A short review on microlocal sheaf theory

Pierre Schapira

January 19, 2016

Abstract

This is a brief survey of the microlocal theory of sheaves of [KS90], with some complements and variants.

Contents

1 Introduction

Between 1960 and 1970, Mikio Sato (see [Sat59, Sat70]) introduced what is now called algebraic analysis and microlocal analysis. The idea of algebraic analysis is to use the tools of algebraic geometry (categories, sheaves) to interpret and to treat problems of analysis and the idea of microlocal analysis is, given a manifold M, to look at its cotangent bundle T^*M to better understand the phenomena on M and to treat some objects living on M (*e.g.,* distributions, hyperfunctions, differential operators) as the projection on M of objects living on T^*M.

The microlocal theory of sheaves, due to Masaki Kashiwara and the author, has appeared in [KS82] and was developed in [KS85, KS90]. It is an illustration of Sato's philosophy since it shows that it is possible to associate to an abelian sheaf F (in the derived sense) on a real manifold M, a closed conic *co-isotropic* subset $\mathrm{SS}(F)$ of the cotangent bundle T^*M, its singular support, which describes the set of non-propagation of F. This theory has applications in various domains, such as singularity theory, D-module theory and symplectic topology.

In these notes we make a very brief survey of this theory.

Denote by $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ the bounded derived category of sheaves of $\mathbf{k}$-modules, for a unital comutative ring $\mathbf{k}$. First we recall the equivalent definitions of $\mathrm{SS}(F)$ and discuss with some proofs the behavior of the singular support with respect to the six operations, with a glance at Morse theory. We recall the construction of the specialization functor and its Fourier–Sato transform, the microlocalization functor, as well as a variant of this last functor, the functor *μhom* which plays a central role in the whole theory. Then we introduce the localization $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; A)$ of the category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ with respect to some subset $A \subset T^*M$ and make *quantized contact transforms* operate on it for $A = \{p\}$, the functor *μhom* playing then the role of the internal $\mathscr{H}om$. Following [GKS12], we briefly show how Hamiltonian isotopies operate and deduce a very short proof of Arnold's non displaceability theorem (after the pioneering work of Tamarkin [Tam08]). We also introduce simple and pure sheaves along a smooth Lagrangian submanifold. Finally, we treat applications of this theory to the study of holomorphic solutions of D-modules, in particular elliptic pairs and hyperbolic systems.

We assume the reader familiar with classical sheaf theory (in the derived setting).

2 Microsupport of sheaves

In this section, we recall some definitions and results from [KS90], following its notations with the exception of slight modifications. We consider a real manifold M of class C^∞.

2.1 Some geometrical notions

For a locally closed subset A of M, one denotes by $\mathrm{Int}(A)$ its interior and by $\overline{A}$ its closure.

One denotes by Δ_M or simply Δ the diagonal of $M \times M$ and by δ_M, or simply δ, the diagonal embedding $M \hookrightarrow M \times M$.

For two manifolds M and N, one denotes by q_1 and q_2 the projections from $M \times N$ to M and N, respectively.

One denotes by pt a set with one element. When necessary, we look at pt as a manifold.

Vector bundles

Let $\tau\colon E \to M$ be a real (finite dimensinal) vector bundle over M.

The antipodal map a_M on E is defined by:

$$a_M\colon E \to E, \quad (x;\xi) \mapsto (x;-\xi). \tag{2.1}$$

If A is a subset of E, we write A^a instead of $a_M(A)$.

One denotes by $\mathbb{R}^+$ the multiplicative group $\mathbb{R}_{>0}$. Then $\mathbb{R}^+$ acts on E. We say that a subset A of E is $\mathbb{R}^+$-conic, or simply conic, if $\mathbb{R}^+ \cdot A = A$.

One denotes by $\tau_M \colon TM \to M$ and $\pi_M \colon T^*M \to M$ the tangent and cotangent bundles to M. If $L \subset M$ is a submanifold[1], we denote by T_LM its normal bundle and by T^*_LM its conormal bundle. They are defined by the exact sequences

$$\begin{aligned} &0 \to TL \to L \times_M TM \to T_LM \to 0, \\ &0 \to T^*_LM \to L \times_M T^*M \to T^*L \to 0. \end{aligned}$$

One identifies M with T^*_MM, the zero-section of T^*M. One sets

$$\dot{T}^*M := T^*M \setminus M, \quad \dot{\pi}_M := \pi_M|_{\dot{T}^*M}. \tag{2.2}$$

If there is no risk of confusion, one simply writes τ and π instead of τ_M and π_M.

Let $f \colon M \to N$ be a morphism of real manifolds. To f are associated the tangent morphisms

$$\begin{tikzcd} TM \arrow[r, "f'"] \arrow[dr, "\tau"'] & M \times_N TN \arrow[r, "f_\tau"] \arrow[d, "\tau"] & TN \arrow[d, "\tau"] \\ & M \arrow[r, "f"] & N. \end{tikzcd} \tag{2.3}$$

By duality, we deduce the diagram:

$$\begin{tikzcd} T^*M \arrow[dr, "\pi"'] & M \times_N T^*N \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*N \arrow[d, "\pi"] \\ & M \arrow[r, "f"] & N. \end{tikzcd} \tag{2.4}$$

One sets

$$T^*_MN := \operatorname{Ker} f_d = f_d^{-1}(T^*_MM).$$

Note that, denoting by Γ_f the graph of f in $M \times N$, the projection $T^*(M \times N) \to M \times T^*N$ identifies $T^*_{\Gamma_f}(M \times N)$ and $M \times_N T^*N$.

[1] in these notes, a submanifold is always smooth and locally closed

Whitney's normal cones

The intrinsic construction of the Whitney's normal cones will be recalled in Definition 4.4.

Let $S \subset M$ be a locally closed subset and let L be a submanifold of M. The Whitney normal cone $C_L(S)$ is a closed conic subset of T_LM given in a local coordinate system $(x) = (x', x'')$ on M with $N = \{x' = 0\}$ by

$$(2.5)\quad \begin{cases} (x_0''; v_0) \in C_N(S) \subset T_NM \text{ if and only if there exists a sequence} \\ \{(x_n, c_n)\}_n \subset S \times \mathbb{R}^+ \text{ with } x_n = (x_n', x_n'') \text{ such that } x_n' \xrightarrow{n} 0, \\ x_n'' \xrightarrow{n} x_0'' \text{ and } c_n(x_n') \xrightarrow{n} v_0. \end{cases}$$

For two subsets $S_1, S_2 \subset M$, their Whitney's normal cone is given in a local coordinate system (x) on M by:

$$(2.6)\quad \begin{cases} (x_0; v_0) \in C(S_1, S_2) \subset TM \text{ if and only if there exists a sequence} \\ \{(x_n, y_n, c_n)\}_n \subset S_1 \times S_2 \times \mathbb{R}^+ \text{ such that } x_n \xrightarrow{n} x_0,\ y_n \xrightarrow{n} x_0 \text{ and} \\ c_n(x_n - y_n) \xrightarrow{n} v_0. \end{cases}$$

Example 2.1. Let $\mathbb{V}$ be a real finite dimensional vector space and let γ be a closed cone (unless otherwise specified, a cone is always centred at the origin). Then $C_0(\gamma) = \gamma$ and $C_0(\gamma, \gamma)$ is the vector space generated by γ.

Liouville form and Hamiltonian isomorphism

The map $\pi_M \colon T^*M \to M$ induces the maps

$$T^*T^*M \leftarrow T^*M \times_M T^*M \to T^*M.$$

By sending T^*M to $T^*M \times_M T^*M$ by the diagonal map, we get a map $\alpha_M \colon T^*M \to T^*T^*M$, that is a section of $T^*(T^*M)$. This is the Liouville 1-form, given in a local homogeneous symplectic coordinate system $(x; \xi)$ on T^*M, by

$$\alpha_M = \sum_{j=1}^{n} \xi_j \, dx_j.$$

The differential $d\alpha_M$ of the Liouville form is the symplectic form ω_M on T^*M given in a local symplectic coordinate system $(x; \xi)$ on T^*M by

$$\omega_M = \sum_{j=1}^{n} d\xi_j \wedge dx_j.$$

Hence T^*M is not only a symplectic manifold, it is a homogeneous (or exact) symplectic manifold.

We shall use the Hamiltonian isomorphism $H\colon T^*(T^*M) \xrightarrow{\sim} T(T^*M)$ given in a local symplectic coordinate system $(x;\xi)$ by

$$H(\langle\lambda, dx\rangle + \langle\mu, d\xi\rangle) = -\langle\lambda, \partial_\xi\rangle + \langle\mu, \partial_x\rangle.$$

Co-isotropic subsets

Definition 2.2 (See [KS90, Def. 6.5.1])**.** A subset S of T^*M is co-isotropic (one also says involutive) at $p \in T^*M$ if $C_p(S,S)^\perp \subset C_p(S)$. Here we identify the orthogonal $C_p(S,S)^\perp$ to a subset of T_pT^*M via the Hamiltonian isomorphism.

When S is smooth, one recovers the usual notion.

2.2 Microsupport of sheaves

References are made to [KS90, §5.1-5.3].

Sheaves

We consider a commutative unital ring $\mathbf{k}$ of finite global dimension (*e.g.* $\mathbf{k} = \mathbb{Z}$ or $\mathbf{k}$ a field). We denote by $\mathbf{k}_M$ the sheaf of $\mathbf{k}$-valued locally constant functions on M and by $\mathsf{D}(\mathbf{k}_M)$ (resp. $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$) the derived category (resp. bounded derived category) of sheaves of $\mathbf{k}$-modules on M. We shall identify $\mathsf{D}(\mathbf{k}_{\mathrm{pt}})$ with $\mathsf{D}(\mathbf{k})$ and we denote by a_M the unique map $M \to \mathrm{pt}$.

We assume that the reader is familiar with the Grothendieck six operations on sheaves.

For $V \in \mathsf{D}(\mathbf{k})$, we set $V_M := a_M^{-1}V$. For a locally closed subset A of M, we still denote by $\mathbf{k}_A$, the sheaf which is $\mathbf{k}_A$ on A and 0 on $M \setminus A$. For $F \in \mathsf{D}(\mathbf{k}_M)$, one sets $F_A := F \otimes \mathbf{k}_A$ and $\mathrm{R}\Gamma_A F := \mathrm{R}\mathscr{H}om\,(\mathbf{k}_A, F)$.

The dualizing complex ω_M is defined as $\omega_M = a_M^!(\mathbf{k})$. One has the isomorphism

$$\omega_M \simeq \mathrm{or}_M\,[\dim M] \tag{2.7}$$

where or_M is the orientation sheaf on M.

The duality functors D'_M and D_M are given by

$$\mathrm{D}'_M F = \mathrm{R}\mathscr{H}om\,(F, \mathbf{k}_M),$$
$$\mathrm{D}_M F = \mathrm{R}\mathscr{H}om\,(F, \omega_M).$$

One also sets $\omega_M^{\otimes -1} = \mathrm{D}'_M \omega_M$.

If $f\colon M \to N$ is a morphism of manifolds, one sets

$$\omega_{M/N} := f^! \mathbf{k}_N \simeq \omega_M \otimes f^{-1}\omega_N^{\otimes -1}.$$

We shall have to consider *cohomologically constructible* sheaves and, in the case M is real analytic, $\mathbb{R}$-*constructible* sheaves. We don't recall here their definitions and refer to [KS90, § 3.4, § 8.4]. In the case $M = \mathrm{pt}$, $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k})$ is cohomologically constructible if and only if it is represented by a bounded complex of finitely generated projective $\mathbf{k}$-modules.

Micro-support

To $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ one associates $\mathrm{SS}(F)$, its *singular support* or *micro-support* as follows.

Definition 2.3. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and let $p \in T^*M$. One says that $p \notin \mathrm{SS}(F)$ if there exists an open neighborhood U of p such that for any $x_0 \in M$ and any real C^1-function φ on M defined in a neighborhood of x_0 satisfying $d\varphi(x_0) \in U$ and $\varphi(x_0) = 0$, one has $(\mathrm{R}\Gamma_{\{x;\varphi(x)\geq 0\}}(F))_{x_0} \simeq 0$.

In other words, $p \notin \mathrm{SS}(F)$ if the sheaf F has no cohomology supported by "half-spaces" whose conormals are contained in a neighborhood of p.

- By its construction, the microsupport is closed and is conic, that is, invariant by the action of $\mathbb{R}^+$ on T^*M.
- $\mathrm{SS}(F) \cap T^*_M M = \pi_M(\mathrm{SS}(F)) = \mathrm{Supp}(F)$.
- $\mathrm{SS}(F) = \mathrm{SS}(F\,[j])$ $(j \in \mathbb{Z})$.
- The microsupport satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.

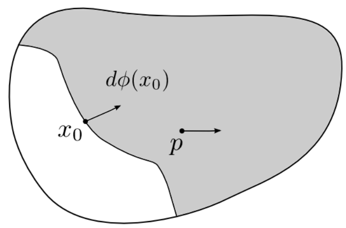

Figure 1: Microsupport

An essential properties of the micro-support is given by the next theorem. The proof is beyond the scope of these notes and will not be even sketched here.

Theorem 2.4 (See [KS90, Th. 6.5.4])**.** *Let* $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$*. Then its micro-support* $\mathrm{SS}(F)$ *is co-isotropic.*

Example 2.5. (i) $\mathrm{SS}(F) \subset T^*_M M$ if and only if F is a local system, that is, $H^j(F)$ is locally constant on M for all $j \in \mathbb{Z}$.
(ii) If N is a smooth closed submanifold of M and $F = \mathbf{k}_N$, then $\mathrm{SS}(F) = T^*_N M$, the conormal bundle to N in M.
(iii) Let φ be C^1-function with $d\varphi(x) \neq 0$ when $\varphi(x) = 0$. Let $U = \{x \in M; \varphi(x) > 0\}$ and let $Z = \{x \in M; \varphi(x) \geq 0\}$. Then

$$\mathrm{SS}(\mathbf{k}_U) = U \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \leq 0\},$$
$$\mathrm{SS}(\mathbf{k}_Z) = Z \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \geq 0\}.$$

(iv) Let $(X, \mathscr{O}_X)$ be a complex manifold and let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module (see § 6.2). Set $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$. Then $\mathrm{SS}(F) = \mathrm{char}(\mathscr{M})$, the characteristic variety of $\mathscr{M}$. See § 6.2 for details. Note that this result together with Theorem 2.4 gives a totally new proof of the involutivity of the characteristic variety of coherent D-modules.

