j = i, \\ 0 \text{ if } & j \neq i. \end{cases} \tag{1.10}$$

The tensor product $N \otimes_A M$ is the $\mathbf{k}$-module defined as the quotient of $\mathbf{k}^{(N\times M)}$ by the submodule generated by the following elements (where $n, n' \in N, m, m' \in M, a \in A, \lambda \in \mathbf{k}$ and $N \times M$ is identified to a subset of $\mathbf{k}^{(N\times M)}$):

$$\begin{cases} (n + n', m) - (n, m) - (n', m) \\ (n, m + m') - (n, m) - (n, m') \\ (na, m) - (n, am) \\ \lambda(n, m) - (n\lambda, m). \end{cases}$$

The image of (n, m) in $N \otimes_A M$ is denoted $n \otimes m$. Hence an element of $N \otimes_A M$ may be written (not uniquely!) as a finite sum $\sum_j n_j \otimes m_j$, $n_j \in N$, $m_j \in M$ and:

$$\begin{cases} (n + n') \otimes m = n \otimes m + n' \otimes m \\ n \otimes (m + m') = n \otimes m + n \otimes m' \\ na \otimes m = n \otimes am \\ \lambda(n \otimes m) = n\lambda \otimes m = n \otimes \lambda m. \end{cases}$$

Denote by $\beta\colon N \times M \to N \otimes_A M$ the natural map which associates $n \otimes m$ to (n, m).

Proposition 1.3.1. *The map β is $(A, \mathbf{k})$-bilinear and for any $\mathbf{k}$-module L and any $(A, \mathbf{k})$-bilinear map $f\colon N \times M \to L$, the map f factorizes uniquely through a $\mathbf{k}$-linear map $\varphi\colon N \otimes_A M \to L$.*

The proof is left to the reader.

Proposition 1.3.1 is visualized by the diagram:

$$\begin{array}{ccc} N \times M & \xrightarrow{\beta} & N \otimes_A M \\ & \searrow^{f} & \downarrow{\scriptstyle \varphi} \\ & & L. \end{array}$$

Consider an A-linear map $f\colon M \to L$. It defines a linear map $\mathrm{id}_N \times f\colon N \times M \to N \times L$, hence a $(A, \mathbf{k})$-bilinear map $N \times M \to N \otimes_A L$, and finally a $\mathbf{k}$-linear map

$$\mathrm{id}_N \otimes f\colon N \otimes_A M \to N \otimes_A L.$$

One constructs similarly $g \otimes \mathrm{id}_M$ associated to $g\colon N \to L$.

There is are natural isomorphisms $A \otimes_A M \simeq M$ and $N \otimes_A A \simeq N$.

Denote by $\mathrm{Bil(N{\times}M, L)}$ the $\mathbf{k}$-module of $(A, \mathbf{k})$-bilinear maps from $N \times M$ to L. One has the isomorphisms

$$\begin{aligned} \mathrm{Bil(N \times M, L)} &\simeq \mathrm{Hom}_{\mathbf{k}}(N \otimes_A M, L) \\ &\simeq \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbf{k}}(N, L)) \\ &\simeq \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{k}}(M, L)). \end{aligned} \tag{1.11}$$

For $L \in \mathrm{Mod}(\mathbf{k})$ and $M \in \mathrm{Mod}(A)$, the $\mathbf{k}$-module $L \otimes_{\mathbf{k}} M$ is naturally endowed with a structure of a left A-module. For $M, N \in \mathrm{Mod}(A)$ and $L \in \mathrm{Mod}(\mathbf{k})$, we have the isomorphisms (whose verification is left to the reader):

$$\begin{aligned} \mathrm{Hom}_A(L \otimes_{\mathbf{k}} N, M) &\simeq \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{k}}(L, M)) \\ &\simeq \mathrm{Hom}_{\mathbf{k}}(L, \mathrm{Hom}_A(N, M)). \end{aligned} \tag{1.12}$$

If A is commutative, there is an isomorphism: $N \otimes_A M \simeq M \otimes_A N$ given by $n \otimes m \mapsto m \otimes n$. Moreover, the tensor product is associative, that is, if L, M, N are A-modules, there are natural isomorphisms $L \otimes_A (M \otimes_A N) \simeq (L \otimes_A M) \otimes_A N$. One simply writes $L \otimes_A M \otimes_A N$.

Inductive and projective limits

We shall study inductive and projective limits in a very special situation, sufficient for our purpose.

Definition 1.3.2. Let I be an poset. A projective system β indexed by I with values in $\mathrm{Mod}(A)$, denoted $\beta\colon I^{\mathrm{op}} \to \mathrm{Mod}(A)$, is the data

$$\begin{cases} \text{for any } i \in I \text{ of an } A\text{-module } M_i, \\ \text{for any pair } i \leq j \text{ of an } A\text{-linear map } v_{ij}\colon M_j \to M_i \\ \text{these data satisfying} \\ v_{ii} = \mathrm{id}_{M_i} \text{ for any } i \in I \text{ and } v_{ij} \circ v_{jk} = v_{ik} \text{ for any } i \leq j \leq k. \end{cases}$$

The projective limit of β, denoted $\varprojlim_i M_i$ (or simply $\varprojlim M_i$ if there is no risk of confusion) is the A-module given by:

$$\varprojlim_i M_i = \{x = \{x_i\}_{i\in I} \in \prod_i M_i; u_{ij}(x_j) = x_i \text{ for any } i \leq j\}.$$

Hence, $\varprojlim M_i$ is a submodule of $\prod_i M_i$ and there are natural linear maps $\pi_j\colon \varprojlim_i M_i \to M_j$.

Definition 1.3.3. Let I be a poset. An inductive system α indexed by I with values in $\mathrm{Mod}(A)$, denoted $\alpha\colon I \to \mathrm{Mod}(A)$, is the data

$$\begin{cases} \text{for any } i \in I \text{ of an } A\text{-module } M_i, \\ \text{for any pair } i \leq j \text{ of an } A\text{-linear map } u_{ji}\colon M_i \to M_j \\ \text{these data satisfying} \\ u_{ii} = \mathrm{id}_{M_i} \text{ for any } i \in I \text{ and } u_{kj} \circ u_{ji} = u_{ki} \text{ for any } i \leq j \leq k. \end{cases}$$

Now we assume that I is filtrant. One defines the inducive limit of α, denoted $\varinjlim_i M_i$ (or $\varinjlim M_i$ if there is no risk of confusion), as follows. Consider the submodule N of $\bigoplus_{i\in I} M_i$ given by:

$$N = \{\sum_{j\in J} x_j,\, x_j \in M_j,\ J \text{ finite; there exists } k \geq J \text{ with } \textstyle\sum_{j\in J} u_{kj}(x_j) = 0\}.$$

(Here, we identify M_j to a submodule of $\bigoplus_{i\in I} M_i$, in other words, we do not write the symbols σ_j.) Then

$$\varinjlim_i M_i = \bigoplus_{i\in I} M_i/N.$$

Hence, $\varinjlim M_i$ is a quotient module of $\bigoplus_i M_i$ and there are natural linear maps $\sigma_j \colon M_j \to \varinjlim_i M_i$.

The filtrant inductive limit $\varinjlim_i M_i$ (together with the maps σ_j, $j \in I$ is characterized by the two properties

- if $x \in M_j$ and $\sigma_j(x) = 0$, then there exists $k \geq j$ such that $u_{kj}(x) = 0$,
- for any $y \in \varinjlim_i M_i$ there exists $j \in I$ and $x \in M_j$ such that $y = \sigma_j(x)$.

Consider the set $\bigsqcup_i M_i$ and the relation on this set $M_i \ni x_i \sim x_j \in M_j$ if there exists $k \in I$, $k \geq i$, $k \geq j$ and $u_{ki}(x_i) = u_{kj}(x_j)$. It follows easily from the fact that I is filtrant that $\sim$ is an equivalence relation and one checks that

$$\varinjlim_i M_i \simeq \bigsqcup_i M_i / \sim .$$

Example 1.3.4. Assume that for any $i \leq j$, the map $u_{ji} \colon M_i \to M_j$ is injective, Then, identifying M_i to a submodule of M_j by this map, we have $\varinjlim_i M_i \simeq \bigcup_i M_i$.

The next result is obvious.

Proposition 1.3.5. *Let I be a filtrant poset and let $J \subset I$ be a cofinal subset. Then the natural linear map $\varinjlim_{j\in J} M_j \to \varinjlim_{i \in I} M_i$ is an isomorphism.*

Example 1.3.6. Denote by $\mathbf{k}[x]^{\leq n}$ the submodule of $\mathbf{k}[x]$ consisting of polynomials of degree $\leq n$.

(a) For $i \leq j$, denote by $u_{ji} \colon \mathbf{k}[x]^{\leq i} \to \mathbf{k}[x]^{\leq j}$ the canonical injection. Then

$$\varinjlim_n \mathbf{k}[x]^{\leq n} \xrightarrow{\sim} \mathbf{k}[x].$$

(b) For $i \leq j$, denote by $v_{ij} \colon \mathbf{k}[x]^{\leq j} \to \mathbf{k}[x]^{\leq i}$ the canonical projection. Then

$$\mathbf{k}[[x]] \xrightarrow{\sim} \varprojlim_n \mathbf{k}[x]^{\leq n}.$$

(Recall that $\mathbf{k}[[x]]$ denotes the module of formal series with coefficients in $\mathbf{k}$.)

Example 1.3.7. Let X be a topological space and denote by $C^0(X)$ the $\mathbb{C}$-vector space of $\mathbb{C}$-valued continuous functions on X. Let X_n be an increasing sequence of open subsets of X satisfying $\bigcup_n X_n = X$. For $p \geq n$ we define the linear map $v_{np} \colon C^0(X_p) \to C^0(X_n)$ as the restriction map which, to a continuous function defined on X_p, associates its restriction to X_n. Then

$$C^0(X) \xrightarrow{\sim} \varprojlim_n C^0(X_n).$$

Example 1.3.8. Let X be a topological space and let Z be a closed subset. Consider the poset $(J, \leq)$ of open neighborhoods of Z, ordered by inclusion. Let $(I, \leq) := (J, \leq^{\mathrm{op}})$, the set J with the opposite order. Since $U, V \in I$ implies $U \cap V \in I$, the poset $(I, \leq)$ is filtrant. One sets

$$C^0_X(Z) = \varinjlim_{U \in I} C^0(U),$$

where the map $C^0(U) \to C^0(V)$ ($U \leq V$ in I, that is, $V \subset U$) is again the restriction map. One calls an element of $C^0_X(Z)$ a germ of continuous function on Z. Hence, a germ of continuous function on Z is represented by a pair (U, f) where U is an open neighborhood of Z and $f \in C^0(U)$, with the relation that (U, f) and (V, g) define the same germ on Z if there exists an open neighborhood W of Z with $W \subset U \cap V$ and $f|_W = g|_W$.

This example is particularly important when $Z = \{x\}$ for some $x \in X$. It gives the notion of the germ of a function at a point $x \in X$.

1.4 Complexes and cohomology

Complexes

Definition 1.4.1. (a) A complex of A-modules $(M^\bullet, d^\bullet)$ is a sequence of A-modules $\{M^n\}_{n\in\mathbb{Z}}$ and linear maps $\{d^n_M \colon M_n \to M_{n+1}\}_{n\in\mathbb{Z}}$ satisfying

$$d^n_M \circ d^{n-1}_M = 0 \text{ for all } n \in \mathbb{Z}. \tag{1.13}$$

(Note that this condition means that $\operatorname{Im} d^{n-1}_M \subset \operatorname{Ker} d^n_M$ for all $n \in \mathbb{Z}$).

(b) A morphism of complexes $f^\bullet \colon M^\bullet \to N^\bullet$ is the data of morphisms $f^n \colon M^n \to N^n$ satisfying $f^{n+1} \circ d^n_M = d^n_N \circ f^n$ for all n.

One often writes d^n instead of d^n_M and one visualizes a complex as:

$$\cdots \to M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} M^{n+1} \to \cdots . \tag{1.14}$$

A morphism of complexes $f^\bullet \colon M^\bullet \to N^\bullet$ is visualized by a commutative diagram:

$$\begin{array}{ccccccc} \cdots \longrightarrow & M^n & \xrightarrow{d^n_M} & M^{n+1} & \longrightarrow \cdots \\ & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & \\ \cdots \longrightarrow & N^n & \xrightarrow{d^n_N} & N^{n+1} & \longrightarrow \cdots \end{array} \tag{1.15}$$

- One defines naturally the direct sum of two complexes.
- A complex is bounded (resp. bounded below, bounded above) if $M^n = 0$ for $|n| >> 0$ (resp. $n << 0$, $n >> 0$).
- One also encouters complexes which are only defined for $n \in [a, b]$ where $a \leq b$ are integers:

$$M^\bullet := M^a \to \cdots \to M^b.$$

In this case one identifies $M^\bullet$ with the complex extended by 0:

$$M^\bullet := \cdots \to 0 \to M^a \to \cdots \to M^b \to 0 \cdots .$$

- In particular, one identifies a module M to a complex "concentrated in degree 0":

$$M^\bullet := \cdots \to 0 \to M \to 0 \to \cdots .$$

- Consider modules and linear maps $M' \xrightarrow{f} M \xrightarrow{g} M''$. This sequence is a complex if $g \circ f = 0$, that is, if $\operatorname{Im} f \subset \operatorname{Ker} g$. One says that this sequence is exact if $\operatorname{Im} f = \operatorname{Ker} g$.
- More generally, a sequence of morphisms $X^p \xrightarrow{d^p} \cdots \to X^n$ with $d^{i+1} \circ d^i = 0$ for all $i \in [p, n-1]$ is exact if $\operatorname{Im} d^i \xrightarrow{\sim} \operatorname{Ker} d^{i+1}$ for all $i \in [p, n-1]$.
- A short exact sequence is an exact sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$. Hence, this is a complex such that $\operatorname{Im} f = \operatorname{Ker} g$, f is injective and g is surjective.

Example 1.4.2. Recall that an A-module M is finitely generated if there exists an exact sequence $A^{N_0} \xrightarrow{f} M \to 0$. let us denote by N the kernel of f. This is an A-module. Assume that N is itself finitely generated. Hence there exists an exact sequence $A^{N_1} \to N \to 0$ from which we deduce an exact sequence

$$A^{N_1} \to A^{N_0} \to M \to 0.$$

In this case, one says that M is of finite presentation.

Note that if A is left Noetherian, any finitely generated A-module is of finite presentation. (In fact, this property can be taken as a definition of being Noetherian.) In such a case, one constructs inductively a "finite free resolution" of M:

$$\cdots \to A^{N_r} \to \cdots \to A^{N_1} \to A^{N_0} \to M \to 0.$$

Shift functor

Let $\mathcal{C}$ be an additive category, let $X \in \mathrm{C}(\mathcal{C})$ and let $p \in \mathbb{Z}$. One defines the shifted complex $X[p]$ by:

$$\begin{cases} (X[p])^n = X^{n+p} \\ d^n_{X[p]} = (-1)^p d^{n+p}_X \end{cases}$$

Definition 1.4.3. Consider a complex $(M^\bullet, d^\bullet)$. The n-th group of cohomology of $M^\bullet$ is the A-module $H^n(M^\bullet) := \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}$.

In particular a complex $(M^\bullet, d^\bullet)$ is exact if and only if $H^n(M^\bullet) \simeq 0$ for all $n \in \mathbb{Z}$. Also note that $H^n(M^\bullet)[p] = H^{n+p}(M^\bullet)$.

A morphism of complexes $f^\bullet : M^\bullet \to N^\bullet$ induces for all morphisms for all n (we keep the same notation f^n to denote these morphisms)

$$f^n\colon \operatorname{Ker} d^n_M \to \operatorname{Ker} d^n_M, \quad f^n\colon \operatorname{Im} d^{n-1}_M \to \operatorname{Im} d^{n-1}_N$$

hence morphisms

$$f^n\colon H^n(M^\bullet) \to H^n(N^\bullet).$$

Split exact sequences

Proposition 1.4.4. *Let*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0 \tag{1.16}$$

be a short exact sequence in Mod(A). *Then the conditions* (a) *to* (e) *are equivalent.*

(a) *there exists* $h\colon M'' \to M$ *such that* $g \circ h = \mathrm{id}_{M''}$.

(b) *there exists* $k\colon M \to M'$ *such that* $k \circ f = \mathrm{id}_{M'}$.

(c) *there exists* $\varphi = (k, g)$ *and* $\psi = (f + h)$ *such that* $X \xrightarrow{\varphi} M' \oplus M''$ *and* $M' \oplus M'' \xrightarrow{\psi} M$ *are isomorphisms inverse to each other.*

(d) *The complex* (1.16) *is isomorphic to the complex* $0 \to M' \to M' \oplus M'' \to M'' \to 0$.

Proof. (a) $\Rightarrow$ (c). Since $g = g \circ h \circ g$, we get $g \circ (\mathrm{id}_M - h \circ g) = 0$, which implies that $\mathrm{id}_M - h \circ g$ factors through $\operatorname{Ker} g$, that is, through M'. Hence, there exists $k\colon M \to M'$ such that $\mathrm{id}_M - h \circ g = f \circ k$.
(b) $\Rightarrow$ (c) is proved similarly.

(c) $\Rightarrow$ (a). Since $g \circ f = 0$, we find $g = g \circ h \circ g$, that is $(g \circ h - \mathrm{id}_{X''}) \circ g = 0$. Since g is an epimorphism, this implies $g \circ h - \mathrm{id}_{M''} = 0$.
(c) $\Rightarrow$ (b) is proved similarly.
(d) is obvious by (c). q.e.d.

Definition 1.4.5. In the above situation, one says that the exact sequence splits, or that the sequence is split exact.

Example 1.4.6. (i) If $\mathbf{k}$ is a field, all exact sequences in $\mathrm{Mod}(\mathbf{k})$ split.
(ii) The exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

does not split.

Exactness of limits

Consider a family of exact sequences of A-modules

$$M_i' \to M_i \to M_i'' \tag{1.17}$$

indexed by a set I.

Proposition 1.4.7. *The sequences below are exact:*

$$\bigoplus_i M_i' \to \bigoplus_i M_i \to \bigoplus_i M_i'' \tag{1.18}$$

$$\prod_i M_i' \to \prod_i M_i \to \prod_i M_i''. \tag{1.19}$$

The proof is obvious and left to the reader.

One often translates Proposition 1.4.7 by saying that direct sums and products are exact functors on A-modules.

One defines in an obvious way the notions of a projective or inductive system of complexes.

Proposition 1.4.8. *Consider a projective system of exact sequences*

$$0 \to M_i' \xrightarrow{f_i} M_i \xrightarrow{g_i} M_i'' \tag{1.20}$$

indexed by a poset I. Then the sequence

$$0 \to \varprojlim_i M_i' \xrightarrow{f} \varprojlim_i M_i \xrightarrow{g} \varprojlim_i M_i'' \tag{1.21}$$

is exact.

One often translates Proposition 1.4.8 by saying that projective limits are left exact functors on A-modules.

Proof. (i) Recall that $\varprojlim M_i$ is a submodule of $\prod_i M_i$ and similarly with M_i' instead of M_i. On the other hand, $\prod_i M_i'$ is a submodule of $\prod_i M_i$. It follows that $\varprojlim M_i'$ is a submodule of $\varprojlim M_i$. hence, f is injective.
(ii) Let $x = \{x_i\}_i \in \varprojlim_i M_i$ with $g(x) = 0$. Then $g_i(x_i) = 0$ for all i and by the exactness of (1.19), there exists a unique $y = \{y_i\}_i \in \prod_i M_i'$ with $f_i(y_i) = x_i$. One checks immediately that $y \in \varprojlim_i M_i'$. Hence, $x = f(y)$. q.e.d.

One shall be aware the exactness of the sequence $0 \to M_i' \to M_i \to M_i'' \to 0$ does not imply the exactness of the sequence $0 \to \varprojlim_i M_i' \to \varprojlim_i M_i \to \varprojlim_i M_i'' \to 0$.

Example 1.4.9. Consider the k-algebra $A := \mathbf{k}[x]$ over a field $\mathbf{k}$. Denote by $I = A \cdot x$ the ideal generated by x. Notice that $A/I^{n+1} \simeq \mathbf{k}[x]^{\leq n}$, where $\mathbf{k}[x]^{\leq n}$ denotes the $\mathbf{k}$-vector space consisting of polynomials of degree $\leq n$. For $p \leq n$ denote by $v_{pn}\colon A/I^n \twoheadrightarrow A/I^p$ the natural epimorphisms. They define a projective system of A-modules. We have seen that

$$\varprojlim_n A/I^n \simeq \mathbf{k}[[x]],$$

the ring of formal series with coefficients in $\mathbf{k}$. On the other hand, for $p \leq n$ the monomorphisms $I^n \rightarrowtail I^p$ define a projective system of A-modules and one has

$$\varprojlim_n I^n \simeq 0.$$

Now consider the projective system of exact sequences of A-modules

$$0 \to I^n \to A \to A/I^n \to 0.$$

By taking the projective limit of these exact sequences one gets the sequence $0 \to 0 \to \mathbf{k}[x] \to \mathbf{k}[[x]] \to 0$ which is no more exact.

There is a nice criterion, known as the Mittag-Leffler condition (see [9]), which makes that the projective limit of exact sequences remains exact.

