A finiteness theorem for holonomic DQ-modules on Poisson manifolds

Masaki Kashiwara and Pierre Schapira

March 6, 2020

Abstract

On a complex symplectic manifold, we prove a finiteness result for the global sections of solutions of holonomic DQ-modules in two cases: (a) by assuming that there exists a Poisson compactification (b) in the algebraic case. This extends our previous result of [KS12] in which the symplectic manifold was compact. The main tool is a finiteness theorem for $\mathbb{R}$-constructible sheaves on a real analytic manifold in a non proper situation.

Contents

Key words: deformation quantization, holonomic modules, microlocal sheaf theory, constructible sheaves

MSC: 53D55, 35A27, 19L10, 32C38

The research of M.K was supported by Grant-in-Aid for Scientific Research (X) 15H03608, Japan Society for the Promotion of Science.

The research of P.S was supported by the ANR-15-CE40-0007 “MICROLOCAL”.

1 Introduction and statement of the results

Consider a complex Poisson manifold X of complex dimension d_X endowed with a DQ-algebroid $\mathscr{A}_X$. Recall that $\mathscr{A}_X$ is a $\mathbb{C}[[\hbar]]$-algebroid locally isomorphic to a star algebra $(\mathscr{O}_X[[\hbar]], \star)$ to which the Poisson structure is associated. Denote by $\mathscr{A}_X^{\mathrm{loc}}$ the localization of $\mathscr{A}_X$ with respect to $\hbar$, a $\mathbb{C}((\hbar))$-algebroid. For short, we set

$$\mathbb{C}^{\hbar} := \mathbb{C}[[\hbar]], \quad \mathbb{C}^{\hbar,\mathrm{loc}} := \mathbb{C}((\hbar)).$$

Hence $\mathscr{A}_X^{\mathrm{loc}} \simeq \mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^{\hbar}} \mathscr{A}_X$. The algebroids $\mathscr{A}_X$ and $\mathscr{A}_X^{\mathrm{loc}}$ are right and left Noetherian (in particular coherent) and if $\mathscr{M}$ is a (say left) coherent $\mathscr{A}_X^{\mathrm{loc}}$-module, then its support is a closed complex analytic subvariety of X and it follows from Gabber's theorem that it is co-isotropic. In the extreme case where X is symplectic and the support is Lagrangian, one says that $\mathscr{M}$ is holonomic.

Recall the following definitions (see [KS12, Def. 2.3.14, 2.3.16 and 2.7.2]).

(a) A coherent $\mathscr{A}_X$-submodule $\mathscr{M}_0$ of a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is called an $\mathscr{A}_X$-lattice of $\mathscr{M}$ if $\mathscr{M}_0$ generates $\mathscr{M}$.

(b) A coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is good if, for any relatively compact open subset U of X, there exists an $(\mathscr{A}_X|_U)$-lattice of $\mathscr{M}|_U$.

(c) One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$ consisting of objects with good cohomology.

(d) In the algebraic case (see below) a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is called algebraically good if there exists an $\mathscr{A}_X$-lattice of $\mathscr{M}$. One still denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$ consisting of objects with algebraically good cohomology.

Let $Y \subset X$. We shall consider the hypothesis

(1.1) Y is open, relatively compact, subanalytic in X and the Poisson structure on X is symplectic on Y.

Example 1.1. Denote by X_{ns} the closed complex subvariety of X consisting of points where the Poisson bracket is not symplectic and set $Y = X \setminus X_{\mathrm{ns}}$. Hence Y is an open subanalytic subset of X and is symplectic. If X is compact, then Y satisfies hypothesis (1.1).

In this paper we shall prove the following theorem which extends [KS12, Th. 7.2.3] in which X was symplectic, that is, $Y = X$.

Theorem 1.2. *Assume that Y satisfies hypothesis (1.1). Let $\mathscr{M}$ and $\mathscr{L}$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ and assume that both $\mathscr{M}|_Y$ and $\mathscr{L}|_Y$ are holonomic. Then the two complexes $\mathrm{RHom}_{\mathscr{A}_Y^{\mathrm{loc}}}(\mathscr{M}|_Y,\mathscr{L}|_Y)$ and $\mathrm{R}\Gamma_c(Y;\mathrm{R}\mathscr{H}om_{\mathscr{A}_Y^{\mathrm{loc}}}(\mathscr{L}|_Y,\mathscr{M}|_Y))\,[d_X]$ have finite dimensional cohomology over $\mathbb{C}^{\hbar,\mathrm{loc}}$ and are dual to each other.*

We shall also obtain a similar conclusion under rather different hypotheses, namely that $X=Y$ is symplectic and all data are algebraic (see [KS12, § 2.7]). Let X be a smooth algebraic variety and let $\mathscr{A}_X$ be a DQ-algebroid on X. We denote by X_{an} the associated complex analytic manifold and $\mathscr{A}_{X_{\mathrm{an}}}$ the associated DQ-algebroid on X_{an} (see Lemma 5.1). For a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ we denote by $\mathscr{M}_{\mathrm{an}}$ its image by the natural functor $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})\to\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}^{\mathrm{loc}})$.

Theorem 1.3. . *Let X be a quasi-compact separated smooth symplectic algebraic variety over $\mathbb{C}$ endowed with the Zariski topology. Let $\mathscr{M}$ and $\mathscr{L}$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$. Then the two complexes $\mathrm{RHom}_{\mathscr{A}_{X_{\mathrm{an}}}^{\mathrm{loc}}}(\mathscr{M}_{\mathrm{an}},\mathscr{L}_{\mathrm{an}})$ and $\mathrm{R}\Gamma_c(X_{\mathrm{an}};\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X_{\mathrm{an}}}^{\mathrm{loc}}}(\mathscr{L}_{\mathrm{an}},\mathscr{M}_{\mathrm{an}}))\,[d_X]$ have finite dimensional cohomology over $\mathbb{C}^{\hbar,\mathrm{loc}}$ and are dual to each other.*

The main tool in the proof of both theorems is Theorem 2.2 below which gives a finiteness criterion for $\mathbb{R}$-constructible sheaves on a real analytic manifold in a non proper situation.

This Note is motivated by the paper [GJS19] of Sam Gunningham, David Jordan and Pavel Safronov on Skein algebras, whose main theorem is based over such a finiteness result (see loc. cit. § 3). The proof of these authors uses a kind of Nakayama theorem in the case where $\mathscr{M}$ and $\mathscr{L}$ are simple modules over smooth Lagrangian varieties.

2 Finiteness results for constructible sheaves

In this paper, $\mathbf{k}$ is a Noetherian commutative ring of finite global homological dimension.

We denote by $\mathrm{D}^{\mathrm{b}}_f(\mathbf{k})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k})$ consisting of objects with finitely generated cohomology. We denote by D the duality functor $\mathrm{RHom}\,(\,\bullet\,,\mathbf{k})$ and we say that two objects A and B of $\mathrm{D}^{\mathrm{b}}_f(\mathbf{k})$ are dual to each other if $\mathrm{D}A\simeq B$, which is equivalent to $\mathrm{D}B\simeq A$.

For a sheaf of rings $\mathscr{R}$, one denotes by $\mathrm{D}(\mathscr{R})$ the derived category of left $\mathscr{R}$-modules. We shall also encounter the full triangulated subcategory $\mathrm{D}^+(\mathscr{R})$ or $\mathrm{D}^{\mathrm{b}}(\mathscr{R})$ of complexes whose cohomology is bounded from below or is bounded.

For a real analytic manifold M, one denotes by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ the bounded derived category of sheaves of $\mathbf{k}$-modules on M. We shall use the six Grothendieck operations. In particular, we denote by ω_M the dualizing complex. We also use the notations for $F\in\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$

$$\mathrm{D}'_M F := \mathrm{R}\mathscr{H}om\,(F,\mathbf{k}_M),\quad \mathrm{D}_M F := \mathrm{R}\mathscr{H}om\,(F,\omega_M).$$

Recall that an object F of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ is weakly $\mathbb{R}$-constructible if condition (i) below is satisfied. If moreover condition (ii) is satisfied, then one says that F is $\mathbb{R}$-constructible.

(i) there exists a subanalytic stratification $M=\bigsqcup_{a\in A}M_a$ such that $H^j(F)|_{M_a}$ is locally constant for all $j\in\mathbb{Z}$ and all $a\in A$

(ii) $H^j(F)_x$ is finitely generated for all $x \in M$ and all $j \in \mathbb{Z}$.

One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of $\mathbb{R}$-constructible objects.

