# MICRO-HYPERBOLIC SYSTEMS

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## Introduction

The so-called "Cauchy Problem" has a very long and classical story, from J. Hadamard [7], I. G. Petrovski [22], J. Leray [20], L. Gårding [6], ... to, for example, the last results of Ivrii and Pietkov [13] or L. Hörmander [12], but we will not review here.

The difficulty treated by the last authors lies in the fact that the principal symbol of the operator is not of simple characteristic, and the lower order symbols have an essential role (Levi conditions). However, we know by [4] that with the use of hyperfunctions the situation is simple: "hyperbolicity" is given by the principal symbol. This fact allows us to treat the case of (overdetermined) systems. In this paper, we treat the initial value problem and the problem of propagation for hyperfunction or microfunction solutions of (micro-)hyperbolic systems. The hyperbolicity is just a geometrical property of the complex characteristic variety of the system.

Let $M$ be a real analytic manifold, $X$ a complexification of $M$, and $T_M^*X$ the conormal bundle of $M$ in the cotangent vector bundle $T^*X$ of $X$. Let us denote by $\mathcal{C}_M$ the sheaf of microfunctions on $T_M^*X$, and by $\mathcal{E}_X$ the sheaf of microdifferential operators on $T^*X$ (cf. [24]). Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module and $SS(\mathcal{M})$ its characteristic variety (in $T^*X$). We say (cf. [17]) that a covector $\theta\in T^*(T^*X)$ is micro-hyperbolic for $\mathcal{M}$ if $H(\theta)$ does not belong to the normal cone to $SS(\mathcal{M})$ along $T_M^*X$. We denote this cone by $C(SS(\mathcal{M}); T_M^*X)$. Here $H$ is the identification of $T(T^*M)$ and $T^*(T^*M)$ by the symplectic structure. Recall that $C(SS(\mathcal{M}); T_M^*X)_x\subset T_x(T^*X)$ is the set of limits of sequences (in some local chart) $a_n(x_n-y_n)$, with $a_n\in\mathbf{R}_+$, $x_n\in SS(\mathcal{M})$, $y_n\in T_M^*X$, $x_n\xrightarrow[n]{} x$, $y_n\xrightarrow[n]{} x$ (cf. § 1).

When the system reduces to a single equation, that is when $\mathcal{M}=\mathcal{E}_X/\mathcal{E}_XP$ for a microdifferential operator $P$, the system is micro-hyperbolic in the codirection $\theta$ at $(x_0, i\eta_0)\in T_M^*X$ if, in some local chart we have: $\sigma(P)((x, i\eta)+\varepsilon H(\theta))\neq 0$ for $(x, i\eta)$ in a neighbourhood of $(x_0, i\eta_0)$ in $T_M^*X$ and $0<\varepsilon\ll 1$. $\sigma(P)$ denotes the principal symbol of $P$. Thus in the case of differential operators we get the usual definition of weak hyperbolicity.

We prove the following prolongation theorem: if $Z$ is a conic closed set of $T_M^*X$, and if the exterior conormals to $Z$ at $x\in\partial Z$ are micro-hyperbolic for $\mathcal{M}$ then, for any $j$, $\mathrm{Ext}^j_{\mathcal{E}_X}(\mathcal{M}, \Gamma_Z(\mathcal{C}_M))_x=0$.

Meanwhile we prove the following: let $N$ be an analytic submanifold of $M$, and assume that all conormals to $N$ in $M$ are micro-hyperbolic for $\mathcal{M}$. Let $Y$ be the complexification of $N$ in $X$, $\tilde{\omega}$ and $\varrho$ the natural maps from $T^*X\times_X Y$ to $T^*X$ and $T^*Y$ and $\mathcal{M}_Y$ the system induced by $\mathcal{M}$ on $Y$. We write Hom $(F, G)$ for the sheaf of homomorphisms of the sheaf $F$ in the sheaf $G$, and $\mathrm{Ext}^j$ for the $j$th derived functor of Hom. Then for any $j$ we have a natural isomorphism:

$$\varrho_*\tilde{\omega}^{-1}\,\mathrm{Ext}^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)\simeq \mathrm{Ext}^j_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{C}_N)$$

that is, the Cauchy problem is well posed (for microfunctions) on $N$. If we assume that $\mathcal{M}$ is a module on the ring $\mathcal{D}_X$ of differential operators and that $N$ is a submanifold of $M$, replacing the sheaf $\mathcal{C}$ by the sheaf $\mathcal{B}$ of hyperfunctions, we get the isomorphisms

$$\mathrm{Ext}^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|N\simeq \mathrm{Ext}^j_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$$

In the case of a single equation $\mathcal{M}=\mathcal{D}_X/\mathcal{D}_XP$, where $P$ is of order $m$, $\mathcal{M}_Y$ is a free $\mathcal{D}_Y$-module of rank $m$, and we obtain:

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|N=\mathrm{Ker}\,(\mathcal{B}_M\xrightarrow[P]{}\mathcal{B}_M)|N\simeq \mathcal{B}_N^m$$

$$\mathrm{Ext}^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|N=(\mathcal{B}_M/P\mathcal{B}_M)|N=0$$

which is equivalent to the fact that the usual Cauchy Problem is well posed. When the system $\mathcal{M}$ reduces to a single equation, these theorems were proved, in the differential case, by J. M. Bony and P. Schapira [4], then in the microdifferential case by M. Kashiwara and T. Kawai [17].

To prove these theorems, we represent the sheaf $C_M$ by a sheaf of boundary values of holomorphic functions on a strictly pseudo-convex domain of $\mathbf{C}^n$, and we prove a prolongation theorem for the holomorphic solutions of the system. For this, first we must define the action of microdifferential operators: if $V_0 \subset V_1 \subset\subset D$ are open subsets of $\mathbf{C}^n$, $\Gamma$ is a real convex proper cone of $\mathbf{C}^n$ and if $(V_1+\Gamma)-(V_0+\Gamma)=V_1-V_0$, then we define the ring $\mathcal{E}(\Gamma; D)$ of microdifferential operators defined near $D\times\Gamma^0$ which acts on $\mathcal{O}(V_0)/\mathcal{O}(V_1)$ ($\mathcal{O}(V)$ denotes the space of holomorphic functions on $V$).

The geometry of the prolongation theorem being invariant by a change of real coordinates, we are lead to a geometrical problem quite simple in its nature.

We will give three applications:

1. We give a new proof for our previous result of [19]. Let $Y$ be a complex submanifold of $X$, $x\in T^*X\times_X Y$, and let $\mathcal{M}$ and $\mathcal{N}$ be two coherent $\mathcal{E}_X$-modules defined near $x$. Assume that $Y$ is non microcharacteristic for $(\mathcal{M}, \mathcal{N})$, i.e., the conormals to $Y$ in $X$ do not belong to $C(SS(\mathcal{M}); SS(\mathcal{N}))$. Let $\mathcal{N}^{\mathbf{R}}$ be the module generated by $\mathcal{N}$ over the ring $\mathcal{E}_X^{\mathbf{R}}$ of complex microlocal operators. Then for all $j$, $\mathrm{Ext}^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}^{\mathbf{R}})_x \simeq \mathrm{Ext}^j_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{N}^{\mathbf{R}}_Y)_{\rho(x)}$, that is, the Cauchy problem is well posed for $(\mathcal{M}, \mathcal{N}^{\mathbf{R}})$ on $Y$.

This theorem improves the results of [8], [9] and moreover gives a generalization for systems. At the same time we prove the prolongation theorem which we could not get by the complex method in [19].

2. We extend a theorem of J. M. Bony and P. Schapira [5] on the propagation of singularities to a more general setting where one has systems of equations. Again we emphasize that this gives stronger result for the case of a single equation.

Let $\Lambda$ be a conic involutive submanifold of $T^*_M X$, $\Lambda^{\mathbf{C}}$ the complexification of $\Lambda$ in $T^*X$, and $\tilde{\Lambda}$ the union of bicharacteristic leaves of $\Lambda^{\mathbf{C}}$ issued from $\Lambda$. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module such that any non zero vector of $T_\Lambda(T^*_M X)$ does not belong to $C(SS(\mathcal{M}); \tilde{\Lambda})$. Then, the support of a section of $\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, C_M|\Lambda)$ is a union of bicharacteristic leaves of $\Lambda$ (in fact we give a theorem which treats all the group $\mathrm{Ext}^j_{\mathcal{E}_X}(\mathcal{M}, C_M|\Lambda)$).

3. We extend a result of M. Kashiwara [15] to the microdifferential case, and prove that if $\mathcal{M}$ is a holonomic $\mathcal{E}_X$-module, the sheaves $\mathrm{Ext}^j_{\mathcal{E}_X}(\mathcal{M}, C_M)$ are locally constant along strata of a stratification of $SS(\mathcal{M})\cap T^*_M X$ which satisfies the Whitney conditions.

We shall now give the plan of this paper.

In § 1, we review microdifferential operators, microfunctions, etc.

In § 2 we give the notion of micro-hyperbolicity and announce the main theorems on the initial value problem and on the prolongation.

In § 3, we define the ring $\mathcal{E}(G; D)$ of "micro-differential operators defined near $D\times(-G^0)$" and give the operation of this ring on the relative cohomology group with holomorphic functions as coefficients. Also, we investigate the geometry of conormal cones and $Q$-flat sets.

In § 4, we give the theorem of prolongation in the complex domain.

§ 5 is a preliminary but important part of the proof of the main theorems and uses results from § 4. We finish the proof in § 7.

In § 6, we give the proof of the division theorem for sheaves of microfunctions with holomorphic parameters, which is necessary for the proof of the initial value theorem for micro-hyperbolic systems.

§ 8, § 9, and § 10 are for the applications of micro-hyperbolic systems.

## § 1. Preliminaries

**1.1.** Let $W$ be a manifold of class $C^1$, $TW$ the tangent vector bundle to $W$, and let $V$ and $S$ be two subsets of $W$.

*Definition 1.1.1.* The normal cone $C_x(S;\ V)$ of $S$ along $V$ at $x$ is the subset of $T_xW$ defined by:

$C_x(S;\ V)=\{v\in T_x(W)$; there are sequences $\{x_n\}$ in $S$, $\{y_n\}$ in $V$ and $\{a_n\}$ in $\mathbf{R}_+$ such that $\{x_n\}$ and $\{y_n\}$ converge to $x$ and that $a_n(x_n-y_n)$ converge to $v\}$.

We denote by $C(S;\ V)$ the union of $C_x(S;\ V)$ $(x\in W)$. The definition is free from the choice of local coordinate systems.

Note that:

$C(V; S)$ is a closed cone in $TW$.

$C(S;\ V)=-C(V; S)$.

If V is smooth, $C_x(S;\ V)$ is a closed cone of $T_xW$, invariant by $T_xV$. We sometimes identify it with its image in $T_VW$, and denote it by $C_V(S)$. $C_{\{x\}}(S)=C_x(S;\ \{x\})$ will be denoted by $C_x(S)$.

If we identify $W$ with the diagonal of $W\times W$ and $TW$ with $T_\Delta(W\times W)$ the normal to $\Delta$ by the first component, we have (cf. [19]):

$$C(S;\ V)=C_\Delta(S\times V).$$

If $W$ is open in some vector space, $v\in T_x(W)$ does not belong to $C_x(S;\ V)$ if and only if there exists an open cone $\Gamma$ with $v\in\Gamma$ such that

$$((U\cap V)+\Gamma)\cap U\cap S=\emptyset$$

for some neighborhood $U$ of $x$.

**1.2.** We review in this section and the following ones some constructions of [24].

Let $W$ be a $C^2$-manifold and $V$ a $C^2$-submanifold of $W$. We denote by $T^*W$ the cotangent vector bundle to $W$, and by $T_V^*W$ the conormal bundle to $V$ in $W$. We endow $(W-V)\coprod T_V^*W$ with the topology of the comonoidal transform (cf. [24, chapter 1, § 2] where this topology is only defined on $W\coprod S_V^*W$, $S_V^*W$ being the spherical cotangent bundle, but that of $(W-V)\coprod T_V^*W$ is the inverse image of the preceding by the mapping which identifies two points of $T_V^*W$ on the same orbit of the action of $\mathbf{R}^+$).

Let $\pi$ denote the projection of $(W-V)\coprod T_V^*W$ on $W$, and let $F$ be a sheaf on $W$. We have for $(x,\xi)\in T_V^*W$:

$$(\mathcal{H}^k_{T_V^*W}(\pi^{-1}F))_{(x,\xi)} = \varinjlim_{U,\,G} H^k_G(U,F)$$

where $U$ runs on the family of neighborhoods of $x$ in $W$ and $G$ on the family of closed sets of $W$ such that the normal cone $C_V(G)_x$ to $G$ along $V$ at $x$ is contained in the set $\{0\}\cup\{v\in T_VW_x;\ \langle v,\xi\rangle>0\}$. The sheaf $\mathcal{H}^k_{T_V^*W}(\pi^{-1}F)$ is regarded as a sheaf on $T^*W$ supported by $T_V^*W$. It is locally constant on the orbits of the action of $\mathbf{R}^+$. If we identify $W$ with the zero section $T_W^*W$ of $T^*W$ we have

$$\mathcal{H}^k_{T_V^*W}(\pi^{-1}F)\,|\,T_W^*W = \mathcal{H}^k_V(F).$$

We denote by $a$ the antipodal map on $T^*W$:

$$a\colon (x,\xi)\mapsto(x,\,-\xi).$$

We denote by $\omega_{V|W}$ the sheaf of relative orientation of $V$ in $W$.

**1.3.** Now let $X$ be a complex analytic manifold of dimension $n$, $T^*X$ the complex cotangent vector bundle to $X$ and $\omega$ the canonical 1-form on $T^*X$. For local coordinates $(z_1,\dots,z_n;\ \zeta_1,\dots,\zeta_n)$ on $T^*X$, we have

$$\omega=\sum_{i=1}^n \zeta_i\,dz_i.$$

The isomorphism $H$ of $T^*(T^*X)$ on $T(T^*X)$ associated to $\omega$ is defined by:

$$\langle\theta,v\rangle=\langle d\omega,\,v\wedge H(\theta)\rangle$$

for $v\in T(T^*X)$, $\theta\in T^*(T^*X)$.

We denote by $\mathcal{O}_X$ the sheaf on $X$ of holomorphic functions. If $Y$ is a complex submanifold of codimension $d$, the sheaf $\mathcal{C}^{\mathbf{R}}_{Y|X}$ on $T_Y^*X$ is defined, with the preceding notations, by:

$$\mathcal{C}^{\mathbf{R}}_{Y|X}=\mathcal{H}^d_{T_Y^*X}(\pi^{-1}\mathcal{O}_X)^a.$$

Let $\gamma$ be the projection on the complex projective cotangent bundle:

$$\gamma\colon T^*X - T^*_X X \to P^*X.$$

The sheaf $C^\infty_{Y|X}$ on $T^*_Y X$ is defined by:

$$C^\infty_{Y|X}|T^*X - T^*_X X = \gamma^{-1}\gamma_* C^{\mathbf{R}}_{Y|X}$$

$$C^\infty_{Y|X}|T^*_X X = C^{\mathbf{R}}_{Y|X}|T^*_X X (= \mathcal{H}^d_Y(\mathcal{O}_X)).$$

If we take local coordinates, it is possible to associate a symbol of infinite order with any section of $C^\infty_{Y|X}$ (cf. [24, chapter 2, § 1]). The sheaf $C_{Y|X}$ is then defined as the subsheaf of $C^\infty_{Y|X}$ of sections of finite order.

The sheaf $\mathcal{E}^{\mathbf{R}}_X$ of microlocal operators on $T^*X$ is given by:

$$\mathcal{E}^{\mathbf{R}}_X = C^{\mathbf{R}}_{X|X\times X} \underset{\mathcal{O}_{X\times X}}{\otimes} \Omega^{(0,n)}_{X\times X}$$

where $\Omega^{(0,n)}_{X\times X}$ denotes the sheaf of holomorphic forms of type $(0, n)$ on $X \times X$. The sheaf $\mathcal{E}^{\mathbf{R}}_X$ owns a natural structure of (unitary, non commutative) ring (cf. § 3) and

$$\mathcal{E}^{\mathbf{R}}_X|T^*_X X = \mathcal{D}^\infty_X$$

where $\mathcal{D}^\infty_X$ is the sheaf on $X$ of differential operators of infinite order.

We construct in the same way the sheaf $\mathcal{E}^\infty_X$ (resp. $\mathcal{E}_X$) of microdifferential operators of infinite order (resp. finite order) with $C^\infty_{X|X\times X}$ (resp. $C_{X|X\times X}$) instead of $C^{\mathbf{R}}_{X|X\times X}$. Then, we have

$$\mathcal{E}^\infty_X|X = \mathcal{D}^\infty_X$$

$$\mathcal{E}_X|X = \mathcal{D}_X$$

where $\mathcal{D}_X$ is the sheaf (of rings) of differential operators of finite order on $X$.

A system of microdifferential equations is, by definition, a coherent $\mathcal{E}_X$-module $\mathcal{M}$, defined on an open subset $U$ of $T^*X$. The characteristic variety of the system, denoted by $SS(\mathcal{M})$, is nothing but the support of $\mathcal{M}$ in $U$.

*Example.* Let $(P)$ be an $N\times N$ matrix of microdifferential operators on $U \subset T^*X$. We associate $\mathcal{M} = \mathcal{E}^N_X/\mathcal{E}^N_X\cdot(P)$. Then there exists ([23]) a holomorphic function on $U$, homogeneous in $\zeta$, denoted by $\det(P)$, such that:

$$SS(\mathcal{M}) = \{(z, \zeta) \in U;\ \det(P)(z, \zeta) = 0\}.$$

If $N = 1$, $\det(P)$ coincides with the principal symbol $\sigma(P)$ of $P$.

**1.4.** Let $Y$ be a complex analytic manifold, and $\varphi$ a holomorphic map from $Y$ to $X$. We denote by $\bar{\omega}$ and $\varrho$ the natural mappings:

$$\begin{aligned} &\bar{\omega}\colon T^*X \underset{X}{\times} Y \to T^*X \\ &\varrho\colon T^*X \underset{X}{\times} Y \to T^*Y. \end{aligned} \tag{1.4.1}$$

If we identify $Y$ with the graph of $\varphi$ in $Y\times X$ and $T^*_Y(Y\times X)$ with $T^*X\times_X Y$, the $(\varrho^{-1}\mathcal{E}^{\mathbf{R}}_Y, \bar{\omega}^{-1}\mathcal{E}^{\mathbf{R}}_X)$-bimodule $\mathcal{E}^{\mathbf{R}}_{Y\to X}$ on $T^*X\times_X Y$ is defined by:

$$\mathcal{E}^{\mathbf{R}}_{Y\to X} = \mathcal{C}^{\mathbf{R}}_{Y|Y\times X} \underset{\mathcal{O}_X}{\otimes} \Omega_X^{(\dim X)},$$

and the $(\bar{\omega}^{-1}\mathcal{E}^{\mathbf{R}}_X, \varrho^{-1}\mathcal{E}^{\mathbf{R}}_Y)$-bimodule $\mathcal{E}^{\mathbf{R}}_{X\leftarrow Y}$ is defined by

$$\mathcal{E}^{\mathbf{R}}_{X\leftarrow Y} = \mathcal{C}^{\mathbf{R}}_{Y|Y\times X} \underset{\mathcal{O}_Y}{\otimes} \Omega_Y^{(\dim Y)}.$$

The sheaves $\mathcal{E}^{\infty}_{Y\to X}$, $\mathcal{E}_{Y\to X}$, ... are constructed in the same way, with $\mathcal{C}^{\infty}_{Y|Y\times X}$ or $\mathcal{C}_{Y|Y\times X}$ instead of $\mathcal{C}^{\mathbf{R}}_{Y|Y\times X}$. If $(z_1, ..., z_n)$ is a system of coordinates on $X$ and $Y$ is given by $z_1 = ... = z_l = 0$, there are (non canonical) isomorphisms of $\mathcal{E}_X$-modules:

$$\begin{aligned} \mathcal{E}_{X\leftarrow Y} &\simeq \mathcal{E}_X/\mathcal{E}_X z_1 + ... + \mathcal{E}_X z_l \\ \mathcal{E}_{Y\to X} &\simeq \mathcal{E}_X/z_1\mathcal{E}_X + ... + z_l\mathcal{E}_X. \end{aligned}$$

If $\mathcal{M}$ is a coherent $\mathcal{E}_X$-module, the inverse image of $\mathcal{M}$ by $\varphi$ is by definition

$$\mathcal{M}_Y = \varrho_*(\mathcal{E}_{Y\to X} \underset{\mathcal{E}_X}{\otimes} \mathcal{M}).$$

The map $\varphi$ is non characteristic if $\varrho$ is proper (hence finite) on $\bar{\omega}^{-1}SS(\mathcal{M})$. In this case $\mathcal{M}_Y$ is a coherent $\mathcal{E}_Y$-module with the support $\varrho(\bar{\omega}^{-1}SS(\mathcal{M}))$.

**1.5.** Let now $M$ be a real analytic manifold of dimension $n$, and $X$ a complexification of $M$. The sheaf $\mathcal{C}_M$ of microfunctions is the sheaf on $T^*_M X$ given by:

$$\mathcal{C}_M = \mathcal{H}^n_{T^*_M X}(\pi^{-1}\mathcal{O}_X)^a \otimes \omega_{M|X}.$$

This sheaf is naturally endowed with a structure of left $\mathcal{E}^{\mathbf{R}}_X$-module, and a fortiori, o $\mathcal{E}^{\infty}_X$- and $\mathcal{E}_X$-module. Moreover:

$$\mathcal{C}_M|M = \mathcal{B}_M$$

where $\mathcal{B}_M$ denotes the sheaf of hyperfunctions on $M$.

**1.6.** To end this paragraph, let us remark that in [24] the sheaves $\mathcal{C}_{Y|X}$, $\mathcal{C}_M$, ... are constructed on the projective or spherical cotangent bundles, but it seems more convenient to get all these sheaves on the same space, $T^*X$. Although the letters $\mathcal{C}_{Y|X}$, $\mathcal{C}_M$, do not denote exactly the same objects as in [24] we keep these notations. At the contrary we denote by $\mathcal{E}_X$ and $\mathcal{E}_X^\infty$ the sheaves denoted by $\mathcal{P}_X^f$ and $\mathcal{P}_X$ (on $P^*X$) in [24], and we call the sections microdifferential operators instead of pseudo-differential operators.

## § 2. Statement of main theorems

**2.1.** Let $M$ be a real analytic manifold and $X$ a complexification of $M$. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module on an open set $U$ of $T^*X$, $V$ a closed set in $U$, $x$ a point of $U$ and $\theta$ a vector of $T_x^*(T^*X)$.

*Definition 2.1.1.* (a) We say that $\theta$ is *non microcharacteristic* for $(\mathcal{M}, V)$ if:

$$H(\theta) \notin C_x(SS(\mathcal{M});\ V),$$

(b) if $V$ is the characteristic variety of a coherent $\mathcal{E}_X$-module $\mathcal{N}$, we say in this case $\theta$ is *non microcharacteristic* for $(\mathcal{M}, \mathcal{N})$.

(c) If $V = T_M^*X \cap U$, we say that $\theta$ is *microhyperbolic* for $\mathcal{M}$.

Definition 2.1.1 (a) has been introduced when $\mathcal{M}$ is reduced to a single equation and $V$ is a complex submanifold of $T^*X$, in a different but equivalent way by J. M. Bony [2].

Definition 2.1.1 (b) has been introduced and studied in detail in our previous paper [19].

Definition 2.1.1 (c) has been introduced (under the name of "partial microhyperbolicity") by M. Kashiwara and T. Kawai [17].

Let now $\varphi$ denote a holomorphic map from $Y$ to $X$. We denote, as in the preceding section (1.4.1), by $\tilde{\omega}$ and $\rho$ the natural mappings associated to $\varphi$.

*Definition 2.1.2.* (a) We say that $\varphi$ is *non microcharacteristic* for $(\mathcal{M}, V)$ at $x$ if, for any nonzero covector $\theta \in T_{\pi(x)}^*X$ with $\varphi^*(\theta) = 0$, the covector $\pi^*(\theta)$ is *non microcharacteristic* for $(\mathcal{M}, V)$.

(b) If $V$ is the characteristic variety of a coherent $\mathcal{E}_X$-module $\mathcal{N}$, one may say that $\varphi$ is *non microcharacteristic* for $(\mathcal{M}, \mathcal{N})$ at $x$.

(c) If $\varphi$ is the complexification of a real analytic map from $N$ to $M$, and if $V = T_M^*X$, we say that $\varphi$ is *microhyperbolic* for $\mathcal{M}$ at $x$ when any non zero covector $\theta$ of $M$ with $\varphi^*(\theta) = 0$ is *non microcharacteristic* for $(\mathcal{M}, V)$.