There are other equivalent definitions of the microsupport.

Let $\mathbb{V}$ be a real finite dimensional vector space and let γ be a closed convex cone centred at the origin. The γ-topology on $\mathbb{V}$ is the topology for which the open sets U are the open subsets U such that $U = U + \gamma$. One denotes by $\mathbb{V}_\gamma$ the space $\mathbb{V}$ endowed with the γ-topology. For an open subset $X \subset \mathbb{V}$, one denotes by X_γ the space X endowed with the topology induced by $\mathbb{V}_\gamma$ and one denotes by

$$\Phi_\gamma \colon X \to X_\gamma$$

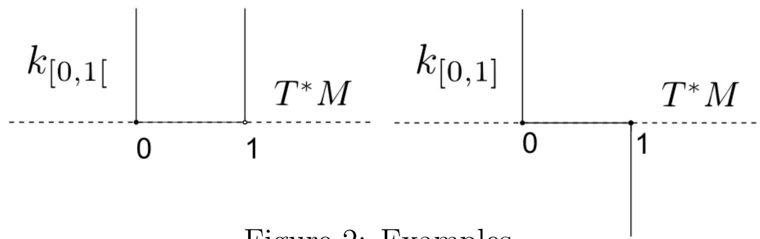

Figure 2: Examples

the continuous map associated with the identity on X.

Theorem 2.6. *Assume that M is an open subset of a vector space $\mathbb{V}$ and let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. Let $p = (x_0; \xi_0) \in T^*M$. Then the following conditions are equivalent*

(a) $p \notin \mathrm{SS}(F)$,

(b) *there exist a neighborhood U of x_0, an $\varepsilon > 0$ and a proper closed convex cone γ with $0 \in \gamma$ such that*

$$\gamma \setminus \{0\} \subset \{v; \langle v, \xi_0 \rangle < 0\} \, (\textit{equivalently, } \xi_0 \in \mathrm{Int}(\gamma^{\circ a})) \tag{2.8}$$

and setting

$$H = \{x \in \mathbb{V}; \langle x - x_0, \xi_0 \rangle \geq -\varepsilon\},$$
$$L = \{x \in \mathbb{V}; \langle x - x_0, \xi_0 \rangle = -\varepsilon\},$$

then $H \cap (U + \gamma) \subset M$ and we have the natural isomorphism for any $x \in U$:

$$\mathrm{R}\Gamma(H \cap (x + \gamma); F) \xrightarrow{\sim} \mathrm{R}\Gamma(L \cap (x + \gamma); F),$$

(c) *there exist a proper closed convex cone γ with $0 \in \gamma$ satisfying (2.8) and $F' \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$ such that $F'|_U \simeq F|_U$ for a neighborhood U of x_0 and* $\mathrm{R}\Phi_{\gamma_*} F' \simeq 0$.

2.3 Functorial operations

References are made to [KS90, §5.4].

Let M and N be two manifolds. We denote by q_i $(i = 1, 2)$ the i-th projection defined on $M \times N$ and by p_i $(i = 1, 2)$ the i-th projection defined on $T^*(M \times N) \simeq T^*M \times T^*N$.

Definition 2.7. Let $f\colon M \to N$ be a morphism of manifolds and let $\Lambda \subset T^*N$ be a closed $\mathbb{R}^+$-conic subset. One says that f is non-characteristic for Λ (or else, Λ is non-characteristic for f, or f and Λ are transverse) if

$$f_\pi^{-1}(\Lambda) \cap T^*_M N \subset M \times_N T^*_N N.$$

- Clearly, if f is submersive then it is non-characteristic for any Λ.
- A morphism $f\colon M \to N$ is non-characteristic for a closed $\mathbb{R}^+$-conic subset Λ of T^*N if and only if $f_d\colon M \times_N T^*N \to T^*M$ is proper on $f_\pi^{-1}(\Lambda)$ and in this case $f_d f_\pi^{-1}(\Lambda)$ is closed and $\mathbb{R}^+$-conic in T^*M.

Recall that $\bullet \overset{\mathrm{L}}{\boxtimes} \bullet := q_1^{-1}(\,\bullet\,) \overset{\mathrm{L}}{\otimes} q_2^{-1}(\,\bullet\,)$.

Theorem 2.8. *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and let $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N)$. One has*

$$\begin{aligned}
&\mathrm{SS}(F \overset{\mathrm{L}}{\boxtimes} G) \subset \mathrm{SS}(F) \times \mathrm{SS}(G),\\
&\mathrm{SS}(\mathrm{R}\mathscr{H}om\,(q_1^{-1}F, q_2^{-1}G)) \subset \mathrm{SS}(F)^a \times \mathrm{SS}(G).
\end{aligned}$$

Theorem 2.9. *Let $f\colon M \to N$ be a morphism of manifolds, let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and assume that f is proper on* $\mathrm{Supp}(F)$. *Then $\mathrm{R}f_!F \xrightarrow{\sim} \mathrm{R}f_*F$ and*

$$\mathrm{SS}(\mathrm{R}f_*F) \subset f_\pi f_d^{-1}\mathrm{SS}(F). \tag{2.9}$$

Moreover, if f is a closed embedding, this inclusion is an equality.

Proof. (i) The isomorphism $\mathrm{R}f_!F \xrightarrow{\sim} \mathrm{R}f_*F$ is obvious.
(ii) Let $y \in N$ and let $\varphi\colon N \to \mathbb{R}$ be a C^1-function such that $\varphi(y) = 0$ and $d(\varphi \circ f)(x) \notin \mathrm{SS}(F)$ for all $x \in f^{-1}(y)$. By the definition of $\mathrm{SS}(F)$ we get

$$\mathrm{R}\Gamma_{\{\varphi\circ f \geq 0\}}(F)|_{f^{-1}(y)} = 0.$$

On the other hand, we have

$$\begin{aligned}
\mathrm{R}\Gamma_{\{\varphi \geq 0\}}(\mathrm{R}f_*F)_y &\simeq (\mathrm{R}f_*\mathrm{R}\Gamma_{\{\varphi\circ f \geq 0\}}(F))_y\\
&\simeq \mathrm{R}\Gamma(f^{-1}(y); \mathrm{R}\Gamma_{\{\varphi\circ f \geq 0\}}(F)) \simeq 0.
\end{aligned}$$

Here, the second isomorphism follows from the fact that one may replace f_* with $f_!$.

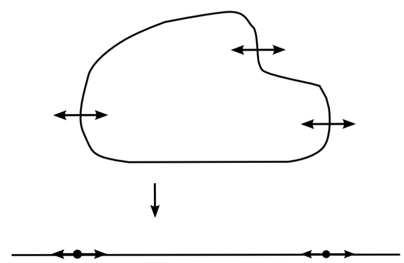

Figure 3: Direct image

(iii) Assume f is a closed embedding and let $p \notin \mathrm{SS}(\mathrm{R}f_*F)$. We may assume that N is a vector space, M is a vector subspace and f is the inclusion. Let H, L, γ, U be as in Theorem 2.6 (b). Hence, we have for $x \in U \cap M$:

$$\mathrm{R}\Gamma(H \cap (x+\gamma); \mathrm{R}f_*F) \xrightarrow{\sim} \mathrm{R}\Gamma(L \cap (x+\gamma); \mathrm{R}f_*F).$$

On the other hand, we have

$$\mathrm{R}\Gamma(H \cap (x+\gamma); \mathrm{R}f_*F) \simeq \mathrm{R}\Gamma((M \cap H) \cap (x + (M \cap \gamma)); F),$$

and a similar formula with H replaced with L. Therefore, $p \notin \mathrm{SS}(F)$ again by Theorem 2.6 (b). Q.E.D.

On Figure 3, one sees that the inclusion in Theorem 2.9 may be strict.

Corollary 2.10. *Let I be an open interval of $\mathbb{R}$, let $q\colon M \times I \to I$ be the projection and let ι_s is the embedding $M \times \{s\} \hookrightarrow M \times I$. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times I})$ such that $\mathrm{SS}(F) \cap (T^*_M M \times T^*I) \subset T^*_{M\times I}(M \times I)$ and q is proper on $\mathrm{Supp}(F)$. Set $F_s := \iota_s^{-1}F$. Then we have isomorphisms $\mathrm{R}\Gamma(M; F_s) \simeq \mathrm{R}\Gamma(M; F_t)$ for any $s, t \in I$.*

Proof. Consider the Cartesian square in which ι_s also denotes the embedding $\{s\} \hookrightarrow I$:

$$\begin{tikzcd} M \times \{s\} \arrow[r, "a_M"] \arrow[d, "\iota_s"'] \arrow[dr, phantom, "\square"] & \{s\} \arrow[d, "\iota_s"] \\ M \times I \arrow[r, "q"'] & I. \end{tikzcd}$$

By the "base change formula" for sheaves, $\mathrm{R}a_{M!}\iota_s^{-1}F \simeq \iota_s^{-1}\mathrm{R}q_!F$. Since q is proper on $\mathrm{Supp}(F)$, we get $\mathrm{R}\Gamma(M; F_s) \simeq (\mathrm{R}q_*F)_s$.

On the other hand, it follows from Theorem 2.9 that $\mathrm{SS}(\mathrm{R}q_*F) \subset T^*_I I$. Hence, there exists $V \in \mathsf{D}^{\mathrm{b}}(\mathbf{k})$ and an isomorphism $\mathrm{R}q_*F \simeq V_I$. Therefore, $(\mathrm{R}q_*F)_s \simeq (\mathrm{R}q_*F)_t$ for any $s, t \in I$. Q.E.D.

Theorem 2.11. *Let $f\colon M \to N$ be a morphism of manifolds, let $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N)$ and assume that f is non-characteristic with respect to $\mathrm{SS}(G)$. Then the natural morphism $f^{-1}G \otimes \omega_{M/N} \to f^!G$ is an isomorphism and*

$$\mathrm{SS}(f^{-1}G) \subset f_d f_\pi^{-1}(\mathrm{SS}(G)). \tag{2.10}$$

Moreover, if f is submersive, this inclusion is an equality.

Note that $f^{-1}G \otimes \omega_{M/N}$ being locally isomorphic to $f^{-1}G$ up to a shift, we get that $\mathrm{SS}(f^!G) = \mathrm{SS}(f^{-1}G)$ in this case.

Sketch of proof. (i) By decomposing f by its graph, it is enough to check separately the case of a closed embedding and the case of a submersion.

(ii) First, we assume that f is submersive.

(ii)-(a) In this case, the isomorphism $f^{-1}G \otimes \omega_{M/N} \xrightarrow{\sim} f^!G$ is well-known. Locally on M, f is isomorphic to the projection $M = N \times T \to N$ and $f^{-1}G \simeq G \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_T$. Hence the inclusion $\mathrm{SS}(f^{-1}G) \subset f_d f_\pi^{-1}(\mathrm{SS}(G))$ is a particular case of Theorem 2.8.

(ii)-(b) The converse inclusion follows from the isomorphism

$$(\mathrm{R}\Gamma_{\{\varphi \geq 0\}}G)_y \simeq (\mathrm{R}\Gamma_{\{\varphi \circ f \geq 0\}}f^{-1}G)_x$$

for any $x \in M$ with $f(x) = y$. Indeed, if $(y_0; \eta_0) \in \mathrm{SS}(G)$, then there exists a sequence $y_n \xrightarrow{n} y_0$ and functions φ_n such that $d\varphi_n(y_n) \xrightarrow{n} \eta_0$ and $(\mathrm{R}\Gamma_{\{\varphi_n \geq 0\}}G)_{y_n} \neq 0$ and the result follows by the definition of the microsupport.

(iii) We may reduce the proof to the case where M is a closed hypersurface of N.

(iii)-(a) Let us prove the isomorphism $f^{-1}G \otimes \omega_{M/N} \xrightarrow{\sim} f^!G$. This is a local problem on M and we may assume that $N = M \sqcup M^+ \sqcup M^-$ for M^+ and M^- two closed half-spaces with boundary M. The exact sequence $0 \to \mathbf{k}_N \to \mathbf{k}_{M^+} \oplus \mathbf{k}_{M^-} \to \mathbf{k}_M \to 0$ gives rise to the distinguished triangle

$$\mathrm{R}\Gamma_M G \xrightarrow{\alpha} (\mathrm{R}\Gamma_{M^+}G)_M \oplus (\mathrm{R}\Gamma_{M^-}G)_M \xrightarrow{\beta} G_M \xrightarrow{+1}.$$

The map β is given by $(u,v) \mapsto \varepsilon u|_M - \varepsilon v|_M$ with $\varepsilon = \pm$, and this sign is given by the relative orientation sheaf $\mathrm{or}_{M/N}$. The hypothesis that M is non-characteristic implies the vanishing of $(\mathrm{R}\Gamma_{M^+}G)_M \oplus (\mathrm{R}\Gamma_{M^-}G)_M$ and we get the isomorphism $f^{-1}G \otimes \mathrm{or}_{M/N}[-1] \xrightarrow{\sim} (\mathrm{R}\Gamma_M G)_M$.

(iii)-(b) The proof of the inclusion (2.10) is more technical and we refer to [KS90, Prop. 5.4.13]. Q.E.D.

Corollary 2.12. *Let $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$.*

(i) *Assume that $\mathrm{SS}(F_1) \cap \mathrm{SS}(F_2)^a \subset T^*_M M$. Then*

$$\mathrm{SS}(F_1 \overset{\mathrm{L}}{\otimes} F_2) \subset \mathrm{SS}(F_1) + \mathrm{SS}(F_2).$$

(ii) *Assume that $\mathrm{SS}(F_1) \cap \mathrm{SS}(F_2) \subset T^*_M M$. Then*

$$\mathrm{SS}(\mathrm{R}\mathscr{H}om\,(F_2, F_1)) \subset \mathrm{SS}(F_2)^a + \mathrm{SS}(F_1).$$

Proof. One has the isomorphisms $\mathrm{R}\mathscr{H}om\,(F_2,F_1) \simeq \delta^! \mathrm{R}\mathscr{H}om\,(q_2^{-1}F_2, q_1^! F_1)$ and $F_1 \overset{\mathrm{L}}{\otimes} F_2 \simeq \delta^{-1}(F_1 \overset{\mathrm{L}}{\boxtimes} F_2)$. Hence, the result follows from Theorems 2.8 and 2.11. Q.E.D.