If X is a complex analytic manifold, one defines similarly the notions of (weakly) $\mathbb{C}$-constructible sheaf, replacing "subanalytic" with "complex analytic" and one denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbf{k}_X)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ consisting of $\mathbb{C}$-constructible objects.

We shall use the following classical result (see [KS90, Prop. 8.4.8 and Exe. VIII.3]).

Proposition 2.1. *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ and assume that F has compact support. Then both objects $\mathrm{R}\Gamma(M;F)$ and $\mathrm{R}\Gamma(M;\mathrm{D}_M F)$ belong to $\mathrm{D}^{\mathrm{b}}_f(\mathbf{k})$ and are dual to each other.*

For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$, one denotes by $\mathrm{SS}(F)$ its microsupport [KS90, Def. 5.1.2], a closed $\mathbb{R}^+$-conic (*i.e.,* invariant by the $\mathbb{R}^+$-action on T^*M) subset of T^*M. Recall that this set is involutive (one also says *co-isotropic*), see loc. cit. Def. 6.5.1.

Theorem 2.2. *Let $j\colon U \hookrightarrow M$ be the embedding of an open subanalytic subset U of M and let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_U)$. Assume that $\mathrm{SS}(F)$ is contained in a closed subanalytic $\mathbb{R}^+$-conic Lagrangian subset Λ of T^*U which is subanalytic in T^*M. Then $\mathrm{R}j_*F$ and $j_!F$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$.*

Proof. (i) Let us treat first $j_!F$. The set Λ is a locally closed subanalytic subset of T^*M and is isotropic. By [KS90, Cor. 8.3.22], there exists a μ-stratification $M = \bigsqcup_{a\in A} M_a$ such that $\Lambda \subset \bigsqcup_{a\in A} T^*_{M_a}M$.

Set $U_a = U \cap M_a$. Then $U = \bigsqcup_{a\in A} U_a$ is a μ-stratification and one can apply loc. cit. Prop. 8.4.1. Hence, for each $a \in A$, $F|_{U_a}$ is locally constant of finite rank. Hence $(j_!F)|_{U_a}$ as well as $(j_!F)_{M\setminus U} \simeq 0$ is locally constant of finite rank. Hence $j_!F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$.
(ii) Set $G = j_!F$. Then $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ by (i) and so does $\mathrm{R}j_*F \simeq \mathrm{R}\mathscr{H}om(\mathbf{k}_U, G)$ (apply [KS90, Prop. 8.4.10]). □

Remark 2.3. One has $\mathrm{SS}(\mathrm{D}_M F) = \mathrm{SS}(F)^a$ where $(\bullet)^a$ is the antipodal map. Hence $\mathrm{D}_M F$ satisfies the same hypotheses as F.

Corollary 2.4. *In the preceding situation, assume moreover that U is relatively compact in M. Then $\mathrm{R}\Gamma(U;F)$ and $\mathrm{R}\Gamma_c(U;\mathrm{D}_U F)$ belong to $\mathrm{D}^{\mathrm{b}}_f(\mathbf{k})$ and are dual to each other.*

Proof. One has $\mathrm{R}\Gamma(U;F) \simeq \mathrm{R}\Gamma(M;\mathrm{R}j_*F)$ and $\mathrm{R}\Gamma_c(U;\mathrm{D}_U F) \simeq \mathrm{R}\Gamma(M;\mathrm{D}_M\mathrm{R}j_*F)$. Since $\mathrm{R}j_*F$ is $\mathbb{R}$-constructible and has compact support, the result follows from Proposition 2.1. □

For a complex analytic manifold X (that we identify with the real underlying manifold if necessary), one denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbf{k}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ consisting of $\mathbb{C}$-constructible sheaves.

In this paper, a smooth algebraic variety X means a quasi-compact smooth algebraic variety over $\mathbb{C}$ endowed with the Zariski topology. We denote by X_{an} the complex analytic manifold underlying X. If X is smooth algebraic variety, we keep the notation $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbf{k}_X)$ to denote the category of algebraically constructible sheaves, that is, object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbf{k}_{X_{\mathrm{an}}})$ locally constant on an algebraic stratifications. Hence, for an algebraic variety X, one shall not confuse $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbf{k}_X)$ and $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbf{k}_{X_{\mathrm{an}}})$, although $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbf{k}_X)$ is a full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbf{k}_{X_{\mathrm{an}}})$.

Corollary 2.5. *Let X be a smooth algebraic variety and let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbf{k}_X)$. Then $\mathrm{R}\Gamma(X_{\mathrm{an}};F)$ and $\mathrm{R}\Gamma_c(X_{\mathrm{an}};\mathrm{D}_{X_{\mathrm{an}}}F)$ have finite dimensional cohomology over $\mathbf{k}$ and are dual to each other.*

Proof. Let Z be a smooth algebraic compactification of X with X open in Z. By the hypothesis, Λ is a closed algebraic subvariety of T^*X. Hence, its closure in T^*Z is a closed algebraic subvariety of T^*Z. Therefore Λ is subanalytic in T^*Z_{an}.

Then the result follows from Corollary 2.4 with $M = Z_{\mathrm{an}}$ and $U = X_{\mathrm{an}}$. □

3 Reminders on DQ-modules, after [KS12]

3.1 Cohomologically complete modules

In this subsection,

(3.1) X denotes a topological space and $\mathscr{R}$ is a sheaf of $\mathbb{Z}[\hbar]$-algebras on X with no $\hbar$-torsion.

Let $\mathscr{M}$ be an $\mathscr{R}$-module. (Hence, a $\mathbb{Z}_X[\hbar]$-module.) One sets

$$\begin{aligned}
&\mathscr{R}^{\mathrm{loc}} := \mathbb{Z}_X[\hbar,\hbar^{-1}] \otimes_{\mathbb{Z}_X[\hbar]} \mathscr{R},\\
&\mathscr{M}^{\mathrm{loc}} := \mathscr{R}^{\mathrm{loc}} \otimes_{\mathscr{R}} \mathscr{M} \simeq \mathbb{Z}_X[\hbar,\hbar^{-1}] \otimes_{\mathbb{Z}_X[\hbar]} \mathscr{M},\\
&\mathrm{gr}_\hbar(\mathscr{R}) := \mathscr{R}/\hbar\mathscr{R},\\
&\mathrm{gr}_\hbar(\mathscr{M}) := \mathrm{gr}_\hbar(\mathscr{R}) \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{M} \simeq \mathbb{Z}_X \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_X[\hbar]} \mathscr{M}.
\end{aligned}$$

Definition 3.1 ([KS12, Def. 1.5.5]). One says that an object $\mathscr{M}$ of $\mathrm{D}(\mathscr{R})$ is cohomologically complete if it belongs to $\mathrm{D}(\mathscr{R}^{\mathrm{loc}})^{\perp r}$, that is, $\mathrm{Hom}_{\mathrm{D}(\mathscr{R})}(\mathscr{N},\mathscr{M}) \simeq 0$ for any $\mathscr{N} \in \mathrm{D}(\mathscr{R}^{\mathrm{loc}})$.

Proposition 3.2 ([KS12, Prop. 1.5.6]). *Let $\mathscr{M} \in \mathrm{D}(\mathscr{R})$. Then the conditions below are equivalent.*

(a) *$\mathscr{M}$ is cohomologically complete,*

(b) $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}},\mathscr{M}) \simeq 0$,

(c) $\varinjlim_{U \ni x} \mathrm{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar,\hbar^{-1}], H^i(U;\mathscr{M})\big) \simeq 0$ *for any $x \in X$, $j = 0,1$ and any $i \in \mathbb{Z}$. Here, U ranges over an open neighborhood system of x.*

Denote by $\mathrm{D}_{\mathrm{cc}}(\mathscr{R})$ the full subcategory of $\mathrm{D}(\mathscr{R})$ consisting of cohomologically complete modules. Then clearly $\mathrm{D}_{\mathrm{cc}}(\mathscr{R})$ is triangulated.