(d) In the preceding situation, if $x$ belongs to $M = M \times_X T^*_X X$ (in this case $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module) we just say that $\varphi$ is *hyperbolic* for $\mathcal{M}$ at $x$.

(e) If $N$ is a submanifold of $M$ and if $\varphi$ is the injection of $N$ in $M$, we also say that $N$ is *(micro-)hyperbolic* for $\mathcal{M}$.

It is clear that if $x$ belongs to $V$, and if $\varphi$ is non microcharacteristic for $(\mathcal{M}, V)$ at $x$, $\varphi$ is non microcharacteristic for $(\mathcal{M}, \{x\})$ at $x$, and this last condition implies that $\varphi$ is non characteristic at $x$ (the converse is false).

*Example 1.* Assume that $\mathcal{M}$ is reduced to a single equation, that is, $\mathcal{M} \simeq \mathcal{E}_X / \mathcal{E}_X . P$, for a microdifferential operator $P$. Let $V$ be a *complex* analytic submanifold of $T^*X$ and let $r$ be the smallest integer such that $\sigma(P)$ vanishes with all its derivatives of order $< r$ on $V$. Let $\theta$ belong to $T^*_x(T^*X)$, $x \in V$. We can prove [19] that $\theta$ is non microcharacteristic for $(\mathcal{M}, V)$ if and only if

$$\sigma(P)(x + \varepsilon H(\theta)) \neq o(\varepsilon^r).$$

Hence, in this case, the definition is that introduced by J. M. Bony [2].

*Example 2.* Assume $\mathcal{M} = \mathcal{E}_X^N / \mathcal{E}_X^N \cdot P$ where $P$ is an $N \times N$ matrix of microdifferential operators on $U \subset T^*X$. Let $(z_1, \ldots, z_n; \zeta_1, \ldots, \zeta_n)$ be a system of local coordinates on $T^*X$, in a neighborhood of $(x^0; i\eta^0) \in T^*_M X$, where $z = x + iy$, $\zeta = \xi + i\eta$. It is easy to prove, with the help of the local Bochner tube theorem [24] that $\mathcal{M}$ is micro-hyperbolic in the $dx_1$-codirection at $(x^0; i\eta^0)$ if and only if we have:

$$(\det P)(x; i\eta + \varepsilon\theta) \neq 0$$

for $0 < \varepsilon \ll 1$, $x$ real, $\eta$ real, $|x - x^0| \ll 1$, $|\eta - \eta^0| \ll 1$, for $\theta = (1, 0, \ldots 0)$. Hence we find at least in the case of one operator, the classical definition of (weak) hyperbolicity [22].

**2.2.** To formulate, and mainly to prove, our theorems, we use systematically the language of derived category, as in [24]. In particular, $\mathbf{R}\,\mathrm{Hom}$, $\mathbf{R}\Gamma$, $\mathbf{R}f_*$, ... stand for the right derived functors of Hom, $\Gamma$, $f_*$, ... and $\otimes^L$ for the left derived functor of $\otimes$.

Let $Z$ be a subset of $T^*_M X$, and $p$ a point in $T^*_M X$. A *conormal* $\theta$ to $Z$ is, by definition, a covector $\theta$ at $p$ satisfying $\langle \theta, v \rangle > 0$ for any $v \notin C_p(T^*_M X - Z, Z)$. If $Z$ is defined by an equation $\varphi \geqslant 0$ with $d\varphi \neq 0$, the conormal to $Z$ at $p \in Z\text{-int}\, Z$ is $cd\varphi(p)$ with $c > 0$. Note that, for $p \notin \bar{Z} - \mathrm{int}\, Z$, no covector is a conormal of $Z$ at $p$. We shall show that $T(T^*_M X) - C(T^*_M X - Z, Z)$ is an open convex cone in § 3.

THEOREM 2.2.1. *Let $\mathcal{M}$ be a left coherent $\mathcal{E}_X$-module defined in a neighborhood of $x \in T_M^* X$. Let $Z$ be a closed subset of $T_M^* X$ such that $x$ is not in the interior of $Z$. If, for any conormal $\theta$ of $Z$ at $x$, $-\theta$ is micro-hyperbolic for $\mathcal{M}$, then:*

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathbf{R}\Gamma_Z(\mathcal{C}_M))_x = 0.$$

If $x$ belongs to $T_M^* M$ we obtain:

COROLLARY 2.2.2. *Let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module defined in a neighborhood of a point $x$ in $M$, and let $Z$ be a closed set in $M$ such that $x$ does not belong to the interior of $Z$. We assume that any conormal of $Z$ at $x$ is hyperbolic for $\mathcal{M}$. Then, we have*

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}; \Gamma_Z(\mathcal{B}_M))_x = 0.$$

Theorem 2.1.1 is of local type. However, once we obtained a theorem of local type, it is not difficult to get a theorem of global type, using only geometric arguments.

Let $Q$ be an open convex cone in $TU$. An open set $\Omega$ in $U$ is called $Q$-flat if $C(U-\Omega, \Omega) \cap Q = \emptyset$.

THEOREM 2.2.3. *Let $\mathcal{M}$ be a left coherent $\mathcal{E}_X$-module defined on an open set $U$ of $T_M^* X$, $Q$ an open convex cone of $TU$. Assume that*

(i) *any codirection $\theta$ such that $\langle \theta, Q \rangle > 0$ is microhyperbolic for $\mathcal{M}$,*

(ii) *there is a $C^1$-function $g$ on $U$ such that $\bigcup_{p \in U}\{v \in T_p U;\ \langle v, dg(p)\rangle > 0\} \supset Q$.*

*Then, for any couple of $Q$-flat open sets $\Omega_1 \supset \Omega_0$ in $U$ such that $\Omega_1 - \Omega_0 \subset\subset U$, we have*

$$\mathrm{Ext}^j_{\mathcal{E}_X}(\Omega_1; \mathcal{M}, \mathcal{C}_M) \xrightarrow{\sim} \mathrm{Ext}^j_{\mathcal{E}_X}(\Omega_0; \mathcal{M}, \mathcal{C}_M).$$

**2.3.** Let $N$ and $M$ be two real analytic manifolds, and let $\varphi$ be a real analytic map from $N$ to $M$, which extends to a holomorphic map from $Y$ to $X$. Here $Y$ and $X$ are complexifications of $N$ and $M$. We denote, as usual, by $\tilde{\omega}$ and $\varrho$ the natural map from $T^*X \times_X Y$ to $T^*X$ and $T^*Y$ associated to $\varphi$.

THEOREM 2.3.1. *Let $\mathcal{M}$ be a left coherent $\mathcal{E}_X$-module on an open set $U \subset T^*X$. We assume that $\varphi$ is micro-hyperbolic for $\mathcal{M}$ on $U$ and we make the extra assumption:*

$$\tilde{\omega}^{-1}(U) \cap \tilde{\omega}^{-1}(SS(\mathcal{M})) \cap \varrho^{-1}(T_N^* Y) \subset \tilde{\omega}^{-1}(T_M^* X). \tag{2.3.1}$$

*Then the natural homomorphism*

$$\varrho_* \tilde{\omega}^{-1}\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M) \to \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{C}_N)$$

*is an isomorphism.*

Let us make the following remark. Since $\varphi$ is microhyperbolic, we can always let the condition (2.3.1) be satisfied by shrinking $U$ without changing $U \cap T_M^* X$. Thus this condition just means that we must not include points of $SS(\mathcal{M})$ outside $T_M^* X$ in the calculation of $\mathcal{M}_Y$. If $U$ contains $T_M^* M$ we obtain:

COROLLARY 2.3.2. *Let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module. Let $\varphi$ be an analytic map from $N$ to $M$, hyperbolic for $\mathcal{M}$ on $M$. Then the natural homomorphism:*

$$\varphi^{-1}(\mathbf{R}\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)) \to \mathbf{R}\,\mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$$

*is an isomorphism.*

It is well known [24] that on a complex manifold of complex dimension $n$, a coherent $\mathcal{E}_X$-module admits, locally, projective resolutions of length $\leqslant n$. Thus we get:

COROLLARY 2.3.3. *Let $N$ be a submanifold of $M$ of dimension $m$, and assume $N$ micro-hyperbolic for $\mathcal{M}$ at $x \in T_M^* X \times_M N$. Then,*

$$\mathrm{Ext}^i_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)_x = 0, \quad i > m.$$

If $\mathcal{M}$ is reduced to a single equation, the induced system on a non characteristic hypersurface is a free module of finite rank, and we get by Theorem 2.3.1 $\mathrm{Ext}^i_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M) = 0$, $i > 0$ in this case. The same is true for $\mathcal{D}_X$ and $\mathcal{B}_M$ instead of $\mathcal{E}_X$ and $\mathcal{C}_M$.

As already mentioned in the introduction, Theorems 2.2.1 and 2.3.1 have been proved, for a single equation, by J. M. Bony and P. Schapira [4] for the differential case, then by M. Kashiwara and T. Kawai [17] for the microdifferential case.

**2.4.** To prove Theorem 2.3.1 we will prove the vanishing theorem of cohomology group in sheaves of microfunctions which are partly holomorphic in some variable (sheaves $\mathcal{C}_{M^+|X}$), and these results can be useful in the study of boundary value problems.

## § 3. Action of microdifferential operators in the complex domain

**3.1.** Let $X$ be a complex manifold of dimension $n$, and take a point $p$ in the cotangent bundle $T^*X$. The ring $\mathcal{E}_p^{\mathbf{R}}$ is defined as the inductive limit:

$$\mathcal{E}_p^{\mathbf{R}} = \varinjlim_{Z, \Omega} H_Z^n(\Omega \times \Omega; \mathcal{O}^{(0,n)}), \tag{3.1}$$

where $\Omega$ runs on the set of neighborhoods of $\pi(p)$ and $Z$ runs on the set of closed sets in $X \times X$ satisfying

$$C_\Delta(Z) \subset \{v \in T_{\pi(p)}X;\ \mathrm{Re}\,\langle v, p\rangle > 0\} \cup \{0\}. \tag{3.2}$$

Here, $\Delta$ means the diagonal set and $\mathcal{O}^{(0,n)}$ is the sheaf of $n$-forms with respect to the second variable. We shall try to provide each cohomology group $H_Z^n(\Omega\times\Omega;\mathcal{O}^{(0,n)})$ with the ring structure so that $\mathcal{E}_p^{\mathbf{R}}$ is obtained by the inductive limit. Recall first how the ring structure on $\mathcal{E}^{\mathbf{R}}$ is defined: denote by $p_{ij}$ the projection from $X\times X\times X$ onto $X\times X$ by the $i$th and $j$th components (i.e. $p_{ij}(x_1, x_2, x_3)=(x_i, x_j)$). Then, the multiplication is obtained as the composition:

$$
\begin{aligned}
(3.3)\qquad & \varinjlim H_Z^n(\Omega\times\Omega;\mathcal{O}^{(0,n)})\times\varinjlim H_Z^n(\Omega\times\Omega;\mathcal{O}^{(0,n)})\\
&\to\varinjlim H^{2n}_{p_{12}^{-1}Z\times p_{23}^{-1}Z}(\Omega\times\Omega\times\Omega;\mathcal{O}^{(0,n,n)})\\
&\to\varinjlim H_Z^n(\Omega\times\Omega;\mathcal{O}^{(0,n)}).
\end{aligned}
$$

The first arrow is defined by the cup-product and the second arrow is induced from $\mathbf{R}p_{13!}\mathcal{O}^{(0,n,n)}\to\mathcal{O}^{(0,n)}[-n]$, where the subscript ! denotes the subsheaf of sections with proper support.

*Definition 3.1.1.* A closed set $Z$ in $X\times X$ is called *proper ordering* on $X$ if $Z$ satisfies the following conditions:

(i) $Z$ contains the diagonal set $\Delta$.

(ii) If $Z$ contains $(x, y)$ and $(y, z)$, then $Z$ contains $(x, z)$, i.e. $p_{13}(p_{12}^{-1}Z\cap p_{23}^{-1}Z)\subset Z$.

(iii) The map $p_{12}^{-1}Z\cap p_{23}^{-1}Z\xrightarrow{p_{13}}Z$ is a proper map.

*Definition 3.1.2.* We say that an open set $\Omega$ in $Z$ is $Z$-round, if $\{y;\ (x, y),\ (y, z)\in Z\}$ is contained in $\Omega$ for any $x$ and $z$ in $\Omega$; equivalently, $p_2(p_1^{-1}\Omega\cap p_3^{-1}\Omega\cap p_{12}^{-1}Z\cap p_{23}^{-1}Z)\subset\Omega$ where $p_j$ are the projection from $X\times X\times X$ onto $X$ by the $j$th component.

*Definition 3.1.3.* We call an open set $\Omega$ in $X$, $Z$-open if $\{y;\ (x, y)\in Z\}$ is contained in $\Omega$ for any $x$ in $\Omega$; equivalently $p_2(Z\cap p_1^{-1}\Omega)\subset\Omega$ where $p_j$ are the projection from $X\times X$ onto $X$ by the $j$th component.

First remark the following proposition:

PROPOSITION 3.1.4. *Let $X$ and $Y$ be locally compact topological spaces, $f$ a continuous map from $X$ into $Y$, $Z$ and $G$ locally closed subsets of $X$ and $Y$ respectively, and let $\mathcal{F}$ be a sheaf on $X$. Suppose that*

(a) *$f^{-1}G\cap Z\to G$ is a proper map,*

(b) *$Z\cap f^{-1}G$ is an open subset of $Z$.*

*Then we can define naturally the following homomorphism:*

$$\mathbf{R}\Gamma_Z(X;\mathcal{F})\to\mathbf{R}\Gamma_G(X;\mathbf{R}f_!\mathcal{F}).$$

*Here $f_!\mathcal{F}$ is the subsheaf of $f_*\mathcal{F}$ of the sections with proper supports on $Y$.*

*Proof*. Let $\Omega$ (resp. $\Omega'$) be an open set in $X$ (resp. $Y$) containing $Z \cap f^{-1}G$ (resp. $G$) as closed set such that $f(\Omega) \subset \Omega'$. Then we obtain the homomorphism:

$$\mathbf{R}\Gamma_Z(X; \mathcal{F}) \to \mathbf{R}\Gamma_{Z \cap f^{-1}G}(\Omega; \mathcal{F}).$$

Since $Z \cap f^{-1}G \to \Omega'$ is a proper map we obtain:

$$\mathbf{R}\Gamma_{Z \cap f^{-1}G}(\Omega; \mathcal{F}) \to \mathbf{R}\Gamma_G(\Omega'; \mathbf{R}f_! \mathcal{F}),$$

and the desired homomorphism is obtained as the composite. Q.E.D.

*Definition 3.1.5*. For a proper relation $Z$ and a $Z$-round open set $D$, we set $\mathcal{E}(Z; D) = H^n_Z(D \times D; \mathcal{O}^{(0,n)})$.

THEOREM 3.1.6. *Suppose that $Z$ is a proper relation on $X$, $D$ is a $Z$-round open set and $\Omega_1 \supset \Omega_0$ two $Z$-open sets Such that $\Omega_1 - \Omega_0 \subset D$ and $\overline{\Omega_1 - \Omega_0}$ is compact. Then we have*

(a) *$\mathcal{E}(Z; D)$ has a canonical ring structure with a unit.*
(b) *$\mathcal{E}(Z; D)$ operates on $H^k_{\Omega_1 - \Omega_0}(\Omega_1; \mathcal{O})$ naturally.*

*Proof*. We have the homomorphism

$$H^n_Z(D \times D; \mathcal{O}^{(0,n)}) \underset{\mathbf{C}}{\otimes} H^n_Z(D \times D; \mathcal{O}^{(0,n)}) \to H^{2n}_{p_{12}^{-1} Z \cap p_{23}^{-1} Z}(D \times D \times D; \mathcal{O}^{(0,n,n)})$$

by the aid of cup-product. Since $p_{12}^{-1}Z \cap p_{23}^{-1}Z \cap D \times D \times D = p_{12}^{-1}Z \cap p_{23}^{-1}Z \cap D \times X \times D$, $p_{13}$: $p_{12}^{-1}Z \cap p_{23}^{-1}Z \cap D \times D \times D \to Z \cap D \times D$ is a proper map. The preceding proposition can be therefore applied to obtain the homomorphism:

$$H^{2n}_{p_{12}^{-1} Z \cap p_{23}^{-1} Z}(D \times D \times D; \mathcal{O}^{(0,n,n)}) \to H^{2n}_Z(D \times D; \mathbf{R}p_{13!}\mathcal{O}^{(0,n,n)}).$$

By applying $\mathbf{R}p_{13!}\mathcal{O}^{(0,n,n)} \to \mathcal{O}^{(0,n)}[-n]$, we obtain $\mathcal{E}(Z; D) \otimes \mathcal{E}(Z; D) \to \mathcal{E}(Z; D)$. It is easy to check this operation makes $\mathcal{E}(Z; D)$ a ring. Also, we can see that $H^n_\Delta(X \times X; \mathcal{O}^{(0,n)})$ contains a canonical element and its image by the homomorphism $H^n_\Delta(X \times X; \mathcal{O}^{(0,n)}) \to H^n_Z(D \times D; \mathcal{O}^{(0,n)})$ is the unit of $\mathcal{E}(Z; D)$. Let us show (b). Setting $S = \Omega_1 - \Omega_0$, we have the homomorphism

$$\begin{aligned}\mathcal{E}(Z; D) \otimes H^k_S(\Omega_1; \mathcal{O}) &\to H^{n+k}_{Z \cap p_2^{-1}S}((D \times D) \cap p_2^{-1}\Omega_1; \mathcal{O}^{(0,n)}) \\ &\to H^k_S(\Omega_1; \mathcal{O})\end{aligned}$$

because the condition in Proposition 3.1.4 is satisfied to apply it.

3.2. We consider the following special case. $X$ is $\mathbf{C}^n$ and $Z$ is $\{(x, y)\in X\times X;\ y-x\in G\}$ where $G$ is a closed proper convex cone containing 0. It is easy to see that $Z$ is a proper ordering on $X$. We shall call an open set $G$-open (resp. $G$-round) if it is $Z$-open (resp. $Z$-round). Therefore, an open set $\Omega$ is $G$-open if and only if $\Omega+G\subset\Omega$ and an open set $D$ is $G$-round if and only if $(D+G)\cap(D+G^a)=D$. Here $G^a=\{-x;\ x\in G\}$ and $D+G=\{x+\gamma;\ x\in D,\ \gamma\in G\}$. For a $G$-round open set $D$, we denote $\mathcal{E}(G; D)$ for $\mathcal{E}(Z; D)$.

Theorem 3.1.6 is translated as follows:

THEOREM 3.2.1. *Let $D$ be a $G$-open set, and $\Omega_1\supset\Omega_0$ two $G$-open sets such that $\Omega_1-\Omega_0\subset D$ and that $\overline{\Omega_1-\Omega_0}$ is compact.*

(a) *$\mathcal{E}(G; D)$ is a ring, and*

(b) *$H^k_{\Omega_1-\Omega_0}(\Omega_1; \mathcal{O})$ is an $\mathcal{E}(G; D)$-module.*

(c) *We have the ring homomorphism*

$$\mathcal{E}(G; D)\to\Gamma(D\times G^{0a}, \mathcal{E})$$

*where $G^{0a}=\{\zeta;\ \mathrm{Re}\,\langle\zeta, z\rangle>0$ for any $z\in G-\{0\}\}$.*

We define the new topology (which we shall call *$G$-topology*) on $X$ as follows: an open set for the $G$-topology is, by definition, a $G$-open set. For a subset $S$ of $X$, we denote by $S_G$ the topological space $S$ with the $G$-topology. Let $\varphi_G$ be the continuous map $\varphi_G$: $X\to X_G$ defined by $x\mapsto x$. If $G_1$ is a closed convex proper cone containing $G$, then we denote by $\varphi_{G_1, G}$ the map $X_G\to X_{G_1}$ defined by $x\mapsto x$.

LEMMA 3.2.2. (i) *$R^k\varphi_{G*}(\mathcal{O}_X)=0$ for $k\neq 0$,*

(ii) *$R^k\varphi_{G_1, G*}(\varphi_{G*}\mathcal{O})=0$ for $k\neq 0$ and equals $\varphi_{G_1*}\mathcal{O}$ for $k=0$.*

*Proof.* Since $\varphi_{G_1}=\varphi_{G_1, G}\circ\varphi_G$, (ii) is an immediate consequence of (i). For any $x\in X$, $R^k\varphi_{G*}(\mathcal{O})_x=\varinjlim_U H^k(U, \mathcal{O})$ where $U$ is a $G$-neighborhood of $x$. However a convex $G$-neighborhood of $x$ forms a fundamental neighborhood system of $x$ (for example, take $V+G$ with a small open ball $V$ containing $x$). Hence, we have $H^k(U, \mathcal{O})=0$ for $k\neq 0$ for such $U$. Q.E.D.

LEMMA 3.2.3. *Let $\Omega\supset\tilde{\omega}$ be two $G$-open sets. Suppose that the following conditions are satisfied.*

(i) *There is a pseudo-convex open set $\omega$ such that $\Omega-\tilde{\omega}$ is an open subset of $\Omega-\omega$.*

(ii) *For any $x\in\Omega$, $(x+G)\cap\tilde{\omega}\neq\varnothing$.*

*Then $R^k\Gamma_{\Omega-\tilde{\omega}}(\varphi_{G*}(\mathcal{O}|_{\Omega_G}))=0$ on $\Omega_G$ for $k\neq 1$.*

*Proof*. Since convex $G$-open sets $U$ in $\Omega$ form a base of open sets in $\Omega_G$, it is enough to show

$$H^k_{U-\tilde{\omega}}(U; \varphi_{G*}\mathcal{O}|_\Omega) = 0 \quad \text{for } k \neq 1.$$

By the preceding lemma, this cohomology group equals $H^k_{U-\tilde{\omega}}(U; \mathcal{O})$. Since $U-\tilde{\omega}$ is closed and open in $U-\omega$ we have

$$H^k_{U-\omega}(U; \mathcal{O}) = H^k_{U-\tilde{\omega}}(U; \mathcal{O}) \oplus H^k_{U\cap(\tilde{\omega}-\omega)}(U; \mathcal{O})$$

which is zero for $k \geqslant 2$, and hence we obtain $H^k_{U-\tilde{\omega}}(U; \mathcal{O}) = 0$ for $k \neq 0, 1$. Since $U \cap \tilde{\omega} \neq \emptyset$ (when $U \neq \emptyset$), $H^0(U; \mathcal{O}) \to H^0(U \cap \tilde{\omega}; \mathcal{O})$ is injective, which implies $H^0_{U-\tilde{\omega}}(U; \mathcal{O}) = 0$.

THEOREM 3.2.4. *Let $\Omega \supset \Omega_0$ be two $G$-open sets, and $D$ a relatively compact $G$-round open set containing $\Omega - \Omega_0$. Suppose that there is a pseudo-convex open set $\omega$ satisfying the following conditions:*

(i) $\omega \cap (\Omega - \Omega_0) = \emptyset$,

(ii) $(\omega + G) \cap \Omega \cap D \subset \omega$,

(iii) $\omega \supset \Omega_0 \cap \partial D$.

*Then $\mathbf{R}\Gamma_{\Omega-\Omega_0}(\varphi_{G*}\mathcal{O})$ is well-defined in the derived category of the abelian category of the sheaves of $\mathcal{E}(G; D)$-modules defined on $\Omega_G$.*

*Proof*. If $G = \{0\}$, then $\mathcal{E}(G; D)$ is nothing but the ring of the differential operators of infinite order defined on $D$ and $\Omega_G = \Omega$. Since $\mathcal{O}$ is an $\mathcal{E}(G; D)$-module, the theorem is evident. Suppose that $G \neq \{0\}$. Since $\Omega \cap D = [(\Omega \cap D) + G] \cap D$, we may assume that $(\Omega \cap D) + G = \Omega$. Set $\tilde{\omega} = (\Omega \cap D \cap \omega) + G = (\Omega_0 \cap D \cap \omega) + G$. It is obvious that $\tilde{\omega} \subset \Omega_0$.

First we shall show that $\tilde{\omega} \cup D \supset \Omega$. Suppose $x \in \Omega - D$. Then there is $y \in \Omega \cap D$ and $\gamma \in G$ such that $x = y + \gamma$. Put $t_0 = \inf\{t \geqslant 0;\ y + t\gamma \notin D\}$. Then, $1 \geqslant t_0 > 0$ and $y + t_0\gamma$ is in $\partial D$ and in $\Omega$, and hence by (iii), it belongs to $\omega$. Therefore, $y + t\gamma$ belongs to $\Omega \cap D \cap \omega$ for $0 < t_0 - t \ll 1$, which implies $x$ belongs to $\tilde{\omega}$.