Proposition 1.4.10. *Let* $0 \to \{M_n'\} \xrightarrow{f_n} \{M_n\} \xrightarrow{g_n} \{M_n''\} \to 0$ *be a projective system of exact sequences of* A*-modules indexed by* $\mathbb{N}$*. Assume that for each* n*, the map* $M_{n+1}' \to M_n'$ *is surjective. Then the sequence*

$$0 \to \varprojlim_n M_n' \xrightarrow{f} \varprojlim_n M_n \xrightarrow{g} \varprojlim_n M_n'' \to 0$$

is exact.

Proof. Let us denote for short by v_p the morphisms $M_p \to M_{p-1}$ which define the projective system $\{M_p\}$, and similarly for v'_p, v''_p. Let $\{x''_p\}_p \in \varprojlim_n M''_n$. Hence $x''_p \in M''_p$, and $v''_p(x''_p) = x''_{p-1}$.

We shall first show that $v_n \colon g_n^{-1}(x''_n) \to g_{n-1}^{-1}(x''_{n-1})$ is surjective. Let $x_{n-1} \in g_{n-1}^{-1}(x''_{n-1})$. Take $x_n \in g_n^{-1}(x''_n)$. Then $g_{n-1}(v_n(x_n) - x_{n-1})) = 0$. Hence $v_n(x_n) - x_{n-1} = f_{n-1}(x'_{n-1})$. By the hypothesis $f_{n-1}(x'_{n-1}) = f_{n-1}(v'_n(x'_n))$ for some x'_n and thus $v_n(x_n - f_n(x'_n)) = x_{n-1}$.

Then we can choose $x_n \in g_n^{-1}(x''_n)$ inductively such that $v_n(x_n) = x_{n-1}$. q.e.d.

Proposition 1.4.11. *Consider an inductive system of exact sequences*

$$0 \to M'_i \xrightarrow{f_i} M_i \xrightarrow{g_i} M''_i \to 0 \tag{1.22}$$

indexed by a filtrant poset I. Then the sequence

$$0 \to \varinjlim_i M'_i \xrightarrow{f} \varinjlim_i M_i \xrightarrow{g} \varinjlim_i M''_i \to 0 \tag{1.23}$$

is exact.

One often translates Proposition 1.4.11 by saying that filtrant inductive limits are left exact functors on A-modules.

Proof. (i) The fact that the sequence

$$\varinjlim_i M'_i \to \varinjlim_i M_i \to \varinjlim_i M''_i \to 0$$

is exact is proved similarly as in Proposition 1.4.8.
(ii) Let us prove that the map f is injective. Consider a finite sequence $\{x'_j\}_{j\in J}$ with $x'_j \in M'_j$ satisfying $f(\sum_j x'_j) = 0$ in $\varinjlim M_i$. Since $f(\sum_j x'_j) = \sum_j f(x'_j)$, there exists k with $k \geq j$ for all $j \in J$ such that $\sum_j f(x'_j) = 0$ in M_k. Therefore, $f_k(\sum_j x'_j) = 0$ in M_k and since f_k is injective, $\sum_j x'_j = 0$ in M'_k and $\sum_j x'_j = 0$ in $\varinjlim_i M'_i$. q.e.d.

1.5 Koszul complexes

If L is a finite free k-module of rank n, one denotes by $\bigwedge^j L$ the k-module consisting of j-multilinear alternate forms on the dual space L^* and calls it the j-th exterior power of L. (Recall that $L^* = \operatorname{Hom}_k(L, k)$.)

Note that $\bigwedge^1 L \simeq L$ and $\bigwedge^n L \simeq k$. One sets $\bigwedge^0 L = k$.

If $(e_1, \dots, e_n)$ is a basis of L and $I = \{i_1 < \cdots < i_j\} \subset \{1, \dots, n\}$, one sets

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_j}.$$

For a subset $I \subset \{1, \dots, n\}$, one denotes by $|I|$ its cardinal. Recall that:

$$\bigwedge^j L \text{ is free with basis } \{e_{i_1} \wedge \cdots \wedge e_{i_j}; 1 \leq i_1 < i_2 < \cdots < i_j \leq n\}.$$

If $i_1, \dots, i_m$ belong to the set $(1, \dots, n)$, one defines $e_{i_1} \wedge \cdots \wedge e_{i_m}$ by reducing to the case where $i_1 < \cdots < i_j$, using the convention $e_i \wedge e_j = -e_j \wedge e_i$.

Let M be an A-module and let $\varphi = (\varphi_1, \dots, \varphi_n)$ be n endomorphisms of M over A which commute with one another:

$$[\varphi_i, \varphi_j] = 0, \ 1 \leq i, j \leq n.$$

(Recall the notation $[a, b] := ab - ba$.) Set $M^{(j)} = M \otimes \bigwedge^j k^n$. Hence $M^{(0)} = M$ and $M^{(n)} \simeq M$. Denote by $(e_1, \dots, e_n)$ the canonical basis of k^n. Hence, any element of $M^{(j)}$ may be written uniquely as a sum

$$m = \sum_{|I|=j} m_I \otimes e_I.$$

One defines $d \in \operatorname{Hom}_A(M^{(j)}, M^{(j+1)})$ by:

$$d(m \otimes e_I) = \sum_{i=1}^{n} \varphi_i(m) \otimes e_i \wedge e_I$$

and extending d by linearity. Using the commutativity of the φ_i's one checks easily that $d \circ d = 0$. Hence we get a complex:

$$K^\bullet(M, \varphi) \colon 0 \to M^{(0)} \xrightarrow{d} \cdots \to M^{(n)} \to 0. \tag{1.24}$$

Definition 1.5.1. The complex $K^\bullet(M, \varphi)$ in (1.24) in which $M^{(0)}$ is in degree 0 is called the Koszul complex of M (associated with the sequence $\varphi = (\varphi_1, \dots, \varphi_n)$).

When $n = 1$, the cohomology of this complex gives the kernel and cokernel of φ_1. More generally,

$$\begin{aligned} H^0(K^\bullet(M, \varphi)) &\simeq \operatorname{Ker} \varphi_1 \cap \ldots \cap \operatorname{Ker} \varphi_n, \\ H^n(K^\bullet(M, \varphi)) &\simeq M/(\varphi_1(M) + \cdots + \varphi_n(M)). \end{aligned}$$

Set $\varphi' = \{\varphi_1, \dots, \varphi_{n-1}\}$ and denote by d' the differential in $K^\bullet(M, \varphi')$. Then φ_n defines a morphism

$$\widetilde{\varphi}_n : K^\bullet(M, \varphi') \to K^\bullet(M, \varphi') \tag{1.25}$$

Main theorem

Theorem 1.5.2. *There exists a long exact sequence*

$$(1.26)\cdots \to H^j(K^\bullet(M,\varphi')) \xrightarrow{\varphi_n} H^j(K^\bullet(M,\varphi')) \to H^{j+1}(K^\bullet(M,\varphi)) \to \cdots$$

Proof. Let us set for short

$$Z^j(\varphi) = \operatorname{Ker}(d^j\colon M \otimes \bigwedge^j \mathbf{k}^n \to M \otimes \bigwedge^{j+1} \mathbf{k}^n),$$
$$B^j(\varphi) = \operatorname{Im}(d^{j-1}\colon M \otimes \bigwedge^{j-1} \mathbf{k}^n \to M \otimes \bigwedge^{j} \mathbf{k}^n),$$
$$H^j(\varphi) := H^j(K^\bullet(M,\varphi)) = Z^j(\varphi)/B^j(\varphi),$$

and define similarly $Z^j(\varphi')$, $B^j(\varphi')$ and $H^j(\varphi')$. We shall construct an exact sequence

$$\cdots \to H^j(\varphi') \xrightarrow{\varphi_n} H^j(\varphi') \xrightarrow{\wedge e_n} H^{j+1}(\varphi) \xrightarrow{\vee e_n} H^{j+1}(\varphi') \xrightarrow{\varphi_n} H^{j+1}(\varphi') \to \cdots .$$

(i) Construction of $\wedge e_n$. Let $a \in Z^j(\varphi')$. We set $\wedge e_n(a) = a \wedge e_n$. We have $\wedge e_n(d'b) = d(b \wedge e_n)$. Hence $\wedge e_n\colon H^j(\varphi') \to H^{j+1}(\varphi)$ is well defined.

(ii) Construction of $\vee e_n$. Let $a = \sum_I a_I e_I \in Z^j(\varphi)$. We set $\vee e_n(a) = \sum_I a'_I e_I$ where $a'_I = a_I$ if $n \notin I$ and $a'_I = 0$ otherwise. We have $\vee e_n(db) = d'(\vee e_n(b))$. Hence $\vee e_n\colon H^{j+1}(\varphi) \to H^{j+1}(\varphi')$ is well defined.

(iii) $\wedge e_n \circ \varphi_n = 0$. Indeed, let $a \in Z^j(\varphi')$. Since $d'a = 0$, we have $\varphi_n(a) \wedge e_n = \varphi_n(a) \wedge e_n + d'a = da$.

(iv) $\varphi_n \circ \vee e_n = 0$. Let $a \in Z^{j+1}(\varphi)$. Let us write $a = a' + a''e_n$. Then $\vee e_n(a) = a'$. We have $0 = da = d'a' + \varphi_n(a') \wedge e_n + d'a'' \wedge e_n$. Hence $d'a' = 0$ and $\varphi_n(a') = d'a''$.

(v) $\operatorname{Ker}(\wedge e_n) = \operatorname{Im} \varphi_n$. Let $a \in Z^j(\varphi')$ and assume that $a \wedge e_n = db$. Set $b = b' + b'' \wedge e_n$. Then $a \wedge e_n = d'b' + d'b'' \wedge e_n + \varphi_n(b') \wedge e_n$. Therefore, $d'b' = 0$ and $d'b'' + \varphi_n(b') = a$, that is, $a - d'b'' = \varphi_n(b')$.

(vi) $\operatorname{Ker} \varphi_n = \operatorname{Im}(\vee e_n)$. Let $a \in Z^{j+1}(\varphi')$ and assume that $\varphi_n(a) = d'b$. Setting $c = a + b \wedge e_n$, we have $\vee e_n(c) = a$ and $dc = d'a + \varphi_n(a) \wedge e_n + d'b \wedge e_n = 0$.

(vii) $\operatorname{Ker}(\vee e_n) = \operatorname{Im}(\wedge e_n)$. Let $a \in Z^{j+1}(\varphi)$ and assume that $\vee e_n(a) = d'b$. Set $a = a' + a'' \wedge e_n$. Then $a' = d'b$ and $a - a'' \wedge e_n = d'b = db - \varphi_n(b) \wedge e_n$. Therefore $a - (a'' + \varphi_n(b)) \wedge e_n = db$. q.e.d.

Definition 1.5.3. (i) If for each j, $1 \le j \le n$, φ_j is injective as an endomorphism of $M/(\varphi_1(M) + \cdots + \varphi_{j-1}(M))$, one says $(\varphi_1, \ldots, \varphi_n)$ is a regular sequence.

(ii) If for each j, $1 \leq j \leq n$, φ_j is surjective as an endomorphism of $\operatorname{Ker}\varphi_1 \cap \ldots \cap \operatorname{Ker}\varphi_{j-1}$, one says $(\varphi_1, \ldots, \varphi_n)$ is a coregular sequence.

Corollary 1.5.4. (i) *Assume $(\varphi_1, \ldots, \varphi_n)$ is a regular sequence. Then $H^j(K^\bullet(M, \varphi)) \simeq 0$ for $j \neq n$.*

(ii) *Assume $(\varphi_1, \ldots, \varphi_n)$ is a coregular sequence. Then $H^j(K^\bullet(M, \varphi)) \simeq 0$ for $j \neq 0$.*

Proof. Assume for example that $(\varphi_1, \ldots, \varphi_n)$ is a regular sequence, and let us argue by induction on n. The cohomology of $K^\bullet(M, \varphi')$ is thus concentrated in degree $n-1$ and is isomorphic to $M/(\varphi_1(M) + \cdots + \varphi_{n-1}(M))$. By the hypothesis, φ_n is injective on this group, and Corollary 1.5.4 follows. q.e.d.

Second proof. Let us give a direct proof of the Corollary in case $n = 2$ for coregular sequences. Hence we consider the complex:

$$0 \to M \xrightarrow{d} M \times M \xrightarrow{d} M \to 0$$

where $d(x) = (\varphi_1(x), \varphi_2(x))$, $d(y, z) = \varphi_2(y) - \varphi_1(z)$ and we assume φ_1 is surjective on M, φ_2 is surjective on $\operatorname{Ker}\varphi_1$.

Let $(y, z) \in M \times M$ with $\varphi_2(y) = \varphi_1(z)$. We look for $x \in M$ solution of $\varphi_1(x) = y, \quad \varphi_2(x) = z$. First choose $x' \in M$ with $\varphi_1(x') = y$. Then $\varphi_2 \circ \varphi_1(x') = \varphi_2(y) = \varphi_1(z) = \varphi_1 \circ \varphi_2(x')$. Thus $\varphi_1(z - \varphi_2(x')) = 0$ and there exists $t \in M$ with $\varphi_1(t) = 0, \quad \varphi_2(t) = z - \varphi_2(x')$. Hence $y = \varphi_1(t + x'), \quad z = \varphi_2(t + x')$ and $x = t + x'$ is a solution to our problem. q.e.d.

Example 1.5.5. Let $\mathbf{k}$ be a field of characteristic 0 and set for short $\mathcal{O}_n := \mathbf{k}[x_1, \ldots, x_n]$.

(i) Denote by $x_i\cdot$ the multiplication by x_i in $\mathcal{O}_n$. We get the Koszul complex:

$$0 \to \mathcal{O}_n^{(0)} \xrightarrow{d} \cdots \to \mathcal{O}_n^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^n \sum_I x_j \cdot a_I \otimes e_j \wedge e_I.$$

The sequence $(x_1\cdot, \ldots, x_n\cdot)$ is a regular sequence in $\mathcal{O}_n$, considered as an $\mathcal{O}_n$-module. Hence the Koszul complex $K^\bullet(\mathcal{O}_n, (x_1\cdot, \ldots, x_n\cdot))$ is exact except in degree n where its cohomology is isomorphic to $\mathbf{k}$.

(ii) Denote by ∂_i the partial derivation with respect to x_i. This is a $\mathbf{k}$-linear map on the $\mathbf{k}$-vector space $\mathcal{O}_n$. We get the Koszul complex

$$0 \to \mathcal{O}_n^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{O}_n^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^{n} \sum_I \partial_j(a_I) \otimes e_j \wedge e_I.$$

The sequence $(\partial_1 \cdot, \dots, \partial_n \cdot)$ is a coregular sequence, and the above complex is exact except in degree 0 where its cohomology is isomorphic to k. Writing dx_j instead of e_j, we recognize the “de Rham complex”.

(iii) Set for short $W_n := W_n(\mathbf{k})$ and denote by $\cdot\,\partial_j$ the multiplication on the right by ∂_j on W_n. These are linear maps on W_n considered as a left W_n-module. We get a Koszul complex $K^\bullet(W_n, (\cdot\,\partial_1, \dots, \cdot\,\partial_n))$

$$0 \to W_n^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} W_n^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^{n} \sum_I a_I \cdot\, \partial_j \otimes e_j \wedge e_I.$$

The sequence $(\cdot\partial_1, \dots, \cdot\partial_n)$ is clearly a regular sequence. Hence the Koszul complex is exact except in degree n where its cohomology is isomorphic to $W_n/(\sum_j W_n \cdot \partial_j) \simeq \mathcal{O}_n$.

(iv) Denote by $\partial_j \cdot$ the multiplication on the left by ∂_j on W_n. These are linear maps on W_n considered as a right W_n-module. We get a Koszul complex $K^\bullet(W_n, (\partial_1 \cdot, \dots, \partial_n \cdot))$

$$0 \to W_n^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} W_n^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^{n} \sum_I \partial_j \,\cdot a_I \otimes e_j \wedge e_I.$$

We have seen that any element P of W_n may be written uniquely as a polynomial $P(x,\partial) = \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha$. Any such a polynomial may also be written uniquely as $P(x,\partial) = \sum_{|\alpha| \leq m} \partial^\alpha b_\alpha(x)$.

It follows that the sequence $(\partial_1 \cdot, \dots, \partial_n \cdot)$ is again a regular sequence. Hence the Koszul complex $K^\bullet(W_n, (\partial_1 \cdot, \dots, \partial_n \cdot))$ is exact except in degree n where its cohomology is isomorphic to the right W_n-module $\Omega_n := W_n/(\sum_j W_n \cdot \partial_j)$.

Co-Koszul complexes

One may also encounter co-Koszul complexes. For $I = (i_1, \dots, i_k)$, introduce

$$e_j \lfloor e_I = \begin{cases} 0 & \text{if } j \notin \{i_1, \dots, i_k\} \\ (-1)^{l+1} e_{I_{\hat{l}}} := (-1)^{l+1} e_{i_1} \wedge \dots \wedge \widehat{e_{i_l}} \wedge \dots \wedge e_{i_k} & \text{if } e_{i_l} = e_j \end{cases}$$

where $e_{i_1} \wedge \dots \wedge \widehat{e_{i_l}} \wedge \dots \wedge e_{i_k}$ means that e_{i_l} should be omitted in $e_{i_1} \wedge \dots \wedge e_{i_k}$. Define δ by:

$$\delta(m \otimes e_I) = \sum_{j=1}^{n} \varphi_j(m) e_j \lfloor e_I .$$

Here again one checks easily that $\delta \circ \delta = 0$, and we get the complex:

$$K_\bullet(M, \varphi)\colon 0 \to M^{(n)} \xrightarrow{\delta} \cdots \to M^{(0)} \to 0, \tag{1.27}$$

Definition 1.5.6. The complex $K_\bullet(M, \varphi)$ in (1.5.7) in which $M^{(n)}$ is in degree 0 is called the co-Koszul complex of M (associated with the sequence $\varphi = (\varphi_1, \dots, \varphi_n)$).

Proposition 1.5.7. *The Koszul complex* (1.24) *and the co-Koszul complex* (1.27) (*in which $M^{(n)}$ is in degree* 0) *are isomorphic.*

Proof. Consider the isomorphism $\bigwedge^j \mathbf{k}^n \simeq \bigwedge^{n-j} \mathbf{k}^n$ which associates $\varepsilon_I m \otimes e_{\widehat{I}}$ to $m \otimes e_I$, where $\widehat{I} = (1, \dots, n) \setminus I$ and ε_I is the signature of the permutation which sends $(1, \dots, n)$ to $I \sqcup \widehat{I}$ (any $i \in I$ is smaller than any $j \in \hat{I}$). Then, up to a sign, $*$ interchanges d and δ. q.e.d.

Proposition 1.5.8. *Let $(a_1, \dots, a_n)$ be n elements of A which commute with one another, that is, $[a_i, a_j] = 0,\ 1 \le i, j \le n$. Let M be an A-module. Then the a_j's define right or left endomorphisms of A and we have*

$$\begin{aligned} K^\bullet(A, (a_1\cdot, \dots, a_n\cdot)) \otimes_A M &\simeq K^\bullet(M, (a_1\cdot, \dots, a_n\cdot)), \\ \operatorname{Hom}_A(K^\bullet(A, (\cdot a_1, \dots, \cdot a_n)), M) &\simeq K_\bullet(M, (a_1\cdot, \dots, a_n\cdot))\,[n] \\ &\simeq K^\bullet(M, (a_1\cdot, \dots, a_n\cdot))\,[n]. \end{aligned}$$

The verification is left to the reader.

Exercises to Chapter 1

Exercise 1.1. Let I be a (non necessarily finite) set and $(X_i)_{i \in I}$ a family of sets indexed by I.

(i) Construct the natural map $\coprod_i \operatorname{Hom}_{\mathbf{Set}}(Y, X_i) \to \operatorname{Hom}_{\mathbf{Set}}(Y, \coprod_i X_i)$ and prove that this map is injective but is not surjective in general. (Hint: use $Y = \emptyset$.).
(iii) Construct the natural map $\coprod_i \operatorname{Hom}_{\mathbf{Set}}(X_i, Y) \to \operatorname{Hom}_{\mathbf{Set}}(\prod_i X_i, Y)$ and prove that this map is neither injective nor surjective in general. (Hint: for the injectivity, use $Y = \mathrm{pt}$.)

Exercise 1.2. Let M be an A-module and denote by I the ordered set of all finitely generated submodules of M. Hence, for $N, L \in I$, $N \leq L$ if and only if $N \subset L$.
(i) Prove that I is filtrant.
(ii) Calculate $\varinjlim\limits_{N \in I} N$.
(iii) Calculate $\varinjlim\limits_{N \in I} M/N$.

Exercise 1.3. We follow the notations of Example 1.3.6. Prove that the natural map

$$\varinjlim_{n} \operatorname{Hom}_{\mathbf{k}}(\mathbf{k}[x]^{\leq n}, \mathbf{k}[x]) \to \operatorname{Hom}_{\mathbf{k}}(\mathbf{k}[x], \mathbf{k}[x])$$

is injective but not surjective.

Exercise 1.4. Let $A = W_2(\mathbf{k})$ be the Weyl algebra in two variables. Construct the Koszul complex associated to $\varphi_1 = \cdot x_1$, $\varphi_2 = \cdot \partial_2$ and calculate its cohomology.