Proposition 3.3 ([KS12, Prop. 1.5.10, Cor. 1.5.9]). *Let $\mathscr{M} \in \mathrm{D}_{\mathrm{cc}}(\mathscr{R})$. Then*

(a) $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{N},\mathscr{M}) \in \mathrm{D}(\mathbb{Z}_X[\hbar])$ *is cohomologically complete for any $\mathscr{N} \in \mathrm{D}(\mathscr{R})$.*

(b) *If $\mathrm{gr}_\hbar(\mathscr{M}) \simeq 0$, then $\mathscr{M} \simeq 0$.*

Proposition 3.4 ([KS12, Prop. 1.5.12])**.** *Let $f\colon X \to Y$ be a continuous map and let $\mathscr{M} \in \mathrm{D}(\mathbb{Z}_X[\hbar])$. If $\mathscr{M}$ is cohomologically complete, then so is $\mathrm{R}f_*\mathscr{M}$.*

Proposition 3.5. *Let $\mathscr{M} \in \mathrm{D}(\mathscr{R})$ be a cohomologically complete object and $a \in \mathbb{Z}$. If $H^i(\mathrm{gr}_\hbar(\mathscr{M})) = 0$ for any $i \geq a$, then $H^i(\mathscr{M}) = 0$ for any $i > a$.*

Proof. The proof is exactly the same as that of [KS12, Prop. 1.5.8] when replacing $i > a$ with $i < a$. □

3.2 Microsupport and constructible sheaves

Let M be a *real analytic* manifold and let $\mathbf{k}$ be a Noetherian commutative ring of finite global homological dimension.

We shall need the next result which does not appear in [KS12].

Proposition 3.6. *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{Z}_M[\hbar])$. Then $\mathrm{SS}(F^{\mathrm{loc}}) \subset \mathrm{SS}(F)$.*

Proof. By using one of the equivalent definitions of the micro-support given in [KS90, Prop. 5.11], it is enough to check that for K compact, $\mathrm{R}\Gamma(K;F)^{\mathrm{loc}} \simeq \mathrm{R}\Gamma(K;F^{\mathrm{loc}})$ which follows from loc. cit. Prop. 2.6.6 and the fact that $\mathbb{Z}[\hbar,\hbar^{-1}]$ is flat over $\mathbb{Z}[\hbar]$. □

Proposition 3.7 ([KS12, Prop. 7.1.6])**.** *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{Z}_M[\hbar])$ and assume that F is cohomologically complete. Then*

$$\mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_\hbar(F)). \tag{3.2}$$

Proof. Let us recall the proof of loc. cit. The inclusion

$$\mathrm{SS}(\mathrm{gr}_\hbar(F)) \subset \mathrm{SS}(F)$$

follows from the distinguished triangle $F \xrightarrow{\hbar} F \to \mathrm{gr}_\hbar(F) \xrightarrow{+1}$. Let us prove the converse inclusion.

Using the definition of the microsupport, it is enough to prove that given two open subsets $U \subset V$ of M, $\mathrm{R}\Gamma(V;F) \to \mathrm{R}\Gamma(U;F)$ is an isomorphism as soon as $\mathrm{R}\Gamma(V;\mathrm{gr}_\hbar(F)) \to \mathrm{R}\Gamma(U;\mathrm{gr}_\hbar(F))$ is an isomorphism. Consider a distinguished triangle $\mathrm{R}\Gamma(V;F) \to \mathrm{R}\Gamma(U;F) \to G \xrightarrow{+1}$. Then we get a distinguished triangle $\mathrm{R}\Gamma(V;\mathrm{gr}_\hbar(F)) \to \mathrm{R}\Gamma(U;\mathrm{gr}_\hbar(F)) \to \mathrm{gr}_\hbar(G) \xrightarrow{+1}$. Therefore, $\mathrm{gr}_\hbar(G) \simeq 0$. On the other hand, G is cohomologically complete, thanks to Proposition 3.4 (applied to $F|_U$ and $F|_V$) and then $G \simeq 0$ by Proposition 3.3 (b). □

Proposition 3.8 ([KS12, Prop. 7.1.7])**.** *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbb{C}^\hbar)$. Then F is cohomologically complete.*

Proof. Let us recall the proof of loc. cit. One has

$$\begin{aligned}
\text{“}\varinjlim_{U \ni x}\text{”}\, \mathrm{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar,\hbar^{-1}], H^i(U;F)\big) &\simeq \mathrm{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar,\hbar^{-1}], \text{“}\varinjlim_{U \ni x}\text{”}\, H^i(U;F)\big) \\
&\simeq \mathrm{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar,\hbar^{-1}], F_x\big) \simeq 0
\end{aligned}$$

where the last isomorphism follows from the fact that F_x is cohomologically complete.

Hence, hypothesis (c) of Proposition 3.2 is satisfied. □

3.3 DQ-modules

In this subsection, X will be a complex manifold (not necessarily symplectic) of complex dimension d_X.

Set $\mathscr{O}_X^\hbar := \mathscr{O}_X[[\hbar]] = \varprojlim_n \mathscr{O}_X \otimes_{\mathbb{C}} (\mathbb{C}^\hbar/\hbar^n\mathbb{C}^\hbar)$. An associative multiplication law $\star$ on $\mathscr{O}_X^\hbar$ is a star-product if it is $\mathbb{C}^\hbar$-bilinear and satisfies

$$f \star g = \sum_{i\geq 0} P_i(f,g)\hbar^i \quad \text{for } f,g \in \mathscr{O}_X, \tag{3.3}$$

where the P_i's are bi-differential operators, $P_0(f,g) = fg$ and $P_i(f,1) = P_i(1,f) = 0$ for $f \in \mathscr{O}_X$ and $i > 0$.

We call $(\mathscr{O}_X[[\hbar]], \star)$ a *star-algebra*. A $\star$-product defines a Poisson structure on $(X, \mathscr{O}_X)$ by the formula

$$\{f,g\} = P_1(f,g) - P_1(g,f) \equiv \hbar^{-1}(f \star g - g \star f) \mod \hbar\mathscr{O}_X[[\hbar]]. \tag{3.4}$$

Definition 3.9. A DQ-algebroid $\mathscr{A}$ on X is a $\mathbb{C}^\hbar$-algebroid locally isomorphic to a star-algebra as a $\mathbb{C}^\hbar$-algebroid.

Remark 3.10. The data of a DQ-algebroid $\mathscr{A}_X$ on X endows X with a structure of a complex Poisson manifold and one says that $\mathscr{A}_X$ is a quantization of the Poisson manifold. Kontsevich's famous theorem [Kon01, Kon03] (see also [Kas96] for the case of contact manifolds) asserts that any complex Poisson manifold may be quantized.

Example 3.11. Assume that M is an open subset of $\mathbb{C}^n$, $X = T^*M$ and denote by $(x;u)$ the symplectic coordinates on X. In this case there is a canonical $\star$-algebra $\mathscr{A}_X$ that is usually denoted by $\widehat{\mathscr{W}}_X(0)$, its localization with respect to $\hbar$ being denoted by $\widehat{\mathscr{W}}_X$.

Let $f,g \in \mathscr{O}_X[[\hbar]]$. Then the DQ-algebra $\widehat{\mathscr{W}}_X(0)$ is the star algebra $(\mathscr{O}_X[[\hbar]], \star)$ where:

$$f \star g \;=\; \sum_{\alpha\in\mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!}(\partial_u^\alpha f)(\partial_x^\alpha g). \tag{3.5}$$

This product is similar to the product of the total symbols of differential operators on X and indeed, the morphism of $\mathbb{C}$-algebras $\pi_M^{-1}\mathscr{D}_M \longrightarrow \widehat{\mathscr{W}}_X$ is given by

$$f(x) \mapsto f(x), \qquad \partial_{x_i} \mapsto \hbar^{-1}u_i,$$

where, as usual, $\mathscr{D}_M$ denotes the ring of finite order holomorphic differential operators and $\pi_M \colon T^*M \to M$ is the projection.