We have

$$(\Omega - \omega) \cap D = \Omega - \tilde{\omega}.$$

In fact, we know already that $(\Omega - \omega) \cap D$ contains $\Omega - \tilde{\omega}$. It is therefore sufficient to prove $\tilde{\omega} \cap D \cap \Omega \subset \omega$, which is an immediate consequence of (ii).

We have also

$$(x + G) \cap \tilde{\omega} \neq \emptyset \quad \text{for any } x \in \Omega.$$

In fact, if $(x + G) \cap \tilde{\omega} = \emptyset$, then $x + G \subset \Omega - \tilde{\omega} \subset D \subset\subset \mathbf{C}^n$, which is a contradiction.

Thus, we may apply Lemma 3.2.3. Set $\mathcal{J}=R^1\Gamma_{\Omega-\tilde{\omega}}(\varphi_{G*}\mathcal{O}|_\Omega)=\mathbf{R}\Gamma_{\Omega-\tilde{\omega}}(\varphi_{G*}\mathcal{O}|_\Omega)$ [1]. Then $\mathcal{J}$ is a sheaf of $\mathcal{E}(G; D)$-modules by Theorem 3.2.1. Hence

$$\mathbf{R}\Gamma_{\Omega-\Omega_0}(\varphi_{G*}\mathcal{O}|_\Omega)=\mathbf{R}\Gamma_{\Omega-\Omega_0}(\mathcal{J})[-1]$$

is well-defined in the derived category of the sheaves of $\mathcal{E}(G; D)$-modules. This is independent of the choice of $\omega$. In fact, if there is another $\omega'$ which satisfies the same condition as $\omega$, then so is $\omega''=\omega\cap\omega'$. We define $\mathcal{J}'$ and $\mathcal{J}''$ in the same way as $\mathcal{J}$ by replacing $\omega$ with $\omega'$ and $\omega''$, respectively. Then it is clear that

$$\mathbf{R}\Gamma_{\Omega-\Omega_0}(\mathcal{J}')=\mathbf{R}\Gamma_{\Omega-\Omega_0}(\mathcal{J}'')=\mathbf{R}\Gamma_{\Omega-\Omega_0}(\mathcal{J}).$$

Q.E.D.

COROLLARY 3.2.5. *Let $D$ be a $G$-round open set and $x_0$ a point in $D$. Then, there exists a $G$-round open neighborhood $U$ of $x_0$ such that, for any $G$-open sets $\Omega_1\supset\Omega_0$ satisfying $\Omega_1-\Omega_0\subset U$, we can define naturally $\mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_{G*}\mathcal{O})$ in the derived category of the abelian category of sheaves of $\mathcal{E}(G; D)$-modules defined on $\Omega_G$.*

*Proof.* By Theorem 3.2.4, it is enough to show there exists a pseudo-convex open set $\omega$ such that $\omega$ is $G$-open and $\omega\cap U=\emptyset$, and that $\omega\supset\Omega_0\cap\partial D$.

By replacing $\Omega_0$ and $\Omega_1$ with $\Omega_0\cap(U+G)$ and $\Omega_1\cap(U+G)$, respectively, we may assume without loss of generality that $\Omega_1\subset U+G$. Then it is evident, that if $U$ is small enough, there exists a convex $G$-open set $\omega$ such that $\omega\cap U=\emptyset$ and $\omega\supset(U+G)\cap\partial D$.

**3.3.** In this section, we shall study the properties of $G$-open sets. First, we shall prove the following propositions, which say that the notion "$G$-open" is almost a local property. $G$ is a closed proper convex set containing $\{0\}$ in this section.

PROPOSITION 3.3.1. *Let $\Omega_1\supset\Omega_0$ be two open sets and $Z=\Omega_1-\Omega_0$. Consider the following conditions.*

(i) *$Z$ is $G$-locally closed (i.e. a difference of $G$-open subsets).*
(ii) *for any point $x$ in $Z$, $x+G^a\subset Z$ in a neighborhood of $x$.*
(iii) *for any point $x\notin Z$, $(x+G^a)\cap Z=\emptyset$ in a neighborhood of $x$.*
(iv) *$\Omega_0$ is $G$-open.*
(v) *$\Omega_1$ is $G$-open.*

*Then* (i) *implies* (ii) *and* (iii). *Under the condition of* (iv) (*resp.* (v)), (i), (ii) *and* (v) (*resp.* (i), (iii), *and* (iv)) *are equivalent.*

*Proof.* First let us show that (i) implies (ii) and (iii). We may assume that $\Omega_0$ and $\Omega_1$ are $G$-open.

(i) $\Rightarrow$ (ii): If $x \in Z$, then $x \notin \Omega_0$ and hence $(x+G^a) \cap \Omega_0 = \emptyset$, which implies $(x+G^a) \cap \Omega_1 \subset Z$.

(i) $\Rightarrow$ (iii): If $x \notin Z$, then $x \in \Omega_0$ or $x \notin \Omega_1$. If $x \in \Omega_0$, then $(x+G^a) \cap Z \cap \Omega_0 = \emptyset$. If $x \notin \Omega_1$, then $(x+G^a) \cap Z \subset (x+G^a) \cap \Omega_1 = \emptyset$.

Thus, it remains to prove that (iv) and (ii) implies (v) (resp. (v) and (iii) implies (iv)).

Let us prove first (iv)+(ii) $\Rightarrow$ (v). Let $x$ be a point in $\Omega_1$ and $\gamma \in G$. We shall show $x+\gamma \in \Omega_1$. Suppose that $x+\gamma \notin \Omega_1$ and we shall see the contradiction. Set $t_0 = \inf\{t \geq 0; x+t\gamma \notin \Omega_1\} > 0$. Then $y = x+t_0\gamma \notin \Omega_1$. Since $y$ does not belong to $Z$, $x+t\gamma$ does not belong to $Z$ for $0 < t_0 - t \ll 1$. For such $t$, $x+t\gamma \in \Omega_1$, and hence $x+t\gamma \in \Omega_0$, which implies $x+t_0\gamma \in \Omega_0$ because $x+t_0\gamma \in (x+t\gamma)+G$. Thus, $x+t_0\gamma \in \Omega_0 \subset \Omega_1$, which is a contradiction.

We shall prove (v)+(iii) $\Rightarrow$ (iv). Take $x \in \Omega_0$ and $\gamma \in G$. If $x+\gamma \notin \Omega_0$, take $t_0 = \inf\{t \geq 0; x+t\gamma \notin \Omega_0\} > 0$. Then $x+t_0\gamma \notin \Omega_0$. Since $x+t_0\gamma \in \Omega_1$, $x+t_0\gamma$ belongs to $Z$. Therefore $x+t\gamma \in Z$ for $0 < t_0 - t \ll 1$. Since $x+t\gamma \in \Omega_1$, $x+t\gamma \notin \Omega_0$, which is a contradiction. Q.E.D.

LEMMA 3.3.2. *Let $\Omega$ be a $G$-open set. Then, for any $x \in \bar{\Omega}$, $x + \operatorname{int} G \subset \Omega$.*

*Proof.* Let $\gamma \in \operatorname{int} G$. Then there is a neighborhood $U$ of 0 such that $U+\gamma \subset G$. Take $y$ in $(x+U^a) \cap \Omega$. Then, $x+\gamma$ is contained in $y+G \subset \Omega$.

LEMMA 3.3.3. *If $\Omega$ is a $G$-open set and if $\operatorname{int} G \neq \emptyset$, then $\Omega$ coincides with the interior of the closure of $\Omega$.*

This is an immediate consequence of the preceding lemma because $\overline{x+\operatorname{int} G}$ contains $x$.

LEMMA 3.3.4. *Let $D$ be a $G$-round open set and $\Omega$ an open subset of $D$. If, $C_x(\Omega) \cap G^a \subset \{0\}$ for any $x \in D - \Omega$, then $D \cap (\Omega + G) = \Omega$; i.e. $\Omega$ is an open set of $D_G$.*

*Proof.* Set $\tilde{\Omega} = (\Omega \cap D) + G$. Let $x$ be a point in $\tilde{\Omega} \cap D$. Then there are $y \in \Omega \cap D$ and $\gamma \in G$ such that $x = y+\gamma$. Set $y_t = y+t\gamma$. Since $D$ is $G$-round, $y_t$ belongs to $D$ for $0 \leq t \leq 1$. Now, we can apply the same argument as in Proposition 3.3.1 to prove that $x$ belongs to $\Omega$. Suppose that $x \notin \Omega$ and set $t_0 = \inf\{t \geq 0; y_t \notin \Omega\}$. Then, $y_{t_0} \notin \Omega$, $t_0 > 0$ and $y_t$ belongs to $\Omega$ for $t < t_0$, $\gamma = \lim_{t \nearrow t_0} (y_{t_0} - y_t)/(t_0 - t)$ belongs to $C_y(\Omega)^a$, which is a contradiction.

LEMMA 3.3.5. *Let $Z$ be a $G$-locally closed set. If $Z$ is open (in the usual topology), then $Z$ is $G$-open.*

*Proof.* Because $Z = Z - \emptyset$, we can apply Proposition 3.3.1.

**3.4.** In the preceding section, we investigated $G$-open sets. We introduce here a notion similar to $G$-openness which is free of the change of coordinates.

*Definition 3.4.1.* Let $M$ be a differentiable manifold of class $C^1$ and $S$ a subset of $M$. We denote $TM-C(M-S; S)$ by $N(S)$ and call it the strict normal cone of $S$.

PROPOSITION 3.4.2. (i) *$N(S)$ is an open convex cone.*

(ii) *When $M=\mathbf{R}^N$, $N_x(S)$ contains a vector $v$ if and only if there are an open convex cone $\Gamma$ containing $v$, a neighborhood $U$ of $x$ and a neighborhood $W$ of $0$ such that $(S\cap U)+(\Gamma\cap W)\subset S$.*

*Proof.* (ii) is evident by the definition. Let us prove (i). Let $v_1$ and $v_2$ be elements of $T_xM-C_x(M-S; S)$. Then, there are open cones $\Gamma_1$ and $\Gamma_2$, a neighborhood $W$ of $0$ and a neighborhood $U$ of $x$ such that $(U\cap S)+(\Gamma_j\cap W)\subset S$ and $\Gamma_j\ni v_j$, $(j=1, 2)$. We may assume that either $\overline{\Gamma_1+\Gamma_2}$ is a proper convex cone or $\Gamma_1\cap\Gamma_2^a\neq\emptyset$. Then, there is a neighborhood $U'$ of $x$ and neighborhoods $W'$ and $W''$ of $0$ such that $(\Gamma_1+\Gamma_2)\cap W''\subset(\Gamma_1\cap W')+(\Gamma_2\cap W')$, $U'+W'\subset U$ and $W'\subset W$. Then

$$(S\cap U')+(\Gamma_1+\Gamma_2)\cap W''\subset(S\cap U')+(\Gamma_1\cap W')+(\Gamma_2\cap W')\subset(S\cap U)+(\Gamma_2\cap W)\subset S.$$

Thus $v_1+v_2$ belongs to $T_xM-C_x(M-S; S)$. Q.E.D.

*Definition 3.4.3.* We call the dual cone $\{\theta\in T^*M;\ \langle\theta, v\rangle>0$ for any $v\in N(S)\}$, *the conormal cone* and denote it by $N^*(S)$.

*Example.* If $\Omega=\{x; f(x)>0\}$ for a $C^1$-function $f$ such that $df\neq0$, then $N_x(\Omega)=\{v\in T_xM;\ \langle v, df(x)\rangle>0\}$ and $N_x^*(\Omega)=\mathbf{R}^+df(x)$ for $x$ with $f(x)=0$.

*Example.* If $Z$ is a cone in $\mathbf{R}^N$,

$$N_0(Z)=\operatorname{int}\{v\in\mathbf{R}^N;\ v+Z\subset Z\}.$$

*Remark.* (I) $N_x(S)\neq T_xM$ if and only if $x\in\bar{S}-\operatorname{int}S$. (II) $N(M-S)=N(S)^a$.

*Definition 3.4.4.* Let $M$ be a differentiable manifold of class $C^1$ and $Q$ an open convex cone in $TM$. Set $Q(x)=Q\cap\tau^{-1}(x)$, where $\tau$ is the projection from $TM$ onto $M$.

(i) An open set $V$ is called *locally $Q$-flat at $x$* if $C_x(M-V, V)\cap Q(x)=\emptyset$, and called *$Q$-flat* if $V$ is locally $Q$-flat at any point.

(ii) A locally closed set $Z$ is called *$Q$-flat on an open set $U$*, if there are two open sets $\Omega_1$ and $\Omega_0$ which are locally $Q$-flat at any point in $U$ such that $U\cap(\Omega_1-\Omega_0)=U\cap Z$. If $Z$ is $Q$-flat on $X$, we say $Z$ is *$Q$-flat*.

**Proposition** 3.4.5. *Let $G$ be a proper closed convex cone in $\mathbf{R}^N$, $D$ a $G$-round open set and $\Omega$ an open set in $D$ and $Q$ an open convex cone in $T\mathbf{R}^N$.*

(i) *If $\Omega$ is $Q$-flat on $D$ and if $Q\supset D\times(G-\{0\})$, then $\Omega$ is an open set in $D_G$.*

(ii) *If $\Omega$ is open in $D_G$ and if $D\times G\supset\bar{Q}\cap\tau^{-1}(D)$, then $\Omega$ is $Q$-flat on $D$.*

*Proof.* (ii) is evident. (i) is an immediate consequence of Lemma 3.3.4. Q.E.D.

**Proposition** 3.4.6. *An open set $V$ is $Q$-flat if and only if $C_x(V)\cap Q(x)^a=\varnothing$* for any *$x\in M-V$.*

*Proof.* Since $C_x(V)\subset C_x(M-V;\,V)^a$, one implication is clear. Let us prove that $V$ is $Q$-flat if $C_x(V)\cap Q(x)^a=\varnothing$ for any $x\in M-V$. We may assume that $M=\mathbf{R}^N$. Suppose $v\in Q(x)$. Take a closed proper convex cone $G$ in $Q(x)\cup\{0\}$ such that $\operatorname{int}G\ni v$. Take a $G$-round open neighborhood $D$ of $x$ such that $Q(y)\supset G-\{0\}$ for $y\in D$. Then, by Lemma 3.3.4, $[(D\cap V)+G]\cap D=V\cap D$, which implies $G\cap C_x(\mathbf{R}^N-V,\,V)=\varnothing$. Q.E.D.

**Proposition** 3.4.7.1. *A union of $Q$-flat open sets is also $Q$-flat.*

*Proof.* Let $\{V_i\}$ be a family of $Q$-flat open sets and set $V=\cup V_i$. For $x\in M\subset\mathbf{R}^N$ and $v\in Q(x)$, take $G$ and $D$ as in the proof of the preceding proposition. Then, all $V_i\cap D$ are open in $D_G$, and hence so is $V\cap D$. This implies immediately $v\notin C_x(\mathbf{R}^N-V;\,V)$. Q.E.D.

**Proposition** 3.4.7.2. *Let $V_j$ be $Q$-flat open sets and $V$ the interior of the intersection of $V_j$'s. Then $V$ is also $Q$-flat.*

*Proof.* For a point $x\in M\subset\mathbf{R}^N$ and $v\in Q(x)$, we take $D$ and $G$ as in the proof of Proposition 3.4.6. Then, $V_j\cap D$ is open in $D_G$; i.e. $[(V_j\cap D)+G]\cap D\subset V_j$, which implies that $[(V\cap D)+G]\cap D\subset V_j$, which implies that $[(V\cap D)+G]\cap D\subset V$. Thus we obtain $C_x(\mathbf{R}^N-V,\,V)\not\ni v$. Q.E.D.

**Proposition** 3.4.8. *Let $Z$ be a locally closed set. If $Z$ is $Q$-flat, then*

(i) $C_y(Z)\cap Q(y)^a=\varnothing$ for $y\in M-Z$

(ii) $C_y(M-Z)\cap Q(y)^a=\varnothing$ for $y\in Z$.

*Proof.* Suppose that $Z$ is $Q$-flat. Then $Z=V_1-V_0$ in a neighborhood of $x$, where $V_1$ and $V_0$ are $Q$-flat in a neighborhood of $x$. Therefore, for $y\in U\cap Z$, $Z=M-V_0$ in a neighborhood of $y$ and hence $C_y(M-Z)=C_y(V_0)\subset C_y(V_0,\,M-V_0)$, which implies $C_y(M-Z)\cap Q(y)^a=\varnothing$. For $y\in U-Z$, if $y\in V_1$ then $y\in V_0$ and hence $Z=\varnothing$ in a neighborhood of $y$ and hence $C_y(Z)=\varnothing$. If $y\notin V_1$, then $C_y(Z)\subset C_y(V_1,\,M-V_1)$, which implies $C_y(Z)^a\cap Q(y)=\varnothing$.

COROLLARY 3.4.9. *Let $Z$ be a closed set. If $Z$ is $Q$-flat, then $M-Z$ is $Q$-flat.*

*Proof.* Set $V=M-Z$. Then we have $C_y(V)\cap Q(y)^a=\emptyset$ for $y\in M-Z$ by Proposition 3.4.8, and hence we can apply Proposition 3.4.6 to show that $V$ is $Q$-flat. Q.E.D.

PROPOSITION 3.4.10. *Let $V_1$ and $V_0$ be two open sets in $M$ such that $V_1\supset V_0$. Suppose that $V_1-V_0$ is $Q$-flat on a neighborhood of $x$. Then, if one of $V_j$ is $Q$-flat in a neighborhood of $x$, then so is the other.*

*Proof.* Suppose that $V_0$ is $Q$-flat. Then for $y\notin V_1$, we have $C_y(V_1)\subset C_y(V_1-V_0)\cup C_y(V_0)$. Hence $C_y(V_1)\cap Q(y)^a=\emptyset$, which implies that $V_1$ is $Q$-flat. Conversely, suppose that $V_1$ is $Q$-flat. For $y\notin V_1$, $C_y(V_0)\subset C_y(V_1)$. For $y\in V_1-V_0$, $C_y(V_0)\subset C_y(M-(V_1-V_0))$. Thus, we have $C_y(V_0)\cap Q(y)^a=\emptyset$ for $y\notin V_0$. Q.E.D.

Let $G$ be a proper closed cone in $\mathbf{C}^n$ and $Q$ an open cone in $T\mathbf{C}^n=\mathbf{C}^n\times\mathbf{C}^n$ containing $\mathbf{C}^n\times(G-0)$. Then the following proposition is an immediate consequence of Proposition 3.3.1, Lemma 3.3.2, Lemma 3.3.4, and Proposition 3.3.10.

PROPOSITION 3.4.11. (i) *A $Q$-flat open set is $G$-open.*

(ii) *Let $\Omega_1$ and $\Omega_0$ be two open sets such that $\Omega_1\supset\Omega_0$. If $\Omega_1-\Omega_0$ is $Q$-flat and if one of them is $G$-open, then so is the other.*

(iii) *For $x$, there is an open neighborhood $U$ of $x$ such that, for any open set $\Omega$ which is $Q$-flat on $U$, $\Omega$ contains $[x+(G-\{0\})]\cap U$ if $x$ is contained in $\overline{\Omega}$.*

## § 4. Prolongation theorem in the complex domain

**4.1.** Let $G$ be a closed convex proper cone in $\mathbf{C}^n$ containing $0$, $D$ a relatively compact $G$-round open set. Let $M^\cdot$ be a bounded complex of free $\mathcal{E}(G;D)$-modules of finite rank (i.e. $M^k=0$ for $k>0$ or $k<0$ and all $M^k$ are free $\mathcal{E}(G;D)$-modules of finite rank). We shall investigate sufficient conditions such that

$$\mathrm{Ext}^j(M^\cdot;\mathbf{R}\Gamma_{\Omega-\Omega_0}(\Omega;\varphi_{G*}(\mathcal{O})))=0.$$

THEOREM 4.1.1. *Let $D$, $G$ and $M^\cdot$ be as above and let $\{\Omega_t\}_{0\leq t\leq 1}$ be a family of open sets in $X=\mathbf{C}^n$, We assume the following conditions.*

(a) *There is an open convex cone $R$ in $TD$ such that, for any $x\in D$, $R(x)$ is non empty and contains $G-\{0\}$ and that either $\Omega_0$ or $\Omega_1$ is $R$-flat on $D$.*

(b) *$\overline{\Omega_1-\Omega_0}$ is a compact set contained in $D$.*

(c) *There is a pseudo-convex open set $\omega$ satisfying*

($c_1$) $\omega \cap (\Omega_1 - \Omega_0) = \varnothing$

($c_2$) $(\omega + G) \cap \Omega_1 \subset \omega$

($c_3$) $\overline{\Omega}_0 \cap \partial D \subset \omega$

(d) $\Omega_{t_0} = \bigcup_{t<t_0} \Omega_t$ *for any* $t_0$ *such that* $0 < t_0 \leqslant 1$ *and* $\overline{\Omega}_{t_0} \supset \bigcup_{t>t_0} \Omega_t$ *for any* $t_0$ *such that* $0 \leqslant t_0 < 1$.

(e) *There is an open convex cone* $Q$ *in* $T\mathbf{C}^n$ *satisfying the following conditions*:

($e_0$) $Q(x)$ *is not empty and contains* $G - \{0\}$ *for* $x \in D$.

($e_1$) $\Omega_{t_1} - \Omega_{t_0}$ *is* $Q$*-flat on a neighborhood of any point in* $\overline{\Omega_1 - \Omega_0}$ *for any* $0 \leqslant t_0 \leqslant t_1 < 1$.

($e_2$) $\mathcal{E}^{\mathbf{R}}_p \otimes_{\mathcal{E}(G;D)} M^{\cdot}$ *is exact for any* $p = (x, \xi) \in T^*D$ *with* $\langle \xi, Q(x) \rangle < 0$.

*Then all* $\Omega_t \cap D$ *are open in* $D_G$ *and we have*

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}(G;D)}(M^{\cdot};\, \mathbf{R}\Gamma_{\Omega_t - \Omega_0}(\Omega_t;\, \varphi_{G*}\mathcal{O})) = 0$$

*for* $0 \leqslant t \leqslant 1$.

The proof of this theorem is rather long and ends at § 4.4.

$\Omega_1 \cap D$ and $\Omega_0 \cap D$ are open in $D_G$ by Proposition 3.4.11 and (a). Applying the same proposition $\Omega_t$ is open in $D_G$ by (e). Therefore, by replacing $\Omega_t$ with $(\Omega_t \cap D) + G$, we may assume that (a′) $\Omega_t$ are $G$-open sets and ($e_0'$) $Q \supset R$. Condition (c) assures that $\mathbf{R}\Gamma_{\Omega_t - \Omega_0}(\varphi_{G*}\mathcal{O}))$ is well-defined by Theorem 3.2.4.

LEMMA 4.1.2. *For any* $0 \leqslant t_0 < 1$, $\bigcap_{t>t_0} \overline{\Omega}_t$ *is contained in* $\overline{\Omega}_{t_0}$.

*Proof*. Take $x \in \cap \overline{\Omega}_t$. Since $\overline{\Omega}_t - D = \overline{\Omega}_{t_0} - D$, $x$ is contained in $\overline{\Omega}_{t_0}$ if $x \notin D$. Suppose that $x \in D$. Take a proper closed convex cone $G_1$ such that $G_1 \subset R(x) \cup \{0\}$ and that $\mathrm{int}\, G_1 \neq \varnothing$. There is a $G_1$-round open neighborhood $D_1$ such that $R(y) \supset G_1 - \{0\}$ for any $y \in D_1$. Then, all $\Omega_t \cap D$ are open in $D_{1G_1}$. Therefore, we have $(x + \mathrm{int}\, G_1) \cap D_1 \subset \Omega_t$ which implies $(x + \mathrm{int}\, G_1) \cap D_1 \subset \overline{\Omega}_{t_0}$. Thus we obtain $x \in \overline{\Omega}_{t_0}$. Q.E.D.

**4.2.** Set
$$K_{t_0} = \bigcap_{t>t_0} \overline{\Omega_t - \Omega_{t_0}} \quad \text{for } 0 \leqslant t_0 < 1.$$

LEMMA 4.2.1. *For any point* $x$ *in* $K_{t_0}$ *and any neighborhood* $U$ *of* $x$, $U - \Omega_{t_0}$ *is a neighborhood of* $x$ *in* $(X - \Omega_{t_0})_G$.