Exercise 1.5. Let $\mathbf{k}$ be a field, $A = \mathbf{k}[x, y]$ and consider the A-module $M = \bigoplus_{i \geq 1} \mathbf{k}[x]t^i$, where the action of $x \in A$ is the usual one and the action of $y \in A$ is defined by $y \cdot x^n t^{j+1} = x^n t^j$ for $j \geq 1$, $y \cdot x^n t = 0$. Define the endomorphisms of M, $\varphi_1(m) = x \cdot m$ and $\varphi_2(m) = y \cdot m$. Calculate the cohomology of the Kozsul complex $K^\bullet(M, \varphi)$.

Chapter 2

The language of categories

In this chapter we introduce some basic notions of category theory which are of constant use in various fields of Mathematics, without spending too much time on this language.
Some references: [4, 5, 8, 14, 15, 16, 18, 19].

2.1 Categories

Definition 2.1.1. A category $\mathcal{C}$ consists of:

(i) a set $\mathrm{Ob}(\mathcal{C})$ whose elements are called the objects of $\mathcal{C}$,

(ii) for each $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ whose elements are called the morphisms from X to Y,

(iii) for any $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map, called the composition, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, and denoted $(f, g) \mapsto g \circ f$,

these data satisfying:

(a) $\circ$ is associative,

(b) for each $X \in \mathrm{Ob}(\mathcal{C})$, there exists $\mathrm{id}_X \in \mathrm{Hom}\,(X, X)$ such that for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, $f \circ \mathrm{id}_X = f$, $\mathrm{id}_X \circ g = g$.

Remark 2.1.2. There are some set-theoretical dangers, illustrated in Remark 2.1.10, and one should mention in which "universe" we are working.

We do not give in these Notes the definition of a universe, only recalling that a universe $\mathcal{U}$ is a set (a very big one) stable by many operations and containing $\mathbb{N}$.

Although we skip this point, when taking products, direct sums or, more generally, limits, we should mention that these limits are indexed by "small" categories.

Notation 2.1.3. One often writes $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$ and $f\colon X \to Y$ (or else $f\colon Y \leftarrow X$) instead of $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. One calls X the source and Y the target of f.

A morphism $f\colon X \to Y$ is an *isomorphism* if there exists $g\colon X \leftarrow Y$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In such a case, one writes $f\colon X \xrightarrow{\sim} Y$ or simply $X \simeq Y$. Of course g is unique, and one also denotes it by f^{-1}.

A morphism $f\colon X \to Y$ is a *monomorphism* (resp. an *epimorphism*) if for any morphisms g_1 and g_2, $f \circ g_1 = f \circ g_2$ (resp. $g_1 \circ f = g_2 \circ f$) implies $g_1 = g_2$. One sometimes writes $f\colon X \rightarrowtail Y$ or else $X \hookrightarrow Y$ (resp. $f\colon X \twoheadrightarrow Y$) to denote a monomorphism (resp. an epimorphism).

Two morphisms f and g are parallel if they have the same sources and targets, visualized by $f, g\colon X \rightrightarrows Y$.

One introduces the *opposite category* $\mathcal{C}^{\mathrm{op}}$:

$$\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C}), \quad \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X),$$

the identity morphisms and the composition of morphisms being the obvious ones.

A category $\mathcal{C}'$ is a *subcategory* of $\mathcal{C}$, denoted $\mathcal{C}' \subset \mathcal{C}$, if: $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}'}(X,Y) \subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for any $X, Y \in \mathcal{C}'$, the composition $\circ$ in $\mathcal{C}'$ is induced by the composition in $\mathcal{C}$ and the identity morphisms in $\mathcal{C}'$ are induced by those in $\mathcal{C}$. One says that $\mathcal{C}'$ is a *full* subcategory if for all $X, Y \in \mathcal{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

A category is *discrete* if the only morphisms are the identity morphisms. Note that a set is naturally identified with a discrete category.

A category $\mathcal{C}$ is *finite* if the family of all morphisms in $\mathcal{C}$ (hence, in particular, the family of objects) is a finite set.

A category $\mathcal{C}$ is a *groupoid* if all morphisms are isomorphisms.

Examples 2.1.4. (i) **Set** is the category of sets and maps (in a given universe), $\mathbf{Set}^f$ is the full subcategory consisting of finite sets.
(ii) **Rel** is defined by: $\mathrm{Ob}(\mathbf{Rel}) = \mathrm{Ob}(\mathbf{Set})$ and $\mathrm{Hom}_{\mathbf{Rel}}(X,Y) = \mathcal{P}(X \times Y)$, the set of subsets of $X \times Y$. The composition law is defined as follows. For $f\colon X \to Y$ and $g\colon Y \to Z$, $g \circ f$ is the set

$$\{(x,z) \in X \times Z;\ \text{there exists } y \in Y \text{ with } (x,y) \in f, (y,z) \in g\}.$$

Of course, $\mathrm{id}_X = \Delta \subset X \times X$, the diagonal of $X \times X$.

(iii) Let A be a ring. The category of left A-modules and A-linear maps is denoted $\mathrm{Mod}(A)$. In particular $\mathrm{Mod}(\mathbb{Z})$ is the category of abelian groups.

We shall often use the notations **Ab** instead of $\mathrm{Mod}(\mathbb{Z})$ and $\mathrm{Hom}_A(\,\cdot\,,\,\cdot\,)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(A)}(\,\cdot\,,\,\cdot\,)$.

One denotes by $\mathrm{Mod}^{\mathrm{f}}(A)$ the full subcategory of $\mathrm{Mod}(A)$ consisting of finitely generated A-modules.

(iv) One associates to a pre-ordered set $(I, \leq)$ a category, still denoted by I for short, as follows. $\mathrm{Ob}(I) = I$, and the set of morphisms from i to j has a single element if $i \leq j$, and is empty otherwise. Note that I^{op} is the category associated with I endowed with the opposite order.

(v) We denote by **Top** the category of topological spaces and continuous maps.

(vi) We shall often represent by the diagram $\bullet \to \bullet$ the category which consists of two objects, say $\{a, b\}$, and one morphism $a \to b$ other than id_a and id_b. We denote this category by **Arr**.

(vii) We represent by $\bullet \rightrightarrows \bullet$ the category with two objects, say $\{a, b\}$, and two parallel morphisms $a \rightrightarrows b$ other than id_a and id_b.

(viii) Let G be a group. We may attach to it the groupoid $\mathcal{G}$ with one object, say $\{a\}$ and morphisms $\mathrm{Hom}_{\mathcal{G}}(a, a) = G$.

(ix) Let X be a topological space locally arcwise connected. We attach to it a category $\widetilde{X}$ as follows: $\mathrm{Ob}(\widetilde{X}) = X$ and for $x, y \in X$, a morphism $f\colon x \to y$ is a path form x to y.

(x) let $\mathcal{C}$ be a category. There is a category $\mathrm{Mor}(\mathcal{C})$ whose objects are the morphisms in $\mathcal{C}$ and morphisms are defined as follows. For two objects $f\colon X \to Y$ and $g\colon V \to W$ in $\mathrm{Mor}(\mathcal{C})$, a morphism $u\colon f \to g$ is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ V & \xrightarrow{g} & W. \end{array}$$

Definition 2.1.5. (i) An object $P \in \mathcal{C}$ is called initial if for all $X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(P, X) \simeq \{\mathrm{pt}\}$. One often denotes by $\emptyset_{\mathcal{C}}$ an initial object in $\mathcal{C}$.

(ii) One says that P is terminal if P is initial in $\mathcal{C}^{\mathrm{op}}$, *i.e.,* for all $X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, P) \simeq \{\mathrm{pt}\}$. One often denotes by $\mathrm{pt}_{\mathcal{C}}$ a terminal object in $\mathcal{C}$.

(iii) One says that P is a zero-object if it is both initial and terminal. In such a case, one often denotes it by 0. If $\mathcal{C}$ has a zero object, for any objects $X, Y \in \mathcal{C}$, the morphism obtained as the composition $X \to 0 \to Y$ is still denoted by $0\colon X \to Y$.

Note that initial (resp. terminal) objects are unique up to unique isomorphisms.

Examples 2.1.6. (i) In the category **Set**, $\emptyset$ is initial and $\{\mathrm{pt}\}$ is terminal.
(ii) The zero module 0 is a zero-object in $\mathrm{Mod}(A)$.
(iii) The category $\underline{\mathbb{Z}}$ associated with the ordered set $(\mathbb{Z}, \leq)$ has neither initial nor terminal object.

Products and coproducts

Let $\mathcal{C}$ be a category and consider a family $\{X_i\}_{i\in I}$ of objects of $\mathcal{C}$ indexed by a set I.

Definition 2.1.7. (a) The product of the family $\{X_i\}_{i\in I}$, if it exists, is the data of an object $Z \in \mathcal{C}$ together with morphisms $\pi_i \colon Z \to X_i$ $(i \in I)$ such that, for any $Y \in \mathcal{C}$, the natural morphism

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \prod_i \mathrm{Hom}_{\mathcal{C}}(Y, X_i)$$

given by $(f \colon Y \to Z) \mapsto \{\pi_i \circ f \colon Y \to X_i\}_{i\in I}$ is an isomorphism.

(b) If $(Z, \{\pi_i\}_{i\in I})$ exists, it is unique up to unique isomorphism (see below) and Z is denoted by $\prod_i X_i$.

(c) In case I has two elements, say $I = \{1, 2\}$, one simply denotes this object by $X_1 \times X_2$. In case $X_i = X$ for all $i \in I$, one writes: $X^I := \prod_i X_i$.

Let us prove the unicity of $(Z, \{\pi_i\}_{i\in I})$. Consider the category $\mathcal{A}$ defined as follows.

- the objects $\widetilde{Y}$ are the families $\widetilde{Y} = \{f_i \colon Y \to X_i\}_{i\in I}$ with $Y \in \mathcal{C}$,
- given two objects $\widetilde{Y} = \{f_i \colon Y \to X_i\}_i$ and $\widetilde{W} = \{g_i \colon W \to X_i\}_i$, a morphism $\widetilde{u} \colon \widetilde{Y} \to \widetilde{W}$ is a morphism $u \colon Y \to W$ such that $f_i = g_i \circ u$ for all i.

Then $(Z, \{\pi_i\}_{i\in I})$ is a terminal object in $\mathcal{A}$.

The coproduct in $\mathcal{C}$ is the product in $\mathcal{C}^{\mathrm{op}}$. Hence:

Definition 2.1.8. (a) The coproduct of the family $\{X_i\}_{i\in I}$, if it exists, is the data of an object $Z \in \mathcal{C}$ together with morphisms $\sigma_i \colon X_i \to Z$ $(i \in I)$ such such that, for any $Y \in \mathcal{C}$, the natural morphism

$$\mathrm{Hom}_{\mathcal{C}}(Z, Y) \to \prod_i \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$$

given by $(f \colon Z \to Y) \mapsto \{f \circ \sigma_i \colon X_i \to Y\}_{i\in I}$ is an isomorphism.

(b) If $(Z, \{\sigma_i\}_i)$ exists, it is unique up to unique isomorphism and it is denoted by $\coprod_i X_i$.

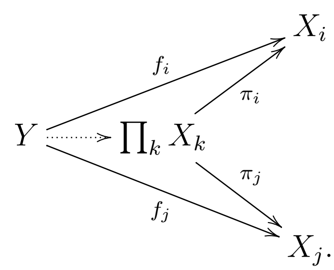
(c) In case I has two elements, say $I = \{1, 2\}$, one simply denotes this object by $X_1 \sqcup X_2$. In case $X_i = X$ for all $i \in I$, one writes: $X^{(I)} := \coprod_i X_i$.

By this definition, the product or the coproduct exist if and only if one has the isomorphisms, functorial with respect to $Y \in \mathcal{C}$:

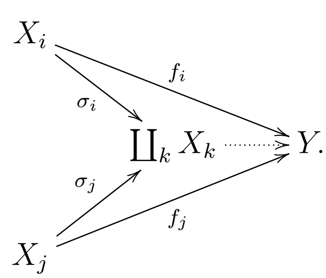
$$\mathrm{Hom}_{\mathcal{C}}(Y, \prod_i X_i) \simeq \prod_i \mathrm{Hom}_{\mathcal{C}}(Y, X_i), \tag{2.1}$$

$$\mathrm{Hom}_{\mathcal{C}}(\coprod_i X_i, Y) \simeq \prod_i \mathrm{Hom}_{\mathcal{C}}(X_i, Y). \tag{2.2}$$

The isomorphism (2.1) may be translated as follows. Given an object Y and a family of morphisms $f_i \colon Y \to X_i$, this family factorizes uniquely through $\prod_i X_i$. This is visualized by the diagram



The isomorphism (2.2) may be translated as follows. Given an object Y and a family of morphisms $f_i \colon X_i \to Y$, this family factorizes uniquely through $\coprod_i X_i$. This is visualized by the diagram



Example 2.1.9. (i) The category **Set** admits products and the two definitions (that given in (1.1) and that given in Definition 2.1.7) coincide.
(ii) The category **Set** admits coproducts namely, the disjoint union.
(iii) Let A be a ring. The category $\mathrm{Mod}(A)$ admits products, as defined in § 1.2. The category $\mathrm{Mod}(A)$ also admits coproducts, which are the direct sums defined in § 1.2. and are denoted $\bigoplus$.

(iv) Let X be a set and denote by $\mathfrak{X}$ the category of subsets of X. (The set $\mathfrak{X}$ is ordered by inclusion, hence defines a category.) For $S_1, S_2 \in \mathfrak{X}$, their product in the category $\mathfrak{X}$ is their intersection and their coproduct is their union.
(v) The category $\underline{\mathbb{Z}}$ associated with the ordered set $(\mathbb{Z}, \leq)$ admits products and coproducts of two objects. For $a, b \in \underline{\mathbb{Z}}$, one has $a \times b = \inf(a, b)$ and $a \sqcup b = \sup(a, b)$.

Remark 2.1.10. In these notes, we have skipped problems related to questions of cardinality and universes but this is dangerous. In particular, when taking products or coproducts. Let us give an example.

Let $\mathcal{C}$ be a category which admits products and assume there exist $X, Y \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has more than one element. Set $M = \mathrm{Mor}(\mathcal{C})$, where $\mathrm{Mor}(\mathcal{C})$ denotes the set of all morphisms in $\mathcal{C}$, and let $\pi = \mathrm{card}(M)$, the cardinal of the set M. We have $\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y)^M$ and therefore $\mathrm{card}(\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \geq 2^{\pi}$. On the other hand, $\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \subset \mathrm{Mor}(\mathcal{C})$ which implies $\mathrm{card}(\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \leq \pi$.

The "contradiction" comes from the fact that $\mathcal{C}$ does not admit products indexed by such a big set as $\mathrm{Mor}(\mathcal{C})$. (The remark was found in [5].)

2.2 Functors

Definition 2.2.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A functor $F \colon \mathcal{C} \to \mathcal{C}'$ consists of a map $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ and for all $X, Y \in \mathcal{C}$, of a map still denoted by F: $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ such that

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \quad F(f \circ g) = F(f) \circ F(g).$$

A contravariant functor from $\mathcal{C}$ to $\mathcal{C}'$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}'$. In other words, it satisfies $F(g \circ f) = F(f) \circ F(g)$. If one wishes to put the emphasis on the fact that a functor is not contravariant, one says it is covariant.

One denotes by $\mathrm{op} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ the contravariant functor, associated with $\mathrm{id}_{\mathcal{C}^{\mathrm{op}}}$.

Definition 2.2.2. Let $F \colon \mathcal{C} \to \mathcal{C}'$ be a functor.

(i) One says that F is faithful (resp. full, resp. fully faithful) if for $X, Y \in \mathcal{C}$ $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ is injective (resp. surjective, resp. bijective).

(ii) One says that F is essentially surjective if for each $Y \in \mathcal{C}'$ there exists $X \in \mathcal{C}$ and an isomorphism $F(X) \simeq Y$.

(iii) One says that F is conservative if any morphism $f\colon X \to Y$ in $\mathcal{C}$ is an isomorphism as soon as $F(f)$ is an isomorphism.

Clearly, a fully faithful functor is conservative (see Exercise 2.2).

Examples 2.2.3. (i) Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$. Then $\mathrm{Hom}_{\mathcal{C}}(X, \bullet)$ is a functor from $\mathcal{C}$ to **Set** and $\mathrm{Hom}_{\mathcal{C}}(\bullet, X)$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to **Set**.
(ii) Let $\mathcal{C}$ be a category and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. One can look at f as a functor from the category $\bullet \to \bullet$ to $\mathcal{C}$.
(iii) Let A be a $\mathbf{k}$-algebra and let N be a right A-module. Then $N \otimes_A \bullet\colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ is a functor. Clearly, the functor $N \otimes_A \bullet$ commutes with direct sums, that is,

$$N \otimes_A (\bigoplus_i M_i) \;\simeq\; \bigoplus_i (N \otimes_A M_i),$$

and similarly for the functor $\bullet \otimes_A M$.
(iv) Let I be a set. The map $\{M_i\}_{i\in I} \mapsto \prod_{i\in I} M_i$ defines a functor from $(\mathrm{Mod}(A))^I$ to $\mathrm{Mod}(A)$.
(v) Let I be a poset. An inductive system of A-modules indexed by I (see § 1.3) is nothing but a functor $I \to \mathrm{Mod}(A)$ and a projective system is a functor $I^{\mathrm{op}} \to \mathrm{Mod}(A)$.
(vi) The forgetful functor *for*: $\mathrm{Mod}(A) \to$ **Set** associates to an A-module M the set M, and to a linear map f the map f. The functor *for* is faithful and conservative but not fully faithful.
(vii) The forgetful functor *for*: **Top** $\to$ **Set** (defined similarly as in (ii)) is faithful. It is neither fully faithful nor conservative.
(viii) The forgetful functor $\Gamma\colon$ **Set** $\to$ **Rel** is defined as follows. For $X \in$ **Set**, $\Gamma(X) = X$. For a morphism $f\colon X \to Y$ in **Set**, $\Gamma(f) \subset X \times Y$ is the graph of f. Then Γ is faithful and conservative.

The Yoneda lemma

Let $X \in \mathcal{C}$. Then X defines a functor

$$h_{\mathcal{C}}(X)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set} \quad Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

and we get a functor

$$h_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}), \quad X \mapsto h_{\mathcal{C}}(X). \tag{2.3}$$

We state without proof the main result of category theory:

Theorem 2.2.4. (The Yoneda lemma) *The functor $h_{\mathcal{C}}$ in* (2.3) *is fully faithful.*

Bifunctors

One defines in an obvious way the product of two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ by setting

$$\begin{aligned}&\mathrm{Ob}(\mathcal{C}_1 \times \mathcal{C}_2) = \mathrm{Ob}(\mathcal{C}_1) \times \mathrm{Ob}(\mathcal{C}_2),\\ &\mathrm{Hom}_{\mathcal{C}_1 \times \mathcal{C}_2}((X_1, X_2), (Y_1, Y_2)) = \mathrm{Hom}_{\mathcal{C}_1}(X_1, Y_1) \times \mathrm{Hom}_{\mathcal{C}_2}(X_2, Y_2).\end{aligned}$$

A bifunctor $F\colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}'$ is a functor on the product category.

Examples 2.2.5. (i) $\mathrm{Hom}_{\mathcal{C}}(\,\bullet\,,\,\bullet\,) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is a bifunctor.
(ii) If A is a $\mathbf{k}$-algebra, $\mathrm{Hom}_A(\,\bullet\,,\,\bullet\,)\colon \mathrm{Mod}(A)^{\mathrm{op}} \times \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ and $\bullet \otimes_A \bullet\colon \mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ are bifunctors.

Morphisms of functors

Definition 2.2.6. Let F_1, F_2 are two functors from $\mathcal{C}$ to $\mathcal{C}'$. A morphism of functors $\theta\colon F_1 \to F_2$ is the data for all $X \in \mathcal{C}$ of a morphism $\theta(X) : F_1(X) \to F_2(X)$ such that for all $f\colon X \to Y$, the diagram below commutes:

$$\begin{array}{ccc} F_1(X) & \xrightarrow{\theta(X)} & F_2(X) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\ F_1(Y) & \xrightarrow{\theta(Y)} & F_2(Y) \end{array}$$

Notation 2.2.7. We denote by $\mathrm{Fct}(\mathcal{C}, \mathcal{C}')$ the category of functors from $\mathcal{C}$ to $\mathcal{C}'$.

Let I be a set. Then $\mathcal{C}^I \simeq \mathrm{Fct}(I, \mathcal{C})$ where the set I is considered as a discrete category.

Examples 2.2.8. Let $\mathbf{k}$ be a field and consider the functor

$$\begin{aligned}&{}^*\colon \mathrm{Mod}(\mathbf{k})^{\mathrm{op}} \to \mathrm{Mod}(\mathbf{k}),\\ &V \mapsto V^* = \mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k}).\end{aligned}$$

Then there is a morphism of functors $\mathrm{id} \to {}^* \circ {}^*$ in $\mathrm{Fct}(\mathrm{Mod}(\mathbf{k}), \mathrm{Mod}(\mathbf{k}))$.
(ii) We shall encounter morphisms of functors when considering pairs of adjoint functors (see (2.7)).

In particular we have the notion of an isomorphism of categories. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ is an isomorphism of categories if there exists $G : \mathcal{C}' \to \mathcal{C}$ such that: $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{C}'}$. In particular, for all $X \in \mathcal{C}$, $G \circ F(X) = X$. In practice, such a situation rarely occurs and is not really interesting. There is a weaker notion that we introduce below.