For a DQ-algebroid $\mathscr{A}_X$, there is locally an isomorphism of $\mathbb{C}$-algebroids $\mathscr{A}_X/\hbar\mathscr{A}_X \xrightarrow{\sim} \mathscr{O}_X$. Moreover there exists a unique isomorphism of $\mathbb{C}$-algebras

$$\mathscr{E}nd(\mathrm{id}_{\mathrm{gr}_\hbar\mathscr{A}_X}) \simeq \mathscr{O}_X. \tag{3.6}$$

Therefore, there is a well-defined functor

$$\bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \bullet \colon \mathrm{D^b}(\mathscr{O}_X) \times \mathrm{D^b}(\mathrm{gr}_\hbar\mathscr{A}_X) \to \mathrm{D^b}(\mathrm{gr}_\hbar\mathscr{A}_X). \tag{3.7}$$

Theorem 3.12 ([KS12, Th. 1.2.5])**.** *For a* DQ*-algebroid* $\mathscr{A}_X$*, both* $\mathscr{A}_X$ *and* $\mathscr{A}_X^{\mathrm{loc}}$ *are right and left Noetherian (in particular, coherent).*

One defines the functors

$$\begin{aligned}
&\mathrm{gr}_\hbar\colon \mathrm{D^b}(\mathscr{A}_X) \to \mathrm{D^b}(\mathrm{gr}_\hbar\mathscr{A}_X), \quad \mathscr{M} \mapsto \mathbb{C}_X \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} \mathscr{M},\\
&(\,\bullet\,)^{\mathrm{loc}}\colon \mathrm{D^b}(\mathscr{A}_X) \to \mathrm{D^b}(\mathscr{A}_X^{\mathrm{loc}}), \quad \mathscr{M} \mapsto \mathbb{C}_X^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}_X^\hbar} \mathscr{M},\\
&\mathit{for}\colon \mathrm{D^b}(\mathrm{gr}_\hbar(\mathscr{A}_X)) \to \mathrm{D^b}(\mathscr{A}_X) \text{ associated with } \sigma_0\colon \mathscr{A}_X \to \mathrm{gr}_\hbar(\mathscr{A}_X).
\end{aligned}$$

The functor $(\,\bullet\,)^{\mathrm{loc}}$ is exact on $\mathrm{Mod}(\mathscr{A}_X)$. The category $\mathrm{Mod}(\mathrm{gr}_\hbar(\mathscr{A}_X))$ is equivalent to the full subcategory of $\mathrm{Mod}(\mathscr{A}_X)$ consisting of objects M such that $\hbar\colon M \to M$ vanishes.

Theorem 3.13 ([KS12, Th. 1.6.1 and 1.6.4])**.** *Let* $\mathscr{M} \in \mathrm{D}^+(\mathscr{A}_X)$*. Then the two conditions below are equivalent:*

(a) *$\mathscr{M}$ is cohomologically complete and* $\mathrm{gr}_\hbar(\mathscr{M}) \in \mathrm{D}^+_{\mathrm{coh}}(\mathrm{gr}_\hbar\mathscr{A}_X)$*,*

(b) $\mathscr{M} \in \mathrm{D}^+_{\mathrm{coh}}(\mathscr{A}_X)$*.*

The next result follows from Gabber's theorem [Gab81].

Proposition 3.14 ([KS12, Prop. 2.3.18])**.** *Let* $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$*. Then* $\mathrm{supp}(\mathscr{M})$ *(the support of* $\mathscr{M}$*) is a closed complex analytic subset of* X*, involutive* (i.e., *co-isotropic*) *for the Poisson bracket on* X*.*

Remark 3.15. One shall be aware that the support of a coherent $\mathscr{A}_X$-module is not involutive in general. Indeed, any coherent $\mathrm{gr}_\hbar\mathscr{A}_X$-module may be regarded as an $\mathscr{A}_X$-module. Hence any closed analytic subset can be the support of a coherent $\mathscr{A}_X$-module.

4 DQ-modules along Λ

4.1 A variation on a theorem of [Kas03]

In order to prove Lemma 4.6 below, we need a slight modification of a result of [Kas03].

Let $\mathscr{R}$ be a ring on a topological space X, and let $\{F_n(\mathscr{R})\}_{n\in\mathbb{Z}}$ be a filtration of $\mathscr{R}$ which satisfies

(a) $\mathscr{R} = \bigcup_{n\in\mathbb{Z}} F_n(\mathscr{R})$,

(b) $1 \in F_0(\mathscr{R})$,

(c) $F_m(\mathscr{R}) \cdot F_n(\mathscr{R}) \subset F_{m+n}(\mathscr{R})$.

We set

$$\mathrm{gr}^F_{\geq 0}(\mathscr{R}) = \bigoplus_{n\geq 0} \mathrm{gr}^F_n(\mathscr{R}).$$

Proposition 4.1. *Assume that*

(a) $F_0(\mathscr{R})$ *and* $\mathrm{gr}^F_{\geq 0}(\mathscr{R})$ *are Noetherian rings,*

(b) $\mathrm{gr}^F_n(\mathscr{R})$ *is a coherent* $F_0(\mathscr{R})$*-module for any* $n \geq 0$.

Then $\mathscr{R}$ *is Noetherian.*

Proof. Define $\widetilde{F}_n(\mathscr{R})$ by

$$\widetilde{F}_n(\mathscr{R}) = \begin{cases} F_n(\mathscr{R}) & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

We shall apply [Kas03, Theorem A.20] to $\widetilde{F}_k(\mathscr{R})$. Hence in order to prove the theorem, it is enough to show

(4.1) for any positive integer m and an open subset U of X, if an $\mathscr{R}|_U$-submodule $\mathscr{N}$ of $\mathscr{R}^{\oplus m}|_U$ has the property that $F_k(\mathscr{N}) := \mathscr{N} \cap F_k(\mathscr{R})^{\oplus m}|_U$ is a coherent $F_0(\mathscr{R})|_U$-module for any $k \geq 0$, then $\mathscr{N}$ is a locally finitely generated $\mathscr{R}|_U$-module.

Since $\mathrm{gr}^F_{\geq 0}(\mathscr{R})$ is a Noetherian ring, $\mathrm{gr}^F_{\geq 0}(\mathscr{N}) := \bigoplus_{n \geq 0} \mathrm{gr}^F_n(\mathscr{N})$ is a coherent $\mathrm{gr}^F_{\geq 0}(\mathscr{R})$-module. Hence there exists locally a finitely generated $\mathscr{R}$-submodule $\mathscr{N}'$ of $\mathscr{N}$ such that $\mathrm{gr}^F_{\geq 0}(\mathscr{N}') = \mathrm{gr}^F_{\geq 0}(\mathscr{N})$. Hence we have $\mathscr{N} = \mathscr{N}' + F_0(\mathscr{N})$. Since $F_0(\mathscr{N})$ is a locally finitely generated $F_0(\mathscr{R})$-module, $\mathscr{N}$ is locally finitely generated $\mathscr{R}$-module. ☐

4.2 The algebroid $\mathscr{A}_{\Lambda/X}$

From now on, X is a complex manifold endowed with a DQ-algebroid $\mathscr{A}_X$.

Definition 4.2 ([KS12, Def. 2.3.10])**.** Let Λ be a smooth submanifold of X and let $\mathscr{L}$ be a coherent $\mathscr{A}_X$-module supported by Λ. One says that $\mathscr{L}$ is simple along Λ if $\mathrm{gr}_\hbar(\mathscr{L})$ is concentrated in degree 0 and $H^0(\mathrm{gr}_\hbar(\mathscr{L}))$ is an invertible $\mathscr{O}_\Lambda \otimes_{\mathscr{O}_X} \mathrm{gr}_\hbar(\mathscr{A}_X)$-module. (In particular, $\mathscr{L}$ has no $\hbar$-torsion.)

Let Λ be a smooth submanifold of X and let $\mathscr{L}$ be a coherent $\mathscr{A}_X$-module simple along Λ. We set for short

$$\begin{aligned} \mathscr{O}^\hbar_\Lambda &:= \mathscr{O}_\Lambda[[\hbar]], & \mathscr{O}^{\hbar,\mathrm{loc}}_\Lambda &:= \mathscr{O}_\Lambda((\hbar)), \\ \mathscr{D}^\hbar_\Lambda &:= \mathscr{D}_\Lambda[[\hbar]], & \mathscr{D}^{\hbar,\mathrm{loc}}_\Lambda &:= \mathscr{D}_\Lambda((\hbar)). \end{aligned}$$

One proves that there is a natural isomorphism of algebroids $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{L}) \simeq \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}^\hbar_\Lambda)$ ([KS12, Lem. 2.1.12]). Then the subalgebroid of $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{L})$ corresponding to the subring $\mathscr{D}_\Lambda[[\hbar]]$ of $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}^\hbar_\Lambda)$ is well-defined. We denote it by $\mathscr{D}_\mathscr{L}$. Then (see [KS12, Lem. 7.1.1]):

(a) $\mathscr{D}_\mathscr{L}$ is isomorphic to $\mathscr{D}^\hbar_\Lambda$ as a $\mathbb{C}^\hbar$-algebroid and $\mathrm{gr}_\hbar(\mathscr{D}_\mathscr{L}) \simeq \mathscr{D}_\Lambda$.

(b) The $\mathbb{C}^\hbar$-algebra $\mathscr{D}_\mathscr{L}$ is right and left Noetherian.