*Proof*. Take $G_1$ and $D_1$ as in the proof of Lemma 4.1.2. Moreover, we assume that $\mathrm{int}\, G_1 \supset G - \{0\}$. Then, $\Omega_{t_0}$ contains $[D_1 \cap (x + \mathrm{int}\, G_1)] + G = W$. It is evident that there is an open neighborhood $V$ of $x$ such that $(V + G) - W \subset U$. Then $(V + G) - \Omega_{t_0}$ is a neighborhood of $x$ in $(X - \Omega_{t_0})_G$ and is contained in $U$. Q.E.D.

This lemma immediately implies the following:

LEMMA 4.2.2. (i) *The topology on $K_{t_0}$ induced from G-topology coincides with the usual topology.*

(ii) *$K_{t_0}$ is relatively separated in $(X-\Omega_{t_0})_G$ (i.e. for $x\neq y$ in $K_{t_0}$, there are neighborhoods $U$, $V$ of $x$, $y$ in $(X-\Omega_{t_0})_G$ such that $U\cap V=\emptyset$). In particular, for a sheaf $\mathcal{L}$ on $(X-\Omega_{t_0})_G$, $H^k(K_{t_0};\mathcal{L})\leftarrow\!\!\!\sim\varinjlim_U H^k(U;\mathcal{L})$ where $U$ runs on a neighborhood system of $K_{t_0}$.*

Let $j_t$ be the inclusion map

$$(\Omega_t-\Omega_{t_0})_G\to(X-\Omega_{t_0})_G.$$

Let $T$ be the functor $\mathcal{L}\mapsto\varinjlim_{t>t_0} j_{t*}j_t^{-1}\mathcal{L}$ from the category of the sheaves on $(\Omega_1-\Omega_0)_G$ into the category of the sheaves on $(X-\Omega_0)_G$.

LEMMA 4.2.3. *For any complex $\mathcal{L}^\cdot$ of sheaves on $(\Omega_1-\Omega_{t_0})_G$, we have*

$$\text{(i)}\qquad \varinjlim_{t>t_0} H^k(\Omega_t-\Omega_{t_0};\mathcal{L}^\cdot)\xrightarrow{\sim} H^k(K_{t_0};\mathbf{R}T_*(\mathcal{L}^\cdot))$$

$$\text{(ii)}\qquad \varinjlim_{t>t_0} R^kj_{t*}(j_t^{-1}\mathcal{L}^\cdot)_x\xrightarrow{\sim} R^kT_*(\mathcal{L}^\cdot)_x$$

*for any $x\in K_{t_0}$.*

*Proof.* First observe that any open set $U$ in $(X-\Omega_{t_0})_G$ containing $K_{t_0}$ contains $\Omega_t-\Omega_{t_0}$ for some $t>t_0$. We may assume that $\mathcal{L}^\cdot$ is a complex of injective sheaves. Then

$$\begin{aligned}\varinjlim_{t>t_0} H^k(\Omega_t-\Omega_{t_0};\mathcal{L}^\cdot)&\simeq\varinjlim_{t>t_0} H^k(\Gamma(\Omega_t-\Omega_{t_0};\mathcal{L}^\cdot))\\&\simeq H^k(\varinjlim_{t>t_0}\Gamma(\Omega_t-\Omega_{t_0};\mathcal{L}^\cdot))\\&\simeq H^k(\varinjlim_{U\supset K_{t_0}}\varinjlim_{t>t_0}\Gamma(U\cap(\Omega_t-\Omega_{t_0});\mathcal{L}^\cdot))\\&\simeq H^k(\varinjlim_{U\supset K_{t_0}}\varinjlim_{t>t_0}\Gamma(U;j_{t*}j_t^{-1}\mathcal{L}^\cdot))\\&\simeq H^k(\varinjlim_{t>t_0}\Gamma(K_{t_0};j_{t*}j_t^{-1}\mathcal{L}^\cdot))\\&\simeq H^k(\Gamma(K_{t_0};\varinjlim_{t>t_0}j_{t*}j_t^{-1}\mathcal{L}^\cdot))\\&\simeq H^k(\Gamma(K_{t_0};T\mathcal{L}^\cdot)).\end{aligned}$$

Thus (i) is proved.

(ii) is proved in the same way. Q.E.D.

PROPOSITION 4.2.4. *If we have*

$$\varinjlim_{t>t_0}\varinjlim_{U\ni x}\mathrm{Ext}^i(M^\cdot;\mathbf{R}\Gamma_{U-\Omega_{t_0}}(U\cap\Omega_t;\varphi_{G*}\mathcal{O}))=0$$

*for any $i$, $0\leq t_0<1$ and $x\in K_{t_0}$, then the conclusion of Theorem 4.1.1 holds. Here, $U$ runs on a neighborhood system of $x$.*

*Proof.* Setting $\mathcal{L}^{\cdot}=\mathbf{R}\operatorname{Hom}(M^{\cdot};\mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_{G*}\mathcal{O}))$, we have $R^kT(\mathbf{R}\Gamma_{\Omega_1-\Omega_{t_0}}(\mathcal{L}^{\cdot}))|_{K_{t_0}}=0$ by the preceding lemma. The same lemma implies also $\varinjlim_{t>t_0}H^k_{\Omega_t-\Omega_{t_0}}(\Omega_t;\mathcal{L}^{\cdot})=0$, or equivalently the homomorphisms

$$\varinjlim_{t>t_0} H^k(\Omega_t;\mathcal{L}^{\cdot})\to H^k(\Omega_{t_0};\mathcal{L}^{\cdot})$$

are isomorphisms.

Note the following lemma.

LEMMA 4.2.5. *Let $X$ be a topological space $\{\Omega_n\}_{n\in Z}$ an increasing sequence of open sets in $X$ such that $X=\cup\,\Omega_n$, and $\mathcal{F}^{\cdot}$ a complex of sheaves on $X$. Then*

(i) *$\varphi_k$: $H^k(X;\mathcal{F}^{\cdot})\to\varprojlim H^k(\Omega_n;\mathcal{F}^{\cdot})$ are surjective for any $k$.*

(ii) *If $\{H^{k-1}(\Omega_n;\mathcal{F}^{\cdot})\}_n$ satisfies the condition of Mittag-Leffler, then $\varphi_k$ is bijective.*

For the Mittag-Leffler condition and for the proof, see [15], [26].

Let us prove that $H^k(\Omega_{t_1};\mathcal{L}^{\cdot})\to H^k(\Omega_{t_0};\mathcal{L}^{\cdot})$ are bijective for $0\leqslant t_0\leqslant t_1\leqslant 1$ by induction on $k$. $H^{k-1}(\Omega_{t_1};\mathcal{L}^{\cdot})\to H^{k-1}(\Omega_{t_0};\mathcal{L}^{\cdot})$ are bijective for $0\leqslant t_0\leqslant t_1\leqslant 1$. Then $\{H^{k-1}(\Omega_t;\mathcal{L}^{\cdot})\}_{t<t_0}$ satisfies the condition of Mittag-Leffler for $1\geqslant t_0>0$, and hence, by the preceding lemma.

$$H^k(\Omega_{t_0};\mathcal{L}^{\cdot})\xrightarrow{\sim}\varprojlim_{t<t_0} H^k(\Omega_t;\mathcal{L}^{\cdot}).$$

Thus, Proposition 4.2.4 is an immediate consequence of the following lemma.

LEMMA 4.2.6. *Let $\{V_t\}_{0\leqslant t\leqslant 1}$ be a family of abelian groups, $\varrho_{t,t'}$ a homomorphism from $V_{t'}$ to $V_t$ for $1\geqslant t'\geqslant t\geqslant 0$. Suppose that*

(i) *$\varrho_{t,t'}\circ\varrho_{t',t''}=\varrho_{t,t''}$ for $1\geqslant t''\geqslant t'\geqslant t\geqslant 0$,*

(ii) *$V_{t_0}\to\varprojlim_{t<t_0}V_t$ is bijective for $0<t_0\leqslant 1$,*

(iii) *$\varinjlim_{t>t_0}V_t\to V_{t_0}$ is bijective, for $0\leqslant t_0<1$.*

*Then all $\varrho_{t,t'}$ are bijective.*

**4.3.** In order to show that the condition in Proposition 4.2.4 is verified, we investigate the following special case.

Let $G$ be a closed convex proper cone with non empty interior, $f(x)$ a linear form ($\mathbf{R}$-valued) such that $f(x)>0$ for $x\in G-\{0\}$. Let $\Omega=\operatorname{int}G$ and $\omega=\{x\in\Omega;\ f(x)>1\}$. Let $D$ be a $G$-round open set containing $\Omega-\omega$ and $M^{\cdot}$ a bounded complex of free $\mathcal{E}(G;D)$-modules of finite rank. Set $\mathcal{F}^{\cdot}=\operatorname{Hom}(M^{\cdot};\mathbf{R}\Gamma_{\Omega-\omega}(\varphi_{G*}\mathcal{O}))$.

PROPOSITION 4.3.1. *Suppose that $\mathcal{E}^{\mathbf{R}}_p\otimes M^{\cdot}$ is exact for any $p=(x,\xi)$ with $x\in D$, $\langle\xi,G\rangle\leqslant 0$, $\xi\neq 0$. Then*

$$\mathbf{R}\operatorname{Hom}(M^{\cdot};\mathbf{R}\Gamma_{\Omega-\omega}(\varphi_{G*}\mathcal{O}))=0.$$

*Proof.* Since $x+\mathrm{int}\, G$ $(x\in\Omega)$ forms a base of open sets in $\Omega_G$, it is enough to show that $\mathbf{R}\Gamma(x+\mathrm{int}\, G; \mathcal{J}^{\cdot})=0$. Replacing $x+\mathrm{int}\, G$ with $\Omega$, the proposition is a consequence of the following propositions.

PROPOSITION 4.3.2. *$H^k(\Omega; \mathcal{J}^{\cdot})=0$ for any $k$.*

*Proof.* We will need the lemmas 4.3.3–4.3.7.

First fix a point $x_0$ in $\Omega$ such that $f(x_0)=1$. For $\varepsilon>0$, we set

$$U_\varepsilon=\{(s_1, s_2)\in\mathbf{R}^2;\ s_1<0\}\cup\{(s_1, s_2)\in\mathbf{R}^2;\ s_2<\varepsilon\}$$
$$\cup\{(s_1, s_2)\in\mathbf{R}^2;\ 0\leqslant s_1<\varepsilon,\ 0\leqslant s_2<2\varepsilon,\ (\varepsilon-s_1)^2+(2\varepsilon-s_2)^2>\varepsilon^2\}.$$

$U_\varepsilon$ is an open set with $C^1$-boundary and $U_\varepsilon+\{(s_1, s_2);\ s_1\leqslant 0,\ s_2\leqslant 0\}=U_\varepsilon$. Set $\delta(x, G)=\min\{|x-y|;\ y\in G\}$. Then, we have

$$\delta(x, G)^2=\delta(x_1, G)^2+2(x-x_1, x_1-y)+o(|x_1-x|)$$

with $y\in G$ such that $\delta(x_1, G)=|x_1-y|$. Therefore, $\delta(x, G)$ is a continuous function on $X$ and $C^1$ on $X-G$. Moreover $d\delta(x, G)\in G^{0a}$ for $x\notin G$. Set $\delta_t(x)=\delta(x-(1-t)x_0, G)=\delta(x, (1-t)x_0+G)$. Then, $\delta_t(x)$ is continuous on $(t, x)$ and we have $\delta_t(x)\leqslant\delta_{t'}(x)$ for $t\geqslant t'$, $\delta_t(x)<\delta_{t'}(x)$ for $t>t'$ and $x\notin(1-t')x_0+G$. There is a constant $c>0$ such that

$$\{x;\ \delta_t(x)<2\varepsilon\}\subset\mathrm{int}\, G\quad\text{for } \varepsilon>0 \text{ and } t<1-c\varepsilon.$$

Set

$$\Omega_t=\{x\in\mathrm{int}\, G;\ (1-f(x), \delta_t(x))\in U_\varepsilon\}=\omega\cup\{x;\ f(x)\geqslant 1,\ (1-f(x), \delta_t(x))\in U_\varepsilon\}$$

for $t<1-c\varepsilon$.

We have $\Omega_t\subset\omega$ for $t<-c\varepsilon$.

LEMMA 4.3.3. (i) $\Omega_{t_0}=\bigcup_{t<t_0}\Omega_t$ and $\overline{\Omega}_{t_0}\supset\bigcap_{t>t_0}\Omega_t$,

(ii) *$\Omega_t$ are $G$-open*

(iii) $\Omega_t+\lambda x_0\subset\Omega_{t-\lambda}$ *for $\lambda>0$.*

*Proof.* First let us prove (ii). For $\gamma\in G-\{0\}$

$$(1-f(x+\gamma), \delta_t(x+\gamma))=(1-f(x), \delta_t(x))+(-f(\gamma), \delta_t(x+\gamma)-\delta_t(x)).$$

Since $-f(\gamma)$, $\delta_t(x+\gamma)-\delta_t(x)$ are non positive, $(1-f(x+\gamma), \delta_t(x+\gamma))$ is in $U_\varepsilon$ if so is $(1-f(x), \delta_t(x))$. (iii) is clear. The relation $\Omega_{t_0}=\cup\,\Omega_t$ is also clear. Let us prove that $\bigcap_{t>t_0}\Omega_t$ is contained in $\overline{\Omega}_{t_0}$. If $x\in\cap\,\Omega_t$, then $x+(t-t_0)x_0$ is contained in $\Omega_{t_0}$ for any $t>t_0$. Thus $x$ is contained in $\overline{\Omega}_{t_0}$. Q.E.D.

LEMMA 4.3.4. *$\overline{\Omega_{t_1}-\Omega_{t_0}}\subset\Omega_{t_2}\cap\{x; f(x)\leqslant 1\}$ for $t_0<t_1<t_2<1-\varepsilon$.*

In fact, if $x\in\Omega_{t_1}-\Omega_{t_0}$ we have $f(x)\leqslant 1$ and hence $\Omega_{t_1}-\Omega_{t_0}$ is contained in $\{x; f(x)\leqslant 1, \delta_{t_1}(x)<2\varepsilon\}$. We have therefore

$$\begin{aligned}\overline{\Omega_{t_1}-\Omega_{t_0}}&\subset\{x\in G; f(x)\leqslant 1, \delta_{t_1}(x)\leqslant 2\varepsilon, (1-f(x), \delta_{t_1}(x))\in\bar{U}_\varepsilon\}\\&=\{x\in G; f(x)\leqslant 1, 0<\delta_{t_1}(x)\leqslant 2\varepsilon, (1-f(x), \delta_{t_1}(x))\in\bar{U}_\varepsilon\}\\&\cup\{x\in(1-t_1)x_0+G; f(x)\leqslant 1\}.\end{aligned}$$

Since $\delta_{t_2}(x)<\delta_{t_1}(x)$ for $x$ such that $\delta_{t_1}(x)>0$, and since $\delta_{t_1}(x)\leqslant 2\varepsilon$ implies $x\in\operatorname{int}G$, $\overline{\Omega_{t_1}-\Omega_{t_0}}$ is contained in $\Omega_{t_2}$. Q.E.D.

LEMMA 4.3.5. *$K_{t_0}=\bigcap_{t>t_0}\overline{\Omega_t-\Omega_{t_0}}$ coincides with $\bigcap_{t>t_0}(\Omega_t-\Omega_{t_0})$. Moreover, $K_{t_0}$ is contained in $\partial\Omega_{t_0}$ and $\partial\Omega_{t_0}$ is a $C^1$-manifold in a neighborhood of $K_{t_0}$.*

In fact, $\partial\Omega_{t_0}\cap\Omega=\{x\in\Omega; (1-f(x), \delta_{t_0}(x))\in\partial U_\varepsilon\}$, any positive linear combination of $d(1-f(x))$, and $d\delta_{t_0}(x)$ does not vanish for $x\notin(1-t_0)x_0+G$. Therefore, $\partial\Omega_{t_0}\cap\Omega$ is a $C^1$-manifold.

LEMMA 4.3.6. *For $x\in\Omega\cap\partial\Omega_{t_0}$, we have*

$$\operatorname{Ext}^j(M^\cdot; \mathbf{R}\Gamma_{\Omega-\Omega_{t_0}}(\varphi_{G*}\mathcal{O}))_x=0.$$

*Proof.* Note that $\partial\Omega_{t_0}$ is a $C^1$-manifold at $x$ whose conormal $p=(x, \xi)$ is contained in the antipodal of the polar of $G$. Consider the following spectral sequence

$$E_2^{pq}=H^p(\operatorname{Hom}(M^\cdot; \mathcal{H}^q_{\Omega-\Omega_{t_0}}(\mathcal{O})_x))\underset{p}{\Rightarrow}H^{p+q}(\operatorname{Hom}(M^\cdot; \mathbf{R}\Gamma_{\Omega-\Omega_{t_0}}(\varphi_{G*}\mathcal{O})_x)).$$

Since $\mathcal{H}^q_{\Omega-\Omega_{t_0}}((\mathcal{O})_x$ is an $\mathcal{E}_p^{\mathbf{R}}$-module and $\mathcal{E}_p^{\mathbf{R}}\otimes M^\cdot$ is exact, $E_2^{pq}=0$ for all $p, q$ and hence $H^j(\operatorname{Hom}(M^\cdot; \mathbf{R}\Gamma_{\Omega-\Omega_{t_0}}(\varphi_{G*}\mathcal{O})_x))=0$. Q.E.D.

LEMMA 4.3.7. *$H^j(\Omega_{1-c\varepsilon}; \mathcal{F}^\cdot)\to H^j(\omega; \mathcal{F}^\cdot)$ are bijective.*

*Proof.* The conditions in Theorem 4.1.1 are all satisfied, we can apply Proposition 4.2.4, and therefore it is enough to show that, for all $x\in K_{t_0}$,

$$\varinjlim_{t>t_0}\varinjlim_{U}H^j_{U-\Omega_{t_0}}(U_t; \mathcal{F}^\cdot)\quad\text{vanishes,}$$

where $U$ runs on a neighborhood system of $x$. Since $x$ is contained in $\Omega_t$ for $t>t_0$, these cohomology groups coincide with $H^j(\operatorname{Hom}(M^\cdot; \mathbf{R}\Gamma_{\Omega-\Omega_{t_0}}(\varphi_{G*}\mathcal{O})_x))$, which is zero by the preceding lemma. Q.E.D.

Thus, if we set

$$\Omega_\varepsilon = \{x \in \Omega;\ (1-f(x),\ \delta(x,\ G+c\varepsilon x_0) \in U_\varepsilon\},$$

we have an isomorphism

$$H^j(\Omega_\varepsilon;\ \mathcal{J}^\cdot) \xrightarrow{\sim} H^j(\omega;\ \mathcal{J}^\cdot).$$

$\Omega = \bigcup_{\varepsilon>0} \Omega_\varepsilon$ and, for any $\varepsilon_1, \varepsilon_2 > 0$ there exists $\varepsilon > 0$ such that $\Omega_\varepsilon \supset \Omega_{\varepsilon_1} \cup \Omega_{\varepsilon_2}$. Therefore $H^j(\Omega;\ \mathcal{J}^\cdot) \xrightarrow{\sim} \varprojlim_\varepsilon H^j(\Omega_\varepsilon;\ \mathcal{J}^\cdot)$ if $\{H^{j-1}(\Omega_\varepsilon;\ \mathcal{J}^\cdot)\}$ satisfies the condition of Mittag-Leffler (Lemma 4.2.5), which is obviously satisfied. Thus, we obtain

$$H^j(\Omega;\ \mathcal{J}^\cdot) \xrightarrow{\sim} H^j(\omega;\ \mathcal{J}^\cdot).$$

This completes the proof of Proposition 4.3.2 and hence the proof of Proposition 4.3.1.

**4.4.** Now, let us return to the proof of Theorem 4.1.1. We already observed in Proposition 4.2.4 that it is enough to show

$$\varinjlim_U \mathrm{Ext}^k(M^\cdot;\ \mathbf{R}\Gamma_{U-\Omega_{t_0}}(U \cap \Omega_t;\ \varphi_{G*}\mathcal{O})) = 0$$

for $x_0 \in K_{t_0}$ and $t > t_0$, where $U$ runs on a system of neighborhoods of $x_0$. Let $U$ be a neighborhood of $x_0$. By shrinking $U$, we may assume that $\mathcal{E}_p^{\mathbf{R}} \otimes M^\cdot$ is exact for $p = (x, \xi)$ with $x \in U$ and $\xi \neq 0$ such that $\langle \xi, G_1 \rangle \leq 0$ for some closed convex proper cone $G_1$ contained in $Q(x_0) \cup \{0\}$. Since $\Omega_t - \Omega_{t_0}$ is $Q$-flat, for any $y$ sufficiently near $x_0$, $U \cap (y + \mathrm{int}\, G_1) \cap (\Omega_t - \Omega_{t_0}) \subset\subset U$. Set $V = y + \mathrm{int}\, G_1$ and take $f(x)$ such that $\{x \in \bar{V};\ f(x) \geq f(x_0) - 1\} \subset U$. Set $\omega = \{x \in V;\ f(x) < f(x_0) - 1\}$. Then, by Proposition 4.3.1, we have

$$\mathbf{R}\,\mathrm{Hom}\,(M^\cdot;\ \mathbf{R}\Gamma_{V-\omega}(\varphi_{G_1*}\mathcal{O}|_V)) = 0.$$

In particular, we have

$$H^k(\Omega_t \cap V - \omega;\ \mathbf{R}\,\mathrm{Hom}\,(M^\cdot;\ \mathbf{R}\Gamma_{(V-\omega)-\Omega_{t_0}}(\varphi_{G*}\mathcal{O})) = 0$$

for $V = y + \mathrm{int}\, G_1$.

Note that, if $y \in x + (\mathrm{int}\, G_1)^a$, $V$ is a neighborhood of $x_0$. Thus, taking an inductive limit, we obtain

$$\varinjlim_U \mathrm{Ext}^k(M^\cdot;\ \mathbf{R}\Gamma_{U \cap (\Omega_t - \Omega_{t_0})}(U \cap \Omega_t;\ \varphi_{G*}\mathcal{O})) = 0.$$

This completes the proof of Theorem 4.1.1.

**4.5.** We reformulate Theorem 4.1.1 in the following way, which we use in the later sections.

THEOREM 4.5.1. *Let $G$, $D$ and $M^{\cdot}$ be as in § 4.1, and $x_0$ a point in $D$. Then, there is an open neighborhood $U$ of $x_0$ satisfying the following property: let $\Omega_1$ and $\Omega_0$ be two open sets satisfying the following conditions:*

(a) $\Omega_1 \supset \Omega_0$ *and* $\Omega_1 - \Omega_0 \subset\subset U$.

(b) *There is an open convex cone $R$ in $TU$ such that $R \supset D \times (G - \{0\})$, the projection $R \to U$ is surjective and $\Omega_1$ and $\Omega_0$ are $R$-flat on $U$.*

(c) *There are an open convex cone $Q$ of $TX$ and a $C^1$-function $g$ on $U$ with*

 ($c_1$) $\{v; \langle v, dg(x)\rangle > 0\} \supset Q(x) \supset G - \{0\}$ *for any* $x \in U$.

 ($c_2$) $\Omega_1 - \Omega_0$ *is $Q$-flat in a neighborhood of* $\overline{\Omega_1 - \Omega_0}$.

 ($c_3$) $\mathcal{E}_p^{\mathbf{R}} \otimes M^{\cdot}$ *is exact for $p = (x, \xi)$ such that $x \in U$ and* $\langle \xi, Q(x)\rangle < 0$.