Definition 2.2.9. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if there exists $G\colon \mathcal{C}' \to \mathcal{C}$ such that: $G \circ F$ is isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $F \circ G$ is isomorphic to $\mathrm{id}_{\mathcal{C}'}$.

We shall not give the proof of the following important result below.

Theorem 2.2.10. *The functor $F\colon \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

If two categories are equivalent, all results and concepts in one of them have their counterparts in the other one. This is why this notion of equivalence of categories plays an important role in Mathematics.

Examples 2.2.11. (i) Let $\mathbf{k}$ be a field and let $\mathcal{C}$ denote the category defined by $\mathrm{Ob}(\mathcal{C}) = \mathbb{N}$ and $\mathrm{Hom}_{\mathcal{C}}(n,m) = M_{m,n}(\mathbf{k})$, the set of matrices of type (m,n) with entries in $\mathbf{k}$, the composition being the usual composition of matrices. Define the functor $F\colon \mathcal{C} \to \mathrm{Mod}^f(\mathbf{k})$ as follows. For $n \in \mathbb{N}$, set $F(n) = \mathbf{k}^n \in \mathrm{Mod}^f(\mathbf{k})$. For a matrix $A \in M_{m,n}(\mathbf{k})$, set $F(A) = u$ where $u\colon \mathbf{k}^n \to \mathbf{k}^m$ is the linear map represented by the matrix A. Clearly F is fully faithful. Since any finite dimensional vector space admits a basis, it is isomorphic to $\mathbf{k}^n$ for some n. It follows that F is essentially surjective. In conclusion, F is an equivalence of categories.
(ii) let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. There is an equivalence

$$\mathrm{Fct}(\mathcal{C},\mathcal{C}')^{\mathrm{op}} \simeq \mathrm{Fct}(\mathcal{C}^{\mathrm{op}},(\mathcal{C}')^{\mathrm{op}}). \tag{2.4}$$

(iii) Let I, J and $\mathcal{C}$ be categories. There are equivalences

$$\mathrm{Fct}(I \times J,\mathcal{C}) \simeq \mathrm{Fct}(J,\mathrm{Fct}(I,\mathcal{C})) \simeq \mathrm{Fct}(I,\mathrm{Fct}(J,\mathcal{C})). \tag{2.5}$$

Adjoint functors

Definition 2.2.12. Let $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{C}' \to \mathcal{C}$ be two functors. One says that (F,G) is a pair of adjoint functors or that F is a left adjoint to G, or that G is a right adjoint to F if there exists an isomorphism of bifunctors:

$$\mathrm{Hom}_{\mathcal{C}'}(F(\,\bullet\,),\,\bullet\,) \simeq \mathrm{Hom}_{\mathcal{C}}(\,\bullet\,,G(\,\bullet\,)) \tag{2.6}$$

If G is an adjoint to F, then G is unique up to isomorphism. In fact, $G(Y)$ is a representative of the functor $X \mapsto \mathrm{Hom}_{\mathcal{C}}(F(X),Y)$.

The isomorphism (2.6) gives the isomorphisms

$$\mathrm{Hom}_{\mathcal{C}'}(F \circ G(\,\bullet\,),\,\bullet\,) \simeq \mathrm{Hom}_{\mathcal{C}}(G(\,\bullet\,),G(\,\bullet\,)),$$
$$\mathrm{Hom}_{\mathcal{C}'}(F(\,\bullet\,),F(\,\bullet\,)) \simeq \mathrm{Hom}_{\mathcal{C}}(\,\bullet\,,G \circ F(\,\bullet\,)).$$

In particular, we have morphisms $X \to G \circ F(X)$, functorial in $X \in \mathcal{C}$, and morphisms $F \circ G(Y) \to Y$, functorial in $Y \in \mathcal{C}'$. In other words, we have morphisms of functors

$$F \circ G \to \mathrm{id}_{\mathcal{C}'}, \qquad \mathrm{id}_{\mathcal{C}} \to G \circ F. \tag{2.7}$$

Examples 2.2.13. (i) Let $X \in \mathbf{Set}$. Using the bijection (1.4), we get that the functor $\mathrm{Hom}_{\mathbf{Set}}(X, \bullet) \colon \mathbf{Set} \to \mathbf{Set}$ is right adjoint to the functor $\bullet \times X$.
(ii) Let A be a $\mathbf{k}$-algebra and let $L \in \mathrm{Mod}(\mathbf{k})$. Using the first isomorphism in (1.12), we get that the functor $\mathrm{Hom}_k(L, \bullet) \colon \mathrm{Mod}(A)$ to $\mathrm{Mod}(A)$ is right adjoint to the functor $\bullet \otimes_{\mathbf{k}} L$.
(iii) Let A be a $\mathbf{k}$-algebra. Using the isomorphisms in (1.12) with $N = A$, we get that the functor $\mathit{for} \colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ which, to an A-module associates the underlying $\mathbf{k}$-module, is right adjoint to the functor $A \otimes_{\mathbf{k}} \bullet \colon \mathrm{Mod}(\mathbf{k}) \to \mathrm{Mod}(A)$ (extension of scalars).

2.3 Additive and abelian categories

Additive categories

Definition 2.3.1. A category $\mathcal{C}$ is additive if it satisfies conditions (i)-(v) below:

(i) for any $X, Y \in \mathcal{C}$, $\quad \mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathbf{Ab}$,

(ii) the composition law $\circ$ is bilinear,

(iii) there exists a zero object in $\mathcal{C}$,

(iv) the category $\mathcal{C}$ admits finite coproducts,

(v) the category $\mathcal{C}$ admits finite products.

Note that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq \emptyset$ since it is a group and for all $X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, 0) = \mathrm{Hom}_{\mathcal{C}}(0, X) = 0$. (The morphism 0 should not be confused with the object 0.)

Notation 2.3.2. If X and Y are two objects of $\mathcal{C}$, one denotes by $X \oplus Y$ (instead of $X \sqcup Y$) their coproduct, and calls it their direct sum. One denotes as usual by $X \times Y$ their product. This change of notations is motivated by the fact that if A is a ring, the forgetful functor $\mathrm{Mod}(A) \to \mathbf{Set}$ does not commute with coproducts.

One easily proves that if $\mathcal{C}$ satisfies the axioms (i)-(ii)-(iii), then the conditions (iv) and (v) are equivalent and moreover the objects $X \oplus Y$ and $X \times Y$ are isomorphic. Setting $Z = X \oplus Y \simeq X \times Y$ there exist morphisms morphisms $i_1 \colon X \to Z$, $i_2 \colon Y \to Z$, $p_1 \colon Z \to X$ and $p_2 \colon Z \to Y$ satisfying

$$\begin{aligned} &p_1 \circ i_1 = \mathrm{id}_X, \quad p_1 \circ i_2 = 0 \\ &p_2 \circ i_2 = \mathrm{id}_Y, \quad p_2 \circ i_1 = 0, \\ &i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_Z . \end{aligned}$$

Example 2.3.3. (i) If A is a ring, $\mathrm{Mod}(A)$ and $\mathrm{Mod}^{\mathrm{f}}(A)$ are additive categories.
(ii) **Ban**, the category of $\mathbb{C}$-Banach spaces and linear continuous maps is additive.
(iii) If $\mathcal{C}$ is additive, then $\mathcal{C}^{\mathrm{op}}$ is additive.
(iv) Let I be category. If $\mathcal{C}$ is additive, the category $\mathrm{Fct}(I, \mathcal{C})$ of functors from I to $\mathcal{C}$, is additive.
(v) If $\mathcal{C}$ and $\mathcal{C}'$ are additive, then $\mathcal{C} \times \mathcal{C}'$ is additive.

Let $F \colon \mathcal{C} \to \mathcal{C}'$ be a functor of additive categories. One says that F is additive if for $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ is a morphism of groups. We shall not prove here the following result.

Proposition 2.3.4. *Let $F \colon \mathcal{C} \to \mathcal{C}'$ be a functor of additive categories. Then F is additive if and only if it commutes with direct sums, that is, for X and Y in $\mathcal{C}$:*

$$\begin{aligned} F(0) &\simeq 0 \\ F(X \oplus Y) &\simeq F(X) \oplus F(Y). \end{aligned}$$

Unless otherwise specified, functors between additive categories will be assumed to be additive.
Generalization. Let $\mathbf{k}$ be a commutative ring. One defines the notion of a $\mathbf{k}$-additive category by assuming that for X and Y in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a $\mathbf{k}$-module and the composition is $\mathbf{k}$-bilinear.

Complexes in additive categories

The notions of complexes introduced in § 1.4 extend to additive categories.

Let $\mathcal{C}$ denote an additive category. A complex $(X^\bullet, d_X^\bullet)$ in $\mathcal{C}$ is a sequence of objects X^n and morphisms d^n $(n \in \mathbb{Z})$:

$$\cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to \cdots$$

such that $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$.

A morphism of complexes is visualized by a commutative diagram similar to (1.15):

$$\begin{array}{ccccccc} \cdots \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1} & \longrightarrow \cdots \\ & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & \\ \cdots \longrightarrow & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} & \longrightarrow \cdots \end{array} \tag{2.8}$$

One defines naturally the direct sum of two complexes and we get a new additive category, the category $\mathrm{C}(\mathcal{C})$ of complexes in $\mathcal{C}$.

A complex is bounded (resp. bounded below, bounded above) if $X^n = 0$ for $|n| >> 0$ (resp. $n << 0$, $n >> 0$). One denotes by $\mathrm{C}^*(\mathcal{C})(* = b, +, -)$ the full additive subcategory of $\mathrm{C}(\mathcal{C})$ consisting of bounded complexes (resp. bounded below, bounded above).

One considers $\mathcal{C}$ as a full subcategory of $\mathrm{C}^b(\mathcal{C})$ by identifying an object $X \in \mathcal{C}$ with the complex $X^\bullet$ "concentrated in degree 0":

$$X^\bullet := \cdots \to 0 \to X \to 0 \to \cdots$$

where X stands in degree 0.

Definition 2.3.5. let $(X^\bullet, d_X^\bullet)$ be a complex in $\mathcal{C}$. For $r \in \mathbb{Z}$, one defines the shifted comnplex $(X^\bullet[r], d_{X[r]}^\bullet)$ by setting

$$(X^\bullet[r])^i = X^{i+r}, \quad d_{X[r]}^i = (-1)^r d_X^{i+r}.$$

Kernels and cokernels

Let $\mathcal{C}$ be an additive category and consider a morphism $f\colon X_0 \to X_1$ in $\mathcal{C}$.

Definition 2.3.6. The kernel of f, if it exists, is the data of an object $\mathrm{Ker}(f) \in \mathcal{C}$ together with a morphism $h\colon \mathrm{Ker}(f) \to X_0$ such that, for any $Y \in \mathcal{C}$ and any morphism $u\colon Y \to X_0$ satisfying $f \circ u = 0$, the the natural morphism

$$\mathrm{Hom}_{\mathcal{C}}(Y, \mathrm{Ker}(f)) \to \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(Y, X_0) \xrightarrow{f\circ} \mathrm{Hom}_{\mathcal{C}}(Y, X_1) \tag{2.9}$$

is an isomorphism.

The terminology $\mathrm{Ker}(f)$ is justified by the next result.

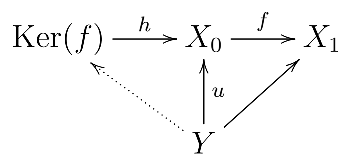
Lemma 2.3.7. *If* $(\mathrm{Ker}(f), h)$ *exists, it is unique up to unique isomorphism.*

Proof. Let $\mathcal{C}$ denote the category defined as follows.

- The objects of $\mathcal{C}$ are the pairs (Y, u) where $u\colon Y \to X_0$ satisfies $f \circ u = 0$,
- a morphism $w\colon (Y, u) \to (Y', u')$ in $\mathcal{C}$ is a morphism $v\colon Y \to Y'$ such that $u' \circ w = u$.

Then $(\mathrm{Ker}(f), h)$ is a terminal object in $\mathcal{C}$. q.e.d.

The isomorphism (2.9) may be translated as follows. Given an objet Y and a morphism $u\colon Y \to X_0$ such that $f \circ u = 0$, the morphism u factors uniquely through $\mathrm{Ker}(f)$. This is visualized by the diagram



Lemma 2.3.8. *Let* $(\mathrm{Ker}(f), h)$ *be the kernel of* f*. Then* h *is a monomorphism.*

Proof. Consider a pair of parallel arrows $a, b\colon Z \rightrightarrows \mathrm{Ker}(f)$ such that $h \circ a = h \circ b$. Then $h \circ (a - b) = 0$ and in particular, $f \circ h \circ (a - b) = 0$. Therefore $h \circ (a - b)$ factors uniquely through $\mathrm{Ker}(f)$. The unicity implies $a - b = 0$. q.e.d.

The cokernel in $\mathcal{C}$ is the kernel in $\mathcal{C}^{\mathrm{op}}$. Hence:

Definition 2.3.9. The cokernel of f, if it exists, is the data of an object $\mathrm{Coker}(f) \in \mathcal{C}$ together with a morphism $k\colon X_1 \to \mathrm{Coker}(f)$ such that, for any $Y \in \mathcal{C}$ and any morphism $w\colon X_1 \to Y$ satisfying $w \circ f = 0$, the the natural morphism

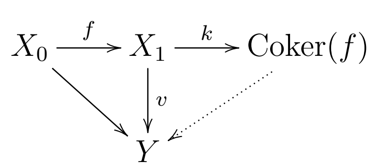
$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{Coker}(f), Y) \to \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(X_0, Y) \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(X_1, Y) \tag{2.10}$$

is an isomorphism.

If $(\mathrm{Coker}(f), k)$ exists, it is unique up to unique isomorphism.

If $(\mathrm{Coker}(f), k)$ exists then k is an epimorphism.

The isomorphism (2.10) may be translated as follows. Given an objet Y and a morphism $v\colon X_1 \to Y$ such that $v \circ f = v \circ g$, the morphism v factors uniquely through $\mathrm{Coker}(f)$. This is visualized by diagram:

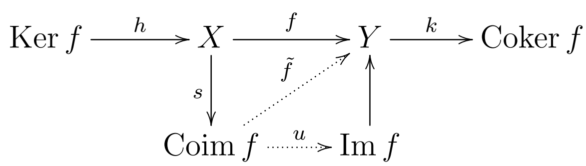


Example 2.3.10. (i) Let A be a ring. The category $\mathrm{Mod}(A)$ admits kernels and cokernels. As already mentioned, the kernel of a linear map $f\colon M \to N$ is the A-module $f^{-1}(0)$ and the cokernel is the quotient module $M/\operatorname{Im} f$.
(ii) Assume that A is not Noetherian, that is, there exists an ideal I of A which is not finitely generated. Then A and A/I belong to $\mathrm{Mod}^{\mathrm{f}}(A)$ but the natural map $A \to A/I$ does not have a kernel in $\mathrm{Mod}^{\mathrm{f}}(A)$.

Let $\mathcal{C}$ be an additive category which admits kernels and cokernels. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. One defines:

$$\begin{aligned} \operatorname{Coim} f &:= \operatorname{Coker} h, \text{ where } h\colon \operatorname{Ker} f \to X \\ \operatorname{Im} f &:= \operatorname{Ker} k, \text{ where } k\colon Y \to \operatorname{Coker} f. \end{aligned}$$

Consider the diagram:



Since $f \circ h = 0$, f factors uniquely through $\tilde{f}$, and $k \circ f$ factors through $k \circ \tilde{f}$. Since $k \circ f = k \circ \tilde{f} \circ s = 0$ and s is an epimorphism, we get that $k \circ \tilde{f} = 0$. Hence $\tilde{f}$ factors through $\operatorname{Ker} k = \operatorname{Im} f$. We have thus constructed a canonical morphism:

$$\operatorname{Coim} f \xrightarrow{u} \operatorname{Im} f. \tag{2.11}$$

Examples 2.3.11. (i) For a ring A and a morphism f in $\mathrm{Mod}(A)$, (2.11) is an isomorphism.
(ii) The category **Ban** admits kernels and cokernels. If $f\colon X \to Y$ is a morphism of Banach spaces, define $\operatorname{Ker} f = f^{-1}(0)$ and $\operatorname{Coker} f = Y/\overline{\operatorname{Im} f}$ where $\overline{\operatorname{Im} f}$ denotes the closure of the space $\operatorname{Im} f$. It is well-known that there exist continuous linear maps $f\colon X \to Y$ which are injective, with dense and non closed image. For such an f, $\operatorname{Ker} f = \operatorname{Coker} f = 0$ although f is not an isomorphism. Thus $\operatorname{Coim} f \simeq X$ and $\operatorname{Im} f \simeq Y$. Hence, the morphism (2.11) is not an isomorphism.

Definition 2.3.12. Let $\mathcal{C}$ be an additive category. One says that $\mathcal{C}$ is abelian if:

(i) any $f\colon X \to Y$ admits a kernel and a cokernel,

(ii) for any morphism f in $\mathcal{C}$, the natural morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism.

In an abelian category, a morphism f is a monomorphism (resp. an epimorphism) if and only if $\operatorname{Ker} f \simeq 0$ (resp. $\operatorname{Coker} f \simeq 0$). If f is both a monomorphism and an epimorphism, then it is an isomorphism.

Examples 2.3.13. (i) If A is a ring, $\mathrm{Mod}(A)$ is an abelian category. If A is Noetherian, then $\mathrm{Mod}^{\mathrm{f}}(A)$ is abelian.
(ii) The category **Ban** admits kernels and cokernels but is not abelian. (See Examples 2.3.11 (ii).)
(iii) If $\mathcal{C}$ is abelian, then $\mathcal{C}^{\mathrm{op}}$ is abelian. (Recall that for a morhism $f\colon X \to Y$ in $\mathcal{C}$, $\operatorname{Ker} f^{\mathrm{op}} \simeq \operatorname{Coker} f$, where $f^{\mathrm{op}}\colon Y \to X$ is the morphism in $\mathcal{C}^{\mathrm{op}}$ associated with f.)
(iv) If $\mathcal{C}$ is abelian, then the categories of complexes $\mathrm{C}^*(\mathcal{C})$ $(* = \mathrm{ub}, \mathrm{b}, +, -)$ are abelian. For example, if $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathcal{C})$, the complex Z defined by $Z^n = \operatorname{Ker}(f^n\colon X^n \to Y^n)$, with differential induced by those of X, will be a kernel for f, and similarly for $\operatorname{Coker} f$.
(v) Let I be category. Then if $\mathcal{C}$ is abelian, the category $\mathrm{Fct}(I, \mathcal{C})$ of functors from I to $\mathcal{C}$, is abelian. If $F, G\colon I \to \mathcal{C}$ are two functors and $\varphi\colon F \to G$ is a morphism of functors, the functor $\operatorname{Ker}\varphi$ is given by $\operatorname{Ker}\varphi(X) = \operatorname{Ker}(F(X) \to G(X))$ and similarly with $\operatorname{Coker}\varphi$. Then the natural morphism $\operatorname{Coim}\varphi \to \operatorname{Im}\varphi$ is an isomorphism.
(vi) If $\mathcal{C}$ and $\mathcal{C}'$ are abelian, then $\mathcal{C} \times \mathcal{C}'$ is abelian.

Consider a complex in an abelian category: $X' \xrightarrow{f} X \xrightarrow{g} X''$. Since $g \circ f = 0$, the morphism g factorizes as

$$\operatorname{Im} f \to \operatorname{Ker} g. \tag{2.12}$$

Definition 2.3.14. (i) One says that a complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ is exact if $\operatorname{Im} f \xrightarrow{\sim} \operatorname{Ker} g$.

(ii) More generally, a sequence of morphisms $X^p \xrightarrow{d^p} \cdots \to X^n$ with $d^{i+1} \circ d^i = 0$ for all $i \in [p, n-1]$ is exact if $\operatorname{Im} d^i \xrightarrow{\sim} \operatorname{Ker} d^{i+1}$ for all $i \in [p, n-1]$.

(iii) A short exact sequence is an exact sequence $0 \to X' \to X \to X'' \to 0$

Any morphism $f\colon X \to Y$ may be decomposed into short exact sequences:

$$0 \to \operatorname{Ker} f \to X \to \operatorname{Coim} f \to 0,$$
$$0 \to \operatorname{Im} f \to Y \to \operatorname{Coker} f \to 0,$$

with $\operatorname{Coim} f \simeq \operatorname{Im} f$.

Proposition 2.3.15. *Let*

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0 \tag{2.13}$$

be a short exact sequence in $\mathcal{C}$. *Then the conditions* (a) *to* (e) *are equivalent.*

(a) *there exists* $h\colon X'' \to X$ *such that* $g \circ h = \mathrm{id}_{X''}$.

(b) *there exists* $k\colon X \to X'$ *such that* $k \circ f = \mathrm{id}_{X'}$.

(c) *there exists* $\varphi = (k, g)$ *and* $\psi = (f + h)$ *such that* $X \xrightarrow{\varphi} X' \oplus X''$ *and* $X' \oplus X'' \xrightarrow{\psi} X$ *are isomorphisms inverse to each other.*

(d) *The complex* (2.13) *is isomorphic to the complex* $0 \to X' \to X' \oplus X'' \to X'' \to 0$.