We denote by $I_\Lambda \subset \mathscr{O}_X$ the defining ideal of Λ. Let $\mathscr{I}$ be the kernel of the composition

$$\hbar^{-1}\mathscr{A}_X \xrightarrow{\hbar} \mathscr{A}_X \longrightarrow \mathrm{gr}_\hbar \mathscr{A}_X \to \mathscr{O}_\Lambda \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathrm{gr}_\hbar \mathscr{A}_X. \tag{4.2}$$

Then we have

$$\mathscr{I}/\mathscr{A}_X \simeq I_\Lambda \otimes_{\mathscr{O}_X} \mathrm{gr}_\hbar \mathscr{A}_X. \tag{4.3}$$

Remark 4.3. In [KS12, Ch. 7, § 1] we have used the symbol map $\sigma\colon \mathscr{A}_X \to \mathscr{O}_X$. This map is only defined locally, but all results of this chapter are of local nature. If nevertheless, one wants a global construction, then one has to replace the sequence two lines above Definition 7.1.2 of loc. cit. with (4.2).

Definition 4.4 ([KS12, Def. 7.1.2])**.** One denotes by $\mathscr{A}_{\Lambda/X}$ the $\mathbb{C}^\hbar$-subalgebroid of $\mathscr{A}_X^{\mathrm{loc}}$ generated by $\mathscr{I}$.

The ideal $\hbar\mathscr{I}$ is contained in $\mathscr{A}_X$, hence acts on $\mathscr{L}$ and one sees easily that $\hbar\mathscr{I}$ sends $\mathscr{L}$ to $\hbar\mathscr{L}$. Hence, $\mathscr{I}$ acts on $\mathscr{L}$ and defines a functor $\mathscr{A}_{\Lambda/X} \to \mathscr{D}_{\mathscr{L}}$. We thus have the morphisms of algebroids

$$\begin{array}{ccccc} \mathscr{A}_X|_\Lambda & \longrightarrow & \mathscr{A}_{\Lambda/X}|_\Lambda & \longrightarrow & \mathscr{A}_X^{\mathrm{loc}}|_\Lambda \\ & \searrow & \downarrow & & \downarrow \\ & & \mathscr{D}_{\mathscr{L}} & \longrightarrow & \mathscr{D}_{\mathscr{L}}^{\mathrm{loc}}. \end{array}$$

In particular, $\mathscr{L}$ is naturally an $\mathscr{A}_{\Lambda/X}$-module and $\mathrm{gr}_\hbar(\mathscr{D}_{\mathscr{L}}) \simeq \mathscr{D}_\Lambda$ is a $\mathrm{gr}_\hbar(\mathscr{A}_{\Lambda/X})$-module.

Example 4.5. We follow the notations of Example 3.11. Let $\Lambda = M$. Then $\mathscr{L} := \widehat{\mathscr{W}_X}(0)/\big(\sum_i \widehat{\mathscr{W}_X}(0)u_i\big) \simeq \mathscr{O}_\Lambda^\hbar$ is simple along Λ and $\mathscr{I} \subset \hbar^{-1}\mathscr{A}_X = \mathscr{A}_X(-1)$ is generated by $\hbar^{-1}u = (\hbar^{-1}u_1, \dots, \hbar^{-1}u_n)$. Identifying $\hbar^{-1}u_i$ with $\frac{\partial}{\partial x_i}$ we get an isomorphism $\mathscr{D}_{\mathscr{L}} \simeq \mathscr{D}_\Lambda[[\hbar]]$.

From now on, and until the end of the proof of Proposition 4.8 we work locally on X and thus we may assume that there is an isomorphism $\mathrm{gr}_\hbar \mathscr{A}_X \xrightarrow{\sim} \mathscr{O}_X$.

We introduce a filtration $F\mathscr{A}_X^{\mathrm{loc}}$ on $\mathscr{A}_X^{\mathrm{loc}}$ by setting

$$F_k \mathscr{A}_X^{\mathrm{loc}} = \hbar^{-k}\mathscr{A}_X \text{ for } k \in \mathbb{Z}. \tag{4.4}$$

Therefore, there is a natural isomorphism

$$\mathrm{gr}_k^F \mathscr{A}_X^{\mathrm{loc}} \simeq T^{-k}\mathscr{O}_X \text{ given by } \hbar \longleftrightarrow T. \tag{4.5}$$

We endow $\mathscr{A}_{\Lambda/X}$ with the induced filtration, that is,

$$F_k\mathscr{A}_{\Lambda/X} = \mathscr{A}_{\Lambda/X} \cap F_k\mathscr{A}_X^{\mathrm{loc}}.$$

Recall (see [KS12, § 1.4]) that for a left Noetherian $\mathbb{C}^\hbar$-algebra $\mathscr{R}$, one says that a coherent $\mathscr{R}$-module $\mathscr{P}$ is locally projective if the functor

$$\mathscr{H}om_{\mathscr{R}}(\mathscr{P}, \bullet)\colon \mathrm{Mod}_{\mathrm{coh}}(\mathscr{R}) \to \mathrm{Mod}(\mathbb{C}^\hbar_X)$$

is exact. This is equivalent to each of the following conditions: (i) for each $x \in X$, the stalk $\mathscr{P}_x$ is projective as an $\mathscr{R}_x$-module, (ii) for each $x \in X$, the stalk $\mathscr{P}_x$ is a flat $\mathscr{R}_x$-module, (iii) $\mathscr{P}$ is locally a direct summand of a free $\mathscr{R}$-module of finite rank.

Recall that one says that a ring R has global homological dimension $\leq d$ if both $\mathrm{Mod}(R)$ and $\mathrm{Mod}(R^{\mathrm{op}})$ have homological dimension $\leq d$ (see [KS90, Exe. I.28]). In such a case, we shall write for short $\mathrm{ghd}(R) \leq d$.

Also recall that d_X denotes the complex dimension of X.

Lemma 4.6. *One has*

(a) $(\mathscr{A}_{\Lambda/X})^{\mathrm{loc}} \simeq \mathscr{A}_X^{\mathrm{loc}}$.

(b) *The algebra* $\mathrm{gr}^F \mathscr{A}_{\Lambda/X}$ *is a graded commutative subalgebra of* $\mathrm{gr}^F \mathscr{A}_X^{\mathrm{loc}}$.

(c) *There are natural isomorphisms*

$$\mathrm{gr}^F \mathscr{A}_{\Lambda/X} \simeq \bigoplus_{k\in\mathbb{Z}} T^{-k} I_\Lambda^k \quad \textit{and} \quad \mathrm{gr}^F_{\geq 0} \mathscr{A}_{\Lambda/X} \simeq \bigoplus_{k\geq 0} T^{-k} I_\Lambda^k,$$

where $I_\Lambda^k := \mathscr{O}_X$ *for* $k \leq 0$.

(d) *The sheaves of algebras* $\mathrm{gr}^F \mathscr{A}_{\Lambda/X}$ *and* $\mathrm{gr}^F_{\geq 0} \mathscr{A}_{\Lambda/X}$ *are Noetherian.*

(e) *For any* $x \in X$*, one has* $\mathrm{ghd}(\mathrm{gr}^F \mathscr{A}_{\Lambda/X})_x \leq d_X + 1$.

Proof. (a) is obvious since $\mathscr{A}_X \subset \mathscr{A}_{\Lambda/X} \subset \mathscr{A}_X^{\mathrm{loc}}$.
(b) is obvious.
(c) $\mathrm{gr}^F_1(\mathscr{A}_{\Lambda/X}) \simeq I_\Lambda$. Hence, $\mathrm{gr}^F_k \mathscr{A}_{\Lambda/X} \simeq I_\Lambda^k$.
(d) The commutative algebras $\mathrm{gr}^F \mathscr{A}_{\Lambda/X}$ and $\mathrm{gr}^F_{\geq 0} \mathscr{A}_{\Lambda/X}$ are locally finitely presented $\mathscr{O}_X$-algebras. Hence they are Noetherian. (Note that the associated variety with $\mathrm{gr}^F \mathscr{A}_{\Lambda/X}$ is the deformation of normal bundle to Λ.)
(e) For $x \in X$, set $R_x = (\mathrm{gr}^F \mathscr{A}_{\Lambda/X})_x$. If $x \notin \Lambda$, then $R_x \simeq \mathscr{O}_{X,x}[T, T^{-1}]$ and $\mathrm{ghd}(R_x) \leq d_X + 1$. Assume now that $x \in \Lambda$. Then $R_x/TR_x \simeq \mathscr{O}_{\Lambda,x}[y_1, \dots, y_n]$ (with $n = \mathrm{codim}_X \Lambda$) has global homological dimension d_X and $R_x[T^{-1}] \simeq \mathscr{O}_{X,x}[T, T^{-1}]$ has global homological dimension $d_X + 1$. Hence, $\mathrm{ghd}(R_x) \leq d_X + 1$ by the classical Lemma 4.7 below. □

Lemma 4.7. *Let* R *be a commutative Noetherian ring and let* $t \in R$ *be a non-zero divisor. Assume that* R/tR *has global homological dimension* $\leq d$ *and the localization* $R[t^{-1}]$ *has global homological dimension* $\leq d + 1$*. Then* R *has global homological dimension* $\leq d + 1$.