2.4 Kernels

References for this subsection are made to [KS90, §3.6].

Let M_i $(i = 1,2,3)$ be manifolds. For short, we write $M_{ij} := M_i \times M_j$ $(1 \le i,j \le 3)$ and $M_{123} = M_1 \times M_2 \times M_3$. We denote by q_i the projection $M_{ij} \to M_i$ or the projection $M_{123} \to M_i$ and by q_{ij} the projection $M_{123} \to M_{ij}$. Similarly, we denote by p_i the projection $T^*M_{ij} \to T^*M_i$ or the projection $T^*M_{123} \to T^*M_i$ and by p_{ij} the projection $T^*M_{123} \to T^*M_{ij}$. We also need to introduce the map p_{12^a}, the composition of p_{12} and the antipodal map on T^*M_2.

Let $\Lambda_1 \subset T^*M_{12}$ and $\Lambda_2 \subset T^*M_{23}$. We set

$$\Lambda_1 \overset{a}{\circ} \Lambda_2 := p_{13}({p_{12^a}}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2). \tag{2.11}$$

We consider the operation of convolution of kernels:

$$\begin{aligned} \underset{2}{\circ}\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}}) \times \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}}) &\to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{13}}) \\ (K_1, K_2) &\mapsto K_1 \underset{2}{\circ} K_2 := \mathrm{R}q_{13!}(q_{12}^{-1}K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_2). \end{aligned}$$

Let $\Lambda_i = \mathrm{SS}(K_i) \subset T^*M_{i,i+1}$ and assume that

$$(2.12)\quad \begin{cases} \text{(i) } q_{13} \text{ is proper on } q_{12}^{-1}\operatorname{Supp}(K_1) \cap q_{23}^{-1}\operatorname{Supp}(K_2), \\ \text{(ii) } \; p_{12^a}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2 \cap (T^*_{M_1}M_1 \times T^*M_2 \times T^*_{M_3}M_3) \\ \qquad\qquad \subset T^*_{M_1\times M_2\times M_3}(M_1\times M_2\times M_3). \end{cases}$$

It follows from the functorial properties of the microsupport, namely Theorems 2.8, 2.9 and 2.11, that under the assumption (2.12) we have:

$$(2.13)\qquad \mathrm{SS}(K_1 \underset{2}{\circ} K_2) \subset \Lambda_1 \overset{a}{\circ} \Lambda_2.$$

If there is no risk of confusion, we write $\circ$ instead of $\underset{2}{\circ}$.

3 Morse theory for sheaves

References are made to [KS90, § 1.12, § 5.4].

3.1 A basic lemma

The next lemma, although elementary, is extremely useful. It is due to M. Kashiwara.

Let $\{X_s, \rho_{s,t}\}_{s\in\mathbb{R}}$ be a projective system of sets indexed by $\mathbb{R}$. Hence, the X_s are sets and $\rho_{s,t}\colon X_t \to X_s$ are maps defined for $s \le t$, satisfying the natural compatibility conditions. Set

$$\lambda_s\colon X_s \to \varprojlim_{r<s} X_r, \quad \mu_s\colon \varinjlim_{t>s} X_t \to X_s.$$

Lemma 3.1 (See [KS90, Pro. 1.12.6])**.** *Assume that for each $s \in \mathbb{R}$, both maps λ_s and μ_s are injective (resp. surjective). Then all maps ρ_{s_0,s_1} $(s_0 \le s_1)$ are injective (resp. surjective).*

3.2 Mittag-Leffler theorem

References are made to [Gro61] (see [KS90, § 1.12]). Consider a projective system of abelian groups indexed by $\mathbb{N}$, $\{M_n, \rho_{n,p}\}_{n\in\mathbb{N}}$, with $\rho_{n,p}\colon M_p \to M_n$ $(p \ge n)$. (In the sequel we shall simply denote such a system by $\{M_n\}_n$.) Recall that one says that this system satisfies the Mittag-Leffler condition (ML for short) if for any $n \in \mathbb{N}$ the decreasing sequence $\{\rho_{n,p} M_p\}$ of subgroups of M_n is stationary.

Of course, this condition is in particular satisfied if all maps $\rho_{n,p}$ are surjective.

Notation 3.2. For a projective system of abelian groups $\{M_n\}_n$, we set $M_\infty = \varprojlim_n M_n$.

Consider a projective system of exact sequences of abelian groups indexed by $\mathbb{N}$. For each $n \in \mathbb{N}$ we have an exact sequence

$$E_n \colon 0 \to M'_n \to M_n \to M''_n \to 0, \tag{3.1}$$

and we have morphisms $\rho_{n,p} \colon E_p \to E_n$ satisfying the compatibility conditions. Recall:

Lemma 3.3. *If the projective system $\{M'_n\}_n$ satisfies the* ML *condition, then the sequence*

$$E_\infty \colon 0 \to M'_\infty \to M_\infty \to M''_\infty \to 0 \tag{3.2}$$

is exact.

Now consider a projective system of complexes

$$E_n^\bullet \colon \cdots \to M_n^{j-1} \to M_n^j \to M_n^{j+1} \to \cdots, \tag{3.3}$$

and its projective limit

$$E_\infty^\bullet \colon \cdots \to M_\infty^{j-1} \to M_\infty^j \to M_\infty^{j+1} \to \cdots, \tag{3.4}$$

Denote by

$$\Phi_k \colon H^k(E_\infty^\bullet) \to \varprojlim_n H^k(E_n^\bullet) \tag{3.5}$$

the natural morphism.

Lemma 3.4. *Assume that for all $j \in \mathbb{Z}$, the system $\{M_n^j\}_n$ satisfies the* ML *condition. Then*

(a) *for each $k \in \mathbb{Z}$, the map Φ_k in (3.5) is surjective,*

(b) *if moreover, for a given i the system $\{H^{i-1}(E_n^\bullet)\}_n$ satisfies the* ML *condition, then Φ_i is bijective.*

3.3 Morse lemma in dimension one

Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, let $U \subset M$ be an open subset, ∂U its boundary, $\overline{U}$ its closure. The exact sequence $0 \to \mathbf{k}_U \to \mathbf{k}_M \to \mathbf{k}_{\overline{U}} \to 0$ gives rise to the distinguished triangle $\mathrm{R}\Gamma_{M\setminus U}F \to F \to \mathrm{R}\Gamma_U F \xrightarrow{+1}$. Applying the functor $(\,\bullet\,)_{\overline{U}}$ we get the distinguished triangle

$$(\mathrm{R}\Gamma_{M\setminus U}F)|_{\partial U} \to F_{\overline{U}} \to \mathrm{R}\Gamma_U F \xrightarrow{+1} . \tag{3.6}$$

Until the end of this subsection, $M = \mathbb{R}$. For $t \in \mathbb{R}$, we set

$$Z_t =]-\infty, t], \quad I_t =]-\infty, t[.$$

Lemma 3.5. *Let $-\infty < a < b < +\infty$. Let $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}})$ and assume that $(\mathrm{R}\Gamma_{[t,+\infty[}(G))_t \simeq 0$ for all $t \in [a, b[$. Then one has the natural isomorphism*

$$\mathrm{R}\Gamma(I_b; G) \xrightarrow{\sim} \mathrm{R}\Gamma(I_a; G).$$

Note that if $(t; dt) \notin \mathrm{SS}(G)$ for all $t \in [a, b[$, then $(\mathrm{R}\Gamma_{[t,+\infty[}G)_t \simeq 0$ for all $t \in [a, b[$.

Proof. As a particular case of (3.6) with $U = I_t$, we have the distinguished triangle

$$(\mathrm{R}\Gamma_{[t,+\infty[}G)_t \to G_{Z_t} \to \mathrm{R}\Gamma_{I_t}G \xrightarrow{+1} . \tag{3.7}$$

Applying the functor $\mathrm{R}\Gamma(\mathbb{R}; \bullet)$, we deduce the distinguished triangle

$$(\mathrm{R}\Gamma_{[t,+\infty[}G)_t \to \mathrm{R}\Gamma(Z_t; G) \to \mathrm{R}\Gamma(I_t; G) \xrightarrow{+1} . \tag{3.8}$$

Set

$$E_s^k = H^k(I_s; G).$$

Consider the assertions

$$\varinjlim_{t>s} E_t^k \xrightarrow{\sim} E_s^k \text{ for all } s \in [a, b[, \tag{3.9}$$

$$\varprojlim_{s<t} E_t^k \xleftarrow{\sim} E_t^k \text{ for all } t \in]a, b]. \tag{3.10}$$

By the hypothesis, $\mathrm{R}\Gamma(Z_t;G) \xrightarrow{\sim} \mathrm{R}\Gamma(I_t;G)$ for $t \in [a,b[$. Therefore, (3.9) holds for any $k \in \mathbb{Z}$. Moreover, (3.10) holds for $k \ll 0$. Let us argue by induction on k and assume (3.10) holds for all $t \in]a,b[$ and all $k \leq k_0$. Applying Lemma 3.1, we find the isomorphisms

$$(3.11)\; H^k(I_s;G) \xrightarrow{\sim} H^k(I_t;G) \text{ for all } k \leq k_0 \text{ and all } a < s \leq t < b.$$

On the other hand, we may represent G by a complex of flabby sheaves $G^\bullet$. Let $t \in]a,b]$ be given. Consider the complex

$$E_n^\bullet = \Gamma(I_{t-1/n};G^\bullet).$$

Since $G^\bullet$ is a complex of flabby sheaves, the projective systems $\{\Gamma(I_{t-1/n};G^j)\}_n$ satisfies the ML condition for all $j \in \mathbb{Z}$.

By (3.11), the projective system $\{H^{k_0}(E_n^\bullet)\}_n$ satisfies the ML condition. Applying Lemma 3.4 to $\{E_n^\bullet\}_n$, we get that (3.10) is satisfied for $k = k_0 + 1$ and the induction proceeds. Again by Lemma 3.1, we get the isomorphisms $H^k(I_s;G) \xrightarrow{\sim} H^k(I_t;G)$ for all k all $a < s \leq t \leq b$.

Finally, using (3.8) and the hypothesis, we have the isomorphims

$$H^k(I_a;G) \xleftarrow{\sim} H^k(Z_a;G) \xrightarrow{\sim} H^k(I_s;G) \text{ for all } k \in \mathbb{Z},\ a < s \leq b.$$

Q.E.D.

3.4 Morse theorem

We consider a function $\psi\colon M \to \mathbb{R}$ of class C^1.

Theorem 3.6. *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, let $\psi\colon M \to \mathbb{R}$ be a function of class C^1 and assume that ψ is proper on* $\mathrm{Supp}(F)$. *Let $a < b$ in $\mathbb{R}$ and assume that $d\psi(x) \notin \mathrm{SS}(F)$ for $a \leq \psi(x) < b$. For $t \in \mathbb{R}$, set $M_t = \psi^{-1}(]-\infty,t[)$. Then the restriction morphism $\mathrm{R}\Gamma(M_b;F) \to \mathrm{R}\Gamma(M_a;F)$ is an isomorphism.*

The classical Morse theorem corresponds to the constant sheaf $F = \mathbf{k}_M$.

Proof. Set $G = \mathrm{R}\psi_* F$. Then $\mathrm{R}\Gamma(M_t;F) \simeq \mathrm{R}\Gamma(]-\infty,t[;G)$. By Theorem 2.9, $\mathrm{SS}(G) \subset \psi_d \psi_\pi^{-1}(\mathrm{SS}(F))$. In other words,

$$\mathrm{SS}(G) \subset \{(t;\tau); \text{there exists } x \in M, \psi(x) = t, d\psi(x) \in \mathrm{SS}(F)\}.$$

Therefore, $(t;dt) \notin \mathrm{SS}(G)$ for $a \leq t < b$ and it remains to apply Lemma 3.5. Q.E.D.

Set

$$\Lambda_\psi = \{(x; d\psi(x))\} \subset T^*M. \tag{3.12}$$

The next corollary will be an essential tool when proving non-displaceability theorems (see Theorem 5.20). below.

Corollary 3.7. *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and let $\psi\colon M \to \mathbb{R}$ be a function of class C^1. Let Λ_ψ be given by (3.12). Assume that*

(i) $\mathrm{Supp}(F)$ *is compact,*

(ii) $\mathrm{R}\Gamma(M; F) \neq 0$.

Then $\Lambda_\psi \cap \mathrm{SS}(F) \neq \emptyset$.

Proof. Assume that $\Lambda_\psi \cap \mathrm{SS}(F) = \emptyset$. It follows from Theorem 3.6 that $\mathrm{R}\Gamma(M_t; F)$ does not depend on $t \in \mathbb{R}$. Since F has compact support, we get $\mathrm{R}\Gamma(M_t; F) \simeq 0$ for $t \ll 0$ and $\mathrm{R}\Gamma(M; F) \simeq \mathrm{R}\Gamma(M_t; F)$ for $t \gg 0$. This is a contradiction. Q.E.D.

3.5 Propagation

References are made to [KS90, §5.2].

The microsupport is a tool to obtain global propagation results.

Theorem 3.8. *Let $\mathbb{V}$ be a real finite dimensional vector space and let γ be a proper closed convex cone centred at $0 \in \mathbb{V}$. Let $U \subset \mathbb{V}$ be an open subset and let $\Omega_0 \subset \Omega_1$ be two γ-open subsets of $\mathbb{V}$. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_U)$. Assume*

$$\begin{cases} \mathrm{SS}(F) \cap (U \times \mathrm{Int}(\gamma^{\circ a})) = \emptyset, \\ \Omega_1 \setminus \Omega_0 \subset U, \\ \textit{for any } x \in \Omega_1,\ (x+\gamma) \setminus \Omega_0 \textit{ is compact.} \end{cases}$$

Then

$$(\mathrm{R}\Phi_{\gamma_*}\mathrm{R}\Gamma_{U\setminus\Omega_0}F)|_{\Omega_1} \simeq 0 \tag{3.13}$$

and the natural morphism

$$\mathrm{R}\Gamma(\Omega_1 \cap U; F) \to \mathrm{R}\Gamma(\Omega_0 \cap U; F) \tag{3.14}$$

is an isomorphism.