The proof is the same as that of Proposition 1.4.4.

Definition 2.3.16. As in the case of modules, in the above situation, one says that the exact sequence splits, or that the sequence is split exact.

Note that an additive functor of abelian categories sends split exact sequences into split exact sequences.

Cohomology

The cohomology objects of a complex in an abelian category are defined similarly as in § 1.4.

Consider a complex $(X^\bullet, d^\bullet)$ in $\mathcal{C}$, that is, an object of $\mathrm{C}(\mathcal{C})$. Recall from (2.12) that there are natural morphisms

$$\operatorname{Im} d^{n-1} \to \operatorname{Ker} d^n. \tag{2.14}$$

The n-th group of cohomology of $X^\bullet$ is the object of $\mathcal{C}$ given by

$$H^n(X^\bullet) := \operatorname{Coker}(\operatorname{Im} d^{n-1} \to \operatorname{Ker} d^n) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}.$$

One says that a complex is exact in degree n if $H^n(X^\bullet) \simeq 0$ and that a complex is exact if it is exact in all degrees.

Long exact sequence associated with a short exact sequence

Theorem 2.3.17. *Let* $0 \to X'^{\bullet} \xrightarrow{f} X^{\bullet} \xrightarrow{g} X''^{\bullet} \to 0$ *be an exact sequence in* $\mathrm{C}(\mathcal{C})$*. Then there exists a long sequence*

$$(2.15)\quad \cdots \xrightarrow{\delta^i} H^i(X'^{\bullet}) \xrightarrow{H^i(f)} H^i(X^{\bullet}) \xrightarrow{H^i(g)} H^i(X''^{\bullet}) \xrightarrow{\delta^{i+1}} H^{i+1}(X') \to \cdots .$$

We shall only give the proof when $\mathcal{C} = \mathrm{Mod}(A)$.

Sketch of proof. Let us represent the exact sequence of the statement as a double complex :

$$(2.16)\quad \begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X'^{i-1} & \xrightarrow{f^{i-1}} & X^{i-1} & \xrightarrow{g^{i-1}} & X''^{i-1} & \to & 0 \\ & & \downarrow{\scriptstyle d^{i-1}_{X'}} & & \downarrow{\scriptstyle d^{i-1}_{X}} & & \downarrow{\scriptstyle d^{i-1}_{X''}} & & \\ 0 & \to & X'^{i} & \xrightarrow{f^{i}} & X^{i} & \xrightarrow{g^{i}} & X''^{i} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$

Hence, in this double complex, the rows are exact and the columns are the complexes $X'^{\bullet}$, $X^{\bullet}$ and $X''^{\bullet}$. The morphisms f and g define for each i a sequence

$$H^i(X') \xrightarrow{H^i(f)} H^i(X) \xrightarrow{H^i(g)} H^i(X'')$$

and one easily checks that this sequence is exact.

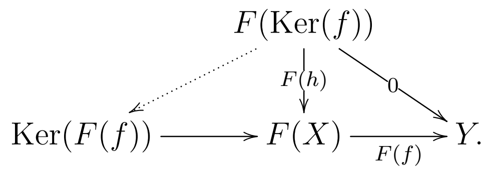
Let us explain how to construct the maps δ^i. Let $x''^{i-1} \in X''^{i-1}$ with $d''x''^{i-1} = 0$ which represents an element of $H^{i-1}(X''^{\bullet})$. Since the rows of the diagram (2.16) are exact, there exists x^{i-1} with $g(x^{i-1}) = x''^{i-1}$. Then $g^i \circ d_X^{i-1}(x^{i-1}) = 0$ and it follow that there exists $x'^i \in X'^i$ with $f^i(x'^i = x^i$ and $d'x'^i = 0$. Then the class of $x'^i \in H^i(X'^{\bullet})$ will depend only on the class of $x''^{i-1} \in H^{i-1}(X''^{\bullet})$ and the maps δ^i's so constructed will have the required properties. q.e.d.

2.4 Exact functors

Let $F\colon \mathcal{C} \to \mathcal{C}'$ be an additive functor of abelian categories and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. Recall that we have an exact sequence $\mathrm{Ker}(f) \xrightarrow{h} X \xrightarrow{f}$

Y. Since F is a functor, $F(f) \circ F(h) = 0$ and it follows that the morphism $F(\mathrm{Ker}(f)) \to F(X)$ factorizes through $\mathrm{Ker}(F(f))$:

(2.17)



In other words, there is a natural morphism

$$F(\mathrm{Ker}(f)) \to \mathrm{Ker}(F(f)). \tag{2.18}$$

Similarly, there exists a natural morphism

$$\mathrm{Coker}(F(f)) \to F(\mathrm{Coker}(f)). \tag{2.19}$$

Remark 2.4.1. In general, the morphisms in (2.17) and (2.19) are not isomorphisms (see Example 2.4.5). In particular, an additive functor of abelian categories $F\colon \mathcal{C} \to \mathcal{C}'$ does not send exact sequences to exact sequences. However, F being additive, it sends split exact sequences to split exact sequences.

Definition 2.4.2. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor of abelian categories. One says that:

(i) F is left exact if it commutes kernels, that is, for any morphism $f\colon X \to Y$, $F(\mathrm{Ker}(f)) \xrightarrow{\sim} \mathrm{Ker}(F(f))$,

(ii) F is right exact if it commutes with cokernels, that is, for any morphism $f\colon X \to Y$, $\mathrm{Coker}(F(f)) \xrightarrow{\sim} F(\mathrm{Coker}(f)$.

(iii) F is exact if it is both left and right exact.

Lemma 2.4.3. *Consider an additive functor $F\colon \mathcal{C} \to \mathcal{C}'$.*

(a) *The conditions below are equivalent:*

   (i) *F is left exact,*

   (ii) *for any exact sequence $0 \to X' \to X \to X''$ in $\mathcal{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathcal{C}'$,*

   (iii) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathcal{C}'$.*

(b) *The conditions below are equivalent:*

(i) *F is exact,*

(ii) *for any exact sequence $X' \to X \to X''$ in $\mathcal{C}$, the sequence $F(X') \to F(X) \to F(X'')$ is exact in $\mathcal{C}'$,*

(iii) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact in $\mathcal{C}'$.*

There is a similar result to (a) for right exact functors.

Proof. The proof is left as an exercise. q.e.d.

Proposition 2.4.4. (i) *The functor $\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Mod}(\mathbb{Z})$ is left exact with respect to each of its arguments.*

(ii) *Consider a pair of functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}'$ and assume (F, G) are adjoint. Then F is right exact and G is left exact.*

(iii) *Let I be a category and let $i \in I$. The functor $\mathrm{Fct}(I, \mathcal{C}) \to \mathcal{C}$, $F \mapsto F(i)$ is exact.*

(iv) *Let A be a ring and let I be a set. The two functors $\prod$ and $\bigoplus$ from $\mathrm{Mod}(A)^I$ to $\mathrm{Mod}(A)$ are exact.*

(v) *Let A be a ring and I a poset. The functor $\varprojlim$ from $\mathrm{Fct}(I^{\mathrm{op}}, \mathrm{Mod}(A))$ to $\mathrm{Mod}(A)$ is left exact.*

(vi) *Let A be a ring and let I a be filtrant poset. The functor $\varinjlim$ from $\mathrm{Fct}(I, \mathrm{Mod}(A))$ to $\mathrm{Mod}(A)$ is exact.*

Proof. (i) follows from (2.9) and (2.10).
(ii) Let us prove that G is left exact. Let $f\colon V \to W$ be a morphism in $\mathcal{C}'$ and let $X \in \mathcal{C}$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(X, G(\mathrm{Ker}\, f)) &\simeq \mathrm{Hom}_{\mathcal{C}}(F(X), \mathrm{Ker}\, f) \\
&\simeq \mathrm{Ker}\,\mathrm{Hom}_{\mathcal{C}}(F(X), f) \\
&\simeq \mathrm{Ker}\,\mathrm{Hom}_{\mathcal{C}}(X, G(f)) \simeq \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{Ker}\, G(f)).
\end{aligned}$$

To conclude, we apply Theorem 2.2.4.
The proof that F is right exact follows by reversing the arrows.
(iii) is obvious and left as an exercise.
(iv) is Proposition 1.4.7.
(v) is Proposition 1.4.8.
(vi) is Proposition 1.4.11. q.e.d.

Note that it follows from Example 1.4.9 that the functor $\varprojlim$ is not right exact.

Example 2.4.5. Let A be a ring and let N be a right A-module. Since the functor $N \otimes_A \bullet$ admits a right adjoint, it is right exact. Let us show that the functors $\mathrm{Hom}_A(\bullet, \bullet)$ and $N \otimes_A \bullet$ are not exact in general. In the sequel, we choose $A = k[x]$, with k a field, and we consider the exact sequence of A-modules:

$$0 \to A \xrightarrow{\cdot x} A \to A/Ax \to 0, \tag{2.20}$$

where $\cdot x$ means multiplication by x.
(i) Apply the functor $\mathrm{Hom}_A(\bullet, A)$ to the exact sequence (2.20). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to A \xrightarrow{x\cdot} A \to 0$$

which is not exact since $x\cdot$ is not surjective. On the other hand, since $x\cdot$ is injective and $\mathrm{Hom}_A(\bullet, A)$ is left exact, we find that $\mathrm{Hom}_A(A/Ax, A) = 0$.
(ii) Apply $\mathrm{Hom}_A(A/Ax, \bullet)$ to the exact sequence (2.20). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A/Ax) \to 0.$$

Since $\mathrm{Hom}_A(A/Ax, A) = 0$ and $\mathrm{Hom}_A(A/Ax, A/Ax) \neq 0$, this sequence is not exact.
(iii) Apply $\bullet \otimes_A A/Ax$ to the exact sequence (2.20). We get the sequence:

$$0 \to A/Ax \xrightarrow{x\cdot} A/Ax \to A/xA \otimes_A A/Ax \to 0.$$

Multiplication by x is 0 on A/Ax. Hence this sequence is the same as:

$$0 \to A/Ax \xrightarrow{0} A/Ax \to A/Ax \otimes_A A/Ax \to 0$$

which shows that $A/Ax \otimes_A A/Ax \simeq A/Ax$ and moreover that this sequence is not exact.
(iv) Notice that the functor $\mathrm{Hom}_A(\bullet, A)$ being additive, it sends split exact sequences to split exact sequences. This shows that (2.20) does not split.

Injective and projective objects

Definition 2.4.6. Let $\mathcal{C}$ be an abelian category.

(i) An object $I \in \mathcal{C}$ is injective if the functor $\mathrm{Hom}_{\mathcal{C}}(\bullet, I) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}(\mathbb{Z})$ is exact.

(ii) An object $P \in \mathcal{C}$ is projective if the functor $\operatorname{Hom}_{\mathcal{C}}(P, \bullet)\colon \mathcal{C} \to \operatorname{Mod}(\mathbb{Z})$ is exact.

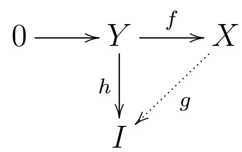
Hence, I is injective in $\mathcal{C}$ if and only if I is projective in $\mathcal{C}^{\mathrm{op}}$.

Example 2.4.7. Let A be a ring. Then free A-modules are projective objects in the category $\operatorname{Mod}(A)$. (See Exercise 2.8.)

Injective objects are useful, thanks to the next result.

Lemma 2.4.8. *Consider the diagram of solid arrows in which the row is exact:*

(2.21)



and assume that I is injective. Then the doted arrow may be completed making the diagram commutative.

Proof. Let us apply the exact functor $\operatorname{Hom}_{\mathcal{C}}(\bullet, I)$ to the sequence $0 \to Y \to X$. We get that the map $\operatorname{Hom}_{\mathcal{C}}(X, I) \xrightarrow{\circ f} \operatorname{Hom}_{\mathcal{C}}(Y, I)$ is surjective. Therefore, there exists $g \in \operatorname{Hom}_{\mathcal{C}}(X, I)$ such that $g \circ f = h$. q.e.d.

Proposition 2.4.9. *Consider an exact sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ and assume that X' is injective. Then the sequence splits.*

Of course there is a similar result when assuming X'' is projective.

Proof. By Lemma 2.4.8 applied with $Y = I = X'$ and $h = \mathrm{id}_{X'}$, we get a morphism $h\colon X \to X'$ such that $h \circ f = \mathrm{id}_{X'}$. Then apply Proposition 2.3.15. q.e.d.

Corollary 2.4.10. *Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories and let $F\colon \mathcal{C} \to \mathcal{C}'$ be an additive functor. Consider an exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ and assume that X' is injective. Then the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.*

2.5 Derived functors

In this section we explain the construction of the derived functor of a left exact functor and give its main properties, without proofs.

Let $\mathcal{C}$ be an abelian category and denote by $\mathcal{I}$ the additive category of injective objects of $\mathcal{C}$.

Definition 2.5.1. One says that $\mathcal{C}$ admits enough injective objects if for any $X \in \mathcal{C}$ there exists $I^0 \in \mathcal{I}$ and an exact sequence $0 \to X \to I^0$.

Assume that $\mathcal{C}$ admits enough injective objects and denote by Z^1 the cokernel of the morphism $X \to I^0$. There exists $I^1 \in \mathcal{I}$ and an exact sequence $0 \to Z^1 \to I^1$. By composing the morphisms $I^) \to Z^1$ and $Z^1 \to I^1$ we get an exact sequence

$$0 \to X \to I^0 \to I^1$$

and by iterating this construction we get a long exact sequence

$$0 \to X \to I^0 \to I^1 \to \cdots \to I^n \to \cdots$$

in which all I^j's are injective objects.

Denote by $I^\bullet$ the complex

$$I^\bullet := 0 \to I^0 \to I^1 \to \cdots \to I^n \to \cdots$$

One says that $I^\bullet$ is an injective complex and that X is quasi-isomorphic to $I^\bullet$, or, for short, that X is qis to $I^\bullet$ or that $X \to I^\bullet$ is a qis.

Let $X, Y \in \mathcal{C}$ and let $X \to I^\bullet$ and $Y \to J^\bullet$ be two qis, with $I^\bullet$ and $J^\bullet$ injective compelexes. One shows that if $f\colon X \to Y$ is a morphism in $\mathcal{C}$, then there exists a morphism of complexes $f^\bullet : I_X^\bullet \to I_Y^\bullet$ making the diagram below commutative:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & X & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots & \longrightarrow & I^n & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle f^n} & & \\ 0 & \longrightarrow & Y & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots & \longrightarrow & J^n & \longrightarrow & \cdots . \end{array} \tag{2.22}$$

Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a left exact functor of abelian categories and assume that $\mathcal{C}$ admits enough injective objects.

Definition 2.5.2. Let $j \in \mathbb{Z}$. The j-th derived functor of F is defined as follows.

(i) For $X \in \mathcal{C}$, choose an injective complex $I_X^\bullet$ and a qis $X \to I_X^\bullet$. One sets $R^jF(X) = H^j(F(I_X^\bullet))$.

(ii) For a morphism $f\colon X \to Y$, choose a morphism $f^\bullet : I_X^\bullet \to I_Y^\bullet$ making the diagram 2.22 commutative and set $R^jF(f) = H^j(F(f^\bullet))$.

One can prove that,

(i) up to isomorphism, $R^jF(X)$ depends only of X and not of the choice of the injective resolution $I^\bullet_X$,

(ii) if $g^\bullet$ is another morphism making the diagram (2.22) commutative, the morphisms $H^j(F(f^\bullet))$ and $H^j(F(g^\bullet))$ are be the same.

One deduces that there exists a well-defined functor $R^jF\colon \mathcal{C} \to \mathcal{C}'$ such that, for any X and any qis $X \to I^\bullet$ where $I^\bullet$ is an injective complex, $R^jF(X)$ is isomorphic to $H^j(F(I^\bullet))$.

By its construction, we have:

- R^jF is an additive functor from $\mathcal{C}$ to $\mathcal{C}'$,
- $R^jF(X) \simeq 0$ for $j < 0$ since $I^j_X = 0$ for $j < 0$,
- $R^0F(X) \simeq F(X)$ since F being left exact, it commutes with kernels,
- $R^jF(X) \simeq 0$ for $j \neq 0$ if F is exact,
- $R^jF(X) \simeq 0$ for $j \neq 0$ if X is injective, by the construction of $R^jF(X)$.

Definition 2.5.3. An object X of $\mathcal{C}$ such that $R^jF(X) \simeq 0$ for all $j > 0$ is called F-acyclic.

Hence, injective objects are F-acyclic for all left exact functors F.

Theorem 2.5.4. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathcal{C}$. Then there exists a long exact sequence:*

$$0 \to F(X') \to F(X) \to \cdots \to R^kF(X') \to R^kF(X) \to R^kF(X'') \to \cdots$$

Sketch of the proof. One constructs an exact sequence of complexes $0 \to X'^\bullet \to X^\bullet \to X''^\bullet \to 0$ whose objects are injective and this sequence is quasi-isomorphic to the sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ in $\mathrm{C}(\mathcal{C})$. Since the objects X'^j are injective, we get a short exact sequence in $\mathrm{C}(\mathcal{C}')$:

$$0 \to F(X'^\bullet) \to F(X^\bullet) \to F(X''^\bullet) \to 0$$

Then one applies Theorem 2.3.17. q.e.d.

Definition 2.5.5. Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. One says that $\mathcal{J}$ is F-injective if:

(i) for any $X \in \mathcal{C}$ there exists $J^0 \in \mathcal{J}$ and an exact sequence $0 \to X \to J^0$.

(ii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X' \in \mathcal{J}, X \in \mathcal{J}$, then $X'' \in \mathcal{J}$,

(iii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X' \in \mathcal{J}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

By considering $\mathcal{C}^{\mathrm{op}}$, one obtains the notion of an F-projective subcategory, F being right exact.

Theorem 2.5.6. *Assume $\mathcal{J}$ is F-injective and contains the category $\mathcal{I}_{\mathcal{C}}$ of injective objects. Let $X \in \mathcal{C}$ and let $0 \to X \to Y^\bullet$ be a resolution of X with $Y^\bullet \in \mathrm{C}^+(\mathcal{J})$. Then for each n, there is an isomorphism $R^nF(X) \simeq H^n(F(Y^\bullet))$.*

In other words, in order to calculate the derived functors $R^nF(X)$, it is enough to replace X with resolution by F-injective objects.

The Ext and Tor groups

Assume that $\mathcal{C}$ has enough injectives and let $Y \in \mathcal{C}$. The j-th right derived functor of the left exact functor $\mathrm{Hom}_{\mathcal{C}}(Y, \bullet)\colon \mathcal{C} \to \mathrm{Mod}(\mathbb{Z})$ is denoted $\mathrm{Ext}^j_{\mathcal{C}}(Y, \bullet)$. Hence,

$$\mathrm{Ext}^j_{\mathcal{C}}(Y, X) \simeq H^j(\mathrm{Hom}_{\mathcal{C}}(Y, I^\bullet_X)),$$

where $I^\bullet_X$ is an injective resolution of X.

If $\mathcal{C}$ has enough projectives, one can also define the j-th right derived functor of the left exact functor $\mathrm{Hom}_{\mathcal{C}}(\bullet, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}(\mathbb{Z})$. One denotes it again by $\mathrm{Ext}^j_{\mathcal{C}}(\bullet, X)$. Hence,

$$\mathrm{Ext}^j_{\mathcal{C}}(Y, X) \simeq H^j(\mathrm{Hom}_{\mathcal{C}}(P^\bullet_Y, X)),$$

where $P^\bullet_Y$ is an projective resolution of Y.

When $\mathcal{C}$ admits both enough injective and projective resolutions, these two constructions coincide. In other words, there are isomorphisms

$$H^j(\mathrm{Hom}_{\mathcal{C}}(Y, I^\bullet_X)), \simeq H^j(\mathrm{Hom}_{\mathcal{C}}(P^\bullet_Y, X)).$$

let $N \in \mathrm{Mod}(A^{\mathrm{op}})$. The left derived functor of the right exact $N \otimes_A \bullet$, denoted $\mathrm{Tor}^A_j(N, \bullet)$ is calculated as follows. Let $M \in \mathrm{Mod}(A)$. Choose a projective resolution $P^\bullet_M$ of M. Then

$$\mathrm{Tor}^A_j(N, M) \simeq H^{-j}(N \otimes_A P^\bullet_M).$$

In fact, it is enough to take flat (see Exercise 2.8) resolutions instead of projective ones.

One can also calculate $\mathrm{Tor}_j^A(N, M)$ by choosing a projective resolution $P_N^\bullet$ of N. In fact, one has the isomorphism

$$H^{-j}(P_N^\bullet \otimes_A M) \simeq H^{-j}(N \otimes_A P_M^\bullet).$$

Example 2.5.7. Let $\mathbf{k}$ be a field and let $A = \mathbf{k}[x_1, \ldots, x_n]$. We identify $\mathbf{k}$ with the A-module $A/(A \cdot x_1 + \cdots + A \cdot x_n)$. By Example 1.5.5, there is a qis $K^\bullet(A, (x_1, \ldots, x_n))\,[n] \to \mathbf{k}$. Since the components of this Koszul complex are free A-modules, we get:

$$\begin{aligned} \mathrm{Ext}_A^j(\mathbf{k}, A) &\simeq H^j(\mathrm{Hom}_A(K^\bullet(A, (x_1, \ldots, x_n))\,[n], A)) \\ &\simeq K^\bullet(A, (x_1, \ldots, x_n)), \end{aligned}$$

where the second isomorphism follows from Proposition 1.5.8. Therefore, $\mathrm{Ext}_A^j(\mathbf{k}, A)$ is zero for $j \neq n$ and is isomorphic to $\mathbf{k}$ for $j = n$.