Proof. (i) Let Spec(R) denote as usual the set of prime ideals of R. For $\mathfrak{p} \in \mathrm{Spec}(R)$, denote by $R_\mathfrak{p}$ the localization of R at $\mathfrak{p}$. It is well-known that R has global homological dimension $\leq d$ if and only if for any $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_\mathfrak{p}$ has global homological dimension $\leq d$.

(ii) Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and assume that $t \notin \mathfrak{p}$. Then $R_\mathfrak{p} \simeq (R[t^{-1}])_\mathfrak{p}$ has global homological dimension $\leq d+1$.

(iii) Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and assume that $t \in \mathfrak{p}$. In this case, $R_\mathfrak{p}/tR_\mathfrak{p} \simeq (R/tR)_\mathfrak{p}$ has global homological dimension $\leq d$. This implies that $R_\mathfrak{p}$ is a regular local ring of global homological dimension $\leq d+1$. □

Proposition 4.8 (see [KS12, Lem. 7.1.3] in the symplectic case)**.** *One has*

(a) *the $\mathbb{C}^\hbar$-algebroid $\mathscr{A}_{\Lambda/X}$ is right and left Noetherian,*

(b) $\mathrm{gr}_\hbar(\mathscr{N}) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}_\hbar \mathscr{A}_{\Lambda/X})$ *for any* $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{\Lambda/X})$.

Proof. (a) follows from Proposition 4.1 since $\mathscr{A}$ is Noetherian by Theorem 3.12, $\mathrm{gr}_{\geq 0}\mathscr{A}_{\Lambda/X}$ is Noetherian by Lemma 4.6 and the I_Λ^k's are coherent $\mathscr{A}_X$-modules since they are coherent $\mathscr{O}_X$-modules.

(b) Let us represent $\mathscr{N}$ by a complex $\mathscr{L}^\bullet$ bounded from above of locally free $\mathscr{A}_{\Lambda/X}$-modules of finite rank. Then $H^i(\mathscr{L}^\bullet) \simeq 0$ for $i \ll 0$. Replacing $\mathscr{L}^\bullet$ with $\tau^{\geq j}\mathscr{L}^\bullet$ for $j \ll 0$ we find a bounded complex $\mathscr{L}^\bullet$ of coherent $\mathscr{A}_{\Lambda/X}$-modules for which $\hbar$ is injective. Now $\mathrm{gr}_\hbar(\mathscr{N})$ is represented by the complex $\mathscr{L}^\bullet/\hbar\mathscr{L}^\bullet$ and the result follows.

(c) Let d denote the projective dimension □

In the sequel, for $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{\Lambda/X})$ we set

$$\mathrm{gr}_\Lambda(\mathscr{N}) := \mathrm{gr}_\hbar(\mathscr{D}_\mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\Lambda/X}} \mathscr{N}) \simeq \mathscr{D}_\Lambda \overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar(\mathscr{A}_{\Lambda/X})} \mathrm{gr}_\hbar(\mathscr{N}). \tag{4.6}$$

Corollary 4.9. *If $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{\Lambda/X})$, then $\mathrm{gr}_\Lambda(\mathscr{N}) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_\Lambda)$ and $\mathrm{char}(\mathrm{gr}_\Lambda(\mathscr{N}))$ is a closed $\mathbb{C}^\times$-conic complex analytic subset of $T^*\Lambda$.*

Proof. By Proposition 4.8 (b) and Lemma 4.6 (e), $\mathrm{gr}_\hbar\mathscr{N}$ is locally quasi-isomorphic to a bounded complex of projective $\mathrm{gr}_\hbar\mathscr{A}_{\Lambda/X}$-modules of finite type. To conclude, note that if $\mathscr{P}$ is a projective $\mathrm{gr}_\hbar\mathscr{A}_{\Lambda/X}$-modules of finite type, then $\mathscr{D}_\Lambda \overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar(\mathscr{A}_{\Lambda/X})} \mathrm{gr}_\hbar(\mathscr{P})$ is concentrated in degree 0 and is $\mathscr{D}_\Lambda$-coherent. The result for $\mathrm{char}(\mathrm{gr}_\Lambda(\mathscr{N}))$ follows. □

Proposition 4.10 (see [KS12, Prop. 7.1.8] in the symplectic case)**.** *Let $\mathscr{N}$ be a coherent $\mathscr{A}_{\Lambda/X}$-module. Then*

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N}, \mathscr{L}) \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}^\hbar_X), \tag{4.7}$$

$$\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N}, \mathscr{L})) = \mathrm{char}(\mathrm{gr}_\Lambda\mathscr{N}). \tag{4.8}$$

Proof. (i) One has

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N}, \mathscr{L}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_\mathscr{L}}(\mathscr{D}_\mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\Lambda/X}} \mathscr{N}, \mathscr{L}).$$

Set $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_{\mathscr{L}}}(\mathscr{D}_{\mathscr{L}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\Lambda/X}} \mathscr{N}, \mathscr{L})$. Then $F \in \mathrm{D}^+(\mathbb{C}^\hbar_X)$, F is cohomologically complete by Proposition 3.3 and $\mathrm{gr}_\hbar(F) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_\Lambda}(\mathrm{gr}_\Lambda \mathscr{N}, \mathscr{O}_\Lambda)$.

(ii) We have $\mathrm{gr}_\hbar F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}^\hbar_X)$ by Lemma 4.6 (c). This implies (4.7) by Proposition 3.5.

(iii) We have $\mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_\hbar(F))$ by Proposition 3.7. On the other hand, $\mathrm{gr}_\hbar(F) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_\Lambda}(\mathrm{gr}_\Lambda \mathscr{N}, \mathscr{O}_\Lambda)$ and the microsupport of this complex is equal to $\mathrm{char}(\mathrm{gr}_\Lambda \mathscr{N})$ by [KS90, Th 11.3.3]. □

Definition 4.11. A coherent $\mathscr{A}_{\Lambda/X}$-submodule $\mathscr{N}$ of a coherent $\mathscr{A}^{\mathrm{loc}}_X$-module $\mathscr{M}$ is called an $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$ if $\mathscr{N}$ generates $\mathscr{M}$ as an $\mathscr{A}^{\mathrm{loc}}_X$-module.

One easily proves that if $\mathscr{N}$ is an $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$, then $\mathrm{char}(\mathrm{gr}_\hbar \mathscr{N})$ depends only on $\mathscr{M}$.

Notation 4.12. For a coherent $\mathscr{A}^{\mathrm{loc}}_X$-module $\mathscr{M}$, one sets $\mathrm{char}_\Lambda(\mathscr{M}) := \mathrm{char}(\mathrm{gr}_\Lambda \mathscr{N})$ for $\mathscr{N}$ a (locally defined) $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$.

4.3 Reminders on holonomic DQmodules

We shall recall here the main results of [KS12, Ch. 7].

In this subsection, we assume that X is symplectic and that Λ is Lagrangian. In this case, $\mathrm{gr}_\hbar(\mathscr{A}_{\Lambda/X}) \simeq \mathscr{D}_\Lambda$ as an algebroid and thus $\mathrm{gr}_\Lambda(\mathscr{N}) \simeq \mathrm{gr}_\hbar(\mathscr{N})$.

Definition 4.13. Assume that X is symplectic and Λ is Lagrangian. An object $\mathscr{N}$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{\Lambda/X})$ is holonomic if $\mathrm{gr}_\hbar(\mathscr{N})$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_\Lambda)$.

Theorem 4.14 (see [KS12, Th. 7.1.10])**.** *Assume that X is symplectic. Let $\mathscr{N}$ be a holonomic $\mathscr{A}_{\Lambda/X}$-module.*

(a) *The objects $\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N}, \mathscr{L})$ and $\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{L}, \mathscr{N})$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}^\hbar_\Lambda)$ and their microsupports are contained in $\mathrm{char}(\mathrm{gr}_\hbar \mathscr{N})$.*

(b) *There is a natural isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}^\hbar_\Lambda)$*

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N}, \mathscr{L}) \xrightarrow{\sim} \mathrm{D}'_X\big(\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{L}, \mathscr{N})\big)\,[d_X]. \tag{4.9}$$

The crucial result in order to prove Theorem 4.16 below is the following.