*Then $\Omega_1 - \Omega_0$ is locally closed in $D_G$ and we have*

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}(G;D)}(M^{\cdot}; \mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_{G*}\mathcal{O})) = 0.$$

*Proof.* We may assume that $U$ is $D$-round and so small that Corollary 3.2.5 holds. Thus, the conditions (a), (b), (c) in Theorem 4.1.1 hold. Now, we may assume that $0 < g(x) < 1$ on $\overline{\Omega_1 - \Omega_0}$. Set $\Omega_t = \Omega_0 \cup \{x \in \Omega_1 \cap U;\ g(x) < t\}$. Then, it is evident that conditions (d) and (e) in Theorem 4.1.1 are satisfied. Thus, we can apply Theorem 4.1.1 and we obtain

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}(G;D)}(M^{\cdot}; \mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_{G*}\mathcal{O})) = 0.$$

## § 5. Micro-Hyperbolic systems on the boundary of an open set of a complex analytic manifold

**5.1.** Let $X$ be a complex manifold of dimension $n$, $S$ a real hypersurface of class $C^1$ and $\Omega^+$ a pseudo-convex open set with $S$ as its boundary. We define the sheaf $\mathcal{C}_S$ on $T_S^*X$ by

$$\mathcal{C}_S = \mathbf{R}\Gamma_{T_S^*X}(\pi^{-1}\mathcal{O}_S)^a[1],$$

where, as usual, $\pi$ denotes the projection of $(X - S) \coprod T_S^*X$ onto $X$, the first space being endowed with the topology of the comonoidal transformation. Suppose that $\Omega^+$ is given by

$$\Omega^+ = \{x \in X;\ s(x) > 0\}$$

for a differentiable function $s(x)$ of class $C^1$ with $ds(x) \neq 0$ on $S$. We call the conormal of $\Omega^+$ at $x \in S$, a covector of the form $\lambda ds(x)$ with $\lambda > 0$. We are just interested in the restriction of the sheaf $\mathcal{C}_S$ to the negative part $(T_S^*X)^-$ of $T_S^*X$ (i.e. $(T_S^*X)^- = \{a\,ds(x);\ a < 0,\ x \in S\}$)

and we denote this restriction by $\mathcal{C}_S^-$. As the sheaf $\mathcal{C}_S^-$ is locally constant on the orbit of the action of $\mathbf{R}^+$, we can regard $\mathcal{C}_S^-$ as a sheaf on $S$ and we have $\mathcal{C}_S^- = \mathbf{R}\Gamma_{X-\Omega^+}(\mathcal{O}_X)|_S$ [1]. If we denote by $j$ the inclusion of $\Omega^+$ in $X$, $\mathcal{C}_S^-$ is the restriction of $(j_*\mathcal{O})/\mathcal{O}$ to $S$; that is, the sheaf $\mathcal{C}_S^-$ is the sheaf of boundary values on $S$ of holomorphic functions defined on $\Omega^+$, modulo functions which extend holomorphically across $S$. The sheaf $\mathcal{C}_S^-$ is naturally endowed with a structure of $\mathcal{E}_X^{\mathbf{R}}$-module.

*Definition 5.1.1.* Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module defined on a neighborhood of a point $x \in T_S^*X$. Let $\theta$ be a covector on $T^*X$ at $x$. We say that $\theta$ is microhyperbolic for $\mathcal{M}$ on $S$, if $\theta$ is not micro-characteristic for $(\mathcal{M}, T_S^*X)$.

Let $\omega$ be the canonical 1-form on the complex homogeneous symplectic manifold $T^*X$. We denote by $(T^*X)^{\mathbf{R}}$ the real homogeneous symplectic manifold $T^*X$ endowed with the 1-form $\omega^{\mathbf{R}} = \omega + \bar{\omega}$. We denote by $H$ and $H^{\mathbf{R}}$ the isomorphisms between the tangent and cotangent spaces of $T^*X$ and $(T^*X)^{\mathbf{R}}$ associated with $\omega$ and $\omega^{\mathbf{R}}$. If $f$ is a holomorphic function on $X$, we have

$$\operatorname{Re} H_f = H^{\mathbf{R}}_{\operatorname{Re} f}.$$

THEOREM 5.1.2. *Assume that $S$ is of class $C^2$. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module defined in a neighborhood of $x \in (T_S^*X)^-$ and let $Z$ be a closed set defined by $\{x \in S;\ \varphi(x) \geq 0\}$ with a differentiable function $\varphi$ of class $C^1$. If $-d\varphi(x)$ is micro-hyperbolic for $\mathcal{M}$ on $S$ and if $\varphi(x) = 0$, then we have*

$$\mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathbf{R}\Gamma_Z(\mathcal{C}_S^-))_x = 0$$

**5.2.** We prove Theorem 5.1.2 by using the theory developed in § 3 and § 4. Representing $\mathcal{M}$ as a $\mathcal{E}(G; D)$-module, we reduce this to the vanishing theorem of the relative cohomology.

We may assume that $X$ is an open set in $\mathbf{C}^n$.

Let us take a free resolution of $\mathcal{M}$ in a neighborhood of $x$:

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{E}_X^{N_0} \leftarrow \mathcal{E}_X^{N_1} \leftarrow \ldots \leftarrow \mathcal{E}_X^{N_r} \leftarrow 0$$

Since $\mathcal{E}_{X,x}^{\mathbf{R}}$ is an inductive limit of the $\mathcal{E}(G; D)$'s with a closed proper convex cone $G$ and a $G$-round open set $D$ such that $D \times (-G^0)$ is a conical neighborhood of $x$, we can find some $G$, $D$ and a complex of free $\mathcal{E}(G; D)$-modules of finite rank

$$M.: 0 \leftarrow \mathcal{E}(G; D)^{N_0} \leftarrow \mathcal{E}(G; D)^{N_1} \leftarrow \ldots \leftarrow \mathcal{E}(G; D)^{N_r} \leftarrow 0$$

such that the complex

$$0 \leftarrow \mathcal{E}_X^{\mathbf{R}N_0} \leftarrow \ldots \leftarrow \mathcal{E}_X^{\mathbf{R}N_r} \leftarrow 0 \quad \text{is isomorphic to } \mathcal{E}_X^{\mathbf{R}} \otimes_{\mathcal{E}(G;D)} M.$$

on $D \times (-G^0)$.

Therefore, $\mathcal{E}_{X,p}^{\mathbf{R}} \otimes M.$ is exact for $p \in D \times (-G^0) - \operatorname{Supp} \mathcal{M}$.

**5.3.** We take a *real* system of local coordinates of class $C^2$ which we denote by $x_1, \ldots, x_N$ $(N=2n)$, such that

$$x=(0;\ -dx_N),\quad \Omega^+=\{x\in X;\ x_N>0\} \tag{5.3.1}$$

and $d\varphi(x)=dx_1$.

Let us denote by $(x_1, \ldots, x_N; \xi_1, \ldots, \xi_N)$ the coordinates on $(T^*X)^{\mathbf{R}}$ with $\omega^{\mathbf{R}}=\Sigma\, \xi_j\, dx_j$. We set: $x=(x_1, x', x_N)$ and $\xi=(\xi_1, \xi', \xi_N)$. In these new coordinates, we may assume that $D\times(-G^\circ)\subset T^*X$ (in the old coordinates) contains the set defined by the conditions

$$-\xi_N>h_0(|\xi_1|+|\xi'|),\quad x\in D \tag{5.3.2}$$

for some constant $h_0>0$.

The condition of micro-hyperbolicity is invariant by a change of coordinates of class $C^2$. If we write the condition that $\partial/\partial\xi_1=-H_{x_1}$ does not belong to $C_{T^*_S X}(SS(\mathcal{M}))$, we find some $h_1>h_0$ such that any $(x, \xi)\in T^*X$ which satisfies the conditions

$$\xi_1>h_1(|x_N|\,|\xi_N|+|\xi'|),\ -\xi_N>h_1(|\xi_1|+|\xi'|)\quad\text{and}\quad |x|\ll 1 \tag{5.3.3}$$

does not belong to $SS(\mathcal{M})$.

Fix $h>2h_1, 1$ and we denote by $Q'_\varrho$ $(\varrho>0)$, the antipodal of the dual cone of the cone given by

$$\xi_1\geqslant h(|x_N|+\varrho)|\xi_N|+h_1|\xi'|,\quad -\xi_N\geqslant h\xi_1+h_1|\xi'|.$$

Therefore, $Q'_\varrho$ has the form:

$$\begin{aligned}Q'_\varrho=\{(x_1, x', x_N; v_1, v', v_N)\in TX;\ -v_1+hv_N>h_1^{-1}(1-h^2(|x_N|+\varrho))|v'|\quad\text{and}\\ v_N-h(|x_N|+\varrho)v_1>h_1^{-1}(1-h^2(|x_N|+\varrho))\,|v'|\}\cup\{(x; v);\ 1<h^2(|x_N|+\varrho)\}.\end{aligned}$$

Let us define the open convex cone $Q_\varrho$ by

$$\begin{aligned}Q_\varrho=\{(x_1, x', x_N; v_1, v', v_N)\in TX;\ -v_1+hv_N>h^{-1}|v'|\quad\text{and}\\ v_N-h(|x_N|+\varrho)v_1>h^{-1}|v'|\}\cup\{(x, v);\ |x_N|>h^{-2}/2-\varrho\}.\end{aligned} \tag{5.3.4}$$

Then $Q'_\varrho$ is contained in $Q_\varrho$. Therefore, $Q_\varrho$-flat sets are $Q'_\varrho$-flat. Let $R$ be the antipodal of the dual cone of

$$-\xi_N\geqslant h\max(|\xi_1|, |\xi'|).$$

$R$ is given by

$$hv_N>|v_1|+|v'|. \tag{5.3.5}$$

Then, $R$ contains $D\times(G-\{0\})\subset TX$. If we take $g(x)=-x_N+x_1/h$, then the conditions $(c_1)$ and $(c_3)$ in Theorem 4.5.1 are satisfied for $|x_N|<h^{-2}/2-\varrho$. Therefore, we can apply Theorem 4.5.1 and obtain the following lemma.

LEMMA 5.3.1. *There is an open neighborhood $U$ of 0 such that*

$$\mathbf{R}\operatorname{Hom}_{\mathcal{E}(G;D)}(\mathcal{M}.;\mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_{G*}\mathcal{O}))=0$$

*for any open sets $\Omega_1\supset\Omega_0$ satisfying the following properties:*

(i) $\Omega_1-\Omega_0\subset\subset U_\varrho=\{x\in U;\ |x_N|<h^{-2}/2-\varrho\}$,

(ii) $\Omega_1$ *and* $\Omega_0$ *are $R$-flat on* $U$,

(iii) $\Omega_1$ *and* $\Omega_0$ *are $Q_\varrho$-flat on a neighborhood of* $\overline{\Omega_1-\Omega_0}$.

LEMMA 5.3.2. *We have*

$$Q_\varrho(x)\subset\{v;\ qv_N-v_1>h^{-1}|v'|\}\quad\text{for } h\leqslant q,\ h(|x_N|+\varrho)q<1.$$

*Proof.* Set $\varepsilon=|x_N|+\varrho$. Since $h^2\varepsilon<\frac{1}{2}$, the inequalities $v_N-h\varepsilon v_1>h^{-1}|v'|$ and $hv_N-v_1>h^{-1}|v'|$ hold on $Q_\varrho(x)$. Therefore, we have

$$(1-h\varepsilon q)(hv_N-v_1)+(q-h)(v_N-h\varepsilon v_1)>(1-h\varepsilon q+q-h)h^{-1}|v'|\geqslant h^{-1}(1-h^2\varepsilon)|v'|,$$

which implies $(1-h^2\varepsilon)(qv_N-v_1)>h^{-1}(1-h^2\varepsilon)|v'|$. Since $\frac{1}{2}>h^2\varepsilon$, we have the desired result.

This lemma immediately implies the following

LEMMA 5.3.3. $\{x;\ qx_N-x_1>\beta(|x'|-c)\}$ *is $Q_\varrho$-flat on* $\{x;\ |x_N|<(1/2qh)-\varrho\}$ *and $R$-flat for $q\geqslant h$ and $\beta\leqslant h^{-1}$.*

LEMMA 5.3.4. $\{x;\ x_Ne^{-x_1h}>c\}$ *is $Q_\varrho$-flat on* $\{x;\ |x_N|<h^{-2}/2-\varrho\}$ *for any $\varrho>0$ and* $c$.

*Proof.* Since the conormal of this set is $(-hx_N,0,1)$ and since $-hx_Nv_1+v_N>0$ on $Q_\varrho$, we obtain the lemma.

LEMMA 5.3.5. *Let $K$ be a compact subset of $S$. Let $\{U_\iota\}_{\iota\in I}$ be a family of open subsets of $X$ such that* (a) $U_\iota\supset K$, (b) $\{U_\iota-\Omega^+\}_\iota$ *is a fundamental system of neighborhoods of $K$ in* $X-\Omega^+$.

*Then, we have*

$$\mathbf{R}\Gamma(K,\mathcal{C}_S^-)=\varinjlim_{\iota}\mathbf{R}\Gamma_{U_\iota-(U_\iota\cap\Omega^+)}(U_\iota;\mathcal{O})[1].$$

This lemma is evident because we have $\mathcal{C}_S^-=\mathbf{R}\Gamma_{X-S}(\mathcal{O})[1]|_S$.

LEMMA 5.3.6. *Set*

$$K_1(a, b) = \{x \in S;\ 0 \leqslant x_1,\ |x'| + hx_1 \leqslant a\}$$

*and*

$$K_0(a, b) = \{x \in K_1(a, b);\ x_1 \leqslant b\}.$$

*Then, there exists* $a_0 > 0$ *such that*

$$\mathbf{R}\Gamma(K_1(a, b);\ \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, C_S^-)) \xrightarrow{\sim} \mathbf{R}\Gamma(K_0(a, b);\ \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, C_S^-))$$

*for* $0 \leqslant a \leqslant a_0$ *and* $0 \leqslant b$.

*Proof*. We shall prove this by using the vanishing theorem given in Lemma 5.3.1. We may assume that $U$ contains $\{x;\ |x_1| < 2h^{-1},\ |x'| < a_0,\ |x_N| < h^{-2}\}$ exchanging $h$ with a bigger one. We may assume that $hb \leqslant a$. For $1/2 > a_1 > a \geqslant 0$, $h^{-1}a_1 \geqslant b_1 > b \geqslant 0$, $\alpha > \max(2h, (3a_1/b_1) - h))$ and $\delta > 0$, we set

$$\Omega_1(\alpha, a_1, b_1) = \{x;\ \alpha x_N > x_1 + h^{-1}(|x'| - a_1),\ hx_N > x_1 + h^{-1}(|x'| - a_1)\}$$

$$\Omega_0(\alpha, a_1, b_1) = \{x \in \Omega_1(\alpha, a_1, b_1);\ 2hx_N > x_1 - b_1\},$$

$$\Omega_j^+(\alpha, a_1, b_1) = \Omega^+ \cap \Omega_j(\alpha, a_1, b_1) \quad (j = 0, 1)$$

$$\tilde{\Omega}_j(\alpha, a_1, b_1, \delta) = \{x \in \Omega_j(\alpha, a_1, b_1);\ hx_N + x_1 > -\delta\}$$

and

$$\tilde{\Omega}_j^+(\alpha, a_1, b_1, \delta) = \{x \in \Omega_j^+(\alpha, a_1, b_1);\ hx_N + x_1 > -\delta\}.$$

Note that $\tilde{\Omega}_j$ and $\tilde{\Omega}_j^+$ are $R$-flat. Then we have

$$\mathbf{R}\Gamma(K_j(a, b), C_S^-) = \varinjlim_{\substack{a_1 \searrow a,\ b_1 \searrow b \\ \delta \searrow 0,\ \alpha \to +\infty}} \mathbf{R}\Gamma_{\tilde{\Omega}_j(\alpha, a_1, b_1, \delta) - \tilde{\Omega}_j^+(\alpha, a_1, b_1, \delta)}(\tilde{\Omega}_j(\alpha, a_1, b_1, \delta), \varphi_{G*}(\mathcal{O}))\,[1],$$

and hence we have

$$\mathbf{R}\Gamma(K_j(a, b), \mathbf{R}\,\mathrm{Hom}(\mathcal{M}, C_S^-)) = \varinjlim \mathbf{R}\,\mathrm{Hom}(M.;\ \mathbf{R}\Gamma_{\tilde{\Omega}_j(\alpha, a_1, b_1, \delta) - \tilde{\Omega}_j^+(\alpha, a_1, b_1, \delta)}(\varphi_{G*}\mathcal{O}))\,[1].$$

Therefore, in order to prove Lemma 5.3.6, it is sufficient to show that

$$\mathbf{R}\,\mathrm{Hom}(M.;\ \mathbf{R}\Gamma_{\tilde{\Omega}_1(\alpha, a_1, b_1, \delta) - \tilde{\Omega}_0(\alpha, a_1, b_1, \delta)}(\tilde{\Omega}_1(\alpha, a_1, b_1, \delta);\ \varphi_{G*}\mathcal{O})) = 0, \tag{5.3.6}$$

$$\mathbf{R}\,\mathrm{Hom}(M.;\ \mathbf{R}\Gamma_{\tilde{\Omega}_1^+(\alpha, a_1, b_1, \delta) - \tilde{\Omega}_0^+(\alpha, a_1, b_1, \delta)}(\tilde{\Omega}_1^+(\alpha, a_1, b_1, \delta);\ \varphi_{G*}\mathcal{O})) = 0. \tag{5.3.7}$$

Let us denote by $\Omega_1$, $\Omega_0$, etc. instead of $\Omega_1(\alpha, a_1, b_1, \delta), \dots$.

LEMMA 5.3.7. (i) $\tilde{\Omega}_1 - \tilde{\Omega}_0 = \Omega_1 - \Omega_0$ *and* $\tilde{\Omega}_1^+ - \tilde{\Omega}_0^+ = \tilde{\Omega}_1^+ - \tilde{\Omega}_0^+$.

(ii) *We have* $-1/\alpha h < x_N < \frac{1}{2}h^{-2}$ *for* $x \in \overline{\Omega_1 - \Omega_0}$.

(iii) $\Omega_1 - \Omega_0 \subset \{x;\ |x'| < a_1 - hb_1,\ |x_1| < 2h^{-1}\}$.

*Proof*. In order to prove (i), it is enough to show that $hx_N+x_1>0$ for $x\in\Omega_1-\Omega_0$. Since $\alpha x_N-x_1>-h^{-1}a_1$ and $x_1-2hx_N\geqslant b_1$, we have

$$(\alpha-2h)(hx_N+x_1)=3h(\alpha x_N-x_1)+(\alpha+h)(x_1-2hx_N)>-3a_1+(\alpha+h)b_1>0.$$

Thus we obtain $hx_N+x_1>0$.

Let us prove (ii). Since, for $x\in\Omega_1-\Omega_0$, $x_1-2hx_N\geqslant b_1$ and $hx_N-x_1>h^{-1}(|x'|-a_1)$, $-hx_N>b_1+h^{-1}(|x'|-a_1)$. Thus $x_N<(1/h^2)a_1-(b_1/h)<(1/2h^2)$. Therefore, we obtain $x_N<\frac{1}{2}h^{-2}$ for $x\in\overline{\Omega_1-\Omega_0}$. For $x\in\Omega_1-\Omega_0$, $(\alpha-2h)x_N=(\alpha x_N-x_1)+(x_1-2hx_N)>b_1+h^{-1}(|x'|-a_1)>b_1-h^{-1}a_1$. Therefore, we have $x_N>-((h^{-1}a_1-b_1)/(\alpha-2h))>-1/\alpha h$.

Last we shall prove (iii). Since $2hx_N>x_1+h^{-1}(|x'|-a_1)$ on $\Omega_1$, we have $b_1\leqslant x_1-2hx_N\leqslant -h^{-1}(|x'|-a_1)$ on $\Omega_1-\Omega_0$, and $-2/\alpha+b_1<b_1+2hx_N\leqslant b_1\leqslant 2hx_N+h^{-1}a_1\leqslant \mathrm{h}^{-1}+h^{-1}a_1$, which shows the desired result. Q.E.D.

By this lemma, $\Omega_1-\Omega_0$ is contained in $U$. Let us prove $\Omega_j(\alpha, a_1, b_1)$ are $Q_\varrho$-flat on $V=\{x;\ -\alpha^{-1}h^{-1}<x_N<2^{-1}h^{-2}\}$ for $0<\varrho\ll 1$. By Lemma 5.3.3, $\Omega_j$ are $Q_\varrho$-flat on a neighborhood of $x_N=0$. Since $\{x_N>0\}\cap\Omega_1=\{x;\ x_N>0,\ hx_N>x_1+h^{-1}(|x'|-a_1)\}$ and since $\{x_N<0\}\cap\Omega_1=\{x;\ x_N<0,\ \alpha x_N>x_1+h^{-1}(|x'|-a_1)\}$, $\Omega_1$ is $Q_\varrho$-flat on $V$. Since $\{x;\ 2hx_N>x_1-b_1\}$ is $Q_\varrho$-flat on $V$, $\Omega_0$ is $Q$-flat on $V$. Thus, the conditions in Lemma 5.3.3 are satisfied and we obtain $\mathbf{R}\operatorname{Hom}(M.;\ \mathbf{R}\Gamma_{\Omega_1-\Omega_0}(\varphi_{G*}\mathcal{O}))=0$, which shows (5.3.6) together with (i) is Lemma 5.3.7.

Now let us prove (5.3.7). Set $\Omega_j(\varepsilon)=\{x\in\Omega_j;\ x_Ne^{-hx_1}>\varepsilon\}$. Then $\tilde{\Omega}_1(\varepsilon)-\Omega_0(\varepsilon)$ is $Q_\varrho$-flat in $V$ by Lemma 5.3.4. Thus, we can apply Lemma 5.3.1 and we get

$$\mathbf{R}\operatorname{Hom}(M.;\ \mathbf{R}\Gamma_{\Omega_1(\varepsilon)-\Omega_0(\varepsilon)}(\varphi_{G*}\mathcal{O}))=0,$$

for $\varepsilon>0$. Thus, we have

$$\mathbf{R}\operatorname{Hom}(M.;\ \mathbf{R}\Gamma_{\Omega_1^+-\Omega_0^+}(\varphi_{G*}\mathcal{O}))=\varprojlim_{\varepsilon}\mathbf{R}\operatorname{Hom}(M.;\ \mathbf{R}\Gamma_{\Omega_1(\varepsilon)-\Omega_0(\varepsilon)}(\varphi_{G*}\mathcal{O}))=0.$$

This shows (5.3.7). Q.E.D.

**5.4.** Now, we resume to prove Theorem 5.1.2. First let us remark that Lemma 5.3.6 is also true if we replace $K_1(a, b)$ and $K_0(a, b)$ with $K_1(a, b)+w$, $K_0(a, b)+w$ for a sufficient small $w$ in $S$.

Set
$$G=\{x\in S;\ hx_1+|x'|\leqslant 0\},$$
$$\tilde{K}_1(a, b)=\{x;\ hx_1+|x'|<a,\ x_1>0\}$$
and
$$\tilde{K}_0(a, b)=\{x;\ hx_1+|x'|<a,\ 0<x_1<b\}.$$

Then we have

$$\mathbf{R}\Gamma(\tilde{K}_j(a,b); \mathbf{R}\operatorname{Hom}(\mathcal{M}, C_S^-)) = \varinjlim_{\varepsilon\to 0} \mathbf{R}\Gamma(K_j(a-2\varepsilon, b-2\varepsilon)+\varepsilon w_0; \mathbf{R}\operatorname{Hom}(\mathcal{M}, C_S^-))$$

for $w_0=(1, 0, ..., 0)$.

This shows immediately

$$\mathbf{R}\Gamma(\tilde{K}_1(a,b); \mathbf{R}\operatorname{Hom}(\mathcal{M}, C_S^-)) \xrightarrow{\sim} \mathbf{R}\Gamma(\tilde{K}_0(a,b); \mathbf{R}\operatorname{Hom}(\mathcal{M}, C_S^-)). \tag{5.4.1}$$

This is also true if we replace $\tilde{K}_j(a, b)$ with their translates.

We shall prove the theorem by using the argument employed in § 4.

Set $Z_0=\{x\in S; x_1\geqslant -\varepsilon\}$ and let $\varphi_G$ be the canonical map from $Z_0$ to $(Z_0)_G$ (i.e. the topological space $Z_0$ with $G$-topology). Then, (5.4.1) shows immediately

$$\mathbf{R}\varphi_{G*}\mathbf{R}\Gamma_{Z_0}\mathbf{R}\operatorname{Hom}(\mathcal{M}, C_S^-)=0$$

in a neighborhood $U$ of 0. Thus, for any $G$-open set $\Omega$ such that $\Omega-Z_0\subset U$ we have $\mathbf{R}\Gamma_{\Omega\cap Z_0}(\Omega; \mathbf{R}\operatorname{Hom}(\mathcal{M}, C_S^-))=0$. Therefore $\mathbf{R}\Gamma_{\Omega\cap Z}(\Omega; \mathbf{R}\operatorname{Hom}(\mathcal{M}, C_S^-))=0$ because $S-Z$ is $G$-open in a neighborhood of $U$. Taking the inductive limit on $\Omega$, we obtain the desired result

$$\mathbf{R}\Gamma_Z\mathbf{R}\operatorname{Hom}(\mathcal{M}, C_S^-)_x=0.$$

**5.5.** Let $S, \Omega^+$ denote the same subsets of $\mathbf{R}^N$ as in § 5.3; that is

$$S=\{x; x_N=0\},\quad \Omega^+=\{x; x_N>0\}.$$

Let $U^-$ be the open set

$$\{x; x_N>0, x_1\geqslant 0\}\cup\{x; x_N>-x_1^2, x_1\leqslant 0\}$$

and $\Sigma^-$ be the boundary of $U^-$:

$$\Sigma^-=\{x; x_N=0, x_1\geqslant 0\}\cup\{x; x_N=-x_1^2, x_1\leqslant 0\}.$$

We define the sheaf $C_{\Sigma^-}^-$ on $\Sigma^-$ by

$$C_{\Sigma^-}^-=\mathbf{R}\Gamma_{\Sigma^-}(\mathcal{O}_X|_{\bar{U}^-})[1]=\mathbf{R}\Gamma_{X-U^-}(\mathcal{O}_X)[1]|_{\Sigma^-}.$$

PROPOSITION 5.5.1. *In this situation, we have:*

$$\mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{M}, C_{\Sigma^-}^-)_0=0.$$

LEMMA 5.5.2. *$U^-$ is $Q_\varrho$-flat on $\{x; -(1/2h)<x_1<-\varrho/(2h-1),\ |x_N|<h^{-2}/2-\varrho\}$ and $R$-flat on $\{x; x_1>-1/2h\}$.*

*Proof.* The boundary of $U^-$ is $x_N=-x_1^2$. Therefore, the conormal at the boundary is $(2x_1, 0, 1)$. For $v\in Q_\varrho(x)$, $v_N+2x_1v_1>0$ if $|x_1|+\varrho<-2hx_1<1$. The second statement is also evident. Q.E.D.