Sketch of proof. (i) Let us prove (3.13) and set $\widetilde{F} := \mathrm{R}\Gamma_{U\setminus\Omega_0}F$. In order to check that $(\mathrm{R}\Phi_{\gamma_*}\widetilde{F})|_{\Omega_1} \simeq 0$, one reduces to the case where $\Omega_0 = \{x \in \mathbb{V}; \langle x, \xi_0\rangle < 0\}$, for some $\xi_0 \in \mathbb{V}^*$, $\xi_0 \neq 0$. Then one is reduced to prove

$$\mathrm{R}\Gamma((x+\gamma)\cap\{\langle x,\xi_0\rangle \geq 0\}; F) \xrightarrow{\sim} \mathrm{R}\Gamma((x+\gamma)\cap\{\langle x,\xi_0\rangle = 0\}; F).$$

This last isomorphism is obtained by constructing a family of open sets $\{U_t\}_{t\geq 0}$ such that U_0 is a neighborhood of $(x+\gamma)\cap\langle x,\xi_0\rangle = 0$, U_1 is a neighborhood of $(x+\gamma)\cap\langle x,\xi_0\rangle \geq 0$ and the conormals to the boundary of U_t do not belong to $\mathrm{SS}(F)$ on $\Omega_1 \setminus \Omega_0$.
(ii) We have to prove that $\mathrm{R}\Gamma_{\Omega_1\setminus\Omega_0}(U; F) \simeq 0$. This follows from

$$\begin{aligned}\mathrm{R}\Gamma_{\Omega_1\setminus\Omega_0}(U;F) &\simeq \mathrm{R}\Gamma(U; \mathrm{R}\Gamma_{\Omega_1\setminus\Omega_0}F)\\ &\simeq \mathrm{R}\Gamma(\Omega_1;\widetilde{F}) \simeq \mathrm{R}\Gamma(\Omega_1; \mathrm{R}\Phi_{\gamma_*}\widetilde{F}|_{\Omega_1}).\end{aligned}$$

Q.E.D.

4 The functor μhom

References for this section are made to [KS90, § 3.7, Ch. 4, §6.2 §7.2].

4.1 Fourier-Sato transform

The classical Fourier transform interchanges (generalized) functions on a vector space $\mathbb{V}$ and (generalized) functions on the dual vector space $\mathbb{V}^*$. The idea of extending this formalism to sheaves, hence of replacing an isomorphism of spaces with an equivalence of categories, seems to have appeared first in Mikio Sato's construction of microfunctions in [Sat70].

Let $\tau : E \to M$ be a finite dimensional real vector bundle over a real manifold M with fiber dimension n and let $\pi : E^* \to M$ be the dual vector bundle. Denote by p_1 and p_2 the first and second projection defined on $E \times_M E^*$, and define:

$$P = \{(x,y) \in E \times_M E^*; \langle x,y\rangle \geq 0\},$$
$$P' = \{(x,y) \in E \times_M E^*; \langle x,y\rangle \leq 0\}.$$

Consider the diagram:

$$\begin{tikzcd} & E \times_M E^* \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ E \arrow[dr, "\tau"'] & & E^* \arrow[dl, "\pi"] \\ & M. & \end{tikzcd}$$

Denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ consisting of conic sheaves, that is, objects with locally constant cohomology on the orbits of the action of $\mathbb{R}^+$.

Definition 4.1. Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$, $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}^*_E)$. One sets:

$$\begin{aligned} F^\wedge &:= \mathrm{R}p_{2!}(p_1^{-1}F)_{P'} \simeq \mathrm{R}p_{2*}(\mathrm{R}\Gamma_P p_1^{-1}F), \\ G^\vee &:= \mathrm{R}p_{1*}(\mathrm{R}\Gamma_{P'} p_2^! G) \simeq \mathrm{R}p_{1!}(p_2^! G)_P. \end{aligned}$$

The main result of the theory is the following.

Theorem 4.2. *The two functors $(\cdot)^\wedge$ and $(\cdot)^\vee$ are inverse to each other, hence define an equivalence of categories $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E) \simeq \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{E^*})$ and for $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$, one has the isomorphism*

$$\mathrm{RHom}\,(F_1^\wedge, F_2^\wedge) \simeq \mathrm{RHom}\,(F_1, F_2). \tag{4.1}$$

Example 4.3. (i) Let γ be a closed proper convex cone in E with $M \subset \gamma$. Then:

$$(\mathbf{k}_\gamma)^\wedge \simeq \mathbf{k}_{\mathrm{Int}(\gamma^\circ)}.$$

Here γ° is the polar cone to γ, a closed convex cone in E^* and $\mathrm{Int}\gamma^\circ$ denotes its interior.
(ii) Let γ be an open convex cone in E. Then:

$$(\mathbf{k}_\gamma)^\wedge \simeq \mathbf{k}_{\gamma^{\circ a}} \otimes \mathrm{or}_{E^*/M}\,[-n].$$

Here $\lambda^a = -\lambda$, the image of λ by the antipodal map.
(iii) Let $(x) = (x', x'')$ be coordinates on $\mathbb{R}^n$ with $(x') = (x_1, \dots, x_p)$ and $(x'') = (x_{p+1}, \dots, x_n)$. Denote by $(y) = (y', y'')$ the dual coordinates on $(\mathbb{R}^n)^*$. Set

$$\gamma = \{x; x'^2 - x''^2 \geq 0\}, \quad \lambda = \{y; y'^2 - y''^2 \leq 0\}.$$

Then $(\mathbf{k}_\gamma)^\wedge \simeq \mathbf{k}_\lambda[-p]$. (See [KS97].)

4.2 Specialization

Let $\iota\colon N \hookrightarrow M$ be the embedding of a closed submanifold N of M. Denote by $\tau_M\colon T_N M \to M$ the normal bundle to N.

If F is a sheaf on M, its restriction to N, denoted $F|_N$, may be viewed as a global object, namely the direct image by τ_M of a sheaf $\nu_N F$ on $T_N M$, called the specialization of F along N. Intuitively, $T_N M$ is the set of light rays issued from N in M and the germ of $\nu_N F$ at a normal vector $(x;v) \in T_N M$ is the germ at x of the restriction of F along the light ray v.

One constructs a new manifold $\widetilde{M}_N$, called the normal deformation of M along N, together with the maps

$$
(4.2) \qquad \begin{tikzcd} T_N M \arrow[r, "s"] \arrow[d, "\tau_M"'] & \widetilde{M}_N \arrow[d, "p"] & \Omega \arrow[l, "j"'] \arrow[dl, "\widetilde{p}"] \\ N \arrow[r, "\iota"'] & M & \end{tikzcd}, \quad t\colon \widetilde{M}_N \to \mathbb{R},\ \Omega = \{t^{-1}(\mathbb{R}_{>0})\}
$$

with the following properties. Locally, after choosing a local coordinate system (x', x'') on M such that $N = \{x' = 0\}$, we have $\widetilde{M}_N = M \times \mathbb{R}$, $t\colon \widetilde{M}_N \to \mathbb{R}$ is the projection, $\Omega = \{(x;t) \in M \times \mathbb{R}; t > 0\}$, $p(x', x'', t) = (tx', x'')$, $T_N M = \{t = 0\}$.

Definition 4.4. (a) Let $S \subset M$ be a locally closed subset. The Whitney normal cone $C_N(S)$ is a closed conic subset of $T_N M$ given by

$$C_N(S) = \overline{\widetilde{p}^{-1}(S)} \cap T_N M.$$

(b) For two subsets $S_1, S_2 \subset M$, their Whitney's normal cone is given by

$$C(S_1, S_2) = C_\Delta(S_1 \times S_2) \tag{4.3}$$

where Δ is the diagonal of $M \times M$ and TM is identified to $T_\Delta(M \times M)$ by the first projection $T(M \times M) \to TM$.

One defines the specialization functor

$$\nu_N\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{T_N M})$$

by a formula mimicking Definition 4.4, namely:

$$\nu_N F := s^{-1} \mathrm{R} j_* \widetilde{p}^{-1} F.$$

Clearly, $\nu_N F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T_N M})$, that is, $\nu_N F$ is a conic sheaf for the $\mathbb{R}^+$-action on $T_N M$. Moreover,

$$\mathrm{R}\tau_{M*}\nu_N F \simeq \nu_N F|_N \simeq F|_N.$$

For an open cone $V \subset T_N M$, one has

$$H^j(V;\nu_N F) \simeq \varinjlim_U H^j(U;F)$$

where U ranges through the family of open subsets of M such that

$$C_N(M \setminus U) \cap V = \emptyset.$$

4.3 Microlocalization

Denote by $\pi_M\colon T^*_N M \to M$ the conormal bundle to N in M, that is, the dual bundle to $\tau_M\colon T_N M \to M$.

If F is a sheaf on M, the sheaf of sections of F supported by N, denoted $\mathrm{R}\Gamma_N F$, may be viewed as a global object, namely the direct image by π_M of a sheaf $\mu_M F$ on $T^*_N M$. Intuitively, $T^*_N M$ is the set of "walls" (half-spaces) in M containing N in their boundary and the germ of $\mu_N F$ at a conormal vector $(x;\xi) \in T^*_N M$ is the germ at x of the sheaf of sections of F supported by closed tubes with edge N and which are almost the half-space associated with ξ.

More precisely, the microlocalization of F along N, denoted $\mu_N F$, is the Fourier-Sato transform of $\nu_N F$, hence is an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T^*_N M})$. It satisfies:

$$\mathrm{R}\pi_{M*}\mu_N F \simeq \mu_N F|_N \simeq \mathrm{R}\Gamma_N F.$$

For a convex open cone $V \subset T^*_N M$, one has

$$H^j(V;\mu_N F) \simeq \varinjlim_{U,Z} H^j_{U\cap Z}(U;F),$$

where U ranges over the family of open subsets of M such that $U \cap N = \pi_M(V)$ and Z ranges over the family of closed subsets of M such that $C_M(Z) \subset V^\circ$ where V° is the polar cone to V.

If $H \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T^*M})$ is a conic sheaf on T^*M, then $\mathrm{R}\pi_{M!}H \simeq \mathrm{R}\Gamma_M H$ and one gets Sato's distinguished triangle

$$\mathrm{R}\pi_{M!}H \to \mathrm{R}\pi_{M*}H \to R\dot{\pi}_{M*}H \xrightarrow{+1} . \tag{4.4}$$

Applying this result to the conic sheaf $\mu_N F$, one gets the distinguished triangle

$$F|_N \otimes \omega_{N/M} \to \mathrm{R}\Gamma_N F|_N \to R\dot{\pi}_{M*}\mu_N F \xrightarrow{+1} . \tag{4.5}$$

4.4 The functor μhom

Let us briefly recall the main properties of the functor μhom, a variant of Sato's microlocalization functor.

Recall that Δ denotes the diagonal of $M \times M$. We shall denote by $\widetilde{\delta}$ the isomorphism

$$\widetilde{\delta}\colon T^*M \xrightarrow{\sim} T^*_M(M \times M), \quad (x;\xi) \mapsto (x, x; \xi, -\xi).$$

Definition 4.5. One defines the functor $\mu hom\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{T^*M})$ by

$$\mu hom(F_2, F_1) = \widetilde{\delta}^{-1}\mu_\Delta \mathrm{R}\mathscr{H}om\,(q_2^{-1}F_2, q_1^! F_1)$$

where q_i $(i = 1, 2)$ denotes the i-th projection on $M \times M$.

Note that

- $\mathrm{R}\pi_{M*}\mu hom(F_2, F_1) \simeq \mathrm{R}\mathscr{H}om\,(F_2, F_1)$,
- $\mu hom(\mathbf{k}_N, F) \simeq \mu_N(F)$ for N a closed submanifold of M,
- assuming that F_2 is cohomologically constructible, there is a distinguished triangle $\mathrm{D}'F_2 \otimes F_1 \to \mathrm{R}\mathscr{H}om\,(F_2, F_1) \to R\dot{\pi}_{M*}\mu hom(F_2, F_1) \xrightarrow{+1}$.

Moreover

$$\mathrm{Supp}\,\mu hom(F_2, F_1) \subset \mathrm{SS}(F_2) \cap \mathrm{SS}(F_1). \tag{4.6}$$

We shall see in the next section that, in some sense, μhom is the sheaf of microlocal morphisms.

Corollary 4.6. (The Petrowsky theorem for sheaves.) *Assume that F_2 is cohomologically constructible and $\mathrm{SS}(F_2) \cap \mathrm{SS}(F_1) \subset T^*_M M$. Then the natural morphism*

$$\mathrm{R}\mathscr{H}om\,(F_2, \mathbf{k}_M) \otimes F_1 \to \mathrm{R}\mathscr{H}om\,(F_2, F_1)$$

is an isomorphism.

For two subsets A and B of T^*M, we still denote by $C(A,B)$ the inverse image in T^*T^*M of their Whithney normal cone by the Hamiltonian isomorphism $H\colon T^*T^*M \xrightarrow{\sim} TT^*M$.

Theorem 4.7 (See [KS90, Cor. 6.4.3])**.** *Let $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. Then*

$$\mathrm{SS}(\mu hom(F_2,F_1)) \subset C(\mathrm{SS}(F_2),\mathrm{SS}(F_1)). \tag{4.7}$$

Consider a vector bundle $\tau\colon E \to N$ over a manifold N. It gives rise to a morphism of vector bundles over N, $\tau'\colon TE \to E \times_N TN$ which by duality gives the map $\tau_d\colon E \times_N T^*N \to T^*E$. By restricting to the zero-section of E, we get the map:

$$T^*N \hookrightarrow T^*E.$$

Applying this construction to the bundle $T^*_N M$ above N, and using the Hamiltonian isomorphism we get the maps

$$T^*N \hookrightarrow T^*T^*_N M \simeq T_{T^*_N M}T^*M. \tag{4.8}$$

Corollary 4.8. (See [KS90, Cor. 6.4.4].) *One has*

$$\begin{aligned}
&\mathrm{SS}(\mathrm{R}\Gamma_N F) \subset T^*N \cap C_{T^*_N M}(\mathrm{SS}(F)),\\
&\mathrm{SS}(F|_N) \subset T^*N \cap C_{T^*_N M}(\mathrm{SS}(F)).
\end{aligned}$$

Microlocal Serre functor

There is an interesting phenomenon which holds with μhom and not with $\mathrm{R}\mathscr{H}om$. Indeed, assume M is real analytic. Then, although the category $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ of $\mathbb{R}$-constructible sheaves does not admit a Serre functor, it admits a kind of microlocal Serre functor, as shown by the isomorphism, functorial in F_1 and F_2 (see [KS90, Prop. 8.4.14]):

$$\mathrm{D}_{T^*M}\mu hom(F_2,F_1) \simeq \mu hom(F_1,F_2) \otimes \pi_M^{-1}\omega_M.$$

This confirms the fact that to fully understand $\mathbb{R}$-constructible sheaves, it is natural to look at them microlocally, that is, in T^*M. This is also in accordance with the "philosophy" of Mirror Symmetry which interchanges the category of coherent $\mathscr{O}_X$-modules on a complex manifold X with the Fukaya category on a symplectic manifold Y. In case of $Y = T^*M$, the Fukaya category is equivalent to the category of $\mathbb{R}$-constructible sheaves on M, according to Nadler-Zaslow [Nad09, NZ09].