Proposition 4.15 (see [KS12, Prop. 7.1.16])**.** *Assume that X is symplectic and Λ is Lagrangian. For a coherent $\mathscr{A}^{\mathrm{loc}}_X$-module $\mathscr{M}$, we have*

$$\mathrm{codim}\,\mathrm{char}_\Lambda(\mathscr{M}) \geq \mathrm{codim}\,\mathrm{Supp}(\mathscr{M}).$$

The next result is a variation on a classical theorem of [Kas75] on holonomic D-modules.

Theorem 4.16 (see [KS12, Th. 7.2.3])**.** *Assume that X is symplectic. Let $\mathscr{M}$ and $\mathscr{N}$ be two holonomic $\mathscr{A}^{\mathrm{loc}}_X$-modules. Then*

(i) *the object* $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{N})$ *belongs to* $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X^{\hbar,\mathrm{loc}})$,

(ii) *there is a canonical isomorphism:*

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{N}) \xrightarrow{\sim} \left(\mathrm{D}'_X\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{N},\mathscr{M})\right)[d_X], \tag{4.10}$$

(iii) *the object* $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{N})[d_X/2]$ *is perverse.*

5 Proof of the main theorems and an example

5.1 Proof of Theorem 1.2

In this subsection, X is again a complex Poisson manifold endowed with a DQ-algebroid $\mathscr{A}_X$.

By using the diagonal procedure, we may assume that $\mathscr{L}=\mathscr{L}_0^{\mathrm{loc}}$ with $\mathscr{L}_0$ an $\mathscr{A}_X$-module simple along Λ. By the hypothesis, we may find an $\mathscr{A}_{\Lambda/X}$-lattice $\mathscr{N}$ of $\mathscr{M}$. Set

$$F_0 := \mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N},\mathscr{L}_0), \quad F := \mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L}) \simeq F^{\mathrm{loc}}. \tag{5.1}$$

One knows by Theorem 4.16 that $F|_Y \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_{Y\cap\Lambda}^{\hbar,\mathrm{loc}})$ and one knows by Proposition 4.10 and Corollary 4.9 that $\mathrm{SS}(F_0)\times_\Lambda(\Lambda\cap Y)$ is Lagrangian and subanalytic in $T^*\Lambda$. Since $\mathrm{SS}(F)\subset\mathrm{SS}(F_0)$ by Proposition 3.6, it remains to apply Corollary 2.4.

5.2 Proof of Theorem 1.3

In this subsection, X is a quasi-compact separated smooth algebraic variety over $\mathbb{C}$ endowed with the Zariski topology. For an algebraic variety X, one denotes by X_{an} the complex analytic manifold associated with X and by $\rho\colon X_{\mathrm{an}}\to X$ the natural map. There is a natural morphism $\rho^{-1}\mathscr{O}_X\to\mathscr{O}_{X_{\mathrm{an}}}$ and it is well-known that this morphism is faithfully flat (cf [Ser56]).

Lemma 5.1. *Let* $\mathscr{A}_X$ *be a* DQ-*algebroid on* X. *Then there exists a* DQ-*algebroid* $\mathscr{A}_{X_{\mathrm{an}}}$ *on* X_{an} *together with a functor* $\rho^{-1}\mathscr{A}_X\to\mathscr{A}_{X_{\mathrm{an}}}$. *Moreover such an* $\mathscr{A}_{X_{\mathrm{an}}}$ *is unique up to a unique isomorphism.*

Proof. First, consider a star algebra $\mathscr{A}=(\mathscr{O}_X^\hbar,\star)$ on a smooth algebraic variety X. The star product is defined by a sequence of algebraic bidifferential operators $\{P_i\}_i$ (see [KS12, Def. 2.2.2]) and one defines a star algebra $\mathscr{A}^{\mathrm{an}}=(\mathscr{O}_{X_{\mathrm{an}}}^\hbar,\star)$ on X_{an} by using the same bidifferential operators.

There exists an open (for the Zariski topology) covering $X=\bigcup_{i\in I}U_i$ such that, for each i, there exists an object s_i of the category $\mathscr{A}_X(U_i)$. Then $\mathscr{A}_i:=\mathscr{E}nd(s_i)$ is a star algebra. For $i,j\in I$, since $s_i|_{U_i\cap U_j}$ and $s_j|_{U_i\cap U_j}$ are locally isomorphic, there exists an open covering $U_i\cap U_j=\bigcup_{a\in A_{ij}}U_{ij}^a$ such that setting $U_{ij}=\bigsqcup_{a\in A_{ij}}U_{ij}^a$, there exist an isomorphism $\alpha_{ij}\colon s_i|_{U_{ij}}\xrightarrow{\sim}s_j|_{U_{ij}}$. Then we have

$$a_{ijk}:=\alpha_{ij}\alpha_{jk}\alpha_{ki}\in\mathrm{End}(s_i|_{U_{ijk}})=\mathscr{A}_i(U_{ijk}),$$

where $U_{ijk} = U_{ij} \times_X U_{jk} \times_X U_{ki}$.

Hence we have an isomorphism $\beta_{ij}\colon \mathscr{A}_i|_{U_{ij}} \xrightarrow{\sim} \mathscr{A}_j|_{U_{ij}}$ defined by $\mathscr{A}_i \ni a \mapsto \alpha_{ij} \circ a \circ \alpha_{ij}^{-1} \in \mathscr{A}_j$. Moreover they satisfy the compatibility condition:

$$\beta_{ij}\beta_{jk}\beta_{ki} = \mathrm{Ad}(a_{ijk}) \in \mathrm{End}(\mathscr{A}_i|_{U_{ijk}}).$$

Then the data $(\{U_i\}, \{U_{ij}\}, \{\mathscr{A}_i\}, \{\beta_{i.j}\}, \{a_{ijk}\})$ satisfies the compatibility condition. Conversely, we can recover $\mathscr{A}_X$ from such data (see [KS12]).

On $(U_i)_{\mathrm{an}}$ we can define $\mathscr{A}_i^{\mathrm{an}}$. Similarly we can extend β_{ij} to $\beta_{ij}^{\mathrm{an}}\colon \mathscr{A}_i^{\mathrm{an}}|_{(U_{ij})_{\mathrm{an}}} \xrightarrow{\sim} \mathscr{A}_j^{\mathrm{an}}|_{(U_{ij})_{\mathrm{an}}}$ Finally we have $a_{ijk} \in \mathscr{A}_i(U_{ijk}) \subset \mathscr{A}_i^{\mathrm{an}}\big((U_{ijk})_{\mathrm{an}}\big)$, Then the data

$$(\{(U_i)_{\mathrm{an}}\}, \{(U_{ij})_{\mathrm{an}}\}, \{\mathscr{A}_i^{\mathrm{an}}\}, \{\beta_{i.j}^{\mathrm{an}}\}, \{a_{ijk}\})$$

satisfies the compatibility condition, and it defines a DQ-algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ on X_{an}. $\square$

Proposition 5.2. *The algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ is faithfully flat over $\rho^{-1}\mathscr{A}_X$.*

Proof. It is enough to prove that for each $x \in X$, $\mathscr{A}_{X_{\mathrm{an}},x}$ is faithfully flat over $\mathscr{A}_{X,x}$. This follows from [KS12, Cor. 1.6.7] since $\mathscr{A}_{X,x}/\hbar\mathscr{A}_{X,x} \simeq \mathscr{O}_{X,x}$ is Noetherian, $\mathscr{A}_{X_{\mathrm{an}},x}$ is cohomologically complete and finally $\mathscr{A}_{X_{\mathrm{an}},x}/\hbar\mathscr{A}_{X_{\mathrm{an}},x} \simeq \mathscr{O}_{X_{\mathrm{an}},x}$ is faithfully flat over $\mathscr{O}_{X,x}$. $\square$

For an $\mathscr{A}_X$-module $\mathscr{M}$ we set

$$\mathscr{M}_{\mathrm{an}} := \mathscr{A}_{X_{\mathrm{an}}} \otimes_{\rho^{-1}\mathscr{A}_X} \rho^{-1}\mathscr{M}.$$