Let us prove Proposition 5.5.1. Set

$$\Omega_1(a, \varepsilon)=\{x;\ hx_N-x_1>h^{-1}(|x'|-a),\ 3hx_N-x_1>h^{-1}(|x'|-a)\}$$
$$\cap\,(U^-\cup\{x;\ \tfrac{1}{2}h(x_N+\varepsilon^2)>\varepsilon^2(x_1+\varepsilon)\}).$$

Since the first set is $Q_\varrho$-flat on $|x_N|<1/4h^2$ and the second set is $Q_\varrho$-flat on $|x_N|<1/4h^2$ for $0<\varrho\ll 1$, $\Omega_1(a, \varepsilon)$ is $Q_\varrho$-flat on $|x_N|<1/4h^2$. Set

$$\Omega_0(a, \varepsilon, \delta)=(\{x\in U^-;\ 2hx_N>x_1+\delta\}\cup\{x;\ x_Ne^{-hx_1}>\delta\})\cap\Omega_1(a, \varepsilon).$$

Then $\Omega_0(a, \varepsilon, \delta)$ is also $Q_\varrho$-flat on $|x_N|<1/4h^2$ for $0<\varrho\ll 1$. Set $\Omega_0(a, \varepsilon)=\bigcup_{\delta>0}\Omega_0(a, \varepsilon, \delta)=U^-\cap\Omega_1(a, \varepsilon)$. It is easy to check that any neighborhood of 0 contains $\Omega_1(a, \varepsilon)-\Omega_0(a, \varepsilon, \delta)$ for $0<a, \varepsilon, \delta\ll 1$.

Therefore, we can apply Lemma 5.3.1 and we get

$$\mathbf{R}\,\mathrm{Hom}\,(\mathcal{M}.;\ \mathbf{R}\Gamma_{\Omega_1(a,\varepsilon)-\Omega_0(a,\varepsilon,\delta)}(\Omega_1(a, \varepsilon), \varphi_{G*}\mathcal{O}))=0$$

for $0<a, \varepsilon, \delta\ll 1$.

Taking the projective limit with respect to $\delta$, we obtain

$$\mathbf{R}\,\mathrm{Hom}\,(\mathcal{M}.;\ \mathbf{R}\Gamma_{\Omega_1(a,\varepsilon)-\Omega_0(a,\varepsilon)}(\Omega_1(a, \varepsilon); \varphi_{G*}\mathcal{O}))=0.$$

Since $\mathcal{C}_{\Sigma^-,0}^{-}=\varinjlim_{a,\varepsilon\searrow 0}\mathbf{R}\Gamma_{\Omega_1(a,\varepsilon)-\Omega_0(a,\varepsilon)}(\Omega_1(a, \varepsilon); \varphi_{G*}\mathcal{O})[1]$, we obtain the desired result.

## § 6. Division theorem for sheaves of microfunctions with holomorphic parameters

**6.1.** We recall in this paragraph some results briefly announced in [16].

Let $X$ be a complex analytic manifold of dimension $n$, $N$ a real analytic submanifold of $X$. We say that $N$ is of local type $(p, q)$ if, at any point of $N$, there exists a system of local holomorphic coordinates $(z_1, \ldots, z_n)$ such that:

$$N=\{z\in X;\ \mathrm{Im}\, z_1=\ldots=\mathrm{Im}\, z_p=0,\ z_{p+1}=\ldots=z_{n-q}=0\}.$$

*Definition 6.1.1.* Let $N$ be a real analytic submanifold of $X$ of local type $(p, q)$. We define the sheaf $\mathcal{C}_{N|X}$ on $T_N^*X$ by

$$\mathcal{C}_{N|X}=\mathcal{H}^{n-q}_{T_N^*X}(\pi^{-1}\mathcal{O}_X)^a\otimes\omega_{N|X}.$$

(cf. § 1 for the notations).

To be consistent with the notations of [24] we write $\mathcal{C}^{\mathbf{R}}_{N|X}$ instead of $\mathcal{C}_{N|X}$ if $p=0$, and $\mathcal{C}_N$ if $p=n$.

*Examples:*

(a) $p=n$, $q=0$. $X$ is a complexification of $N$ and we find the sheaf $\mathcal{C}_N$ of microfunctions.

(b) $p=0$. $N$ is a complex submanifold of $X$ and we find the sheaf of "holomorphic microfunctions" defined in [24, Chapter 2].

(c) $p<n$, $q=0$: The sheaf $\mathcal{C}_{N|X}$ is considered in [16] and plays an important part in [16].

(c) $p<n$, $p+q=n$: We find a sheaf of microfunctions with holomorphic parameters. This sheaf is used for example in [5] (cf. § 9).

The sheaves $\mathcal{C}_{N|X}$ are sheaves on $T^*X$ supported by $T^*_N X$. They are locally constant on the orbits of the action of $\mathbf{R}_+$, and are naturally endowed with a structure of $\mathcal{E}^{\mathbf{R}}_X$-module.

THEOREM 6.1.2. *Let $N$ and $L$ be two real analytic submanifolds of $X$ of respective local type $(p, 0)$ and $(p, q)$. Let $Y_0$ (resp. $Y_1$) be the complex submanifold of $X$ of dimension $p$ (resp. $p+q$) which contains $N$ (resp. $L$).*

(a) *there exists, locally on $T^*_N X - T^*_{Y_0} X$ and $T^*_L X - T^*_{Y_1} X$, a complex homogeneous canonical transformation which exchanges $T^*_N X$ and $T^*_L X$,*

(b) *if $\varphi$ is such a transformation, $\varphi$ can be extended as an isomorphism of $\mathcal{E}_X$-module of $\mathcal{C}_{N|X}$ to $\mathcal{C}_{L|X}$.*

*Proof.* We choose two systems of local coordinates such that:

$$X = \mathbf{C}^p \times \mathbf{C}^l \times \mathbf{C}^q$$
$$N = \mathbf{R}^p \times \{0\} \times \{0\}$$
$$Y_0 = \mathbf{C}^p \times \{0\} \times \{0\}$$

and

$$X = \mathbf{C}^p \times \mathbf{C}^l \times \mathbf{C}^q$$
$$L = \mathbf{R}^p \times \{0\} \times \mathbf{C}^q$$
$$Y_1 = \mathbf{C}^p \times \{0\} \times \mathbf{C}^q.$$

It is clear that a partial Legendre canonical transformation will exchange $T^*_N X - T^*_{Y_0} X$, with $T^*_L X - T^*_{Y_1} X$. We set

$$\tilde{X} = \mathbf{C}^p \times \mathbf{C}^l \times \bar{\mathbf{C}}^l \times \mathbf{C}^q \times \bar{\mathbf{C}}^q$$
$$\tilde{M} = \mathbf{R}^p \times \underbrace{\mathbf{C}^l \times \bar{\mathbf{C}}^l}_{\mathbf{C}^l} \times \underbrace{\mathbf{C}^q \times \bar{\mathbf{C}}^q}_{\mathbf{C}^q}$$
$$\tilde{Y}_0 = \mathbf{C}^p \times \{0\} \times \{0\} \times \{0\} \times \{0\}$$
$$\tilde{Y}_1 = \mathbf{C}^p \times \{0\} \times \{0\} \times \mathbf{C}^q \times \bar{\mathbf{C}}^q$$

and we identify $X$ with

$$\Delta = \mathbf{C}^p \times \underset{\mathbf{C}^l}{\mathbf{C}^l \times \bar{\mathbf{C}}^l} \times \underset{\mathbf{C}^q}{\mathbf{C}^q \times \bar{\mathbf{C}}^q}.$$

In other words, if we denote by $(z, v, w)$ the coordinates in $X$, we look at $X$ as a real manifold on $v$, $w$, and complexify it in these variables. We denote by $(z, v, \bar{v}, w, \bar{w})$ the coordinates on $\tilde{X}$, and consider the $\mathcal{E}_{\tilde{X}}$-modules $\mathcal{L}_0$ and $\mathcal{L}_1$ given by the relations:

$$\mathcal{L}_0\colon \bar{v}f = 0, \quad \bar{w}f = 0$$

$$\mathcal{L}_1\colon \bar{v}f = 0, \quad (\partial/\partial\bar{w})f = 0.$$

It is not difficult to prove the isomorphisms:

$$\mathcal{C}_{N|X} \simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{\bar{\mathbf{C}}^l \times \bar{\mathbf{C}}^q}}(\mathcal{L}_0, \mathcal{C}_{\tilde{M}})$$

$$\mathcal{C}_{L|X} \simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{\bar{\mathbf{C}}^l \times \bar{\mathbf{C}}^q}}(\mathcal{L}_1, \mathcal{C}_{\tilde{M}}).$$

In fact the case of equations of type $\partial/\partial\bar{w}$ is treated in [24, chapter 3, Th. 2.2.5], but we can get the general case with slight modifications. The modules $\mathcal{L}_0$ and $\mathcal{L}_1$ are equivalent as $\mathcal{E}_{\tilde{X}}$-modules, by a real quantized transformation which exchanges $T^*_{\tilde{M}}\tilde{X} - T^*_{\tilde{Y}_0}\tilde{X}$ with $T^*_{\tilde{M}}\tilde{X} - T^*_{\tilde{Y}_1}\tilde{X}$ (cf. [24, chapter 3]). If $\phi$ is such a transformation, $\phi$ defines an isomorphism of the $\mathrm{End}_{\mathcal{E}_{\tilde{X}}}(\mathcal{L}_0)$-module $\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{\tilde{X}}}(\mathcal{L}_0, \mathcal{C}_{\tilde{M}})$ with the $\mathrm{End}_{\mathcal{E}_{\tilde{X}}}(\mathcal{L}_1)$-module $\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{\tilde{X}}}(\mathcal{L}_1, \mathcal{C}_{\tilde{M}})$, and it remains to remark that:

$$\mathrm{End}_{\mathcal{E}_{\tilde{X}}}(\mathcal{L}_0) = \mathrm{End}_{\mathcal{E}_{\tilde{X}}}(\mathcal{L}_1) \simeq \mathcal{E}_X.$$

**6.2.** In this section we denote by $(t, x_2, \dots, x_n) = (t, x)$ the coordinates in $\mathbf{C}^n$, and we define the submanifold $N$ by the equations $x_2 = \dots x_{n-p} = \mathrm{Im}\, x_{n-p+1} = \dots = \mathrm{Im}\, x_n = 0$. Thus $N$ is of local type $(p, 1)$, and $t$ is an holomorphic coordinate in $N$. Let $P$ be a micro-differential operator in a neighborhood of the point $p = (0, 0; 0, \dots, 1)$, with:

$$\frac{\partial^j}{\partial t^j}\sigma(P)(p) = 0 \quad j < m$$

$$\neq 0 \quad j = m.$$

LEMMA 6.2.1. *In the preceding situation any $u \in (\mathcal{C}_{N|X})_p$ can be written in a unique way*

$$u = Pv + w$$

*with $v$, $w \in (\mathcal{C}_{N|X})_p$, and $(\partial^m/\partial t^m)w = 0$.*

*Proof.* By the division theorem we can write

$$\frac{1}{2\pi i}\frac{1}{s-t} = P(t, x, D_t, D_x)\, G(t, s, x, D_t, D_s, D_x) + K(t, s, x, D_t, D_s, D_x)$$

with $(\mathrm{ad}\, D_t)^m K=0$. We may assume $K$ and $G$ defined bor $|s|>|t|$. Then:

$$\begin{aligned} u(t,x) &= \frac{1}{2\pi i}\oint \frac{1}{s-t} u(s,t)\,ds \\ &= P(t,x,D_t,D_x)\oint G(t,s,x,D_t,D_s,D_x)\,u(s,x)\,ds \\ &\quad + \oint K(t,s,x,D_t,D_s,D_x)\,u(s,x)\,ds \end{aligned}$$

Here $\oint$ is a contour integral around $|s|=\varrho>|t|$. We set

$$v=\oint Gu(s,x)\,ds$$

$$w=\oint Ku(s,x)\,ds,$$

then $v$ and $w$ belong to $\mathcal{C}_{N|X}$ and $D_t^m w=\oint[(\mathrm{ad}\, D_t)^m K]u(s,x)ds=0$. We shall show the uniqueness. We may assume $P$ of Weierstrass type in $t$:

$$P(t,x,D_t,D_x)=\sum_{j=0}^{m} A_j(x,D_t,D_x)\,t^j$$

with ord $A_j\leqslant 0$, $A_m=1$. Assume

$$P(t,x,D_t,D_x)\,v(t,x)=w(t,x)=\sum_{j=0}^{m-1} t^j w_j(x)$$

then:

$$v(t,x)=\frac{1}{2\pi i}\oint_{|s|=\varrho\gg 0}\frac{1}{s-t}P^{-1}(s,x,D_s,D_x)\,w(s,x)\,ds.$$

By the change of variables $\lambda=1/s$, the operator $P(s,x,D_s,D_x)$ becomes

$$Q(\lambda,x,D_\lambda,D_x)=\sum_{j=0}^{m} A_j(x,-\lambda^2 D_\lambda,D_x)\,\lambda^{-j}$$

and $\lambda^{m-1}Q\lambda$ is well defined and invertible at $\lambda=0$. Thus

$$v(t,x)=\frac{1}{2\pi i}\oint_{|\lambda|=1/\varrho}\frac{\lambda}{\lambda^2(1/\lambda-t)}\,(\lambda^{m-1}Q\lambda)^{-1}\,(\lambda^{m-1}w(1/\lambda,x))\,d\lambda$$

and the term we integrate being holomorphic, we get $v=0$.

**6.3.** Let $V$ be an involutive submanifold of $T^*X$, of codimension $p$. Let $x$ belong to $V$, and let $b(x)$ be the unique bicharacteristic leaf of dimension $p$ of $V$ passing through $x$. We say that $V$ is non characteristic for a coherent $\mathcal{E}_X$-module $\mathcal{M}$ at $x$ if:

$$b(x)\cap SS(\mathcal{M})\subset\{x\}$$

is a neighborhood of $x$. Recall that $V$ is said to be regular at $x$, if $\omega|_V$ is nonzero at $x$, where $\omega$ is the canonical 1-form on $T^*X$.

THEOREM 6.3.1. [16]. *Let $N$ be a real analytic submanifold of $X$ of local type $(p, q)$, $Y_0$ the complex submanifold of dimension $p+q$ containg $N$. Let $\mathcal{M}$ be a left coherent $\mathcal{E}_X$-module on an open set $U \subset T^*X$, and let $V$ be a complex involutive submanifold of $U$ which contains $T_N^*X$. Let $\mathcal{L}$ be a left coherent $\mathcal{E}_X$-module on $U$ such that:*

(6.3.1) $$SS(\mathcal{L}) = V$$

(6.3.2) *$\mathcal{L}$ has simple characteristics on $V$.*

*We assume $V$ non characteristic for $\mathcal{M}$. Then the natural homomorphism:*

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathbf{C}_{N|X}) \leftarrow \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{L}) \overset{L}{\underset{\mathrm{End}_{\mathcal{E}_X}(\mathcal{L})}{\otimes}} \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{L}, \mathbf{C}_{N|X})$$

*is an isomorphism outside $T_N^*X \cap T_{Y_0}^*X$.*

*Proof.* The involutive manifold $T^*X \times_X Y_0 = V_0$ is regular outside $T_{Y_0}^*X$. We may assume by a complex canonical transformation that $V$ and $V_0$ are given in some local coordinates $(z_1, \ldots, z_n; \zeta_1, \ldots, \zeta_n)$ of $T^*X$ by:

$$V\colon\ z_1 = \ldots = z_l = 0$$

$$V_0\colon z_1 = \ldots = z_l = \ldots = z_r = 0$$

then $T_N^*X$ will be of the type:

$$T_N^*X\colon \{z_1 = \ldots = z_r = 0,\ (z'; \zeta') \in \Lambda'\}$$

where $z' = (z_{r+1}, \ldots, z_n)$, $\zeta' = (\zeta_{r+1}, \ldots, \zeta_n)$, and $\Lambda'$ is a real Lagrangean manifold whose complexification is $T^*(\mathbf{C}^{n-r})$. Thus a complex canonical transformation in the $(z'; \zeta')$ variables, and Theorem 6.1.2 reduces the situation to $N$ of local type $(p, 0)$, and $V = T^*X \times_X Y$, for a complex submanifold $Y$ of $X$ containing $N$. As all $\mathcal{E}_X$-modules which satisfy (6.3.1) and (6.3.2) are locally isomorphic, we may assumed $\mathcal{L} = \mathcal{E}_{X \leftarrow Y}$. Now we use the method of [16] to reduce the problem to the case where $\mathcal{M}$ is a single equation. Let $x$ belong to $V$. By the hypothesis that $V$ is non characteristic, we may assume

$$SS(\mathcal{M}) \cap \varrho^{-1}\varrho(x) \subset \{x\}$$

where $\varrho$ denotes, as usual, the projection $T^*X \times_X Y \to T^*Y$. Then it is enough to prove

$$\begin{aligned}\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathbf{C}_{N|X})_x &\simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_{X\leftarrow Y})_x \overset{L}{\underset{\varrho^{-1}\mathcal{E}_Y}{\otimes}} \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{E}_{X\leftarrow Y}, \mathbf{C}_{N|X})_x \\ &\simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathbf{C}_{N|Y})_y[-d]\end{aligned}$$

where $y = \varrho(x)$, $d = \mathrm{codim}_X\, Y$. By induction on $d$ we may assume that $Y$ is a hypersurface.

Let $u_1, \ldots, u_r$ be generators of $\mathcal{M}$ near $x$. We can find microdifferential operators $P_1, \ldots, P_r$ near $x$, such that:

$$P_i u_i = 0$$

$$Y \text{ is non characteristic for each } P_i \quad (i = 1, \ldots, r).$$

Let $\mathcal{M}'$ be the module $\oplus_{i=1}^{r} \mathcal{E}_X / \mathcal{E}_X \cdot P_i$ and $\mathcal{M}''$ the module defined by the exact sequence

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{M}' \leftarrow \mathcal{M}'' \leftarrow 0. \tag{6.3}$$

Then the sequence

$$0 \leftarrow \mathcal{M}_Y \leftarrow \mathcal{M}'_Y \leftarrow \mathcal{M}''_Y \leftarrow 0 \tag{6.4}$$

is exact. We apply the functor $\mathrm{Hom}_{\mathcal{E}_X}(\cdot, \mathcal{C}_{N|X})_x$ to (6.3) and $\mathrm{Hom}_{\mathcal{E}_X}(\cdot, \mathcal{C}_{N|Y})_y$ to (6.4). As $\mathcal{M}$ and $\mathcal{M}''$ satisfy the same hypothesis, we see by induction on $i$, that it is enough to prove that for any $i$ the natural homomorphisms

$$\mathrm{Ext}^i_{\mathcal{E}_Y}(\mathcal{M}'_Y, \mathcal{C}_{N|Y})_y \to \mathrm{Ext}^{i+1}_{\mathcal{E}_X}(\mathcal{M}', \mathcal{C}_{N|X})_x$$

are isomorphisms, that is to prove the theorem when $\mathcal{M} = \mathcal{E}_X / \mathcal{E}_X \cdot P$ for a microdifferential operator $P$. By Theorem 6.1.2 we can take for $N$ the submanifold of $X$ described in section 6.2 and for $V$ the manifold of equation $\tau = 0$ in $T^*X$, where $(t, x; \tau, \xi)$ are coordinates in $T^*X = T^*(\mathbf{C} \times \mathbf{C}^{n-1})$. Let $m$ be the order of the zero of $\sigma(P) | \varrho^{-1}(\varrho(x))$ at $x$. Then $\mathcal{M}_Y \simeq \mathcal{E}_Y^m$, and it remains to apply Lemma 6.2.1.

*Remark 6.3.2.* The isomorphism of Theorem 6.3.1 remains valid all over $T^*_N X$ when $N$ is a complex submanifold of $X$; this is clear by the proof.

## § 7. Proofs of the main theorems

**7.1.** Let $M$ be a real analytic manifold of dimension $n$ and $X$ a complexification of $M$. Let $U^+$ be a strictly pseudo-convex open set in $X$ with real analytic boundary $S$. It is well known that we can find locally in $T^*_M X - T^*_X X$ and $T^*_S X - T^*_X X$, a complex homogeneous canonical transformation $\varphi$ which exchanges $T^*_M X$ and $T^*_S X$. Moreover it can be proved, with the results of [16], that $\varphi$ can be extended as an isomorphism of $\mathcal{E}_X$-modules of $\mathcal{C}_M$ and $\mathcal{C}_S$ (with the notations of § 5). For example, if $X = \mathbf{C}^n$ with coordinates $(z_1, \ldots, z_n)$, where $z = x + iy$, and

$$M = \{z; y = 0\}$$

$$S = \left\{ z; x_n = \sum_{j=1}^{n-1} x_j^2 \right\}$$

we can define $\varphi$ by: $(z, \zeta) \to (iz + d_\zeta \varphi(\zeta),\ -i\zeta)$ where $\varphi(\zeta) = (\zeta_1^2 + \ldots + \zeta_{n-1}^2)/(-4\zeta_n)$.

**7.2.** Now we prove Theorem 2.2.1. If $Z$ is conic in $T^*_M X$, and $x$ does not belong to $T^*_M M$, the image by $\varphi$ of $Z$ in $T^*_S X$ will arise from a closed set of $S$. Thus in this case the theorem follows from Theorem 5.1.2 by using the same argument as in § 4.4 whose details are left to the reader (replace Lemma 4.3.6 by Theorem 5.1.2). The general case results of the preceding one by the following trick. We define $Z'$ in $T^*_{M\times\mathbf{R}}(M\times\mathbf{C})$ by

$$Z'=\{(x,t;i(\xi,\tau));\ (x,i\xi/\tau)\in Z,\tau>0\}.$$

Let $\delta_t$ be the $\mathcal{E}_{\mathbf{C}}$-module

$$\delta_t=\mathcal{E}_{\mathbf{C}}/\mathcal{E}_{\mathbf{C}}\cdot t.$$

Then we have the isomorphism:

$$\mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{M},\mathbf{C}_M)\simeq\mathbf{R}\operatorname{Hom}_{\mathcal{E}_{X\times\mathbf{C}}}(\mathcal{M}\hat{\otimes}\delta_t,\mathbf{C}_{M\times\mathbf{R}})\Big|_{\substack{t=0\\ \tau=1}}$$

and if $x=(x^0,i\xi^0)$, $y=(x^0,0;i\xi^0,i)$ the conormals to $Z'$ at $y$ are micro-hyperbolic for $\mathcal{M}\hat{\otimes}\delta_t$, which completes the proof of Theorem 2.2.1.

**7.3.** We begin the proof of Theorem 2.3.1. Setting $Z=Y\times X$ we decompose $\varphi$ into $Y\xrightarrow{j}Z\xrightarrow{p}X$ (cf. [24], Chapter 2) where $j$ is the graph map, and $p$ the second projection. It is enough to prove the theorem for $j$ and $p$, because $j$ will be micro-hyperbolic for $\mathcal{M}_Z$, and

$$(\mathcal{M}_Z)_Y=\mathcal{M}_Y.$$

**7.4.** Assume $\varphi$ is smooth. Then $\varphi$ is micro-hyperbolic for any coherent $\mathcal{E}_X$-module $\mathcal{M}$. As the theorem we want to prove is local, we may consider a resolution of $\mathcal{M}$ by free $\mathcal{E}_X$-modules of finite rank. It is then enough to prove the theorem when $\mathcal{M}=\mathcal{E}_X$, that is to verify

$$\varrho_*\tilde{\omega}^{-1}\mathbf{C}_M\simeq\mathbf{R}\operatorname{Hom}_{\mathcal{E}_Y}(\mathcal{E}_{Y\to X},\mathbf{C}_N)$$

which is clear.