Microlocal Fourier-Sato transform

The Fourier-Sato transfom is by no means local: it interchanges sheaves on a vector bundle E and sheaves on E^*. However, this transformation is *microlocal* in the following sense.

Let $E \to Z$ be vector bundle over a manifold Z. There is a natural isomorphism $T^*E \simeq T^*E^*$ given in local coordinates

$$T^*E \ni (z, x; \zeta, \xi) \mapsto (z, \xi; \zeta, -x) \in T^*E^*. \tag{4.9}$$

Theorem 4.9 ([KS90, Exe. VII.2]). *Let $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$. There is a natural isomorphism*

$$\mu hom(F_2, F_1) \simeq \mu hom(F_2^{\wedge}, F_1^{\wedge}). \tag{4.10}$$

5 Microlocal theory

5.1 Localization

Let A be a subset of T^*M and let $Z = T^*M \setminus A$. The full subcategory $\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_M)$ of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of sheaves F such that $\mathrm{SS}(F) \subset Z$ is a triangulated subcategory. One sets

$$\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; A) := \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)/\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_M),$$

the localization of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ by $\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_M)$. Hence, the objects of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; A)$ are those of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ but a morphism $u\colon F_1 \to F_2$ in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ becomes an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; A)$ if, after embedding this morphism in a distinguished triangle $F_1 \to F_2 \to F_3 \xrightarrow{+1}$, one has $\mathrm{SS}(F_3) \cap A = \emptyset$. When $A = \{p\}$ for some $p \in T^*M$, one simply writes $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; p)$ instead of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; \{p\})$.

The functor μhom describes in some sense the microlocal morphisms of the category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. More precisely, for U open in T^*M, it follows from (4.6) that μhom induces a bifunctor:

$$\mu hom\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; U)^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; U) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_U).$$

Moreover, the sequence of morphisms

$$\begin{aligned} \mathrm{RHom}\,(G, F) &\simeq \mathrm{R}\Gamma(M; \mathrm{R}\mathscr{H}om\,(G, F)) \\ &\simeq \mathrm{R}\Gamma(T^*M; \mu hom(G, F)) \\ &\to \mathrm{R}\Gamma(U; \mu hom(G, F)) \end{aligned}$$

define the morphism

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M;U)}(G,F) \to H^0\mathrm{R}\Gamma(U;\mu hom(G,F)). \tag{5.1}$$

The morphism (5.1) is not an isomorphism, but it induces an isomorphism at each $p \in T^*M$:

Theorem 5.1 (See [KS90, Th. 6.1.2])**.** *Let $p \in T^*M$. Then*

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M;p)}(G,F) \simeq H^0(\mu hom(G,F)_p).$$

5.2 Pure and simple sheaves

Let S be a smooth submanifold of M and let $\Lambda = T^*_SM$. Let $p \in \Lambda, p \notin T^*_MM$ and let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M;p)$. Let us say that F is pure at p if $F \simeq V\,[d]$ for some $\mathbf{k}$-module V and some shift d and let us say that F is simple if moreover V is free of rank one. A natural question is to generalize this definition to the case where Λ is a smooth Lagrangian submanifold of $\dot{T}^*M$ but is no more necessarily a conormal bundle. Another natural question would be to calculate the shift d. This last point makes use of the Maslov index and we refer to [KS90, §. 7.5].

Notation 5.2. Let Λ be a smooth $\mathbb{R}^+$-conic Lagrangian locally closed submanifold of $\dot{T}^*M$, closed in an open conic neighborhood W of Λ.

(i) We denote by $\mathsf{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of objects F such that there exists an open neighborhood W of Λ (containing Λ as a closed subset) in T^*M such that $\mathrm{SS}(F) \cap W \subset \Lambda$.

(ii) One denotes by $\mathrm{DLoc}(\mathbf{k}_\Lambda)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_\Lambda)$ consisting of objects F such that for each $j \in \mathbb{Z}$, $H^j(F)$ is a local system on Λ. Equivalently, $\mathrm{DLoc}(\mathbf{k}_\Lambda)$ is the subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_\Lambda)$ consisting of sheaves with microsupport contained in the zero-section $T^*_\Lambda\Lambda$.

Applying Theorem 4.7, we get

Corollary 5.3. *The functor μhom induces a functor*

$$\mu hom\colon \mathsf{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M) \to \mathrm{DLoc}(\mathbf{k}_\Lambda).$$

Lemma 5.4. *Let $L \in \mathsf{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M;W)$. There is a natural morphism $\mathbf{k}_\Lambda \to \mu hom(L,L)$.*

Proof. Represent $L \in \mathsf{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$ by $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. The morphism $\mathbf{k}_M \to \mathrm{R}\mathscr{H}om\,(F,F) \simeq \mathrm{R}\pi_*\mu hom(F,F)$ defines the morphism $\mathbf{k}_{T^*M} \to \mu hom(F,F)$. Since $\mu hom(L,L)$ is supported by Λ in a neighborhoodd of Λ, this last morphism factorizes through $\mathbf{k}_\Lambda$. Q.E.D.

The notions of pure and simple sheaves are introduced and intensively studied in [KS90, § 7.5].

For a C^∞-function φ on M we denote by Λ_φ the (non conic) Lagrangian submanifold of T^*M given by

$$\Lambda_\varphi := \{(x; d\varphi(x)); x \in M\}.$$

Let $p \in \Lambda$. One says that φ is transverse to Λ at p if $\varphi(\pi_M(p)) = 0$ and the manifolds Λ and Λ_φ intersect transversally at p. We define the Lagrangian planes in T_pT^*M:

$$\lambda_0(p) = T_p(\pi_M^{-1}\pi_M(p)), \quad \lambda_\Lambda(p) = T_p\Lambda, \quad \lambda_\varphi(p) = T_p\Lambda_\varphi. \tag{5.2}$$

Lemma 5.5. *Let $p \in \Lambda$ and let φ be transverse to Λ at p. The property that $\mathrm{R}\Gamma_{\{\varphi\geq 0\}}(F)_{\pi_M(p)}$ is concentrated in a single degree* (*resp. and is free of rank one over* $\mathbf{k}$) *does not depend on the choice of φ.*

Proof. See [KS90, Prop. 7.5.3, 7.5.6]. Q.E.D.

In loc. cit., the shift of $\mathrm{R}\Gamma_{\varphi\geq 0}(F)_{\pi_M(p)}$ is related to the Maslov index of the Lagrangian planes of (5.2)

By this lemma, one can state:

Definition 5.6. Let $F \in \mathsf{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$ and let φ be transverse to Λ at p.

(a) One says that F is pure on Λ if $\mathrm{R}\Gamma_{\{\varphi\geq 0\}}(F)_{\pi(p)}$ is concentrated in a single degree. One denotes by $\mathrm{Pure}(\Lambda, \mathbf{k})$ the subcategory of $\mathsf{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$ consisting of pure sheaves.

(b) One says that F simple on Λ if $\mathrm{R}\Gamma_{\{\varphi\geq 0\}}(F)_{\pi(p)}$ is concentrated in a single degree and is free of rank one. One denotes by $\mathrm{Simple}(\Lambda, \mathbf{k})$ the subcategory of $\mathsf{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$ consisting of simple sheaves.

When $\mathbf{k}$ is a field, there is an easy criterium of purity and simplicity.

Proposition 5.7. *Assume that $\mathbf{k}$ is a field and let $F \in \mathsf{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$. Then*

(a) *F is pure on Λ if and only if $\mu hom(F,F)|_\Lambda$ is concentrated in degree 0,*

(b) *F simple on Λ if and only if $\mathbf{k}_\Lambda \xrightarrow{\sim} \mu hom(L,L)|_\Lambda$.*

Sketch of proof. By using a quantized contact transformation (see § 5.3 below) one reduces the problem to the case where $\Lambda = T^*_N M$ for a closed submanifold N of M. Then locally, $F \simeq A_N$ for some $A \in \mathsf{D}^{\mathrm{b}}(\mathbf{k})$ and the result is obvious in this case. Q.E.D.

Remark 5.8. Let $L \in \mathrm{Simple}(\Lambda, \mathbf{k})$. Then the functor

$$\mu hom(L, \bullet)\colon \mathrm{Pure}(\Lambda,\mathbf{k}) \to \mathrm{DLoc}(\mathbf{k}_\Lambda) \tag{5.3}$$

is well-defined. One shall be aware that:
(i) the category $\mathrm{Simple}(\Lambda, \mathbf{k})$ may be empty,
(ii) the functor in (5.3) is not fully faithful in general,
(iii) the categories $\mathrm{Pure}(\Lambda, \mathbf{k})$ and $\mathrm{Simple}(\Lambda, \mathbf{k})$ are not additive.

Proposition 5.9 (see [KS90, Cor.7.5.4])**.** *Let $F \in \mathsf{D}^{\mathrm{b}}_{(\Lambda)}(\mathbf{k}_M)$. Then the set of $p \in \Lambda$ in a neighborhood of which F is pure (resp. simple) is open and closed in Λ.*

Proof. We shall only give a proof when assuming that $\mathbf{k}$ is a field.
One knows by Corollary 5.3 that $L := \mu hom(F,F)$ is a local system on Λ. Then the set of $p \in \Lambda$ in a neighborhood of which L is concentrated in degree 0 (resp. is of rank one) is open and closed in Λ. Q.E.D.

Remark 5.10. Pure sheaves are intensively (and implicitely) used in [STZ14] in their study of Legendrain knots.

5.3 Quantized contact transformations

References for this subsection are made to [KS90, §7.2].

Consider two manifolds M and N, two conic open subsets $U \subset T^*M$ and $V \subset T^*N$ and a homogeneous contact transformation χ:

$$T^*N \supset V \underset{\chi}{\xrightarrow{\sim}} U \subset T^*M. \tag{5.4}$$

Denote by V^a the image of V by the antipodal map a_N on T^*N and by Λ the image of the graph of χ by $\mathrm{id}_U \times a_N$. Hence Λ is a conic Lagrangian submanifold of $U \times V^a$. Consider $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times N})$ and the hypotheses

$$\text{(5.5)} \quad \begin{cases} K \text{ is cohomologically constructible,} \\ K \text{ is simple along } \Lambda, \\ (p_1^{-1}U \cup p_2^{-1}V^a) \cap \mathrm{SS}(K) \subset \Lambda. \end{cases}$$

Theorem 5.11 (See [KS90, Th. 7.2.1])**.** *If K satisfies the hypotheses (5.5), then the functor $K\circ$ induces an equivalence*

$$\text{(5.6)} \qquad K\circ\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N; V) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; U).$$

Moreover, for $G_1, G_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N; V)$

$$\text{(5.7)} \qquad \chi_*(\mu hom(G_1, G_2)|_V) \xrightarrow{\sim} \mu hom(K\circ G_1, K\circ G_2)|_U.$$

One calls (χ, K) a quantized contact transformation (a QCT, for short).

Corollary 5.12. *Let Λ_i be a conic smooth Lagrangian submanifold of $\dot{T}^*M_i$ $(i = 1, 2)$ with $\chi(\Lambda_2) = \Lambda_1$. Then $K\circ$ induces an equivalence* $\mathrm{Pure}(\Lambda_2, \mathbf{k}) \xrightarrow{\sim} \mathrm{Pure}(\Lambda_1; \mathbf{k})$ *and similarly when* Pure *is replaced with* Simple.

Corollary 5.13. *Consider a homogeneous contact transformation $\chi\colon T^*M \supset U \xrightarrow{\sim} V \subset T^*N$. Then for any $p \in U$, there exists a conic open neighborhood W of p in U and a quantized contact transform $(\chi|_W, K)$ where $\chi|_W\colon W \xrightarrow{\sim} \chi(W)$ is the restriction of χ.*

Proof. Locally any contact transform χ is the composition $\chi_1 \circ \chi_2$ where the graph of each χ_i $(i = 1, 2)$ is the Lagrangian manifold associated with the conormal to a hypersurface S. In this case, one can choose $K = \mathbf{k}_S$. Q.E.D.

5.4 Quantization of Hamiltonian isotopies

References for this subsection are made to [GKS12].

Hamiltonian symplectic isotopies

- A symplectic manifold (X, ω_X), or simply X, is a real C^∞-manifold X endowed with a closed non-degenerate 2-form ω_X.

- For two symplectic manifolds (X, ω_X) and (Y, ω_Y), one endows $X \times Y$ with the symplectic form $\omega_X + \omega_Y$.
- One denotes by X^a the symplectic manifold for which $\omega_{X^a} = -\omega_X$.
- The symplectic form ω_X defines the Hamiltonian isomorphism $H^{-1} \colon TX \xrightarrow{\sim} T^*X$ by the formula (up to a sign) $H^{-1}(v) = \iota_v(\omega_X)$ where ι_v is the interior product.
- If $f \colon X \to \mathbb{R}$ is a C^∞-map, the image by H of its differential df is a vector field on X, called the Hamiltonian vector field and denoted H_f. Hence, $H_f = H(df)$.
- A symplectic isomorphism φ is a C^∞ isomorphism $\varphi \colon X \to X$ such that $\varphi^*\omega_X = \omega_X$. Its graph Λ_φ is a Lagrangian submanifold of $X \times X^a$.
- Consider an open interval I and a map $f \colon X \times I \to X$. We shall write for short $f_s = f(\cdot, s)$.