Proof of Theorem 1.3. As in the proof of Theorem 1.2, we may assume that $\mathscr{L} \simeq \mathscr{L}_0^{\mathrm{loc}}$ where $\mathscr{L}_0$ is a simple $\mathscr{A}_X$-module along a smooth algebraic Lagrangian manifold Λ, the module $\mathscr{M}$ remaining algebraically good. Choose an $\mathscr{A}_{\Lambda/X}$-lattice $\mathscr{N}$ of $\mathscr{M}$. Let

$$F_{\mathrm{an}} := \mathrm{R}\mathscr{H}om_{\mathscr{A}^{\mathrm{loc}}_{X_{\mathrm{an}}}}(\mathscr{M}_{\mathrm{an}}, \mathscr{L}_{\mathrm{an}}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X_{\mathrm{an}}}}(\mathscr{N}_{\mathrm{an}}, (\mathscr{L}_0)_{\mathrm{an}})^{\mathrm{loc}}. \tag{5.2}$$

By Proposition 4.10 we know that $\mathrm{SS}(F_{\mathrm{an}}) \subset \mathrm{char}(\mathrm{gr}_\Lambda \mathscr{N}_{\mathrm{an}})$ and this set is contained in $\mathrm{char}(\mathrm{gr}_\Lambda \mathscr{N})$ which is an algebraic Lagrangian subvariety of $T^*\Lambda$. To conclude, apply Corollary 2.5. $\square$

Remark 5.3. (i) If one assumes that $\mathscr{M}$ and $\mathscr{L}$ are simple modules along two smooth algebraic varieties Λ_1 and Λ_2 of X, which is the situation appearing in [GJS19], there is a much simpler proof. Indeed, it follows from [KS12, Th. 7.4.3] that in this case

$$\mathrm{SS}(F) \subset C(\Lambda_1, \Lambda_2), \tag{5.3}$$

the Whitney normal cone of Λ_1 along Λ_2 and this set is algebraic. Hence, it remains to apply Corollary 2.5. Note that Th. 7.4.3 of loc. cit. is a variation on [KS08].

(ii) Also remark that (5.3) is no more true in the general case of irregular holonomic modules and until now, there is no estimate of $\mathrm{SS}(F)$, except of course, the fact that it is a Lagrangian set.

5.3 An example

Consider the Poisson manifold $X = \mathbb{C}^4$ with coordinates (x_1, x_2, y_1, y_2), the Poisson bracket being defined by:

$$\begin{gathered}\{x_1, x_2\} = 0,\ \{y_1, x_1\} = \{y_2, x_2\} = x_1, \\ \{y_1, y_2\} = y_2, \{y_1, x_2\} = y_2, \{y_1, x_2\} = \{y_2, x_1\} = 0.\end{gathered} \tag{5.4}$$

Denote by $\mathscr{A}_X$ the DQ-algebra defined by the relations $y_1 = \hbar x_1 \partial_{x_1}$, $y_2 = \hbar x_1 \partial_{x_2}$, that is,

$$\begin{gathered}[x_1, x_2] = 0,\ [y_1, x_1] = [y_2, x_2] = \hbar x_1, [y_1, y_2] = \hbar y_2, \\ [y_1, x_2] = \hbar y_2, [y_1, x_2] = [y_2, x_1] = 0.\end{gathered} \tag{5.5}$$

Hence, $Y = \{x_1 \neq 0\}$ is the symplectic locus $X \setminus X_{\mathrm{ns}}$ of the Poisson manifold X. Set $\Lambda = \{y_1 = y_2 = 0\}$. Then $\Lambda \cap Y$ is Lagrangian in Y.

Define the $\mathscr{A}_X$-module $\mathscr{L}$ by $\mathscr{L} = \mathscr{A}_X \cdot u$ with the relations $y_1 u = y_2 u = 0$. Then $\mathscr{L} \simeq \mathscr{O}_\Lambda^\hbar$ and for $a(x) \in \mathscr{O}_\Lambda^\hbar$, one has

$$\begin{cases} y_1 a(x) u = \hbar x_1 \frac{\partial a}{\partial x_1} u \\ y_2 a(x) u = \hbar x_1 \frac{\partial a}{\partial x_2} u. \end{cases}$$

Now define the left $\mathscr{A}_X$ module $\mathscr{M}$ by $\mathscr{M} = \mathscr{A}_X \cdot v$ with the relations $(y_1 + \hbar) v = y_2 v = 0$. Then the complex below, in which the operators act on the right

$$0 \longleftarrow \mathscr{M} \longleftarrow \mathscr{A}_X \xleftarrow[\cdot \begin{pmatrix} y_1 + \hbar \\ y_2 \end{pmatrix}]{} \mathscr{A}_X^{\oplus 2} \xleftarrow[\cdot (y_2, -y_1)]{} \mathscr{A}_X \longleftarrow 0 \tag{5.6}$$

is a free resolution of $\mathscr{M}$.

Hence, the object $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M}, \mathscr{L}^{\mathrm{loc}})$ is represented by the complex (the operators act on the left)

$$0 \longrightarrow \mathscr{O}_\Lambda^{\hbar,\mathrm{loc}} \xrightarrow[\begin{pmatrix} x_1 \partial_{x_1} + 1 \\ x_1 \partial_{x_2} \end{pmatrix} \cdot]{} (\mathscr{O}_\Lambda^{\hbar,\mathrm{loc}})^{\oplus 2} \xrightarrow[(x_1 \partial_{x_2}, -x_1 \partial_{x_1}) \cdot]{} \mathscr{O}_\Lambda^{\hbar,\mathrm{loc}} \longrightarrow 0. \tag{5.7}$$

Since $x_1 \partial_{x_1} \mathscr{O}_\Lambda^{\hbar,\mathrm{loc}} + x_1 \partial_{x_2} \mathscr{O}_\Lambda^{\hbar,\mathrm{loc}} = x_1 \mathscr{O}_\Lambda^{\hbar,\mathrm{loc}}$ and $\mathscr{O}_\Lambda^{\hbar,\mathrm{loc}} / x_1 \mathscr{O}_\Lambda^{\hbar,\mathrm{loc}} \simeq \mathscr{O}_{\Lambda \cap \{x_1 = 0\}}^{\hbar,\mathrm{loc}}$, we have

$$\mathscr{E}xt^2_{\mathscr{A}_X}(\mathscr{M}, \mathscr{L}^{\mathrm{loc}}) \simeq \mathscr{O}_{\Lambda \cap \{x_1 = 0\}}^{\hbar,\mathrm{loc}}.$$

This example shows that $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M}, \mathscr{L}^{\mathrm{loc}})$ does not belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}^{\hbar,\mathrm{loc}})$.

References

[Gab81] Ofer Gabber, *The integrability of the characteristic variety*, Amer. Journ. Math. **103** (1981), 445–468.

[GJS19] Sam Gunningham, David Jordan, and Pavel Safronov, *The finiteness conjecture for Skein modules* (2019), available at arXiv:1908.05233.

[Kas75] Masaki Kashiwara, *On the maximally overdetermined systems of linear differential equations I*, Publ. Res. Inst. Math. Sci. **10** (1975), 563-579.

[Kas96] ———, *Quantization of contact manifolds*, Publ. RIMS, Kyoto Univ. **32** (1996), 1–5.

[Kas03] ———, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003.

[Kon03] Maxim Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), 157–216.

[Kon01] ———, *Deformation quantization of algebraic varieties*, Lett. Math. Phys. **56** (2001), 271–294.

[KS90] Masaki Kashiwara and Pierre Schapira, *Sheaves on Manifolds*, Grundlehren der Mathematischen Wissenschaften, vol. 292, Springer-Verlag, Berlin, 1990.

[KS08] ———, *Constructibility and duality for simple holonomic modules on complex symplectic manifolds*, Amer. J. Math **130** (2008), 207–237.

[KS12] ———, *Deformation quantization modules*, Astérisque, vol. 345, Soc. Math. France, 2012.

[Ser56] Jean-Pierre Serre, *Géométrie algébrique et géométrie analytique*, Ann. Institut Fourier de Grenoble **6** (1956), 1-42.

Masaki Kashiwara
Kyoto University Institute for Advanced Study,
Research Institute for Mathematical Sciences
Kyoto University, 606–8502, Japan
and
Department of Mathematical Sciences and School of Mathematics,
Korean Institute for Advanced Studies,
Seoul 130-722, Korea

Pierre Schapira
Sorbonne University, CNRS IMJ-PRG
4, place Jussieu F-75005 Paris France
e-mail: pierre.schapira@imj-prg.fr
http://webusers.imj-prg.fr/~pierre.schapira/