**7.5.** We assume now that $\varphi$ is a closed embedding, and identify $Y$ with its image in $X$. It is clear by induction on the codimension of $Y$ in $X$ that we may assume $Y$ is the complexification in $X$ of a hypersurface $N$ of $M$. We keep the notations of § 5, and consider the hypersurfaces $S$ and $\Sigma$ of $X=\mathbf{C}^n$:

$$S=\left\{z;x_n=\sum_{i=1}^{n-1}x_i^2\right\}$$

$$\Sigma=\left\{z;x_n=\sum_{i=2}^{n-1}x_i^2\right\}$$

$$\Sigma^\pm=\{z\in S;\ \pm x_1\leqslant 0\}\cup\{x\in\Sigma;\ \pm x_1\geqslant 0\}$$

and the sheaves $\mathcal{C}_S^-$, $\mathcal{C}_\Sigma^-$, $\mathcal{C}_{\Sigma^+}^-$ and $\mathcal{C}_{\Sigma^-}^-$ on $(T_S^*X)^-$, $(T_\Sigma^*X)^-$, $(T_{\Sigma^+}^*X)^-$ and $(T_{\Sigma^-}^*X)^-$. If $\varphi$ is a canonical transformation which exchanges $T_M^*X$ with $T_S^*X$, the inverse image by $\varphi$ of $T_S^*X \cap T_\Sigma^*X$ will be a regular hypersurface of $T_M^*X$. Thus by composing $\varphi$ on the right with a real canonical transformation on $T_M^*X$, we may assume that $\varphi$ exchanges $T_N^*X$ with $T_\Sigma^*X$ and $T_M^*X$ with $T_S^*X$. It is not difficult to prove, by the same method as for $\mathcal{C}_M$, looking at $\mathcal{C}_{N|X}$ as a sheaf of microfunctions with one holomorphic parameter, that $\varphi$ extends to an isomorphism of $\mathcal{E}_X$-modules of $\mathcal{C}_{N|X}$ with $\mathcal{C}_\Sigma^-$. It would be tedious to prove the compatibility of the isomorphisms of $\mathcal{C}_{N|X}$ and $\mathcal{C}_\Sigma^-$ with that of $\mathcal{C}_M$ with $\mathcal{C}_S^-$, and we prefer, for the proof of Theorem 2.3.1, to "translate" everything in terms of $\mathcal{E}_X$-modules.

Let $V$ be the complexification (in $T^*X$) of $T_N^*X \cap T_M^*X$, $\Lambda$ the image of $V$ by $\varphi$. If $(z, \zeta)$ are the coordinates in $T^*(\mathbf{C}^n)$, $\Lambda$ is given by:

$$\Lambda: \zeta_1 = 0.$$

Lemma 7.5.1. *Let $\mathcal{L}$ be a coherent $\mathcal{E}_X$-module such that:*

$$SS(\mathcal{L}) = \Lambda \tag{7.5.1}$$

$$\mathcal{L} \text{ has simple characteristics on } \Lambda. \tag{7.5.2}$$

*Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module such that $\Lambda$ is non characteristic for $\mathcal{M}$. Then the natural homomorphism*

$$\mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_\Sigma^-) \leftarrow \mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{L}) \overset{L}{\underset{\operatorname{End}(\mathcal{L})}{\otimes}} \mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{L}, \mathcal{C}_\Sigma^-)$$

*is an isomorphism.*

*Proof.* As $\varphi$ extends to an isomorphism of $\mathcal{E}_X$-modules of $\mathcal{C}_{N|X}$ with $\mathcal{C}_\Sigma^-$, Lemma 7.5.1 follows from Theorem 6.3.1.

Lemma 7.5.2. *Let $\mathcal{L}$ be a coherent $\mathcal{E}_X$-module which satisfies* (7.5.1) *and* (7.5.2). *Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module which is micro-hyperbolic on $S$ at the codirections $dx_1$ and $-dx_1$. Then we have a natural isomorphism:*

$$\mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_S^-)|_{S\cap\Sigma} \simeq \mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{L}) \overset{L}{\underset{\operatorname{End}(\mathcal{L})}{\otimes}} \mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{L}, \mathcal{C}_S^-)[+1].$$

*Proof.* The hypersurface $S \cap \Sigma$ of $S$ defines two closed sets $Z_+$ and $Z_-$ whose boundaries are $S \cap \Sigma$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{C}_\Sigma^-|_{S\cap\Sigma} & \longrightarrow & \mathcal{C}_{\Sigma^+}^-|_{S\cap\Sigma} \oplus \mathcal{C}_{\Sigma^-}^-|_{S\cap\Sigma} & \longrightarrow & \mathcal{C}_S^-|_{S\cap\Sigma} & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \Gamma_{S\cap\Sigma}(\mathcal{C}_S^-) & \to & \Gamma_{Z_+}(\mathcal{C}_S^-)|_{S\cap\Sigma} \oplus \Gamma_{Z_-}(\mathcal{C}_S^-)|_{S\cap\Sigma} & \to & \mathcal{C}_S^-|_{S\cap\Sigma} & \to 0.
\end{array}$$

The prolongation theorem implies

$$\mathbf{R}\operatorname{Hom}(\mathcal{M}, \Gamma_{Z_\pm}(C_S^-)|_{S\cap\Sigma}) = 0$$

and

$$\mathbf{R}\operatorname{Hom}(\mathcal{M}, C_{\Sigma^\pm}^-|_{S\cap\Sigma}) = 0.$$

In this induced commutative diagram

$$\begin{array}{ccc} \mathbf{R}\operatorname{Hom}(\mathcal{M}, C_S^-|_{S\cap\Sigma}) & \xrightarrow{\alpha_1} & \mathbf{R}\operatorname{Hom}(\mathcal{M}, C_\Sigma^-|_{S\cap\Sigma})[1] \\ \downarrow & & \downarrow \beta_2 \\ \mathbf{R}\operatorname{Hom}(\mathcal{M}, C_S^-|_{S\cap\Sigma}) & \xrightarrow{\alpha_2} & \mathbf{R}\operatorname{Hom}(\mathcal{M}, \Gamma_{S\cap\Sigma}(C_S^-)|_{S\cap\Sigma})[1], \end{array}$$

$\alpha_1$ and $\alpha_2$ are isomorphisms and hence so is $\beta_2$. Now, consider the following diagram:

$$\begin{array}{ccc} \mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{L}) \overset{L}{\underset{\mathrm{End}(\mathcal{L})}{\otimes}} \mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{L}, C_\Sigma^-)|_{S\cap\Sigma} & \xrightarrow{\gamma_1} & \mathbf{R}\operatorname{Hom}(\mathcal{M}, C_\Sigma^-|_{S\cap\Sigma}) \\ \downarrow \beta_1 & & \downarrow \beta_2 \\ \mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{L}) \overset{L}{\underset{\mathrm{End}(\mathcal{L})}{\otimes}} \mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{L}, \Gamma_{S\cap\Sigma}(C_S^-)) & \xrightarrow{\gamma_2} & \mathbf{R}\operatorname{Hom}(\Gamma_{S\cap\Sigma}(C_S^-)) \end{array}$$

By Lemma 7.5.1, the homomorphism $\gamma_1$ is an isomorphism.

The module $\mathcal{L}$ being isomorphic to $\mathcal{E}_X/\mathcal{E}_X D_1$, we have $\mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{L}, C_\Sigma^-)|_{S\cap\Sigma} \xrightarrow{\sim} \mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{L}, \Gamma_{S\cap\Sigma}(C_S^-))$, and hence $\beta_1$ is an isomorphism. Since $\beta_2$ is an isomorphism so is $\gamma_2$. Thus we obtain the lemma.

If we reformulate Lemma 7.5.2 by replacing $C_S^-$ by $C_M$ and $\mathcal{L}$ by $\mathcal{E}_{X\leftarrow Y}$ we get:

$$\tilde{\omega}^{-1}\mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{M}, C_M) \simeq \mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_{X\leftarrow Y}) \overset{L}{\underset{\varrho^{-1}\mathcal{E}_Y}{\otimes}} \varrho'^{-1} C_N[1] \tag{7.5.3}$$

where $\varrho'$ denotes the projection

$$\varrho' \colon T_M^*X \underset{M}{\times} N \to T^*Y.$$

The hypothesis (2.3.1) of Theorem 2.3.1 implies

$$\varrho'_*(\mathcal{E}_{Y\to X} \underset{\mathcal{E}_X}{\otimes} \mathcal{M}) = \varrho_*(\mathcal{E}_{Y\to X} \underset{\mathcal{E}_X}{\otimes} \mathcal{M}) = \mathcal{M}_Y.$$

Thus, it remains to take the direct image by $\varrho'$ of isomorphism (7.5.3) to get the theorem.

## § 8. Application I: Cauchy problem for sheaves of coherent $\mathcal{E}_X$-modules

**8.1.** We show in this section how our theorems allow us to give new proofs to the results of [19] and to complete them.

Let $\varphi$ be a holomorohic map from $Y$ to $X$, $\varrho$ and $\tilde{\omega}$ denoting, as usual, the mappings:

$$\varrho \colon T^*X \underset{X}{\times} Y \to T^*Y$$
$$\tilde{\omega} \colon T^*X \underset{X}{\times} Y \to T^*X.$$

If $\mathcal{N}$ is a coherent $\mathcal{E}_X$-module we write $\mathcal{N}^\infty$ and $\mathcal{N}^{\mathbf{R}}$ for $\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{N}$ and $\mathcal{E}_X^{\mathbf{R}} \otimes_{\mathcal{E}_X} \mathcal{N}$. Recall that $\mathcal{E}_X^\infty$ and $\mathcal{E}_X^{\mathbf{R}}$ are flat over $\mathcal{E}_X$. We have given in § 1 the definition of "$\varphi$ non microcharacteristic for $(\mathcal{M}, \mathcal{N})$". Let $d = \dim_{\mathbf{C}} X - \dim_{\mathbf{C}} Y$.

THEOREM 8.1.1. [19, Theorem 3.1]. *Let $\mathcal{M}$ and $\mathcal{N}$ be two coherent $\mathcal{E}_X$-modules on an open set $U \subset T^*X$. Assume $\varphi$ non microcharacteristic for $(\mathcal{M}, \mathcal{N})$. Then the natural homomorphism on $\tilde{\omega}^{-1}(U)$:*

$$\tilde{\omega}^{-1} \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}^{\mathbf{R}}) \to \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_{X \leftarrow Y}) \overset{L}{\underset{\varrho^{-1}\mathcal{E}_Y}{\otimes}} (\mathcal{E}^{\mathbf{R}}_{Y \to X} \underset{\mathcal{E}_X}{\otimes} \mathcal{N})[d]$$

*is an isomorphism.*

*The result remains true if we replace $\mathcal{N}^{\mathbf{R}}$ and $\mathcal{E}^{\mathbf{R}}_{Y\to X}$ by $\mathcal{N}^\infty$ and $\mathcal{E}^\infty_{Y \to X}$.*

*Proof.* As for the proof of Theorem 2.3.1, it is easy to see that it is sufficient to consider the case where $Y$ is a submanifold of $X$. The theorem must be proved at each point $x \in \omega^{-1}(U)$ and $\varphi$ being non characteristic we may assume:

$$SS(\mathcal{M}) \cap \varrho^{-1}\varrho(x) = \{x\}$$
$$SS(\mathcal{M}) \cap \varrho^{-1}\varrho(x) = \{x\}.$$

Then we have to prove:

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}^{\mathbf{R}})_x \xrightarrow{\sim} \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{N}_Y^{\mathbf{R}})_y$$

where $y = \varrho(x)$.

We identify $T^*X$ with the diagonal of $T^*X \times T^*X$ by the first projection, and $T(T^*X)$ with $T_{T^*X}(T^*X \times T^*X)$. Let us denote by $\mathcal{N}^*$ the adjoint system to $\mathcal{N}$

$$\mathcal{N}^* = \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{N}, \mathcal{E}_X) \underset{\mathcal{O}_X}{\otimes} \Omega_X^{\otimes -1}$$

(where $\Omega_X$ is the sheaf of holomorphic $n$-forms on $X$). We have:

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}^{\mathbf{R}}) \simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{X\times X}}(\mathcal{M} \hat{\otimes} \mathcal{N}^*, \mathbf{C}^{\mathbf{R}}_{X|X\times X})$$

and also:

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{N}_Y^{\mathbf{R}}) \simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{Y\times Y}}(\mathcal{M}_Y \hat{\otimes} (\mathcal{N}_Y)^*, \mathbf{C}^{\mathbf{R}}_{Y|Y\times Y}).$$

We first restrict the systems to $Y \times X$, then to $Y \times Y$. We have

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{Y\times X}}(\mathcal{M}_Y \hat{\otimes} \mathcal{N}^*, \mathbf{C}^{\mathbf{R}}_{Y|Y\times X}) \simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{Y\times Y}}(\mathcal{M}_Y \hat{\otimes} (\mathcal{N}^*)_Y, \mathbf{C}^{\mathbf{R}}_{Y|Y\times Y})[d]$$

by Remark 6.3.2, and $(\mathcal{N}^*)_Y[d] = (\mathcal{N}_Y)^*$. The theorem will thus result of the following:

LEMMA 8.1.2. [19, Proposition 3.4]. *Let $Y$ and $Z$ be complex submanifolds of $X$. We assume $Y$ and $Z$ transversal. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module defined near $x \in T^*X \times_X Y$. We assume $Y$ is non microcharacteristic for $\mathcal{M}$ on $T^*_Z X$ and*

$$SS(\mathcal{M}) \cap \varrho^{-1}(\varrho(x)) \subset \{x\}.$$

*Then the natural homomorphism*

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbf{R}}_{Z|X})_x \to \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathcal{C}^{\mathbf{R}}_{Z\cap Y|Y})_{\varrho(x)}$$

*is an isomorphism.*

*Proof.* We can assume that in some local coordinates:

$$X = \mathbf{C}^p \times \mathbf{C}^q$$

$$Z = \{0\} \times \mathbf{C}^q$$

$$Y = \mathbf{C}^p \times \mathbf{C}^{q-d} \times \{0\}.$$

Let $\tilde{X}$ denote the complexification of $X$: $\tilde{X} = X \times \bar{X}$ and we identify $X$ with $\Delta = X \times_{\tilde{X}} \bar{X}$. Let us denote by $\mathcal{L}$ the system on $\tilde{X}$ defined by the equations:

$$\mathcal{L}: \begin{cases} \bar{z}_i u = 0 & i = 1, \dots, p \\ (\partial/\partial \bar{z}_j)\, u = 0 & j = p+1, \dots, p+q \end{cases}$$

We have:

$$(8.1.1) \qquad T^*_\Delta \tilde{X} \cap (T^*X \times SS(\mathcal{L})) \simeq T^*_Z X$$

and an isomorphism (cf. § 6):

$$(8.1.2) \qquad \mathcal{C}^{\mathbf{R}}_{Z|X} \simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{\tilde{X}}}(\mathcal{L}, \mathcal{C}_X)$$

where $\mathcal{C}_X$ is, as usual, the sheaf of microfunctions on the real manifold $X$. Let $\tilde{Y}$ be the complexification of $Y$ in $\tilde{X}$. We have:

$$(\mathcal{M} \hat{\otimes} \mathcal{L})_{\tilde{Y}} \simeq \mathcal{M}_Y \hat{\otimes} \mathcal{L}_{\bar{Y}}$$

and $\mathcal{L}_{\bar{Y}}$ is the system defined by the equations:

$$\bar{z}_i u = 0, \quad i = 1, \dots, p$$

$$(\partial/\partial \bar{z}_j)\, u = 0, \quad j = p+1, \dots, p+q-d.$$

It is thus enough to prove:

$$(8.1.3) \qquad \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{\tilde{X}}}(\mathcal{M} \hat{\otimes} \mathcal{L}, \mathcal{C}_X)_x \simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{\tilde{Y}}}((\mathcal{M} \hat{\otimes} \mathcal{L})_{\tilde{Y}}, \mathcal{C}_Y)_{\varrho(x)}$$

because the first term of (8.1.3) is equal to

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{\bar{X}}}(\mathcal{L}, \mathcal{C}_X))_x \simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbf{R}}_{Z|X})_x$$

and the second term of (8.1.3) is equal to

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_{\bar{Y}}}(\mathcal{L}_{\bar{Y}}, \mathcal{C}_Y))_{\varrho(x)} \simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}_Y, \mathcal{C}^{\mathbf{R}}_{Z\cap Y|Y})_{\varrho(x)}$$

The isomorphism (8.1.3) will be a consequence of Theorem 2.3.1 if we show that $Y$ is microhyperbolic for $\mathcal{M}\hat{\otimes}\mathcal{L}$. Let $\theta$ belong to $T(T^*X)$, with:

$$\theta \notin C(SS(\mathcal{M}); T^*_Z X)$$

it is enough to show that

$$(\theta, 0) \notin C(SS(\mathcal{M}) \times SS(\mathcal{L}); T^*_\Delta \bar{X})$$

in $T(T^*X \times T^*\bar{X})$. Let us denote by $(z, \bar{w})$ a point in $T^*X \times T^*\bar{X}$.

Let $(z_n, \bar{w}_n)$ and $(z'_n, \bar{w}'_n)$ be two sequences in $T^*(X \times \bar{X})$, such that:

$$z_n \in SS(\mathcal{M}), \quad \bar{w}_n \in SS(\mathcal{L}), \quad (z'_n, \bar{w}'_n) \in T^*_\Delta(X \times \bar{X})$$
$$(z_n, \bar{w}_n) \xrightarrow[n]{} (z_0, \bar{w}_0) \qquad (z'_n, \bar{w}'_n) \xrightarrow[n]{} (z_0, \bar{w}_0)$$

and there exist $c_n \in \mathbf{R}_+$, with

$$c_n(z_n - z'_n) \xrightarrow[n]{} \theta, \quad c_n(\bar{w}_n - \bar{w}'_n) \xrightarrow[n]{} 0.$$

We have $\bar{w}'_n = \bar{z}'_n$, $\bar{w}_0 = \bar{z}_0$, hence $z_0$ belongs to $T^*_Z X$ and $c_n(w_n - z'_n) \xrightarrow[n]{} 0$ thus

$$c_n(z_n - w_n) \xrightarrow[n]{} \theta$$

as $\bar{w}_n \in SS(\mathcal{L})$, $w_n \in T^*_Z X$ and $\theta \in C(SS(\mathcal{M}); T^*_Z X)$: this is a contradiction.

**8.2.** If we use Theorem 2.2.1 in place of Theorem 2.3.1 we get, by the same arguments, the following result that we could not obtain directly by the complex method.

THEOREM 8.2.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two coherent $\mathcal{E}_X$-modules on an open set $U \subset T^*X$. Let $Z$ be a closed set of $U$ and $x$ a point outside the interior of $Z$. Assume that any conormal to $Z$ at $x$ is non micro-characteristic for $(\mathcal{M}, \mathcal{N})$. Then*

$$(\mathbf{R}\Gamma_Z \mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}^{\mathbf{R}}))_x = 0.$$

*The result remains true if we replace $\mathcal{N}^{\mathbf{R}}$ by $\mathcal{N}^{\infty}$ and if we assume that $Z$ is invariant by $\mathbf{C}^*$.*

We give many applications of Theorem 8.1.1 in [19] and show in particular how it allows us to extend to (overdetermined) systems the results of Hamada and Hamada-Leray-Wagschal (cf. [8], [9]). But even for a single equation our hypothesis is weaker than those in [9] ("non microcharacteristic" instead of "constant multiplicities").

Let us give another example. Let $X=\mathbf{C}^p\times\mathbf{C}^q$ with coordinates $(x, t)$. Let $\varphi(t)$ be a holomorphic function on $X$, which does not depend on $x$, $\varphi\not\equiv 0$, $S$ the hypersurface of $X$ given by $\varphi=0$ ($S$ may be singular). Let $P$ be a differential operator whose principal part is a polynomial (with holomorphic coefficients on $X$) in $D_{x_1}, \dots, D_{x_p}$, $\varphi(t)D_{t_1}, \dots, \varphi(t)D_{t_q}$. We assume the hypersurface $x_1=0$ non characteristic. We prove in [19] using Theorem 8.1.1 that the Cauchy problem is well posed on $x_1=0$ with holomorphic data on $X-S$ (in a neighborhood of $x_1=0$). If we use Theorem 8.2.1 instead of Theorem 8.1.1 we get: Let $\Omega$ be a pseudo-convex open set with $C^1$-boundary. Assume $0\in\partial\Omega$, and $(1, 0 \dots 0)$ is the conormal of $\Omega$ at 0. Let $f$ be a holomorphic function on $\Omega-S\cap\Omega$, such that $Pf$ extends to $X-S$ in a neighborhood of 0. Then the same is true for $f$. Moreover, if $g$ is holomorphic on $\Omega-S\cap\Omega$, there exists a solution $f$ of the equation $Pf=g$ which is holomorphic on $(\Omega-S\cap\Omega)\cap U$ for a neighborhood $U$ of 0.

## § 9. Application II: Propagation of singularities

**9.1.** In this section, we generalize the results of J. M. Bony and P. Schapira [5] (cf. also [1], [11]) and extend them to systems of micro-differential equations. Let $M$ be a real analytic manifold of dimension $n$, $X$ a complexification of $M$, and $N$ a real analytic submanifold of $X$ of local type $(q, n-q)$ which contains $M$. We set:

$$\Lambda = T_N^*X\cap T_M^*X.$$

It is clear by the definitions that there exists a natural homomorphism:

$$\mathcal{C}_{N|X}|_\Lambda\to\mathcal{C}_{M|X}|_\Lambda$$

and this homomorphism is injective (cf. [5, Theorem 6.2]), but we do not need this fact here.

THEOREM 9.1.1. *With the preceding notations let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module defined on an open set $U\subset T^*X$. We assume that for any $\theta\in T_{\Lambda\cap U}(T_M^*X)$, $\theta\neq 0$, $\theta$ is non micro-characteristic for $(\mathcal{M}, T_N^*X)$. Then the natural homomorphism*

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{N|X}|_\Lambda)\to\mathbf{R}\,\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{M}|_\Lambda)$$

*is an isomorphism.*

*Proof.* We may assume, with $p=n-q$:

$$X=\mathbf{C}^p\times\mathbf{C}^q$$
$$M=\mathbf{R}^p\times\mathbf{R}^q$$
$$N=\mathbf{C}^p\times\mathbf{R}^q.$$

Let us define, as in § 6, the following:

$$\tilde{X}=\bar{\mathbf{C}}^p\times\mathbf{C}^p\times\mathbf{C}^q$$
$$\tilde{M}=\bar{\mathbf{C}}^p\underset{\bar{\mathbf{C}}^p}{\times}\mathbf{C}^p\times\mathbf{R}^q$$
$$\Delta=\bar{\mathbf{C}}^p\underset{\bar{\mathbf{C}}^p}{\times}\mathbf{C}^p\times\mathbf{C}^q.$$

We identify $X$ with its image $\Delta$ in $\tilde{X}$ by the diagonal map. Let $\mathcal{L}$ be the $\mathcal{E}_{\bar{\mathbf{C}}^p}$-module given by the equations:

$$\mathcal{L}:\frac{\partial}{\partial\bar{z}_j}u=0,\qquad j=1,\dots,p.$$

We have seen (§ 6)

$$T_N^*X\simeq T_{\tilde{M}}^*\tilde{X}\cap(SS(\mathcal{L})\times T^*(\mathbf{C}^p\times\mathbf{C}^q)).\tag{9.1.1}$$

$$\mathbf{C}_{N|X}\simeq\mathbf{R}\operatorname{Hom}_{\mathcal{E}_{\bar{\mathbf{C}}^p}}(\mathcal{L},\mathbf{C}_{\tilde{M}})\tag{9.1.2}$$

and it is thus enough to prove:

$$\mathbf{R}\operatorname{Hom}_{\mathcal{E}_X}(\mathcal{M},\mathbf{C}_M)|_\Lambda\simeq\mathbf{R}\operatorname{Hom}_{\mathcal{E}_{\tilde{X}}}(\mathcal{L}\hat{\otimes}\mathcal{M},\mathbf{C}_{\tilde{M}})|_\Lambda.$$

But $(\mathcal{L}\hat{\otimes}\mathcal{M})|_X=\mathcal{M}$, and by Theorem 2.3.1, it is sufficient to show that $X$ (identified with $\Delta$) is microhyperbolic for $\mathcal{L}\hat{\otimes}\mathcal{M}$.

The same argument as in Lemma 8.1.2 shows, by (9.1.1) that if $\theta$ belongs to $T(T^*X)$, and

$$\theta\notin C(SS(\mathcal{M});T_N^*X)$$

then

$$(0,\theta)\notin C(SS(\mathcal{L})\times SS(\mathcal{M});T_{\tilde{M}}^*\tilde{X})$$

which achieves the proof of the theorem.