Example 2.5.8. We follow the notations of Example 1.5.5 and we shall calculate the groups $\mathrm{Tor}_j^{W_n}(\Omega_n, \mathcal{O}_n)$. We have seen that there is a qis

$$K^\bullet(W_n, (\partial_1 \cdot, \ldots, \partial_n \cdot))[n] \to \Omega_n.$$

Since the components of this Koszul complex are free W_n-modules, we get by Proposition 1.5.8:

$$\begin{aligned} \mathrm{Tor}_j^{W_n}(\Omega_n, \mathcal{O}_n) &\simeq H^{-j}(K^\bullet(W_n, (\partial_1 \cdot, \ldots, \partial_n \cdot))[n] \otimes_{W_n} \mathcal{O}_n) \\ &\simeq H^{-j}(K^\bullet(\mathcal{O}_n, (\partial_1 \cdot, \ldots, \partial_n \cdot))[n]). \end{aligned}$$

We find the De Rham complex of $\mathcal{O}_n$ shifted by n. Therefore $\mathrm{Tor}_j^{W_n}(\Omega_n, \mathcal{O}_n)$ is zero for $j \neq n$ and is isomorphic to $\mathbf{k}$ for $j = n$.

Exercises to Chapter 2

Exercise 2.1. Prove that the categories **Set** and $\mathbf{Set}^{\mathrm{op}}$ are not equivalent and similarly with the categories $\mathbf{Set}^f$ and $(\mathbf{Set}^f)^{\mathrm{op}}$.
(Hint: if $F : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ were such an equivalence, then $F(\emptyset) \simeq \{\mathrm{pt}\}$ and $F(\{\mathrm{pt}\}) \simeq \emptyset$. Now compare $\mathrm{Hom}_{\mathbf{Set}}(\{\mathrm{pt}\}, X)$ and $\mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(\{\mathrm{pt}\}), F(X))$ when X is a set with two elements.)

Exercise 2.2. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a faithful functor and let f be a morphism in $\mathcal{C}$. Prove that if $F(f)$ is a monomorphism (resp. an epimorphism), then f is a monomorphism (resp. an epimorphism).

Exercise 2.3. Let $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{C}' \to \mathcal{C}''$ be two functors.
(i) Prove that if $G \circ F$ is faithful, then F is faithful.
(ii) Prove that if $G \circ F$ is fully faithful and G is faithful, then F is fully faithful.

Exercise 2.4. (i) Is the natural functor **Set** $\to$ **Rel**: full, faithful, fully faithful, conservative?
(ii) Prove that the category **Rel** of relations is equivalent to its opposite category.

Exercise 2.5. (i) Prove that in the category **Set**, a morphism f is a monomorphism (resp. an epimorphism) if and only if it is injective (resp. surjective).
(ii) Prove that in the category of rings, the morphism $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.
(iii) In the category **Top**, give an example of a morphism which is both a monomorphism and an epimorphism and which is not an isomorphism.

Exercise 2.6. Let $\mathcal{C}$ be a category. We denote by $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ the identity functor of $\mathcal{C}$ and by $\mathrm{End}\,(\mathrm{id}_{\mathcal{C}})$ the set of endomorphisms of the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, that is, $\mathrm{End}\,(\mathrm{id}_{\mathcal{C}}) = \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$. Prove that the composition law on $\mathrm{End}\,(\mathrm{id}_{\mathcal{C}})$ is commutative.

Exercise 2.7. Consider two complexes in an abelian category $\mathcal{C}$: $X_1' \to X_1 \to X_1''$ and $X_2' \to X_2 \to X_2''$. Prove that the two sequences are exact if and only if the sequence $X_1' \oplus X_2' \to X_1 \oplus X_2 \to X_1'' \oplus X_2''$ is exact.

Exercise 2.8. (i) Prove that a free module is projective.
(ii) Prove that a module P is projective if and only if it is a direct summand of a free module (*i.e.*, there exists a module K such that $P \oplus K$ is free).
(iii) An A-module M is flat if the functor $\bullet \otimes_A M$ is exact. (One defines similarly flat right A-modules.) Deduce from (ii) that projective modules are flat.
(iv) Prove that a filtrant inductive limit of flat modules is flat.

Exercise 2.9. If M is a $\mathbb{Z}$-module, set $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.
(i) Prove that $\mathbb{Q}/\mathbb{Z}$ is injective in $\mathrm{Mod}(\mathbb{Z})$.
(ii) Prove that the map $\mathrm{Hom}_{\mathbb{Z}}(M, N) \to \mathrm{Hom}_{\mathbb{Z}}(N^\vee, M^\vee)$ is injective for any $M, N \in \mathrm{Mod}(\mathbb{Z})$.
(iii) Prove that if P is a right projective A-module, then $P^\vee$ is left A-injective.
(iv) Let M be an A-module. Prove that there exists an injective A-module I and a monomorphism $M \to I$.
(Hint: (iii) Use formula (1.12). (iv) Prove that $M \mapsto M^{\vee\vee}$ is an injective map using (ii), and replace M with $M^{\vee\vee}$.)

Exercise 2.10. Let $\mathcal{C}$ be an abelian category and consider a commutative diagram of complexes

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X_0' & \longrightarrow & X_0 & \longrightarrow & X_0'' \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X_1' & \longrightarrow & X_1 & \longrightarrow & X_1'' \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X_2' & \longrightarrow & X_2 & \longrightarrow & X_2'' \end{array}$$

Assume that all rows are exact as well as the second and third column. Prove that all columns are exact.
(Hint: assume $\mathcal{C} = \mathrm{Mod}(A)$ for a ring A.)

Exercise 2.11. We follow the notations of Examples 1.5.5 and 2.5.8. Calculate $\mathrm{Ext}^j_{W_n}(\mathcal{O}_n, \mathcal{O}_n)$.

Exercise 2.12. We follow the notations of Examples 1.5.5 and 2.5.8 and recall Exercise 1.4. Set $B_1 = W_2/(W_2 \cdot x_1 + W_2 \cdot \partial_2)$ and $B_2 = W_2/(W_2 \cdot x_2 + W_2 \cdot \partial_1)$. Calculate $\mathrm{Ext}^j_{W_2}(B_1, B_2)$.

Chapter 3

Sheaves

In this chapter we expose basic sheaf theory in the framework of topological spaces.
Recall that all along these Notes, $\mathbf{k}$ denotes a commutative unital ring.
Some references: [7, 11, 13, 18].

3.1 Presheaves

Let X be a topological space. The family of open subsets of X is ordered by inclusion. We denote by Op_X the associated category. Hence:

$$\mathrm{Hom}_{\mathrm{Op}_X}(U, V) = \begin{cases} \{\mathrm{pt}\} & \text{if } U \subset V, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that the category Op_X admits a terminal object, namely X, and finite products, namely $U \times V = U \cap V$.

Definition 3.1.1. One sets $\mathrm{PSh}(\mathbf{k}_X) := \mathrm{Fct}((\mathrm{Op}_X)^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k}))$ and calls an object of this category a presheaf of $\mathbf{k}$-modules, or simply a presheaf. In other words, a presheaf on X is a functor from $(\mathrm{Op}_X)^{\mathrm{op}}$ to $\mathrm{Mod}(\mathbf{k})$.

Hence, a presheaf F on X associates to each open subset $U \subset X$ a $\mathbf{k}$-module $F(U)$, and to an open inclusion $V \subset U$, a linear map $\rho_{VU} : F(U) \to F(V)$, such that for each open inclusions $W \subset V \subset U$, one has:

$$\rho_{UU} = \mathrm{id}_U, \qquad \rho_{WU} = \rho_{WV} \circ \rho_{VU}.$$

A morphism of presheaves $\varphi : F \to G$ is thus the data for any open set U of a linear map $\varphi(U) : F(U) \to G(U)$ such that for any open inclusion $V \subset U$, the diagram below commutes:

$$\begin{array}{ccc} F(U) & \xrightarrow{\varphi(U)} & G(U) \\ \downarrow & & \downarrow \\ F(V) & \xrightarrow{\varphi(V)} & G(V) \end{array}$$

The category $\mathrm{PSh}(\mathbf{k}_X)$ inherits of most of the properties of the category $\mathrm{Mod}(\mathbf{k})$. In particular it is abelian. For example, if F and G are two presheaves, the presheaf $U \mapsto F(U) \oplus G(U)$ is the direct sum of F and G in $\mathrm{PSh}(\mathbf{k}_X)$. If $\varphi : F \to G$ is a morphism of presheaves, then $(\mathrm{Ker}\,\varphi)(U) \simeq \mathrm{Ker}\,\varphi(U)$ and $(\mathrm{Coker}\,\varphi)(U) \simeq \mathrm{Coker}\,\varphi(U)$ where $\varphi(U) : F(U) \to G(U)$. Hence, a complex $F' \to F \to F''$ is exact in the category $\mathrm{PSh}(\mathbf{k}_X)$ if and only if, for any $U \in \mathrm{Op}_X$, the sequence $F'(U) \to F(U) \to F''(U)$ is exact in the category $\mathrm{Mod}(\mathbf{k})$. In particular, for $U \in \mathrm{Op}_X$, the functor $\mathrm{PSh}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k})$, $F \mapsto F(U)$ is exact by Proposition 2.4.4.

Notation 3.1.2. (i) One calls the morphisms ρ_{VU}, the restriction morphisms. If $s \in F(U)$, one better writes $s|_V$ instead of $\rho_{VU}(s)$ and calls $s|_V$ the restriction of s to V.
(ii) One denotes by $F|_U$ the presheaf on U defined by $V \mapsto F(V)$, V open in U and calls $F|_U$ the restriction of F to U.

Hence, we have the functor

$$(\bullet)|_U \colon \mathrm{PSh}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_U), \quad F \mapsto F|_U.$$

Clearly, this functor is exact.

Examples 3.1.3. (i) Let $M \in \mathrm{Mod}(\mathbf{k})$. The correspondence $U \mapsto M$ is a presheaf, called the constant presheaf on X with fiber M. For example, if $M = \mathbb{C}$, one gets the presheaf of $\mathbb{C}$-valued constant functions on X.
(ii) Let $\mathcal{C}^0(U)$ denote the $\mathbb{C}$-vector space of $\mathbb{C}$-valued continuous functions on U. Then $U \mapsto \mathcal{C}^0(U)$ (with the usual restriction morphisms) is a presheaf of $\mathbb{C}$-vector spaces, denoted $\mathcal{C}^0_X$.

Definition 3.1.4. Let $x \in X$, and let I_x denote the poset consisting of open neighborhoods of x. Since $U, V \in I_x$ implies $U \cap V \in I_x$, the poset I_x^{op} is filtrant. We consider I_x as a full subcategory of Op_X.

For a presheaf F on X, one sets (see § 1.3):

$$F_x = \varinjlim_{U \in I_x^{\mathrm{op}}} F(U). \tag{3.1}$$

One calls F_x the stalk of F at x.

Let $x \in U$ and let $s \in F(U)$. The image $s_x \in F_x$ of s is called the germ of s at x. Note that any $s_x \in F_x$ is represented by a section $s \in F(U)$ for some open neighborhood U of x, and for $s \in F(U), t \in F(V)$, $s_x = t_x$ means that there exists an open neighborhood W of x with $W \subset U \cap V$ such that $\rho_{WU}(s) = \rho_{WV}(t)$. (See Example 1.3.8.)

Proposition 3.1.5. *The functor $F \mapsto F_x$ from $\mathrm{PSh}(\mathbf{k}_X)$ to $\mathrm{Mod}(\mathbf{k})$ is exact.*

Proof. The functor $F \mapsto F_x$ is the composition

$$\mathrm{PSh}(\mathbf{k}_X) = \mathrm{Fct}(\mathrm{Op}_X^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k})) \to \mathrm{Fct}(I_x^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k})) \to \mathrm{Mod}(\mathbf{k}).$$

The first functor associates to a presheaf F its restriction to the category I_x^{op}. It is clearly exact. Since the poset I_x^{op} is filtrant, the functor $\varinjlim$ is exact by Proposition 1.4.11. q.e.d.

3.2 Sheaves

Notation 3.2.1. For a family $\mathcal{U} := \{U_i\}_{i \in I}$ of open subsets of X indexed by a set I, one sets $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$, etc.

One says that $\mathcal{U}$ is an open covering of U if $\bigcup_i U_i = U$.

Let F be a presheaf on X and consider the two conditions below.

S1 For any open subset $U \subset X$, any open covering $U = \bigcup_i U_i$, any $s \in F(U)$ satisfying $s|_{U_i} = 0$ for all i, one has $s = 0$.

S2 For any open subset $U \subset X$, any open covering $U = \bigcup_i U_i$, any family $\{s_i \in F(U_i), i \in I\}$ satisfying $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all i, j, there exists $s \in F(U)$ with $s|_{U_i} = s_i$ for all i.

Definition 3.2.2. (i) One says that F is separated if it satisfies S1. One says that F is a sheaf if it satisfies S1 and S2.

(ii) One denotes by $\mathrm{Mod}(\mathbf{k}_X)$ the full $\mathbf{k}$-additive subcategory of $\mathrm{PSh}(cor_X)$ whose objects are sheaves and by $\iota_X : \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_X)$ the forgetful functor.

(iii) One writes $\mathrm{Hom}_{\mathbf{k}_X}(\bullet, \bullet)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_X)}(\bullet, \bullet)$.

- If F is a sheaf of $\mathbf{k}$-modules, then $F(\emptyset) = 0$.
- If $\{U_i\}_{i \in I}$ is a family of disjoint open subsets and F is a sheaf, then $F(\bigsqcup_i U_i) = \prod_i F(U_i)$.

- If F is a sheaf on X, then its restriction $F|_U$ to an open subset U is a sheaf.

Notation 3.2.3. Let F be a sheaf of $\mathbf{k}$-modules on X.
(i) One defines its support, denoted by $\operatorname{supp} F$, as the complementary of the union of all open subsets U of X such that $F|_U = 0$. Note that $F|_{X\setminus \operatorname{supp} F} = 0$.
(ii) Let $s \in F(U)$. One can define its support, denoted by $\operatorname{supp} s$, as the complementary of the union of all open subsets U of X such that $s|_U = 0$.

The next result is extremely useful. It says that to check that a morphism of sheaves is an isomorphism, it is enough to do it at each stalk.

Proposition 3.2.4. *Let $\varphi : F \to G$ be a morphism of sheaves.*

(i) *φ is a monomorphism of presheaves if and only if, for all $x \in X$, $\varphi_x : F_x \to G_x$ is injective.*

(ii) *φ is an isomorphism if and only if, for all $x \in X$, $\varphi_x\colon F_x \to G_x$ is an isomorphism.*

Proof. (i) The condition is necessary by Proposition 3.1.5. Assume now φ_x is injective for all $x \in X$ and let us prove that $\varphi\colon F(U) \to G(U)$ is injective. Let $s \in F(U)$ with $\varphi(s) = 0$. Then $(\varphi(s))_x = 0 = \varphi_x(s_x)$, and φ_x being injective, we find $s_x = 0$ for all $x \in U$. This implies that there exists an open covering $U = \cup_i U_i$, with $s|_{U_i} = 0$, and by S1, $s = 0$.
(ii) The condition is clearly necessary. Assume now φ_x is an isomorphism for all $x \in X$ and let us prove that $\varphi\colon F(U) \to G(U)$ is surjective. Let $t \in G(U)$. There exists an open covering $U = \cup_i U_i$ and $s_i \in F(U_i)$ such that $t|_{U_i} = \varphi(s_i)$.

Then, $\varphi(s_i)|_{U_i\cap U_j} = \varphi(s_j)|_{U_i\cap U_j}$, hence by (i), $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ and by S2, there exists $s \in F(U)$ with $s|_{U_i} = s_i$. Since $\varphi(s)|_{U_i} = t|_{U_i}$, we have $\varphi(s) = t$, by S1. q.e.d.

Examples 3.2.5. (i) The presheaf $\mathcal{C}^0_X$ is a sheaf.
(ii) Let $M \in \operatorname{Mod}(\mathbf{k})$. The presheaf of locally constant functions on X with values in M is a sheaf, called the constant sheaf with stalk M and denoted M_X. Note that the constant presheaf with stalk M is not a sheaf except if $M = 0$.
(iii) More generally, let S be a closed subset of X. One defines the constant sheaf M_S with stalk M on S as the sheaf of functions which are locally constant on S with values in M and are 0 on $X \setminus S$. When $S = \{x\}$ for some $x \in X$, the sheaf $M_{\{x\}}$ is called the sky-skrapper sheaf at x with stalk M. Hence, $\Gamma(U; M_{\{x\}})$ is isomorphic to M or 0 according wether $x \in U$ or not.

(iv) On a real manifold X of class C^∞, we have the sheaf $\mathcal{C}_X^\infty$ of complex valued functions of class $\mathcal{C}^\infty$ and the sheaves $\mathcal{C}_X^{\infty,(p)}$ of p-forms with coefficients in $\mathcal{C}^\infty$. These sheaves are also denoted Ω_X^p (hence, $\Omega_X^0 = \mathcal{C}_X^\infty$).

(v) On a complex manifold X, we have the sheaf $\mathcal{O}_X$ of holomorphic functions, and the sheaves Ω_X^p of holomorphic p-forms with coefficients in $\mathcal{O}_X$. (hence, $\Omega_X^0 = \mathcal{O}_X$).

(vi) On a topological space X, the presheaf $U \mapsto \mathcal{C}_X^{0,b}(U)$ of continuous bounded functions is not a sheaf in general. To be bounded is not a local property and axiom S2 is not satisfied.

(vii) Let $X = \mathbb{C}$, and denote by z the holomorphic coordinate. The holomorphic derivation $\frac{\partial}{\partial z}$ is a morphism from $\mathcal{O}_X$ to $\mathcal{O}_X$. Consider the presheaf:

$$F \colon U \mapsto \mathcal{O}(U)/\frac{\partial}{\partial z}\mathcal{O}(U),$$

that is, the presheaf $\mathrm{Coker}(\frac{\partial}{\partial z} : \mathcal{O}_X \to \mathcal{O}_X)$. For U an open disk, $F(U) = 0$ since the equation $\frac{\partial}{\partial z} f = g$ is always solvable. However, if $U = \mathbb{C} \setminus \{0\}$, $F(U) \neq 0$. Hence the presheaf F does not satisfy axiom S1.

Consider the forgetful functor

$$\iota_X : \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_X) \tag{3.2}$$

which, to a sheaf F associates the underlying presheaf. When there is no risk of confusion, we shall often omit the symbol ι_X. In other words, we shall identify a sheaf and the underlying presheaf.

We shall admit the next result.

Theorem 3.2.6. *The forgetful functor ι_X in (3.2) admits a left adjoint*

$$^a : \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_X). \tag{3.3}$$

More precisely, one has the isomorphism, functorial with respect to $F \in \mathrm{PSh}(\mathbf{k}_X)$ and $G \in \mathrm{Mod}(cor_X)$

$$\mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(F, \iota_X G) \simeq \mathrm{Hom}_{\mathbf{k}_X}(F^a, G). \tag{3.4}$$

Moreover (3.4) defines a morphism of presheaves $\theta : F \to F^a$ and $\theta_x : F_x \to F_x^a$ is an isomorphism for all $x \in X$.

Note that if F is locally 0, then $F^a = 0$. If F is a sheaf, then $\theta : F \to F^a$ is an isomorphism.

If F is a presheaf on X, the sheaf F^a is called the sheaf associated with F.

Remark 3.2.7. Assume that the presheaf F is separated, that is, satisfies S1. Then the morphism of presheaves $\theta\colon F \to F^a$ is a monomorphism. Indeed, if $s \in F(U)$ satisfied $\theta(s) = 0$, this implies that $s_x = 0$ for all $x \in U$ and F being separated, $s = 0$.

Example 3.2.8. Let $M \in \mathrm{Mod}(\mathbf{k})$. Then the sheaf associated with the constant presheaf $U \mapsto M$ is the sheaf M_X of M-valued locally constant functions.

Theorem 3.2.9. (a) *The category* $\mathrm{Mod}(\mathbf{k}_X)$ *is abelian and the functor* ι_X : $\mathrm{Mod}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_X)$ *is fully faithful and left exact.*

(b) *The functor* ${}^a\colon \mathrm{PSh}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$ *in* (3.3) *is exact.*

Proof. (a)-(i) Recall that the functor ι_X is fully faithful by the definition of the category $\mathrm{Mod}(\mathbf{k}_X)$.
(a)-(ii) Let $\varphi\colon F \to G$ be a morphism of sheaves and let $\iota_X\varphi\colon \iota_X F \to \iota_X G$ denote the underlying morphism of presheaves. Set $K := \mathrm{Ker}\,\iota_X\varphi$. Hence, K is the presheaf $U \mapsto \mathrm{Ker}(\varphi(U)\colon (F(U) \to G(U))$. Since F is separated, K is separated. Let $U = \bigcup_i U_i$ be an open covering of an open subset U of X and let $\{s_i \in K(U_i), i \in I\}$ satisfying $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all i, j. There exists $s \in F(U)$ with $s|_{U_i} = s_i$ for all i. Since $\varphi(s_i) = 0$ for all i and G is a sheaf, $\varphi(s) = 0$, hence $s \in K(U)$.