Definition 5.14. A Hamiltonian isotopy Φ on X is the data of an open interval I containing 0 and a C^∞-map $\Phi \colon X \times I \to X$ such that

$$(5.8) \quad \begin{cases} \text{(a) } \Phi = \{\varphi_s\}_{s \in I},\ \varphi_s \text{ is a symplectic isomorphism for each } s \in I, \\ \text{(b) } \varphi_0 = \mathrm{id}_X, \\ \text{(c) there exists a } C^\infty\text{-function } f \colon X \times I \to \mathbb{R} \text{ such that } \frac{\partial \Phi}{\partial s} = H_{f_s}. \end{cases}$$

Let Φ be as in (5.8) satisfying conditions (a) and (b). Denote by Λ' its graph in $X \times X^a \times I$. Then Φ is an Hamiltonian isotopy if and only if there exists a Lagrangian manifold

$$(5.9) \qquad \Lambda \subset X \times X^a \times T^*I$$

such that Λ' is the image of Λ by the projection $\pi \colon X \times X^a \times T^*I \to X \times X^a \times I$.

Homogeneous Hamiltonian isotopies

- An exact symplectic manifold (X, α_X), or simply X, is a real C^∞-manifold X endowed with a non-degenerate 1-form α_X such that $\omega_X := d\alpha_X$ is symplectic.

- The Hamiltonian isomorphism on (X, ω_X) sends α_X to a vector field that we call the Euler vector field and denote by eu_X. A submanifold Y of X is homogeneous (or conic) if the Euler vector field is tangent to it.

- A homogeneous symplectic isomorphism φ is a C^∞ isomorphism $\varphi\colon X \to X$ such that $\varphi^*\alpha_X = \alpha_X$. Its graph Λ_φ is a homogeneous Lagrangian submanifold of $X \times X^a$.

Of course a homogeneous symplectic isomorphism induces a symplectic isomorphism.

Example 5.15. Let M be a real manifold of class C^∞. Set $X := \dot{T}^*M$ the space $T^*M \setminus T^*_M M$ and by $\dot{\pi}_M\colon \dot{T}^*M \to M$ the projection. Then X is an exact symplectic manifold when endowed with the Liouville form α_X on $\dot{T}^*M$. If $(x) = (x_1, \dots, x_n)$ is a local coordinate system on M, $(x;\xi)$ the associated coordinate system on T^*M, then

$$\alpha_X = \sum_j \xi_j dx_j, \quad \mathrm{eu}_X = -\sum_j \xi_j \frac{\partial}{\partial \xi_j}.$$

We consider a C^∞-map $\Phi\colon X \times I \to X$.

Definition 5.16. A homogeneous Hamiltonian isotopy Φ on X is the data of an open interval I containing 0 and a C^∞-map $\Phi\colon X \times I \to X$ such that

$$(5.10)\quad \begin{cases} \text{(a) } \Phi = \{\varphi_s\}_{s\in I},\ \varphi_s \text{ is a homogeneous symplectic isomorphism for} \\ \text{each } s \in I, \\ \text{(b) } \varphi_0 = \mathrm{id}_X. \end{cases}$$

Let Φ be a homogeneous Hamiltonian isotopy. Set

$$\begin{aligned} &v_\Phi := \frac{\partial \Phi}{\partial t}\colon X \times I \to TX, \\ &f = \langle \alpha_M, v_\Phi \rangle\colon X \times I \to \mathbb{R},\ f_s = f(\cdot, s). \end{aligned}$$

Then

$$\frac{\partial \Phi}{\partial s} = H_{f_s}.$$

In other words, if Φ satisfies conditions (a) and (b) of Definition 5.16 then it satisfies condition (a), (b) and (c) of Definition 5.14.

Now we assume $X = \dot{T}^*M$. If $\varphi\colon X \to X$ is a homogeneous symplectic isomorphism, its graph Γ_φ is Lagrangean in $X \times X^a$ and we denote by Λ_φ the image of Γ_φ by the antipodal map on the second group of variables, $(x, \xi, y, \eta) \mapsto (x, \xi, y, -\eta)$. Then Λ_φ is Lagrangian in $X \times X$. For short, we call Λ_φ the Lagrangian graph of φ.

Let Φ be a homogeneous Hamiltonian isotopy on $X = \dot{T}^*M$. Then there exists a unique conic Lagrangian submanifold Λ of $\dot{T}^*M \times \dot{T}^*M \times T^*I$ such that

- Λ is closed in $\dot{T}^*(M \times M \times I)$
- for any $s \in I$, the inclusion $i_s\colon M \times M \to M \times M \times I$ is non-characteristic for Λ
- the Lagrangian graph of φ_s is $\Lambda_s = \Lambda \circ T^*_s I$.

Quantization of homogeneous isotopies

When applying kernels associated with homogeneous isotopies we may encounter objects of the derived category of sheaves which are locally bounded but not globally. Hence we denote by $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ the full subcategory of $\mathrm{D}(\mathbf{k}_M)$ consisting of objects F such that for any open relatively compact subset $U \subset\subset M$, $F|_U \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_U)$.

For $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ and $s_0 \in I$, we set

$$K_{s_0} := K|_{s=s_0}.$$

Theorem 5.17 ([GKS12])**.** *Let $\Phi\colon \dot{T}^*M \times I \to \dot{T}^*M$ be a homogeneous Hamiltonian isotopy. Then there exists $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ satisfying*

(a) $\mathrm{SS}(K) \subset \Lambda \cup T^*_{M\times M\times I}(M \times M \times I)$*,*

(b) $K_0 \simeq \mathbf{k}_\Delta$.

Moreover:

(i) *both projections* $\mathrm{Supp}(K) \rightrightarrows M \times I$ *are proper,*

(ii) *setting* $K_s^{-1} := v^{-1}\mathrm{R}\mathscr{H}om(K_s, \omega_M \boxtimes \mathbf{k}_M)$*, we have* $K_s \circ K_s^{-1} \simeq K_s^{-1} \circ K_s \simeq \mathbf{k}_\Delta$ *for all* $s \in I$*,*

(iii) *such a K satisfying the conditions (a) and (b) above is unique up to a unique isomorphism,*

Example 5.18. Let $M = \mathbb{R}^n$ and denote by $(x;\xi)$ the homogeneous symplectic coordinates on $T^*\mathbb{R}^n$. Consider the isotopy $\varphi_s(x;\xi) = (x - s\frac{\xi}{|\xi|};\xi)$, $s \in I = \mathbb{R}$. Then

$$\Lambda_s = \{(x,y,\xi,\eta); |x-y| = |s|,\ \xi = -\eta = \lambda(x-y),\ s\lambda < 0\} \quad \text{for } s \neq 0,$$
$$\Lambda_0 = \dot{T}^*_{\Delta}(M \times M).$$

For $s \in \mathbb{R}$, the morphism $\mathbf{k}_{\{|x-y|\leq s\}} \to \mathbf{k}_{\Delta\times\{s=0\}}$ gives by duality (replacing s with $-s$) $\mathbf{k}_{\Delta\times\{s=0\}} \to \mathbf{k}_{\{|x-y|<-s\}}[n+1]$. We get a morphism $\mathbf{k}_{\{|x-y|\leq s\}} \to \mathbf{k}_{\{|x-y|<-s\}}[n+1]$ and we define K by the distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times I})$:

$$\mathbf{k}_{\{|x-y|<-s\}}[n] \to K \to \mathbf{k}_{\{|x-y|\leq s\}} \xrightarrow{+1}$$

One can show that K is a quantization of the Hamiltonian isotopy $\{\varphi_s\}_s$. We have the isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M})$: $K_s \simeq \mathbf{k}_{\{|x-y|\leq s\}}$ for $s \geq 0$ and $K_s \simeq \mathbf{k}_{\{|x-y|<-s\}}[n]$ for $s < 0$.

Corollary 5.19. *Let Φ be a homogeneous Hamiltnian isotopy as in* Theorem 5.17*. Let Λ_0 be a smooth closed conic Lagrangian submanifold of $\dot{T}^*M$ and let $\Lambda_1 = \varphi_1(\Lambda_0)$. The the categories* $\mathrm{Pure}(\Lambda_0;\mathbf{k})$ *and* $\mathrm{Pure}(\Lambda_1;\mathbf{k})$ *are equivalent. The same result holds with* Simple *instead of* Pure.

5.5 Application to non displaceability

In [Tam08] (see also [GS14] for an exposition and some developments), Dmitry Tamarkin shows that microlocal sheaf theory may be applied to solve some problems of symplectic topology. He gives in particular a new proof of Arnold's non displaceability conjecture/theorem as well as other results of non displaceability. One difficulty is that the objects appearing in microlocal sheaf theory are conic for the $\mathbb{R}^+$-action, or, equivalently, this theory uses the *homogeneous* symplectic structure of the cotangent bundle, contrarily to the problems encountered in classical symplectic topology. This difficulty is overcome by Tamarkin who add a variable t and, denoting by $(t;\tau)$ the coordinates on $T^*\mathbb{R}$, works in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}};\tau > 0)$. However, it is sometimes possible to "translate" non conic problem to conic ones, and then to use the tools of sheaves. This is the approach of [GKS12], that we shall recall now.

Theorem 5.20 ([GKS12])**.** *Consider a homogeneous Hamiltonian isotopy $\Phi = \{\varphi_s\}_{s\in I} : \dot{T}^*M \times I \to \dot{T}^*M$ and a C^1-map $\psi\colon M \to \mathbb{R}$ such that the differential $d\psi(x)$ never vanishes. Set*

$$\Lambda_\psi := \{(x; d\psi(x));\ x \in M\} \subset \dot{T}^*M.$$

Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ with compact support and such that $\mathrm{R}\Gamma(M; F) \neq 0$. Then for any $s \in I$, $\varphi_s(\mathrm{SS}(F) \cap \dot{T}^*M) \cap \Lambda_\psi \neq \emptyset$.

Proof. Let $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times I})$ be the quantization of Φ given by Theorem 5.17.
Set:

$$F_s := K_s \circ F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \quad \text{for } s \in I.$$

We have $F_0 = F$, F_t has compact support and $\mathrm{R}\Gamma(M; F_s) \simeq \mathrm{R}\Gamma(M; F) \neq 0$ by Corollary 2.10. Applying Corollary 3.7, we get $\Lambda_\psi \cap \mathrm{SS}(F_s) \neq \emptyset$. Finally, $\mathrm{SS}(F_s) \cap \dot{T}^*M = \varphi_s(\mathrm{SS}(F) \cap \dot{T}^*M)$. Q.E.D.

Corollary 5.21. *Let $\Phi = \{\varphi_t\}_{t\in I}$ and $\psi\colon M \to \mathbb{R}$ be as in Theorem 5.22. Let N be a non-empty compact submanifold of M. Then for any $t \in I$, $\varphi_t(\dot{T}^*_N M) \cap \Lambda_\psi \neq \emptyset$.*

Consider a compact manifold N and a (no more homogeneous) Hamiltonian isotopy $\Phi = \{\varphi_s\}_{s\in I}$

Theorem 5.22 (Arnold's non displaceability conjecture/theorem)**.** *In the above situation, $\varphi_s(T^*_N N) \cap T^*_N N \neq \emptyset$ for all $s \in I$.*

This theorem can be deduced from Theorem 5.22 by choosing $M = N \times \mathbb{R}$ and $\psi\colon N \times \mathbb{R}$, but this is not totally straightforward.

Remark 5.23. There is now a vast literature in the field of symplectic topology in which microlocal sheaf theory plays an essential role. Let us quote among others [Chi14, Gui12, Gui13, Nad16, Tam15].

6 Applications to Analysis

In this section, $\mathbf{k} = \mathbb{C}$.

6.1 Generalized functions

In the sixties, people were used to work with various spaces of generalized functions constructed with the tools of functional analysis. Sato's construction of hyperfunctions in 59-60 (see [Sat59]) is at the opposite of this practice: he uses purely algebraic tools and complex analysis. The importance of Sato's definition is twofold: first, it is purely algebraic (starting with the analytic object $\mathscr{O}_X$), and second it highlights the link between real and complex geometry. (See [Sat59] and see [Sch07] for an exposition of Sato's work.)

Consider first the case where M is an open subset of the real line $\mathbb{R}$ and let X an open neighborhood of M in the complex line $\mathbb{C}$ satisfying $X \cap \mathbb{R} = M$. The space $\mathscr{B}(M)$ of hyperfunctions on M is given by

$$\mathscr{B}(M) = \mathscr{O}(X \setminus M)/\mathscr{O}(X).$$

It is easily proved, using the solution of the Cousin problem, that this space depends only on M, not on the choice of X, and that the correspondence $U \mapsto \mathscr{B}(U)$ (U open in M) defines a flabby sheaf $\mathscr{B}_M$ on M.

With Sato's definition, the boundary values always exist and are no more a limit in any classical sense.

Example 6.1. (i) The Dirac function at 0 is

$$\delta(0) = \frac{1}{2i\pi}(\frac{1}{x - i0} - \frac{1}{x + i0})$$

Indeed, if φ is a C^0-function on $\mathbb{R}$ with compact support, one has

$$\varphi(0) = \lim_{\varepsilon \longrightarrow 0} \frac{1}{2i\pi} \int_{\mathbb{R}} (\frac{\varphi(x)}{x - i\varepsilon} - \frac{\varphi(x)}{x + i\varepsilon}) dx.$$

(ii) The holomorphic function $\exp(1/z)$ defined on $\mathbb{C} \setminus \{0\}$ has a boundary value as a hyperfunction (supported by $\{0\}$) not as a distribution.