**9.2.** We can now prove the theorem of "propagation of singularities". Note that our method, that is, using an intermediate sheaf of microfunctions with holomorphic parameters, is the same as in [5].

THEOREM 9.2.1. *Let $\Lambda$ be an involutive submanifold of $T^*_M X$, $\Lambda^{\mathbf{C}}$ the complexification of $\Lambda$ in $T^*X$, $\tilde{\Lambda}$ the union of complex bicharacteristic leaves of $\Lambda^{\mathbf{C}}$ issued from $\Lambda$. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module on $U \subset T^*X$. We assume that for any $\theta \in T_\Lambda(T^*_M X)$, $\theta \neq 0$, $\theta$ is non microcharacteristic for $(\mathcal{M}, \tilde{\Lambda})$.*

*Let $u$ be a section of* $\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M|_\Lambda)$. *Then the support of $u$ is a union of bicharacteristic leaves of $\Lambda$.*

*Proof.* We use the same trick as in § 7.2. The section $u \otimes \delta_t$ belongs to

$$\mathrm{Hom}_{\mathcal{E}_{X \times \mathbf{C}}}(\mathcal{M} \hat{\otimes} \delta_t, \mathcal{C}_{M \times \mathbf{R}}|_{\Lambda'})$$

where $\Lambda' = \{(x, t; i(\xi, \theta)); (x, i\xi) \in \Lambda\}$ and if $u \otimes \delta_t$ is zero at some point $(x^0, 0; i(\xi_0, 1))$, then $u$ is zero at $(x^0, i\xi_0)$. As the hypothesis of the theorem are satisfied for $\mathcal{M} \hat{\otimes} \delta_t$ and $\Lambda'$, we may assume from the beginning that $\Lambda$ is regular,that is $\omega|_\Lambda \neq 0$ where $\omega$ is the canonical 1-form on $T^*_M X$. Thus we are, by a real quantized canonical transformation, in the situation of Theorem 9.1.1. It remains to apply Theorem 9.1.1 and to remark that the support of a section of $\mathcal{C}_{N|X}$ in $T^*_N X$ is a union of complex bicharacteristic leaves, by Theorem 2.2.9 of [24], Chapter 3.

*Remark.* Our condition is weaker than the condition of [5], which is equivalent to say that $\theta$ is non microcharacteristic for $(\mathcal{M}, \Lambda^{\mathbf{C}})$.

Let us take an example to see the difference.

Let $M = \mathbf{R}^p \times \mathbf{R}^q$, $P$ be a micro-differential operator whose principal symbol $P_m$ is written with the coordinates $(x, t; i(\xi, \tau))$ on $iT^*M$

$$P_m(x, t; i\xi, i\tau) = Q_m(x, t; i\xi, i\tau) + R_m(t; i\xi, i\tau)$$

with:

$$Q_m(x, t; i\xi, i\tau) \geqslant c|\xi|^2 \quad \text{for some } c > 0,$$
$$Q_m(x, t; i\xi, i\tau) = 0 \quad \text{for } \xi = 0,$$

and

$$R_m(t; i\xi, i\tau) \geqslant 0.$$

There exists $h > 0$ such that

$$|\xi| > h[|\Delta t| + |\Delta\xi| + |\Delta\tau|], \ |y| < 1$$

implies

$$P_m(x + iy, t + i\Delta t; i\xi + \Delta\xi, i\tau + \Delta\tau) \neq 0.$$

If $\Lambda$ denotes the manifold $\xi = 0$ in $T^*_M X$, the conditions of Theorem 9.1.1 are satisfied and we have propagation "in $x$".

For example let us take on $\mathbf{R}^3$:

$$P = D_x^2 + D_{t_1}^2 + (t_1^2 + t_2^2) D_{t_2}^2$$

then the analytic singularities of the hyperfunctions solution of $Pu=0$ will propagate along the line $t_1 = t_2 = 0$.

## § 10. Application III: Holonomic systems

**10.1** Let $M$ be a real analytic manifold, $X$ a complexification of $M$, and $\mathcal{M}$ a holonomic system of micro-differential equations defined on an open set $U$ of $T^*X$. We shall show

THEOREM 10.1.1. *For any $j$, the group* $\mathrm{Ext}^j(\mathcal{M}, \mathcal{C}_M)$ *are constructible sheaves; that is, there is a stratification of $U \cap T_M^*X$ satisfying the condition of Whitney such that* $\mathrm{Ext}^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)$ *is locally constant on each stratum.*

In the case of system of differential equations, this is proved in [15].

**10.2.** Let $X$ be an analytic manifold, $Y$ a submanifold of $X$ (over $\mathbf{C}$ or $\mathbf{R}$), and $T_Y X$ the normal bundle of $Y$. Let $\theta$ be a 1-form on $X$ whose restriction on $Y$ vanishes. Then $\theta$ defines a linear function on $T_Y X$, which we shall denote $l_Y(\theta)$.

Suppose $Y$ is locally defined by $f_1 = \ldots = f_l = 0$. Then $\theta$ is written in the form $\theta = \sum a_j df_j + \sum f_j \eta_j$ with 1-forms $\eta_j$ and functions $a_j$. Set $v_j = \sigma(f_j)$ the linear function corresponding to $f_j$ or equivalently $l_Y(df_j)$. Therefore

$$l_Y(\theta) = \sum a_j \sigma(f_j).$$

*Definition 10.2.1.* We define the 1-form $\sigma_Y(\theta)$ on $T_Y X$

$$\sigma_Y(\theta) = \sum a_j d\sigma(f_j) + \sum \sigma(f_j)\eta_j.$$

PROPOSITION 10.2.2. *$\sigma_Y(\theta)$ is well-defined.*

*Proof.* First, we shall show that $\sigma_Y(\theta)$ does not depend on the choice of $a_j$ and $\eta_j$. Suppose that $\theta$ has two expressions:

$$\theta = \sum a_j df_j + f_j \eta_j = \sum a'_j df_j + f_j \eta'_j.$$

Since $\sum (a_j - a'_j) df_j \equiv 0 \bmod (f_1, \ldots, f_l)$, we have $a'_j = a_j + \sum b_{jk} f_k$ for some functions $b_{jk}$. Therefore, we have

$$\sum f_j \eta_j = \sum_{j,k} b_{jk} f_k df_j + \sum f_j \eta'_j.$$

or equivalently

$$\sum f_j(\eta_j - \sum_k b_{kj} df_k - \eta'_j) = 0.$$

Therefore $\eta_j - \sum b_{kj} df_k - \eta'_j \equiv 0 \bmod (f_1, \dots, f_l)$. Thus we can write

$$\eta'_j = \eta_j - \sum_k b_{kj} df_k + f_j \eta_{jk},$$

for some 1-forms $\eta_{jk}$. Then it is easy to see

$$\sum a_j d\sigma(f_j) + \sigma(f_j)\eta_j = \sum a'_j d\sigma(f_j) + \sigma(f_j)\eta'_j$$

because $a_j|_Y = a'_j|_Y$ and $\eta_j|_Y = \eta'_j|_Y$.

Now, we shall show that the definition of $\sigma(\theta)$ does not depend on the choice of $f_j$.

Choose another $\{f'_1, \dots, f'_l\}$ so that $Y$ is defined by $f'_1 = \dots = f'_l = 0$. Then we can write $f_j = \sum c_{jk} f'_k$. Therefore, if $\theta = \sum a_j df_j + \sum f_j \eta_j$, we have $\theta = \sum_{j,k} a_j c_{jk} df'_k + \sum_k f'_k(\sum_j a_j dc_{jk} + \sum_j c_{jk}\eta_j)$. Then we obtain

$$\begin{aligned}\sum_{j,k} a_j c_{jk} d(\sigma f'_k) &+ \sum \sigma(f'_k)(\sum_j a_j dc_{jk} + \sum_j c_{jk}\eta_j) \\ &= \sum_j a_j d(\sum_k c_{jk}\sigma(f'_k)) + \sum_j(\sum_k c_{jk}\sigma(f'_k))\eta_j \\ &= \sum_j a_j d\sigma(f_j) + \sum \sigma(f_j)\eta_j,\end{aligned}$$

which shows the result. Q.E.D.

Note that $\sigma_Y(\theta) - dl_Y(\theta)$ is zero modulo functions vanishing on the zero section of $T_Y X$.

**10.3.** Let $f$ be a map from $X'$ to $X$ and let $Y'$, $Y$ be submanifolds of $X'$ and $X$ respectively. Suppose that $f(Y') \subset Y$. If a 1-form $\theta$ on $X$ vanishes on $Y$, then $f^*\theta$ vanishes on $Y'$. Let $\tilde{f}$ be the canonical map $T_{Y'}X' \to T_Y X$. Then we have

$$\tilde{f}^*\sigma_Y(\theta) = \sigma_{Y'}(f^*\theta). \tag{10.3.1}$$

In fact, it is enough to check if for $\theta = dg$ or $\theta = g\eta$ where $g$ is a function on $X$ vanishing on $Y$ and $\eta$ is a 1-form on $X$. We have

$$\tilde{f}^*\sigma_Y(dg) = \tilde{f}^* d\sigma_Y(g) = d(\sigma_Y(g)\circ\tilde{f}) = d\sigma_{Y'}(g\circ f) = \sigma_{Y'}(d(g\circ f)),$$

and

$$\tilde{f}^*\sigma_Y(g\eta) = \tilde{f}^*(\sigma_Y(g)\eta) = \sigma_{Y'}(g\circ f)f^*\eta = \sigma_{Y'}(f^*(g\eta)).$$

**10.4.** Let $V$ be a subanalytic set of $X$.

**Proposition** 10.4.1. *Let $X$ be a real analytic manifold, $Y$ a submanifold, $\theta$ a 1-form on $X$ which vanishes on $Y$. If $\theta$ vanishes on a subanalytic set $V$ (i.e. $\theta|_V=0$ at a non-singular locus of $V$), then $\sigma_Y(\theta)|_{C_Y(V)}=0$ and $C_Y(V)\subset\{l_Y(\theta)=0\}$. (For subanalytic sets, we refer to [10].)*

*Proof.* Considering the blowing up of $X$ with center $Y$ and using the result in § 10.3, we may assume without loss of generality that $Y$ is a hypersurface of $X$. The question being local, we assume $X=\{(t,x)\in\mathbf{R}^{n+1}\}$ and $Y=\{(t,x)\in X;\ t=0\}$, and we may assume that $V$ is contained in $t>0$.

Let $\pi$ be the projection from $\{v\in T_YX;\ \sigma(t)(v)\geqslant 0\}$ onto $Y$. Then $C_Y(V)=\pi^{-1}(\overline{V}\cap Y)$.

Let us denote $\theta=a(t,x)dt+t\eta$. It is then enough to show that

$$a(0,x)=0 \tag{10.4.1}$$

on $\overline{V}\cap Y$ and $\eta|_{\overline{V}\cap Y}=0$. Since $\overline{V}$ is a subanalytic set there is a proper map $\varphi$: $W\to X$ such that $\varphi(W)=V$. Let $W_1$ be the union of connected components where $t$ is identically zero. Then $\varphi(W_1)\subset Y$, and hence $\varphi(W-W_1)=\overline{V}$. Therefore, we may assume that $t$ is not identically zero on each connected component of $W$.

Let us show that

$$a|_{\varphi^{-1}(Y)}=\eta|_{\varphi^{-1}(Y)}=0. \tag{10.4.2}$$

In order to show that it is enough to consider a generic point where $t$ has the form $g^m$ with $dg\neq 0$. Then $\varphi^*\theta=ad(g^m)+g^m\eta=mg^{m-1}(adg+g\eta)=0$ and hence $adg+g\eta=0$. It implies $a$ is a multiple of $g$, say $a=bg$. Then $\eta=-bdg$. This shows (10.4.2). Since $\varphi(\varphi^{-1}Y)=\overline{V}\cap Y$, we have $a|_{Y\cap\overline{V}}=\eta|_{Y\cap\overline{V}}=0$ by (10.4.2). Q.E.D.

**10.5.** Let $(X,\omega)$ be a homogeneous symplectic manifold of dimension $2n$; i.e. $\omega$ is a 1-form on a manifold $X$ such that $\omega$ and $(d\omega)^n$ does not vanish at any point.

Let $\Lambda$ be a homogeneous Lagrangian manifold (i.e. a manifold of dimension $n$ on which $\omega$ vanishes). Then, $T_\Lambda X$ and $T^*\Lambda$ are identified by the Hamilton map $H$: $T^*X \xrightarrow{\sim} TX$.

Let $\omega_\Lambda$ be the fundamental 1-form on $T^*_\Lambda X$. Then we have

**Proposition** 10.5.1. $\omega_\Lambda+\sigma_\Lambda(\omega)=dl_\Lambda(\omega)$.

*Proof.* Take a local coordinate system $(x_1,\ldots,x_n,\xi_1,\ldots,\xi_n)$ such that $d\omega=\sum d\xi_j\wedge dx_j$ and $\Lambda=\{\xi=0\}$. If we identify $T_\Lambda X$ with $X$ by this linear structure, $\omega_\Lambda=-\langle\xi,dx\rangle$. We can write $\omega=\langle\xi,dx\rangle+d\varphi$. Since $\omega|_\Lambda=0$, $d\varphi|_\Lambda=0$, and hence we may assume $\varphi|_\Lambda=0$. Then $l_\Lambda(\omega)=\sigma_\Lambda(\varphi)$ and $\sigma_\Lambda(\omega)=\langle\xi,dx\rangle+d\sigma_\Lambda(\varphi)$, which shows the result. Q.E.D.

**Theorem** 10.5.2. *Let $V$ be a homogeneous Lagrangian subanalytic set. Then $C_\Lambda(V)$ is an isotropic subanalytic set of $(T^*\Lambda,\omega_\Lambda)$ and is contained in the zero of $l_\Lambda(\omega)$.*

This is an immediate consequence of Proposition 10.4.1 and the preceding proposition.

**10.6.** Let $X$ be a real analytic manifold.

*Definition 10.6.1.* Let $V$ be a conic subset of $T^*V$. A locally closed set $Y$ of $X$ is called flat at $y \in Y$ with respect to $V$ if

$$C(V; \pi^{-1}(Y))_p \subset \{v \in T_p(T^*X);\ \langle v, \omega(p)\rangle \geqslant 0\},$$

for any point $p$ in $\pi_X^{-1}(y)$.

LEMMA 10.6.2. *If a submanifold $Y$ is flat with respect to $V$, then $\pi^{-1}(Y) \cap V \subset T_Y^*X$.*

*Proof.* Take a point $p$ in $V \cap \pi^{-1}(Y)$. Then $C_{\pi^{-1}(Y)}(V)$ contains $T_p(\pi^{-1}(Y))$. Hence $\omega(p)=0$ on $T_p(\pi^{-1}(Y))$. This is equivalent to say that $p$ belongs to $T_Y^*X$.

PROPOSITION 10.6.3. *Suppose that $X$ is an open set in $\mathbf{R}^N$ and that a subset $Y$ is flat with respect to a conic set $V$ in $T^*X$ at a point $x_0$. Then there is $\varepsilon > 0$ such that $(x; y-x)$ does not belong to $V$ for $x \in X$, $y \in Y$ satisfying $|x-x_0|$, $|y-x_0| < \varepsilon$, $x \neq y$.*

*Proof.* We shall prove the proposition by contradiction.

If the proposition is false, then there are sequences $x_n \in X$ and $y_n \in Y$ which converges to $x_0$ such that $(x_n; y_n-x_n)$ is contained in $V$ and $x_n \neq y_n$. Let $c_n > 0$ be a sequence such that $c_n(y_n-x_n)$ tends to $v \neq 0$. Then, $(x_n; c_n(y_n-x_n))$ is a sequence in $V$ which converges to $p=(x_0; v)$ and $(y_n; c_n(y_n-x_n))$ is a sequence in $\pi^{-1}Y$ which converges to $p$. Since $c_n((x_n; c_n(x_n-y_n))-(y_n; c_n(x_n-y_n)))$ converges to $(-v, 0)$, $(-v, 0)$ belongs to $C_{\pi^{-1}(Y)}(V)$. Thus $\langle(-v, 0), \omega(p)\rangle = -\langle v, v\rangle$, which is a contradiction. Q.E.D.

PROPOSITION 10.6.4. *Let $X = \coprod_\alpha X_\alpha$ be a stratification of Whitney. Then, $V = \coprod_\alpha T_{X_\alpha}^*X$ is a closed subset and each stratum $X_\alpha$ is flat with respect to $V$.*

*Proof.* Let $(x_n; \xi_n)$ be a sequence in $T_{X_\alpha}^*X$ which converges to $(x; \xi)$. We shall prove that $(x; \xi)$ belongs to $T_{X_\beta}^*X$ for $\beta$ such that $X_\beta$ contains $x$. By the condition of Whitney, if $T_{x_n}X_\alpha$ converges to a plane $\tau \subset T_xX$, then $\tau$ contains $T_xX_\beta$. Therefore, the orthogonal $(T_{X_\alpha}^*X)_{x_n}$ converges to $\tau^\perp$ which is contained in $(T_{X_\beta}^*X)_x$. This implies $(x; \xi) \in T_{X_\beta}^*X$. Let us show that $X_\beta$ is flat with respect to $T_{X_\alpha}^*X$. Let $x$ be a point in $X_\beta$, $p=(x, \xi)$ a point in $\pi^{-1}(x)$ and $q$ a point in $C_p(T_{X_\alpha}^*X; \pi^{-1}X_\beta)$. Then there are a sequence $(x_n; \xi_n)$ in $T_{X_\alpha}^*X$, a sequence $(y_n; \eta_n)$ in $\pi^{-1}(X_\beta)$ and a sequence $c_n > 0$ such that $c_n(x_n-y_n; \xi_n-\eta_n)$ converges to $q=(v; w)$ and that $(x_n; \xi_n)$ and $(y_n, \eta_n)$ converge to $p$. Suppose that $T_{x_n}X_\alpha$ converges to a plane $\tau$ in $T_xX$. Then, by the condition of Whitney, $\tau$ contains $v$ and $T_xX_\beta$. Since $p$ is contained in $T_{X_\beta}^*X$, we have $\langle q, \omega(p)\rangle = \langle v, \xi\rangle = 0$. Q.E.D.

*Remark*. Conversely, if $V$ is closed and if each $X_\alpha$ is flat with respect to $V$ then $X=\coprod X_\alpha$ is a stratification of Whitney.

PROPOSITION 10.6.5. *Let $V$ be a closed conic isotropic subanalytic set in $T^*X$. Then there exists a stratification of Whitney $X=\coprod X_\alpha$ of $X$ such that $V$ is contained in $\coprod T^*_{X_\alpha}X$.*

*Proof*. Let $V=\coprod V_\alpha$ be a stratification of Whitney so that $V_\alpha \to Y_\alpha=\pi(Y_\alpha)$ is smooth and $Y_\alpha$ is a submanifold (and subanalytic). Then $V_\alpha$ is contained in $T^*_{Y_\alpha}X$. In fact, let us choose a local coordinate $(x_1, \dots, x_n)$ such that $Y_\alpha$ is defined by $x_1=\dots=x_l=0$. Then on $V_\alpha$, $0=\omega=\sum \xi_j dx_j=\sum_{j=l+1}^n \xi_j dx_j$. The forms $dx_j$ $(l+1\leqslant j)$ are linearly independent on $Y_\alpha$ and hence on $V_\alpha$. This implies $\xi_{l+1}=\dots=\xi_n=0$ on $V_\alpha$. Take a stratification of Whitney $X=\coprod Y'_\beta$ which is a subdivision of $\cup Y_\alpha$. Then this satisfies clearly the required condition. Q.E.D.

**10.7.** Now let us prove Theorem 10.1.1. The method employed here is almost the same as [15]. Let $M$ be a real analytic manifold and $X$ its complexification.

THEOREM 10.7.1. *Let $\mathcal{M}$ be a system of micro-differential equations on $X$, $\Lambda$ the characteristic variety of $\mathcal{M}$ and $V=C_{T^*_MX}(\Lambda)\subset T^*(T^*_MX)$. If a submanifold $Y$ of $T^*_MX$ is flat with respect to $V$, then $\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)|_Y$ is a locally constant sheaf for any $j$.*

*Proof*. Let $(t_1, \dots, t_{2n})$ be a local coordinate system on $T^*_MX$ such that $Y$ is linear, and $y_0$ a point in $Y$. By Proposition 10.6.3, there is $\varepsilon>0$ such that

$$(x; y-x)\notin V \tag{10.7.1}$$

for $x\in T^*_MX$, $y\in Y$ satisfying $|x-y_0|$, $|y-y_0|<2\varepsilon$, $x\neq y$.

Set $U_r(y)=\{x; |x-y|<r\}$. In order to prove the theorem, it is enough to show that

$$\mathrm{Ext}^j(U_\varepsilon(y_0); \mathcal{M}, \mathcal{C}_M) \xrightarrow{\sim} \mathrm{Ext}^j(U_\varrho(y); \mathcal{M}, \mathcal{C}_M) \tag{10.7.2}$$

for $y\in Y$ and $\varrho>0$ such that $|y-y_0|+\varrho<\varepsilon$. In fact, then we have $\mathrm{Ext}^j(U_\varepsilon(y_0); \mathcal{M}, \mathcal{C}_M)\xrightarrow{\sim} \varinjlim \mathrm{Ext}^j(U_\varrho(y); \mathcal{M}, \mathcal{C}_M)\simeq \mathcal{E}xt^j(\mathcal{M}, \mathcal{C}_M)_y$ for any $y\in Y\cap U(y_0)$.

Set $\Omega_t=U_{t\varepsilon+(1-t)\varrho}(ty_0+(1-t)y)$. Then $\Omega_1=U_\varepsilon(y_0)$, $\Omega_0=U_\varrho(y)$. It is easy to check that $\{\Omega_t\}_{0\leqslant t\leqslant 1}$ is an increasing sequence and that

$$\Omega_{t_0}=\bigcup_{t<t_0}\Omega_t \quad 0<t\leqslant 1$$

and

$$\overline{\Omega}_{t_0}=\bigcap_{t>t_0}\Omega_t \quad \text{for } 1>t_0\geqslant 0.$$

Moreover, $\mathcal{M}$ is microhyperbolic by (10.7.1) with respect to $\partial\Omega_t$ and hence

$$\mathrm{Ext}^j_{\Omega_{t_0}-\Omega_t}(\mathcal{M}, C_M)_x = 0 \quad \text{for } x \in \partial\Omega_t,$$

which implies, by the same argument as in [15] or § 4, the desired result (10.7.2). Q.E.D.

Theorem 10.1.1 is immediately proved by this theorem, Theorem 10.5.2 and Proposition 10.6.5.

*Remark.* It has been proven by M. Kashiwara and T. Kawai [18] that for any $x \in T^*_M X \cap U$, any $j$, the vector spaces $\mathrm{Ext}^j_{\mathcal{E}_X}(\mathcal{M}, C_M)_x$ are finite dimensional over $\mathbf{C}$.

**10.8.** If we use Theorem 8.2.1 instead of Theorem 2.2.1 we get, for a complex manifold $X$:

THEOREM 10.8.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two left coherent $\mathcal{E}_X$-modules on an open set $U$ of $T^*X$. We assume that $\mathcal{M}$ and $\mathcal{N}$ are holomonic. Then there is a complex stratification of $U$ satisfying the conditions of Whitney such that for any $j$, the groups* $\mathrm{Ext}^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}^{\mathbf{R}})$ *and* $\mathrm{Ext}^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}^{\infty})$ *are locally constant on each stratum.*

Recall that we set in § 8.

$$\mathcal{N}^{\mathbf{R}} = \mathcal{E}^{\mathbf{R}}_X \underset{\mathcal{E}_X}{\otimes} \mathcal{N}$$

$$\mathcal{N}^{\infty} = \mathcal{E}^{\infty}_X \underset{\mathcal{E}_X}{\otimes} \mathcal{N}.$$

*Note added in proof*

Theorem 2.2.1 is valid in a more general context: we may replace $\mathcal{M}$ with a complex $\mathcal{L}^{\cdot}$, bounded to the left, of free modules of finite rank over $\mathcal{E}^{\infty}_X$ or over $\mathcal{E}^{\mathbf{R}}_X$, and replace the characteristic variety of $\mathcal{M}$ with the union of the closures of the supports of the cohomology groups of $\mathcal{L}^{\cdot}$. This is in fact what we have done in the proof. Let us only notice that the isomorphism of the sheaves $C_M$ and $C_S$ of chapter 7 is compatible with the corresponding isomorphisms of the rings $\mathcal{E}^{\infty}_X$ or $\mathcal{E}^{\mathbf{R}}_X$.

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