We have thus proved that $\mathrm{Ker}\,\iota_X\varphi$ is a sheaf. Let us prove that $\mathrm{Ker}\,\iota_X\varphi$ is the kernel of φ. Consider a morphism of sheaves $\psi\colon H \to F$ such that $\varphi\circ\psi = 0$. The morphism ψ factorizes uniquely through the presheaf $\mathrm{Ker}\,\iota_X\varphi$, that is, through K and it follows that K is the kernel of φ in $\mathrm{Mod}(\mathbf{k}_X)$.
(a)-(iii) Set $L := \mathrm{Coker}\,\iota_X\varphi$. Hence, L is the presheaf $U \mapsto \mathrm{Coker}(\varphi(U))$ where $\varphi(U)$ is the map $F(U) \to G(U)$ associated to φ. Consider a morphism of sheaves $\psi\colon G \to H$ such that $\psi \circ \varphi = 0$. The morphism ψ factorizes uniquely to the presheaf L and it follows from Theorem 3.2.6 that ψ extends uniquely to a morphism of sheaves $L^a \to H$. Therefore, the sheaf L^a is the cokernel of φ in $\mathrm{Mod}(\mathbf{k}_X)$.
(a)-(iv) It follows from (a)-(ii) and (a)-(iii) that for $x \in X$, the germ $(\mathrm{Ker}\,\varphi)_x$ of the kernel of φ is the kernel of $\varphi_x\colon F_x \to G_x$ and similarly, the germ $(\mathrm{Coker}\,\varphi)_x$ of the cokernel of φ is the cokernel of $\varphi_x\colon F_x \to G_x$. It follows that a similar result holds for the image and coimage, and therefore the map $(\mathrm{Coim}\,\varphi)_x \to (\mathrm{Im}\,\varphi)_x$ is an isomorphism for all x. Hence, $\mathrm{Coim}\,\varphi \to \mathrm{Im}\,\varphi$ is an isomorphism by Proposition 3.2.4.
(b)-(i) Let us show that ^{a} commutes with kernels.

The commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Ker}\varphi & \longrightarrow & F & \longrightarrow & G \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{Ker}(\varphi^a) & \longrightarrow & F^a & \longrightarrow & G^a \end{array}$$

defines the morphism $\operatorname{Ker}\varphi \to \operatorname{Ker}\varphi^a$, hence, the morphism $\psi : (\operatorname{Ker}\varphi)^a \to \operatorname{Ker}\varphi^a$. Since the functor $F \mapsto F_x$ commutes both with Ker and with ^{a}, ψ_x is an isomorphism for all x and ψ is an isomorphism by Proposition 3.2.4.
(b)-(ii) Since ^{a} is left adjoint to ι_X, it is right exact. q.e.d.

Recall that the functor $F \mapsto F^a$ commutes with the functors of restriction $F \mapsto F|_U$, as well as with the functor $F \mapsto F_x$.

Proposition 3.2.10. (i) *Let $\varphi : F \to G$ be a morphism of sheaves and let $x \in X$. Then $(\operatorname{Ker}\varphi)_x \simeq \operatorname{Ker}\varphi_x$ and $(\operatorname{Coker}\varphi)_x \simeq \operatorname{Coker}\varphi_x$. In particular the functor $F \mapsto F_x$, from $\operatorname{Mod}(\mathbf{k}_X)$ to $\operatorname{Mod}(\mathbf{k})$ is exact.*

(ii) *Let $F' \xrightarrow{\varphi} F \xrightarrow{\psi} F''$ be a complex of sheaves. Then this complex is exact if and only if for any $x \in X$, the complex $F'_x \xrightarrow{\varphi_x} F_x \xrightarrow{\psi_x} F''_x$ is exact.*

Proof. (i) The result is true in the category of presheaves. Since $\iota_X \operatorname{Ker}\varphi \simeq \operatorname{Ker}\iota_X\varphi$ and $\operatorname{Coker}\varphi \simeq (\operatorname{Coker}\iota_X\varphi)^a$, the result follows.
(ii) By Proposition 3.2.4, $\operatorname{Im}\varphi \simeq \operatorname{Ker}\psi$ if and only if $(\operatorname{Im}\varphi)_x \simeq (\operatorname{Ker}\psi)_x$ for all $x \in X$. Hence the result follows from (i). q.e.d.

By this statement, the complex of sheaves above is exact if and only if for each section $s \in F(U)$ defined in an open neighborhood U of x and satisfying $\psi(s) = 0$, there exists another open neighborhood V of x with $V \subset U$ and a section $t \in F'(V)$ such that $\varphi(t) = s|_V$.

On the other hand, a complex of sheaves $0 \to F' \to F \to F''$ is exact if and only if it is exact as a complex of presheaves, that is, if and only if, for any $U \in \operatorname{Op}_X$, the sequence $0 \to F'(U) \to F(U) \to F''(U)$ is exact.

Examples 3.2.11. Let X be a real manifold of dimension n. The (augmented) de Rham complex is

$$0 \to \mathbb{C}_X \to \mathcal{C}_X^{\infty,(0)} \xrightarrow{d} \cdots \to \mathcal{C}_X^{\infty,(n)} \to 0 \tag{3.5}$$

where d is the differential. This complex of sheaves is exact.
(ii) Let X be a complex manifold of dimension n. The (augmented) holomorphic de Rham complex is

$$0 \to \mathbb{C}_X \to \Omega_X^0 \xrightarrow{d} \cdots \to \Omega_X^n \to 0 \tag{3.6}$$

where d is the holomorphic differential. This complex of sheaves is exact.

Definition 3.2.12. Let $U \in \mathrm{Op}_X$. We denote by $\Gamma(U;\bullet) : \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k})$ the functor $F \mapsto F(U)$.

Proposition 3.2.13. *The functor* $\Gamma(U;\bullet)$ *is left exact.*

Proof. The functor $\Gamma(U;\bullet)$ is the composition

$$\mathrm{Mod}(\mathbf{k}_X) \xrightarrow{\iota_X} \mathrm{PSh}(\mathbf{k}_X) \xrightarrow{\lambda_U} \mathrm{Mod}(\mathbf{k}),$$

where λ_U is the functor $F \mapsto F(U)$. Since ι_X is left exact and λ_U is exact, the result follows. q.e.d.

The functor $\Gamma(U;\bullet)$ is not exact in general. Indeed, consider Example 3.2.5 (v). Recall that $X = \mathbb{C}$, z is a holomorphic coordinate and $U = X \setminus \{0\}$. Then the sequence of sheaves $0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{\partial_z} \mathcal{O}_X \to 0$ is exact. Applying the functor $\Gamma(U;\bullet)$, the sequence one obtains is no more exact.

3.3 $\mathcal{H}om$ and $\otimes$

Definition 3.3.1. Let $F, G \in \mathrm{PSh}(\mathbf{k}_X)$. One denotes by $\mathcal{H}om_{\,\mathrm{PSh}(\mathbf{k}_X)}(F,G)$ or simply $\mathcal{H}om\,(F,G)$ the presheaf on X, $U \mapsto \mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_U)}(F|_U, G|_U)$ and calls it the "internal hom" of F and G.

Proposition 3.3.2. *Let* $F, G \in \mathrm{Mod}(\mathbf{k}_X)$. *Then the presheaf* $\mathcal{H}om\,(F,G)$ *is a sheaf.*

We shall skip the proof.

The functor $\mathrm{Hom}_{\mathbf{k}_X}(\bullet,\bullet)$ being left exact, it follows that

$$\mathcal{H}om\,(\bullet,\bullet) : \mathrm{Mod}(\mathbf{k}_X)^{\mathrm{op}} \times \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$$

is left exact with respect of each of its arguments. Note that

$$\mathrm{Hom}_{\mathbf{k}_X}(\bullet,\bullet) \simeq \Gamma(X;\bullet) \circ \mathcal{H}om\,(\bullet,\bullet).$$

Since a morphism: $\varphi\colon F \to G$ defines a $\mathbf{k}$-linear map $F_x \to G_x$, we get a natural morphism $(\mathcal{H}om\,(F,G))_x \to \mathrm{Hom}\,(F_x, G_x)$. In general, this map is neither injective nor surjective.

Definition 3.3.3. Let $F, G \in \mathrm{Mod}(\mathbf{k}_X)$.

(i) One denotes by $F \overset{\mathrm{psh}}{\otimes} G$ the presheaf on X, $U \mapsto F(U) \otimes_{\mathbf{k}} G(U)$.

(ii) One denotes by $F \otimes_{\mathbf{k}_X} G$ the sheaf associated with the presheaf $F \overset{\text{psh}}{\otimes} G$ and calls it the tensor product of F and G. If there is no risk of confusion, one writes $F \otimes G$ instead of $F \otimes_{\mathbf{k}_X} G$.

The functor

$$\bullet \otimes \bullet : \mathrm{Mod}(\mathbf{k}_X) \times \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$$

is the composition of the right exact functor $\overset{\text{psh}}{\otimes}$ and the exact functor ^{a}. This functor is thus right exact and if $\mathbf{k}$ is a field, it is exact. Note that for $x \in X$ and $U \in \mathrm{Op}_X$:

(i) $(F \otimes G)_x \simeq F_x \otimes G_x$,

(ii) $\mathcal{H}om\,(F, G)|_U \simeq \mathcal{H}om\,(F|_U, G|_U)$,

(iii) $\mathcal{H}om\,(\mathbf{k}_X, F) \simeq F$,

(iv) $\mathbf{k}_X \otimes F \simeq F$.

Example 3.3.4. Let $\mathcal{C}_X^\infty$ denote as above the sheaf of real valued $\mathcal{C}^\infty$-functions on a real manifold X. If V is a finite $\mathbb{R}$-dimensional vector space (e.g., $V = \mathbb{C}$), then the sheaf of V-valued $\mathcal{C}^\infty$-functions is nothing but $\mathcal{C}_X^\infty \otimes_{\mathbb{R}_X} V_X$.

3.4 Locally constant and locally free sheaves

Locally constant sheaves

Definition 3.4.1. (i) Let M be a $\mathbf{k}$-module. Recall that the sheaf M_X is the sheaf of locally constant M-valued functions on X. It is also the sheaf associated with the constant presheaf $U \mapsto M$.

(ii) A sheaf F on X is constant if it is isomorphic to a sheaf M_X, for some $M \in \mathrm{Mod}(\mathbf{k})$.

(iii) A sheaf F on X is locally constant if there exists an open covering $X = \bigcup_i U_i$ such that $F|_{U_i}$ is a constant sheaf of U_i.

Recall that a morphism of sheaves which is locally an isomorphism is an isomorphism of sheaves. However, given two sheaves F and G, it may exist an open covering $\{U_i\}_{i \in I}$ of X and isomorphisms $F|_{U_i} \xrightarrow{\sim} G|_{U_i}$ for all $i \in I$, although these isomorphisms are not induced by a globally defined isomorphism $F \to G$.

Example 3.4.2. Consider $X = \mathbb{R}$ and consider the $\mathbb{C}$-valued function $t \mapsto \exp(t)$, that we simply denote by $\exp(t)$. Consider the sheaf $\mathbb{C}_X \cdot \exp(t)$ consisting of functions which are locally a constant multiple of $\exp(t)$. Clearly $\mathbb{C}_X \cdot \exp(t)$ is isomorphic to the constant sheaf $\mathbb{C}_X$, hence, is a constant sheaf. Note that this sheaf may also be defined by the exact sequence

$$0 \to \mathbb{C}_X \cdot \exp(t) \to C^\infty_X \xrightarrow{P} C^\infty_X \to 0$$

where P is the differential operator $\dfrac{\partial}{\partial t} - 1$.

Examples 3.4.3. (i) If X is not connected it is easy to construct locally constant sheaves which are not constant. Indeed, let $X = U_1 \sqcup U_2$ be a covering by two non-empty open subsets, with $U_1 \cap U_2 = \emptyset$. Let $M \in \mathrm{Mod}(\mathbf{k})$ with $M \neq 0$. Then the sheaf which is 0 on U_1 and M_{U_2} on U_2 is locally constant and not constant.
(ii) Let $X = \mathbb{C} \setminus \{0\}$ with holomorphic coordinate z and consider the differential operator $P = z\frac{\partial}{\partial z} - \alpha$, where $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. Let us denote by K_α the kernel of P acting on $\mathcal{O}_X$.

Let U be an open disk in X centered at z_0, and let $A(z)$ denote a primitive of α/z in U. We have a commutative diagram of sheaves on U:

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{z\frac{\partial}{\partial z} - \alpha} & \mathcal{O}_X \\ {\scriptstyle \exp(-A(z))}\downarrow & & \downarrow{\scriptstyle \frac{1}{z}\exp(-A(z))} \\ \mathcal{O}_X & \xrightarrow{\frac{\partial}{\partial z}} & \mathcal{O}_X \end{array}$$

Therefore, one gets an isomorphism of sheaves $K_\alpha|_U \xrightarrow{\sim} \mathbb{C}_X|_U$, which shows that K_α is locally constant, of rank one.

On the other hand, $f \in \mathcal{O}(X)$ and $Pf = 0$ implies $f = 0$. Hence $\Gamma(X; K_\alpha) = 0$, and K_α is a locally constant sheaf of rank one on $\mathbb{C} \setminus \{0\}$ which is not constant.

Locally free sheaves

A sheaf of $\mathbf{k}$-algebras (or, equivalently, a $\mathbf{k}_X$-algebra) $\mathcal{A}$ on X is a sheaf of $\mathbf{k}$-modules such that for each $U \subset X$, $\mathcal{A}(U)$ is endowed with a structure of a $\mathbf{k}$-algebra, and the operations (addition, multiplication) commute to the restriction morphisms. A sheaf of $\mathbb{Z}$-algebras is simply called a sheaf of rings. If $\mathcal{A}$ is a sheaf of rings, one defines in an obvious way the notion of a sheaf F of (left) $\mathcal{A}$-modules (or simply, an $\mathcal{A}$-module) as follows: for each open set $U \subset X$, $F(U)$ is an $\mathcal{A}(U)$-module and the action of $\mathcal{A}(U)$ on $F(U)$ commutes to the restriction morphisms. One also naturally defines the notion of an $\mathcal{A}$-linear morphism of $\mathcal{A}$-modules. Hence we have defined the category $\mathrm{Mod}(\mathcal{A})$ of $\mathcal{A}$-modules.

Examples 3.4.4. (i) Let A be a $\mathbf{k}$-algebra. The constant sheaf A_X is a sheaf of $\mathbf{k}$-algebras.
(ii) On a topological space, the sheaf $\mathcal{C}^0_X$ is a $\mathbb{C}_X$-algebra. If X is open in $\mathbb{R}^n$, the sheaf $\mathcal{C}^\infty_X$ is a $\mathbb{C}_X$-algebra. The sheaf $\mathcal{D}b_X$ is a $\mathcal{C}^\infty_X$-module.
(iii) If X is open in $\mathbb{C}^n$, the sheaf $\mathcal{O}_X$ is a $\mathbb{C}_X$-algebra.

The category $\mathrm{Mod}(\mathcal{A})$ is clearly an additive subcategory of $\mathrm{Mod}(\mathbf{k}_X)$. Moreover, if $\varphi : F \to G$ is a morphism of $\mathcal{A}$-modules, then $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ will be $\mathcal{A}$-modules. One checks easily that the category $\mathrm{Mod}(\mathcal{A})$ is abelian, and the natural functor $\mathrm{Mod}(\mathcal{A}) \to \mathrm{Mod}(\mathbf{k}_X)$ is exact and faithful (but not fully faithful). Now consider a sheaf of rings $\mathcal{A}$.

Definition 3.4.5. (i) A sheaf $\mathcal{L}$ of $\mathcal{A}$-modules is locally free of rank r (resp. of finite rank) if there exists an open covering $X = \cup_i U_i$ such that $\mathcal{L}|_{U_i}$ is isomorphic to a direct sum of r copies (resp. to a finite direct sum) of $\mathcal{A}|_{U_i}$.

(ii) A locally free sheaf of rank one is called an invertible sheaf.

Gluing sheaves

Let X be a topological space, and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X. One sets $U_{ij} = U_i \cap U_j, U_{ijk} = U_{ij} \cap U_k$. First, consider a sheaf F on X, set $F_i = F|_{U_i}$, $\theta_i \colon F|_{U_i} \xrightarrow{\sim} F_i$, $\theta_{ji} = \theta_j \circ \theta_i^{-1}$. Then clearly:

$$\left.\begin{aligned} \theta_{ii} &= \text{id on } U_i, \\ \theta_{ij}\circ\theta_{jk} &= \theta_{ik} \text{ on } U_{ijk}. \end{aligned}\right\} \tag{3.7}$$

The family of isomorphisms $\{\theta_{ij}\}$ satisfying conditions (3.7) is called a 1-cocycle. Let us show that one can reconstruct F from the data of a 1-cocycle.

Theorem 3.4.6. *Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X and let F_i be a sheaf on U_i. Assume to be given for each pair (i, j) an isomorphism of sheaves $\theta_{ji} : F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$, these isomorphisms satisfying the conditions (3.7).*

Then there exists a sheaf F on X and for each i isomorphisms $\theta_i : F|_{U_i} \xrightarrow{\sim} F_i$ such that $\theta_j = \theta_{ji} \circ \theta_i$. Moreover, $(F, \{\theta_i\}_{i\in I})$ is unique up to unique isomorphism.

This result is out of the scope of the course and we shall not prove it here.

Remark 3.4.7. (i) If the F_i's are locally constant, then F is locally constant.
(ii) In the situation of Theorem 3.4.6, if $\mathcal{A}$ is a sheaf of $\mathbf{k}$-algebras on X and if all F_i's are sheaves of $\mathcal{A}|_{U_i}$ modules and the isomorphisms θ_{ji} are $\mathcal{A}|_{U_{ij}}$-linear, the sheaf F constructed in Theorem 3.4.6 will be naturally endowed with a structure of a sheaf of $\mathcal{A}$-modules.

Example 3.4.8. Assume $\mathbf{k}$ is a field, and recall that $\mathbf{k}^\times$ denote the multiplicative group $\mathbf{k} \setminus \{0\}$. Let $X = \mathbb{S}^1$ be the 1-sphere, and consider a covering of X by two open connected intervals U_1 and U_2. Let $U_{12}^\pm$ denote the two connected components of $U_1 \cap U_2$. Let $\alpha \in \mathbf{k}^\times$. One defines a locally constant sheaf L_α on X of rank one over $\mathbf{k}$ by gluing $\mathbf{k}_{U_1}$ and $\mathbf{k}_{U_2}$ as follows. Let $\theta_\varepsilon : \mathbf{k}_{U_1}|_{U_{12}^\varepsilon} \to \mathbf{k}_{U_2}|_{U_{12}^\varepsilon}$ ($\varepsilon = \pm$) be defined by $\theta_+ = 1$, $\theta_- = \alpha$.

Assume that $\mathbf{k} = \mathbb{C}$. One can give a more intuitive description of the sheaf L_α as follows. Let us identify $\mathbb{S}^1$ with $[0,1]/\sim$, where $\sim$ is the relation which identifies 0 and 1. Choose $\beta \in \mathbb{C}$ with $\exp(2i\pi\beta) = \alpha$. If $\beta \notin \mathbb{Z}$, the function $\theta \mapsto \exp(2i\pi\beta\theta)$ is not well defined on $\mathbb{S}^1$ since it does not take the same value at 0 and at 1. However, the sheaf $\mathbb{C}_X \cdot \exp(2i\pi\beta\theta)$ of functions which are a constant multiple of the function $\exp(2i\pi\beta\theta)$ is well-defined on each of the intervals U_1 and U_2, hence is well defined on $\mathbb{S}^1$, although it does not have any global section.

Example 3.4.9. Consider an n-dimensional real manifold X of class $\mathcal{C}^\infty$, and let $\{X_i, f_i\}$ be an atlas, that is, the X_i are open subsets of X and $f_i : X_i \xrightarrow{\sim} U_i$ is a $\mathcal{C}^\infty$-isomorphism with an open subset U_i of $\mathbb{R}^n$. Let $U_{ij}^i = f_i(X_{ij})$ and denote by f_{ji} the map

$$f_{ji} = f_j|_{X_{ij}} \circ f_i^{-1}|_{U_{ij}^i} : U_{ij}^i \to U_{ij}^j. \tag{3.8}$$

The maps f_{ji} are called the transition functions. They are isomorphisms of class $\mathcal{C}^\infty$. Denote by J_f the Jacobian matrix of a map $f : \mathbb{R}^n \supset U \to V \subset \mathbb{R}^n$. Using the formula $J_{g\circ f}(x) = J_g(f(x)) \circ J_f(x)$, one gets that the locally constant function on X_{ij} defined as the sign of the Jacobian determinant $\det J_{f_{ji}}$ of the f_{ji}'s is a 1-cocycle. It defines a sheaf locally isomorphic to $\mathbb{Z}_X$ called the orientation sheaf on X and denoted by or_X.