On a real analytic manifold M of dimension n with complexification X, Sato first proved that the complex $\mathrm{R}\Gamma_M \mathscr{O}_X\,[n]$ is concentrated in degree 0 and he defined the sheaf $\mathscr{B}_M$ as

$$\mathscr{B}_M = H^n_M(\mathscr{O}_X) \otimes \mathrm{or}_M$$

where or_M is the orientation sheaf on M. Since X is oriented, Poincaré's duality gives the isomorphism$\mathrm{D}'_X(\mathbb{C}_M) \simeq \mathrm{or}_M[-n]$. An equivalent definition of hyperfunctions is thus given by

$$\mathscr{B}_M = \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X(\mathbb{C}_M), \mathscr{O}_X). \tag{6.1}$$

Let us define the notion of "boundary value" in this settings. Consider a subanalytic open subset Ω of X and denote by $\overline{\Omega}$ its closure. Assume that:

$$\begin{cases} \mathrm{D}'_X(\mathbb{C}_\Omega) \simeq \mathbb{C}_{\overline{\Omega}}, \\ M \subset \overline{\Omega}. \end{cases}$$

The morphism $\mathbb{C}_{\overline{\Omega}} \to \mathbb{C}_M$ defines by duality the morphism $\mathrm{D}'_X(\mathbb{C}_M) \to \mathrm{D}'_X(\mathbb{C}_{\overline{\Omega}}) \simeq \mathbb{C}_\Omega$. Applying the functor $\mathrm{RHom}\,(\,\bullet\,, \mathscr{O}_X)$, we get the boundary value morphism

$$\mathrm{b}\colon \mathscr{O}(\Omega) \to \mathscr{B}(M) \text{ where } \mathscr{B}(M) := \Gamma(M; \mathscr{B}_M). \tag{6.2}$$

When considering operations on hyperfunctions such as integral transforms, one is naturally lead to consider more general sheaves of generalized functions such as $\mathrm{R}\mathscr{H}om\,(G, \mathscr{O}_X)$ where G is an $\mathbb{R}$-constructible sheaf.

Similarly as in dimension one, one can represent the sheaf $\mathscr{B}_M$ by using Čech cohomology of coverings of $X \setminus M$. For example, let X be a Stein open subset of $\mathbb{C}^n$ and set $M = \mathbb{R}^n \cap X$. Denote by x the coordinates on $\mathbb{R}^n$ and by $x+iy$ the coordinates on $\mathbb{C}^n$. One can recover $\mathbb{C}^n \setminus \mathbb{R}^n$ by $n+1$ open half-spaces $V_i = \langle y, \xi_i \rangle > 0$ $(i = 1, \dots, n+1)$. For $J \subset \{1, \dots, n+1\}$ set $V_J = \bigcap_{j \in J} V_j$. Assuming $n > 1$, we have the isomorphism $H^n_M(X; \mathscr{O}_X) \simeq H^{n-1}(X \setminus M; \mathscr{O}_X)$. Therefore, setting $U_J = V_J \cap X$, one has

$$\mathscr{B}(M) \simeq \sum_{|J|=n} \mathscr{O}_X(U_J) / \sum_{|K|=n-1} \mathscr{O}_X(U_K).$$

On a real analytic manifold M, any hyperfunction $u \in \Gamma(M; \mathscr{B}_M)$ is a (non unique) sum of boundary values of holomorphic functions defined in tubes with edge M. Such a decomposition leads to the so-called Edge of the Wedge theorem and was intensively studied in the seventies.

Then comes naturally the following problem: how to recognize the directions associated with these tubes? This is at the origin of the construction of Sato's microlocalization functor. Sato introduced in [Sat70] the sheaf $\mathscr{C}_M$ of microfunctions on $T^*_M X$ as

$$\mathscr{C}_M = \mu hom(\mathrm{D}'_X(\mathbb{C}_M), \mathscr{O}_X). \tag{6.3}$$

It is proved that again, this complex is concentrated in degree 0. Thus $\mathscr{C}_M$ is a conic sheaf on $T^*_M X$ and one has by its construction

$$\mathscr{B}_M \xrightarrow{\sim} \pi_{M*}\mathscr{C}_M.$$

Denote by spec the natural map.

$$\mathrm{spec}\colon \Gamma(M;\mathscr{B}_M) \xrightarrow{\sim} \Gamma(T^*_M X;\mathscr{C}_M).$$

Definition 6.2. The (analytic) wave front set $\mathrm{WF}(u)$ of a hyperfunction $u \in \mathscr{B}(M)$ is the support of $\mathrm{spec}(u)$.

Soon after Mikio Sato has defined the analytic wave front set of hyperfunction, Lars Hörmander defined the C^∞-wave front set of distributions, by using classical Fourier transform (see [Hör83]).

6.2 Holomorphic solutions of D-modules

References for D-modules are made to [Kas03].

Characteristic variety

Let X be a complex manifold. One denotes by $\mathscr{D}_X$ the sheaf of rings of holomorphic (finite order) differential operators. It is a right and left coherent ring. A system of linear partial differential equations on X is a left coherent $\mathscr{D}_X$-module $\mathscr{M}$. The link with the intuitive notion of a system of linear partial differential equations is as follows. Locally on X, $\mathscr{M}$ may be represented as the cokernel of a matrix $\cdot P_0$ of differential operators acting on the right:

$$\mathscr{M} \simeq \mathscr{D}_X^{N_0}/\mathscr{D}_X^{N_1}\cdot P_0.$$

By classical arguments of analytic geometry (Hilbert's syzygies theorem), one shows that $\mathscr{M}$ is locally isomorphic to the cohomology of a bounded complex

$$\mathscr{M}^\bullet := 0 \to \mathscr{D}_X^{N_r} \to \cdots \to \mathscr{D}_X^{N_1} \xrightarrow{\cdot P_0} \mathscr{D}_X^{N_0} \to 0. \tag{6.4}$$

For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, one sets for short

$$\mathcal{S}ol(\mathscr{M}) := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X).$$

Representing (locally) $\mathscr{M}$ by a bounded complex $\mathscr{M}^\bullet$, we get

$$\mathcal{S}ol(\mathscr{M}) \simeq 0 \to \mathscr{O}_X^{N_0} \xrightarrow{P_0 \cdot} \mathscr{O}_X^{N_1} \to \cdots \mathscr{O}_X^{N_r} \to 0, \tag{6.5}$$

where now $P_0\cdot$ operates on the left.

One defines naturally the characteristic variety of $\mathscr{M}$, denoted $\mathrm{char}(\mathscr{M})$, a closed complex analytic subset of T^*X, conic with respect to the action of $\mathbb{C}^\times$ on T^*X. For example, if $\mathscr{M}$ has a single generator u with relation $\mathscr{I} u = 0$, where $\mathscr{I}$ is a locally finitely generated left ideal of $\mathscr{D}_X$, then

$$\mathrm{char}(\mathscr{M}) = \{(z;\zeta) \in T^*X; \sigma(P)(z;\zeta) = 0 \text{ for all } P \in \mathscr{I}\},$$

where $\sigma(P)$ denotes the principal symbol of P.

The fundamental result below was first obtained in [SKK73].

Theorem 6.3. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then* $\mathrm{char}(\mathscr{M})$ *is a closed conic complex analytic* involutive (i.e., *co-isotropic*) *subset of T^*X.*

The proof of the involutivity is really difficult: it uses microdifferential operators of infinite order and quantized contact transformations. Later, Gabber [Gab81] gave a purely algebraic (and much simpler) proof of this result and we shall give in Theorem 6.4 below another totally different proof.

After identifying X with its real underlying manifold, the link between the microsupport of sheaves and the characteristic variety of coherent $\mathscr{D}$-modules is given by:

Theorem 6.4. (See [KS90, Th. 11.3.3].) *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then*

$$\mathrm{SS}(F) = \mathrm{char}(\mathscr{M}). \tag{6.6}$$

As a corollary of Theorems 2.4 and 6.4, one recovers the fact that the characteristic variety of a coherent $\mathscr{D}_X$-module is co-isotropic.

We shall only prove the inclusion $\subset$ in (6.6), the most useful for applications.

Sketch of proof. Let $p \notin \mathrm{char}(\mathscr{M})$.
(i) Assume first that $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X \cdot P$ for a section P of $\mathscr{D}_X$, say of order m. Hence, $\sigma(P)(p) \neq 0$, where $\sigma(P)$ is the principal symbol of P. If $p = (x_0; 0) \in T^*_X X$, then P is an invertible function at x_0 and the result is clear.

Assume $p \notin T^*_X X$. We choose a local holomorphic coordinate system $(x) = (x_1, \dots, x_n)$ so that $p = (x_0; \xi_0)$ with $\xi_0 = (1, 0, \dots, 0)$ and we set $x = (x_1, x')$. Set

$$\begin{aligned}
\gamma_\delta &= \{x; \Im x_1 = 0, \Re x_1 \geq \delta |x'|, \\
H_\varepsilon &= \{x; \Re\langle x, \xi_0\rangle \geq -\varepsilon\}, \\
L_\varepsilon &= \{x; \Re\langle x - x_0, \xi_0\rangle = -\varepsilon\},
\end{aligned}$$

We choose $0 < R \ll 1$ and $\delta \gg 0$ such that

$$\sigma(P)(x;\xi) \neq 0 \text{ for } |x - x_0| \leq R, \xi \in \gamma_\delta^\circ \setminus \{0\}.$$

Let K be a compact convex subset of $X = \mathbb{C}^n$. Since $\mathrm{R}\Gamma(K; \mathscr{O}_X)$ is concentrated in degree 0, the object $\mathrm{R}\Gamma(K; \mathcal{S}ol(\mathscr{M}))$ is represented by the complex

$$0 \to \mathscr{O}_X(K) \xrightarrow{P} \mathscr{O}_X(K) \to 0.$$

Applying Theorem 2.6, we are reduced to prove that the two complexes

$$0 \to \mathscr{O}_X((x + \gamma_\delta) \cap H_\varepsilon) \xrightarrow{P} \mathscr{O}_X((x + \gamma_\delta) \cap H_\varepsilon) \to 0$$

and

$$0 \to \mathscr{O}_X((x + \gamma_\delta) \cap L_\varepsilon) \xrightarrow{P} \mathscr{O}_X((x + \gamma_\delta) \cap L_\varepsilon) \to 0$$

are quasi-isomorphic for $|x - x_0| \ll 1$. This follows from the precise version of the Cauchy-Kowalevski[2] theorem of Petrowsky, Leray, Zerner (see [Hör83, Vol 1, Th. 11.4.7]).

(ii) In order to reduce to the case (i), one mimics the proof of the Cauchy-Kowalevski theorem for systems of [Kas95]. In a neighborhood of x_0 the $\mathscr{D}$-module $\mathscr{M}$ admits a system of generators $(u_1, \dots, u_N)$ and $p \notin \mathrm{char}(\mathscr{D}_X \cdot u_j$ $(j = 1, \dots, N)$. For each j there exists a section P_j of $\mathscr{D}_X$ such that p is non characteristic for P_j and $P_j u_j = 0$. Hence there is a natural $\mathscr{D}_X$-linear morphism $\mathscr{D}_X / \mathscr{D}_X \cdot P_j \to \mathscr{D}_X \cdot u_j$. Define the coherent $\mathscr{D}_X$-module $\mathscr{K}$ by the exact sequence

$$0 \to \mathscr{K} \to \bigoplus_{j=1}^{N} (\mathscr{D}_X / \mathscr{D}_X \cdot P_j) \to \mathscr{M} \to 0.$$

[2]we use the name "Kowalevski", according to Sofia Kovalevskaya's practice.

Then $p \notin \mathrm{char}(\mathscr{K})$. Set for short $\mathscr{L} = \bigoplus_{j=1}^{N}(\mathscr{D}_X/\mathscr{D}_X \cdot P_j)$. Let $\varphi\colon X \to \mathbb{R}$ be a C^1-function as in Definition 2.3 and denote for short by $\mathcal{S}ol_\varphi$ the functor $(\mathrm{R}\Gamma_{\{x;\varphi(x)\geq 0\}}\mathcal{S}ol(\,\bullet\,))_{x_0} \simeq 0$. We have a distinguished triangle

$$\mathcal{S}ol_\varphi(\mathscr{M}) \to \mathcal{S}ol_\varphi(\mathscr{L}) \to \mathcal{S}ol_\varphi(\mathscr{K}) \xrightarrow{+1}$$

from which one deduces the long exact sequence

$$0 \to H^0\mathcal{S}ol_\varphi(\mathscr{M}) \to H^0\mathcal{S}ol_\varphi(\mathscr{L}) \to H^0\mathcal{S}ol_\varphi(\mathscr{K}) \to H^1\mathcal{S}ol_\varphi(\mathscr{M}) \to \cdots .$$

It follows from (i) that $\mathcal{S}ol_\varphi(\mathscr{L}) \simeq 0$. Therefore, $H^0\mathcal{S}ol_\varphi(\mathscr{M}) \simeq 0$ and $H^j\mathcal{S}ol_\varphi(\mathscr{K}) \simeq H^{j+1}\mathcal{S}ol_\varphi(\mathscr{M})$. Since $\mathscr{K}$ satisfies the same hypotheses as $\mathscr{M}$, we get by induction that $H^j\mathcal{S}ol_\varphi(\mathscr{M}) \simeq 0$ for all $j \in \mathbb{Z}$. Q.E.D.

Cauchy problem

Let Y be a complex submanifold of the complex manifold X and let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. One can define the induced $\mathscr{D}_Y$-module $\mathscr{M}_Y$, but in general it is an object of the derived category $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)$ which is neither concentrated in degree zero nor coherent. Nevertheless, there is a natural morphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y). \tag{6.7}$$

Recall that one says that Y is non-characteristic for $\mathscr{M}$ if

$$\mathrm{char}(\mathscr{M}) \cap T^*_Y X \subset T^*_X X.$$

With this hypothesis, the induced system $\mathscr{M}_Y$ by $\mathscr{M}$ on Y is a coherent $\mathscr{D}_Y$-module and one has the Cauchy-Kowalevski-Kashiwara theorem [Kas95]:

Theorem 6.5. *Assume Y is non-characteristic for $\mathscr{M}$. Then $\mathscr{M}_Y$ is a coherent $\mathscr{D}_Y$-module and the morphism (6.7) is an isomorphism.*

Sketch of proof. (i) Similarly as in the proof of Theorem 6.4, one reduces to the case where $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X \cdot P$ for a differential operator P of order m and Y is a hypersurface.

(ii) Choose a local coordinate system $z = (z_0, z_1, \ldots, z_n) = (z_0, z')$ on X such that $Y = \{z_0 = 0\}$. Then Y is non-characteristic with respect to P (*i.e.,* for the $\mathscr{D}_X$-module $\mathscr{D}_X/\mathscr{D}_X \cdot P$) if and only if P is written as

$$P(z_0, z'; \partial_{z_0}, \partial_{z'}) = \sum_{0\leq j\leq m} a_j(z_0, z', \partial_{z'})\partial_{z_0}^j \tag{6.8}$$

where $a_j(z_0, z', \partial_{z'})$ is a differential operator not depending on ∂_{z_0} of order $\leq m-j$ and $a_m(z_0, z')$ (which is a holomorphic function on X) satisfies: $a_m(0, z') \neq 0$. By the definition of the induced system $\mathscr{M}_Y$ we obtain

$$\mathscr{M}_Y \simeq \mathscr{D}_X/(z_0 \cdot \mathscr{D}_X + \mathscr{D}_X \cdot P).$$

By the Späth-Weierstrass division theorem for differential operators, any $Q \in \mathscr{D}_X$ may be written uniquely in a neighborhood of Y as

$$Q = R \cdot P + \sum_{j=0}^{m-1} S_j(z, \partial_{z'})\partial_{z_0}^j,$$

hence, as

$$Q = z_0 \cdot Q_0 + R \cdot P + \sum_{j=0}^{m-1} R_j(z', \partial_{z'})\partial_{z_0}^j.$$

Therefore $\mathscr{M}_Y$ is isomorphic to $\mathscr{D}_Y^m$. Theorem 6.5 gives:

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y \simeq \mathscr{O}_Y^m, \quad \mathscr{E}xt^1_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y \simeq 0.$$

In other words, the morphism which to a holomorphic solution f of the homogeneous equation $Pf = 0$ associates its m-first traces on Y is an isomorphism and one can solve the equation $Pf = g$ is a neighborhood of each point of Y.

This is exactly the classical Cauchy-Kowalevski theorem. Q.E.D.

6.3 Elliptic pairs

Let us apply Corollary 4.6 when X is a complex manifold. For $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, set

$$\mathscr{A}_G = \mathscr{O}_X \otimes G, \quad \mathscr{B}_G := \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X G, \mathscr{O}_X).$$

Note that if X is the complexification of a real analytic manifold M and we choose $G = \mathbb{C}_M$, we recover the sheaf of real analytic functions and the sheaf of hyperunctions:

$$\mathscr{A}_{\mathbb{C}_M} = \mathscr{A}_M, \quad \mathscr{B}_{\mathbb{C}_M} = \mathscr{B}_M.$$

Now let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. According to [SS94], one says that the pair $(G, \mathscr{M})$ is elliptic if $\mathrm{char}(\mathscr{M}) \cap \mathrm{SS}(G) \subset T^*_X X$.

Corollary 6.6. [SS94] *Let $(\mathscr{M}, G)$ be an elliptic pair.*

(a) *We have the canonical isomorphism:*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_G) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_G). \tag{6.9}$$

(b) *Assume moreover that* $\mathrm{Supp}(\mathscr{M}) \cap \mathrm{Supp}(G)$ *is compact and* $\mathscr{M}$ *admits a global presentation as in* (6.4). *Then the cohomology of the complex* $\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_G)$ *is finite dimensional.*

Proof. (a) This is a particular case of Corollary 4.6.

(b) One represents the left hand side of the global sections of (6.9) by a complex of topological vector spaces of type DFN and the right hand side by a complex of topological vector spaces of type FN. Q.E.D.

Let us particularize Corollary 4.6 to the usual case of an elliptic system. Let M be a real anaytic manifold, X a complexification of M and let us choose $G = \mathrm{D}'_X\mathbb{C}_M$. Then $(G, \mathscr{M})$ is an elliptic pair if and only if

$$T^*_MX \cap \mathrm{char}(\mathscr{M}) \subset T^*_XX. \tag{6.10}$$

In this case, one simply says that $\mathscr{M}$ is an elliptic system. Then one recovers a classical result:

Corollary 6.7. *Let $\mathscr{M}$ be an elliptic system.*

(a) *We have the canonical isomorphism:*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_M) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M). \tag{6.11}$$

(b) *Assume moreover that M is compact and $\mathscr{M}$ admits a global presentation as in* (6.4). *Then the cohomology of the complex* $\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_M)$ *is finite dimensional.*

There is a more precise result, due to Sato [Sat70].

Proposition 6.8. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, let $j \in \mathbb{Z}$ and let $u \in \mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_M)$. Then* $\mathrm{WF}(u) \subset T^*_MX \cap \mathrm{char}(\mathscr{M})$.

Proof. One has

$$\mathrm{R}\mathscr{H}om_{\pi_M^{-1}\mathscr{D}_X}(\pi_M^{-1}\mathscr{M}, \mathscr{C}_M) \quad \simeq \quad \mu hom(\mathrm{D}'\mathbb{C}_M, \mathrm{R}\mathscr{H}om\,(\mathscr{M}, \mathscr{O}_X))$$

and the support of the right-hand side is contained in $\mathrm{SS}(\mathrm{R}\mathscr{H}om\,(\mathscr{M}, \mathscr{O}_X)) \cap \mathrm{SS}(\mathbb{C}_M)$, that is, in $T^*_MX \cap \mathrm{char}(\mathscr{M})$. Q.E.D.

6.4 Hyperbolic systems

Let again M be a real analytic manifold and X a complexification of M.

Recall that we have constructed in (4.8) (with other notations) the maps

$$T^*M \hookrightarrow T^*T^*_MX \simeq T_{T^*_MX}T^*X. \tag{6.12}$$

Definition 6.9. Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module. We set

$$\mathrm{hypchar}_M(\mathscr{M}) = T^*M \cap C_{T^*_MX}(\mathrm{char}(\mathscr{M}))$$

and call $\mathrm{hypchar}_M(\mathscr{M})$ the hyperbolic characteristic variety of $\mathscr{M}$ along M. A vector $\theta \in T^*M$ such that $\theta \notin \mathrm{hypchar}_M(\mathscr{M})$ is called hyperbolic with respect to $\mathscr{M}$. In case $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X \cdot P$ for a differential operator P, one says that θ is hyperbolic for P.

Example 6.10. Assume we have a local coordinate system $z = x + \sqrt{-1}y$ and $M = \{y = 0\}$. Denote by $(z;\zeta)$ the symplectic coordinates on T^*X with $\zeta = \xi + \sqrt{-1}\eta$. Let $(x_0;\theta_0) \in T^*M$ with $\theta_0 \neq 0$. Let P be a differential operator with principal symbol $\sigma(P)$. Applying the definition of the normal cone, we find that $(x_0;\theta_0)$ is hyperbolic for P if and only if

$$\begin{cases} \text{there exist an open neighborhood } U \text{ of } x_0 \text{ in } M \text{ and an open conic} \\ \text{neighborhood } \gamma \text{ of } \theta_0 \in \mathbb{R}^n \text{ such that } \sigma(P)(x;\theta + \sqrt{-1}\eta) \neq 0 \text{ for} \\ \text{all } \eta \in \mathbb{R}^n,\ x \in U \text{ and } \theta \in \gamma. \end{cases} \tag{6.13}$$

As noticed in the 70's by M. Kashiwara, it follows from the local Bochner's tube theorem that condition (6.13) will be satisfied as soon as $\sigma(P)(x;\theta_0 + \sqrt{-1}\eta) \neq 0$ for all $\eta \in \mathbb{R}^n$ and $x \in U$. Hence, one recovers the classical notion of a (weakly) hyperbolic operator (see [Ler53]).

Theorem 6.11. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then*

$$\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_M)) \subset \mathrm{hypchar}_M(\mathscr{M}).$$

The same result holds with $\mathscr{A}_M$ instead of $\mathscr{B}_M$.

Proof. This follows from Corollary 4.8 and the isomorphisms

$$\begin{aligned} &\mathrm{R}\Gamma_M \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathrm{R}\Gamma_M\mathscr{O}_X), \\ &\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X)|_M \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X|_M). \end{aligned}$$

Q.E.D.

We consider the following situation: M is a real analytic manifold of dimension n, X is a complexification of M, $N \hookrightarrow M$ is a real analytic smooth closed submanifold of M of codimension d and $Y \hookrightarrow X$ is a complexification of N in X.

Theorem 6.12. *Let M, X, N, Y be as above and let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. We assume*

$$T_N^*M \cap \mathrm{hypchar}_M(\mathscr{M}) \subset T_M^*M. \tag{6.14}$$

In other words, any non zero vector $\theta \in T_N^*M$ is hyperbolic for $\mathscr{M}$. Then Y is non characteristic for $\mathscr{M}$ in a neighborhood of N and the isomorphism (6.7) induces the isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)|_N \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N). \tag{6.15}$$

Proof. (i) We shall not prove here that the hypothesis implies that Y is non characteristic for $\mathscr{M}$.
(ii) We have the chain of isomorphisms

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)|_N &\simeq \mathrm{R}\Gamma_N \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M) \otimes \mathrm{or}_{N/M}[d] \\
&\simeq \mathrm{R}\Gamma_N \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{R}\Gamma_M \mathscr{O}_X) \otimes \mathrm{or}_N[n+d] \\
&\simeq \mathrm{R}\Gamma_N \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{R}\Gamma_Y \mathscr{O}_X) \otimes \mathrm{or}_N[n+d] \\
&\simeq \mathrm{R}\Gamma_N \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y \otimes \mathrm{or}_N[n-d] \\
&\simeq \mathrm{R}\Gamma_N \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y) \otimes \mathrm{or}_N[n-d] \\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N).
\end{aligned}$$

Here, the first isomorphism follows from Theorem 6.11 and Corollary 4.6, since the micro-support of $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)$ does not intersect T_N^*M outside the zero-section. The second uses the definition of the sheaf $\mathscr{B}_M$, the third is obvious since N is both contained in M and in Y, the fourth follows from Theorem 6.4 and Corollary 4.6, the fifth is Theorem 6.5 and the last one uses the definition of the sheaf $\mathscr{B}_N$. Q.E.D.

Consider for simplicity the case where $\mathscr{M} = \mathscr{D}_X/\mathscr{I}$ where $\mathscr{I}$ is a coherent left ideal of $\mathscr{D}_X$. A section u of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)$ is a hyperfunction u such that $Qu = 0$ for all $Q \in \mathscr{I}$. It follows that the analytic wave front set of u does not intersect $T_Y^*X \cap T_M^*X$ and this implies that the restriction of u (and its derivative) to N is well-defined as a hyperfunction on N. One can show that the morphism (6.15) is then obtained using this restriction morphism, similarly as in Theorem 6.5.

References

[Chi14] Sheng Fu Chiu, *Non-squeezing property of contact balls* (2014), available at arXiv:1212.5818.

[Gab81] Ofer Gabber, *The integrability of the characteristic variety*, Amer. Journ. Math. **103** (1981), 445–468.

[Gro61] Alexander Grothendieck, *Eléments de Géométrie Algébrique III*, Publ. IHES **11, 17** (1961, 1963).

[Gui12] Stéphane Guillermou, *Quantization of conic Lagrangian submanifolds of cotangent bundles* (2012), available at arXiv:1212.5818.

[Gui13] ———, *The Gromov-Eliashberg theorem by microlocal sheaf theory* (2013), available at arXiv:1311.0187.

[GS14] Stéphane Guillermou and Pierre Schapira, *Microlocal theory of sheaves and Tamarkin's non displaceability theorem*, LN of the UMI (2014), 43–85, available at arXiv:1106.1576.

[GKS12] Stéphane Guillermou, Masaki Kashiwara, and Pierre Schapira, *Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems*, Duke Math Journal **161** (2012), 201245.

[Hör83] Lars Hörmander, *The analysis of linear partial differential operators*, Grundlehren der Mathematischen Wissenschaften, vol. 256, 257 and 274, 275, Springer-Verlag, Berlin, 1983, 1985.

[Kas95] Masaki Kashiwara, *Algebraic study of systems of partial differential equations*, Mémoires SMF, vol. 63, Soc. Math. France, 1995. Translated from author's thesis in Japanese, Tokyo 1970, by A. D'Agnolo and J-P.Schneiders.

[Kas03] ———, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003.

[KS82] Masaki Kashiwara and Pierre Schapira, *Micro-support des faisceaux: applications aux modules différentiels*, C. R. Acad. Sci. Paris **295, 8** (1982), 487–490.

[KS85] ———, *Microlocal study of sheaves*, Astérisque, vol. 128, Soc. Math. France, 1985.

[KS90] ———, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften, vol. 292, Springer-Verlag, Berlin, 1990.

[KS97] ———, *Integral transforms with exponential kernels and Laplace transform* **10** (1997), 939-972.

[KS06] ———, *Categories and Sheaves*, Grundlehren der Mathematischen Wissenschaften, vol. 332, Springer-Verlag, Berlin, 2006.

[Ler53] Jean Leray, *Hyperbolic differential equations*, The Institute for Advanced Study, Princeton, N. J., 1953.

[Nad09] David Nadler, *Microlocal branes are constructible sheaves*, Selecta Math. **15** (2009), 563–619.

[Nad16] ———, *Mirror symmetry for the Landau-Ginzburg A-model* (2016), available at arXiv:1601.02977.

[NZ09] David Nadler and Eric Zaslow, *Constructible sheaves and the Fukaya category*, J. Amer. Math. Soc. **22** (2009), 233–286.

[Sat59] Mikio Sato, *Theory of hyperfunctions, I & II*, Journ. Fac. Sci. Univ. Tokyo **8** (1959, 1960), 139–193, 487–436.

[Sat70] ———, *Regularity of hyperfunctions solutions of partial differential equations* **2** (1970), 785–794.

[SKK73] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara, *Microfunctions and pseudo-differential equations; Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau*, Lecture Notes in Math., vol. 287, Springer, Berlin, 1973.

[Scha] Pierre Schapira, *Categories and Homological Algebra*, available at http://webusers.imj-prg.fr/~pierre.schapira/.

[Schb] ———, *Abelian Sheaves*, available at http://webusers.imj-prg.fr/~pierre.schapira/.

[Sch07] ———, *Mikio Sato, a visionary of mathematics*, Notices of the AMS **54, 2** (2007).

[SS94] Pierre Schapira and Jean-Pierre Schneiders, *Index theorem for elliptic pairs*, Astérisque, vol. 224, Soc. Math. France, 1994.

[STZ14] V Shende, D Treumann, and E Zaslow, *Legendrian knots and constructible sheaves* (2014), available at arXiv:1402.0490.

[Tam08] Dmitry Tamarkin, *Microlocal conditions for non-displaceability* (2008), available at arXiv:0809.1584.

[Tam15] ———, *Microlocal category* (2015), available at arXiv:1511.08961.

Pierre Schapira
Sorbonne Universités, UPMC Univ Paris 6
Institut de Mathématiques de Jussieu
e-mail: pierre.schapira@imj-prg.fr
http://webusers.imj-prg.fr/~pierre.schapira/