Remark 3.4.10. In the situation of Theorem 3.4.6, if $\mathcal{A}$ is a sheaf of $\mathbf{k}$-algebras on X and if all F_i's are sheaves of $\mathcal{A}|_{U_i}$ modules and the isomorphisms θ_{ji} are $\mathcal{A}|_{U_{ij}}$-linear, the sheaf F constructed in Theorem 3.4.6 will be naturally endowed with a structure of a sheaf of $\mathcal{A}$-modules.

Example 3.4.11. (i) Let $X = \mathbb{P}^1(\mathbb{C})$, the Riemann sphere. Then $\Omega_X := \Omega_X^1$ is locally free of rank one over $\mathcal{O}_X$. Since $\Gamma(X; \Omega_X) = 0$, this sheaf is not globally free.

(ii) Consider the covering of X by the two open sets $U_1 = \mathbb{C}$, $U_2 = X \setminus \{0\}$. One can glue $\mathcal{O}_X|_{U_1}$ and $\mathcal{O}_X|_{U_2}$ on $U_1 \cap U_2$ by using the isomorphism $f \mapsto z^p f$ ($p \in \mathbb{Z}$). One gets a locally free sheaf of rank one. For $p \neq 0$ this sheaf is not free.

3.5 Flabby sheaves and soft sheaves

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Flabby sheaves

Definition 3.5.1. On a topological space X, an object $F \in \mathrm{Mod}(k_X)$ is flabby if for any open subset U of X the restriction map $\Gamma(X;F) \to \Gamma(U;F)$ is surjective.

Of course, If F is flabby and U is open in X, then $F|_U$ is flabby on U.

Proposition 3.5.2. *Let $0 \to F' \xrightarrow{\alpha} F \xrightarrow{\beta} F'' \to 0$ be an exact sequence of sheaves, and assume F' is flabby. Then the sequence*

$$0 \to \Gamma(X;F') \xrightarrow{\alpha} \Gamma(X;F) \xrightarrow{\beta} \Gamma(X;F'') \to 0$$

is exact.

Proof. Let $s'' \in \Gamma(X;F'')$ and let $\sigma = \{(U;s);\, U \text{ open in } X,\ s \in \Gamma(U;F),\ \beta(s) = s''|_U\}$. Then σ is naturally inductively ordered. Let $(U;s)$ be a maximal element, and assume $U \neq X$.

Let $x \in X \setminus U$, let V be an open neighborhood of x and let $t \in \Gamma(U;F)$ such that $\beta(t) = s''|_V$. Such a pair $(V;t)$ exists since $\beta : F_x \to F''_x$ is surjective. On $U \cap V$, $s - t \in \Gamma(U \cap V; F')$. Let $r \in \Gamma(X;F')$ which extends $s-t$. Then $s-(t+r) = 0$ on $U \cap V$, hence there exists a section $\tilde{s} \in \Gamma(U \cup V; F)$ with $\tilde{s}|_U = s$, $\tilde{s}|_V = t + r$, and $\beta(\tilde{s}) = s''$. This is a contradiction. q.e.d.

Proposition 3.5.3. *Let $X = \bigcup_{i \in I} U_i$ be an open covering of X and let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Assume that $F|_{U_i}$ is flabby for all $i \in I$. Then F is flabby.*

In other words, flabbyness is a local property.

Proof. Let U be an open subset of X and let $s \in F(U)$. Let us prove that s extends to a global section of F. Let $\mathfrak{S}$ be the family of pairs (t, V) such that V is open and contains U and $t|_U = s$. We order $\mathfrak{S}$ as follows:$(t, V) \leq (t', V')$ if $V \subset V'$ and $t'|_V = t$. Then $\mathfrak{S}$ is inductively ordered. Therefore, there exists a maximal element (t, V). Let us show that $V = X$. Otherwise, there exists

$x \in X \setminus V$ and an $i \in I$ such that $x \in U_i$. Then $t|_{U_i \cap V} \in F(U_i \cap V)$ extends to a section $t_i \in F(U_i)$. Since $t_i|_{U_i \cap V} = t|_{U_i \cap V}$, the section t extends to a section on $V \cup U_i$ which contredicts the fact that V is maximal. q.e.d.

Proposition 3.5.4. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves. Assume F' and F are flabby. Then F'' is flabby.*

Proof. Let U be an open subset of X and consider the diagram:

$$\begin{array}{ccccc} \Gamma(X;F) & \longrightarrow & \Gamma(X;F'') & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \\ \Gamma(U;F) & \xrightarrow{\beta} & \Gamma(U;F'') & \longrightarrow & 0 \end{array}$$

Then α is surjective since F is flabby and β is surjective since F' is flabby, in view of the preceding proposition. This implies γ is surjective, hence F'' is flabby. q.e.d.

Soft sheaves

In this subsection all spaces are assumed to be locally compact. For a compact subset K of X we set

$$F(K) = \Gamma(K;F) := \varinjlim_{K \subset U} \Gamma(U;F).$$

Definition 3.5.5. Assume X is locally compact. A sheaf F on X is soft if for any compact subset K of X, the map $\Gamma(X;F) \to \Gamma(K;F)$ is onto.

Of course, If F is soft and U is open in X, then $F|_U$ is soft on U.

Proposition 3.5.6. *Assume X is locally compact and let $F \in \mathrm{Mod}(\mathbf{k}_X)$ be soft. Let K_1 and K_2 be two compact subsets of X and set for short $K_{12} = K_1 \cap K_2$. Then the sequence*

$$0 \to F(K_1 \cup K_2) \xrightarrow{\alpha} F(K_1) \oplus F(K_2) \xrightarrow{\beta} F(K_1 \cap K_2) \to 0 \tag{3.9}$$

is exact. Here $\alpha(u) = (u|_{K_1}, u|_{K_2})$ and $\beta(v_1, v_2) = v_1|_{K_{12}} - v_2|_{K_{12}}$.

Proof. We have to prove that β is surjective. Since any $s \in F(K_1 \cap K_2)$ extends as a section $\widetilde{s} \in F(X)$, we may choose $s_1 = \widetilde{s}|_{K_1}$ and $s_2 = 0$. q.e.d.

Lemma 3.5.7. *Assume X is locally compact. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves and assume F' is soft. Let K be a compact subset of X. Then the sequence below is exact:*

$$0 \to \Gamma(K;F') \xrightarrow{\alpha} \Gamma(K;F) \xrightarrow{\beta} \Gamma(K;F'') \to 0.$$

Proof. Let $\{K_i\}_{i=1}^n$ be a finite covering of K by compact subsets such that there exist $s_i \in \Gamma(K_i; F)$ with $\beta(s_i) = s''|_{K_i}$. We argue by induction on n, and reduce the proof to the case $n = 2$. Then $s_1|_{K_1 \cap K_2} - s_2|_{K_1 \cap K_2}$ belongs to $\Gamma(K_1 \cap K_2; F')$. We extend this element to $s' \in \Gamma(X; F')$ and replace s_2 by $s_2 + s'$. Hence there exists $t \in \Gamma(K_1 \cup K_2; F)$ with $\beta(t) = s''$ and the induction proceeds. q.e.d.

Proposition 3.5.8. *Assume X is locally compact and countable at infinity. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves and assume that F' is soft. Then the sequence below is exact.*

$$0 \to \Gamma(X; F') \xrightarrow{\alpha} \Gamma(X; F) \xrightarrow{\beta} \Gamma(X; F'') \to 0.$$

Proof. Let $\{K_n\}_{n \in \mathbb{N}}$ be an increasing sequence of compact subsets of X, with $X = \cup_n K_n$ and K_n contained in the interior of K_{n+1}. By Lemma 3.5.7 the sequences

$$0 \to \Gamma(K_n; F') \to \Gamma(K_n; F) \to \Gamma(K_n; F'') \to 0$$

are all exact. Moreover the morphisms $\Gamma(K_{n+1}; F') \to \Gamma(K_n; F')$ are all surjective since F' is soft. Hence the sequence obtained by taking the projective limit will remain exact by Proposition 1.4.10. This completes the proof since for any sheaf G, $G(X) \simeq \varprojlim_K G(K)$, where K ranges over the family of compact subsets of X. q.e.d.

Proposition 3.5.9. . *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves, and assume F' and F are c-soft. Then F'' is soft.*

The proof is similar to that of Proposition 3.5.4.

Proposition 3.5.10. *Assume X is locally compact and countable at infinity. Let $X = \bigcup_{i \in I} U_i$ be an open covering of X and let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Assume that $F|_{U_i}$ is soft for all $i \in I$. Then F is soft.*

In other words, to be soft is a local property.

Proof. The proof is similar to that of Proposition 3.5.3. q.e.d.

Example 3.5.11. (i) On a locally compact space X, any sheaf of C^0_X-modules is soft.
(ii) Let X be a real manifold of class C^∞, let K be a compact subset of X and U an open neighborhood of K in X. By the existence of "partition of unity", there exists a real $\mathcal{C}^\infty$-function φ with compact support contained in U and which is identically 1 in a neighborhood of K. It follows that any sheaf of C^∞_X-modules is soft.
(iii) Flabby sheaves are soft.

3.6 Cohomology of sheaves

We shall admit here that the category $\mathrm{Mod}(\mathbf{k}_X)$ of sheaves of $\mathbf{k}$-modules on X admits enough injective objects and moreover that injective sheaves are flabby. Hence, we may derive any left exact functor defined on this category.

Definition 3.6.1. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$ and let U be an open subset of X. One sets

$$H^j(U; F) := R^j\Gamma(U; \bullet)(F).$$

In other words, $H^j(U; F)$ is the j-th derived functor of the functor $\Gamma(U; \bullet)$ calculated at F.

Recall that the groups $H^j(U; F)$ are calculated as follows. Choose an injective resolution of F:

$$0 \to F \to F^0 \to F^1 \to \cdots$$

and denote by $F^\bullet$ the complex

$$F^\bullet := 0 \to F^0 \to F^1 \to \cdots$$

Then

$$H^j(U; F) \simeq H^j(\Gamma(U; F^\bullet)).$$

Moreover, it follows from the results of § 3.5 and Theorem 2.5.6 that we may replace the injective resolution by a flabby resolution, or, when X is locally compact and countable at infinity, by a soft resolution.

Cousin problem and Mayer-Vietoris sequence

Consider two open subsets U_1 and U_2 of X and set for short $U_{12} := U_1 \cap U_2$. The Cousin problem, which was first formulated for holomorphic functions on the complex line, is translated as follows for a sheaf F on X:

> given $s \in F(U_{12})$, can we write s as $s = s_1|_{U_{12}} - s_2|_{U_{12}}$ with $s_i \in F(U_i)$ $(i = 1, 2)$.

Consider the exact sequence

$$0 \to F(U_1 \cup U_2) \xrightarrow{a} F(U_1) \oplus F(U_2) \xrightarrow{b} F(U_{12})$$

in which $a(s) = (s|_{U_1}, s|_{U_2})$ and $b((s_1, s_2)) = s_1|_{U_{12}} - s_2|_{U_{12}}$. Hence the Cousin problem is that of the surjectivity of the map b. The answer is given by the long exact sequence below.

Theorem 3.6.2. The Mayer-Vietoris long exact sequence. *There exists a long exact sequence*

$$
\begin{aligned}
(3.10)\qquad & 0 \to F(U_1 \cup U_2) \xrightarrow{a} F(U_1) \oplus F(U_2) \xrightarrow{b} F(U_{12}) \to H^1 U_1 \cup U_2; F) \\
& \to H^1(U_1; F) \oplus H^1(U_2; F) \to H^1(U_{12}; F) \to \cdots .
\end{aligned}
$$

Proof. If F is injective, the map b is surjective. It follows that if $F^\bullet$ is a complex of injective sheaves, the sequence of complexes

$$0 \to F^\bullet(U_1 \cup U_2) \xrightarrow{a} F^\bullet(U_1) \oplus F^\bullet(U_2) \xrightarrow{b} F^\bullet(U_{12}) \to 0$$

is exact. Now choose a complex of injective sheaves $F^\bullet$ and a qis $F \to F^\bullet$. Since $H^j(V; F) \simeq H^j(\Gamma(V; F^\bullet))$ for any open set V, the result follows from Theorem 2.3.17 . q.e.d.

De Rham cohomology

Let X be a real $\mathcal{C}^\infty$-manifold of dimension n (this implies in particular that X is locally compact and countable at infinity). If $n > 0$, the sheaf $\mathbb{C}_X$ is not acyclic for the functor $\Gamma(X; \cdot)$ in general. In fact consider two connected open subsets U_1 and U_2 such that $U_1 \cap U_2$ has two connected components, V_1 and V_2. The sequence:

$$0 \to \Gamma(U_1 \cup U_2; \mathbb{C}_X) \to \Gamma(U_1; \mathbb{C}_X) \oplus \Gamma(U_2; \mathbb{C}_X) \to \Gamma(U_1 \cap U_2; \mathbb{C}_X) \to 0$$

is not exact since the locally constant function $\varphi = 1$ on V_1, $\varphi = 2$ on V_2 may not be decomposed as $\varphi = \varphi_1 - \varphi_2$, with φ_j constant on U_j. By the Mayer-Vietoris long exact sequence, this implies:

$$H^1(U_1 \cup U_2; \mathbb{C}_X) \neq 0.$$

On the other hand, we have seen in Example 3.5.11 that any sheaf of $\mathcal{C}^\infty_X$-modules is soft.

Denote by $\mathcal{C}^{\infty,(p)}_X$ or else, Ω^p_X, the sheaf on X of differential forms of degree p with $\mathcal{C}^\infty_X$ coefficients. These sheaves are soft and in particular $\Gamma(X; \cdot)$ acyclic.

Consider the complex of sheaves on X:

$$\mathrm{DR}_X \;\; := \;\; 0 \to \Omega^0_X \xrightarrow{d} \cdots \to \Omega^n_X \to 0.$$

We call it the De Rham complex on X with $\mathcal{C}^\infty$ coefficients.

Lemma 3.6.3. (The Poincaré lemma.) *Let $I = (]0,1[)^n$ be the unit open cube in $\mathbb{R}^n$. The complex below is exact.*

$$0 \to \mathbb{C} \to \mathcal{C}^{\infty,(0)}(I) \xrightarrow{d} \cdots \to \mathcal{C}^{\infty,(n)}(I) \to 0.$$

Proof. Consider the Koszul complex $K^\bullet(M,\varphi)$ over the ring $\mathbb{C}$, where $M = \mathcal{C}^\infty(I)$ and $\varphi = (\partial_1, \ldots, \partial_n)$ (with $\partial_j = \frac{\partial}{\partial x_j}$). This complex is nothing but the complex:

$$0 \to \mathcal{C}^{\infty,(0)}(I) \xrightarrow{d} \cdots \to \mathcal{C}^{\infty,(n)}(I) \to 0.$$

Clearly $H^0(K^\bullet(M,\varphi)) \simeq \mathbb{C}$, and it is enough to prove that the sequence $(\partial_1, \ldots, \partial_n)$ is coregular. Let $M_{j+1} = \operatorname{Ker}(\partial_1) \cap \cdots \cap \operatorname{Ker}(\partial_j)$. This is the space of $\mathcal{C}^\infty$-functions on I constant with respect to the variables $x_1, \ldots, x_j$. Clearly, ∂_{j+1} is surjective on this space. q.e.d.

Lemma 3.6.3 implies:

Lemma 3.6.4. *Let X be a $\mathcal{C}^\infty$-manifold of dimension n. Then the natural morphism $\mathbb{C}_X \to \mathrm{DR}_X$ is a quasi-isomorphism.*

Corollary 3.6.5. (The de Rham theorem.) *Let X be a $\mathcal{C}^\infty$-manifold of dimension n. Then $H^j(X;\mathbb{C}_X)$ is isomorphic to $H^j(\Gamma(X;\mathrm{DR}_X))$.*

Note that this result in particular implies that $H^j(\Gamma(X;\mathrm{DR}_X))$ is a topological invariant of X.

Cohomology of complex manifolds

Assume now that X is a complex manifold of complex dimension n, and let $X^{\mathbb{R}}$ be the real underlying manifold. The real differential d splits as $\partial + \bar\partial$, and one denotes by $\mathcal{C}_X^{\infty,(p,q)}$ the sheaf of $\mathcal{C}^\infty$ forms of type (p,q) with respect to $\partial, \bar\partial$. Consider the complex, called the Dolbeault complex (or also the Dolbeault-Grothendieck complex):

$$\mathrm{DB}_X := 0 \to \mathcal{C}_X^{\infty(0,0)} \xrightarrow{\bar\partial} \cdots \to \mathcal{C}_X^{\infty,(0,n)}.$$

The complex Poincaré lemma (that we shall not prove here) is formulated as:

Lemma 3.6.6. *Let X be a complex manifold. Then the natural morphism $\mathcal{O}_X \to \mathrm{DB}_X$ is a quasi-isomorphism.*

Since the sheaves $\mathcal{C}_X^{\infty,(p,q)}$ are soft, it follows that we have isomorphisms

$$H^j(X;\mathcal{O}_X) \xrightarrow{\sim} H^j(\Gamma(X;\mathrm{DB}_X)). \tag{3.11}$$

In other words, the Dolbeault complex is a tool to calculate the cohomology of the sheaf $\mathcal{O}_X$.

Exercises to Chapter 3

Exercise 3.1. Prove that the category $\mathrm{Mod}(\mathbf{k}_X)$ admits direct sums and products (indexed by small sets).

Exercise 3.2. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Define $\widetilde{F} \in \mathrm{Mod}(\mathbf{k}_X)$ by $\widetilde{F} = \bigoplus_{x \in X} F_{\{x\}}$. (Here, $F_{\{x\}} \in \mathrm{Mod}(\mathbf{k}_X)$ and the direct sum is calculated in $\mathrm{Mod}(\mathbf{k}_X)$, not in $\mathrm{PSh}(\mathbf{k}_X)$.) Prove that F_x and $(\widetilde{F})_x$ are isomorphic for all $x \in X$, although F and $\widetilde{F}$ are not isomorphic in general.

Exercise 3.3. Assume $\mathbf{k}$ is a field, and let L be a locally constant sheaf of rank one over $\mathbf{k}_X$ (hence, L is locally isomorphic to the sheaf $\mathbf{k}_X$). Set $L^* = \mathcal{H}om\,(L, \mathbf{k}_X)$.
(i) Prove the isomorphisms $L^* \otimes L \xrightarrow{\sim} \mathbf{k}_X$ and $\mathbf{k}_X \xrightarrow{\sim} \mathcal{H}om\,(L, L)$.
(ii) Assume that $\mathbf{k}$ is a field, X is connected and $\Gamma(X; L) \neq 0$. Prove that $L \simeq \mathbf{k}_X$. (Hint: $\Gamma(X; L) \simeq \Gamma(X; \mathcal{H}om\,(\mathbf{k}_X, L))$.)

Exercise 3.4. Let $M, N \in \mathrm{Mod}(\mathbf{k})$. Prove that

(i) $(M \otimes N)_X \simeq M_X \otimes N_X$,

(ii) $(\mathrm{Hom}\,(M, N))_X \simeq \mathcal{H}om_{\mathbf{k}_X}(M_X, N_X)$.

Exercise 3.5. let $X = U_1 \cup U_2$ be a covering of X by two open sets. Let F be a sheaf on X and assume that:

(i) $U_{12} = U_1 \cap U_2$ is connected and non empty,

(ii) $F|_{U_i}$ $(i = 1, 2)$ is a constant sheaf.

Prove that F is a constant sheaf.

Exercise 3.6. Let I denote the interval $[0, 1]$. Let F be a locally constant sheaf on I. Prove that F is a constant sheaf.

Exercise 3.7. Let X be a discrete topological space. Prove that any sheaf on X is flabby.

Exercise 3.8. We denote here by X the complex line $\mathbb{C}$ and we shall admit that, although it is not soft, the sheaf $\mathcal{O}_X$ satisfies the Cousin property on any open subset U of X.

(i) Let ω be an open subset of $\mathbb{R}$, and let $U_1 \subset U_2$ be two open subsets of $\mathbb{C}$ containing ω as a closed subset. Prove that the natural map

$\mathcal{O}(U_2 \setminus \omega)/\mathcal{O}(U_2) \to \mathcal{O}(U_1 \setminus \omega)/\mathcal{O}(U_1)$ is an isomorphism. One denotes by $\mathcal{B}(\omega)$ this quotient.
(ii) Construct the restriction morphisms to get the presheaf $\omega \to \mathcal{B}(\omega)$ and prove that this presheaf is a sheaf. (This is the sheaf $\mathcal{B}_{\mathbb{R}}$ of Sato's hyperfunctions on $\mathbb{R}$.)
(iii) Prove that the restriction maps $\mathcal{B}(\mathbb{R}) \to \mathcal{B}(\omega)$ (ω open in $\mathbb{R}$) are surjective, that is, the sheaf $\mathcal{B}_{\mathbb{R}}$ is flabby.
(iv) Let Ω an open subset of $\mathbb{C}$ and let $P = \sum_{j=1}^{m} a_j(z)(\frac{\partial}{\partial z})^j$ be a holomorphic differential operator (the coefficients are holomorphic in Ω). Recall the Cauchy theorem which asserts that if Ω is simply connected and if $a_m(z)$ does not vanish on Ω, then P acting on $\mathcal{O}(\Omega)$ is surjective. Prove that if ω is an open subset of $\mathbb{R}$ and if P is a non identically zero holomorphic differential operator defined in a connected open neighborhood of ω, then P acting on $\mathcal{B}(\omega)$ is surjective.

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