A lemma for microlocal sheaf theory in the ∞-categorical setting

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Abstract

Microlocal sheaf theory of [KS90] makes an essential use of an extension lemma for sheaves due to Kashiwara, and this lemma is based on a criterion of the same author giving conditions in order that a functor defined in $\mathbb{R}^{\mathrm{op}}$ with values in the category S*ets* of sets be constant.

In a first part of this paper, using classical tools, we show how to generalize the extension lemma to the case of the unbounded derived category.

In a second part, we extend Kashiwara's result on constant functors by replacing the category S*ets* with the ∞-category of spaces and apply it to generalize the extension lemma to ∞-sheaves, the ∞-categorical version of sheaves.

Finally, we define the micro-support of sheaves with values in a stable $(\infty, 1)$-category.

1 Introduction

Microlocal sheaf theory appeared in [KS82] and was developed in [KS85, KS90]. However, this theory is constructed in the framework of the bounded (or bounded from below) derived category of sheaves $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ on a real manifold M, for a commutative unital ring $\mathbf{k}$, and it appears necessary in various problems to extend the theory to the unbounded derived category of sheaves $\mathsf{D}(\mathbf{k}_M)$. See in particular [Tam08, Tam15].

A crucial result in this theory is [KS90, Prop. 2.7.2], that we call here the "extension lemma". This lemma, which first appeared in [Kas75, Kas83]),

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asserts that if one has an increasing family of open subsets $\{U_s\}_{s\in\mathbb{R}}$ of a topological Hausdorff space M and an object F of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ such that the cohomology of F on U_s extends through the boundary of U_s for all s, then $\mathrm{R}\Gamma(U_s; F)$ is constant with respect to s. A basic tool for proving this result is the "constant functor criterion", again due to Kashiwara, a result which gives a condition in order that a functor $X\colon \mathbb{R}^{\mathrm{op}} \to \mathrm{S}ets$ is constant, where $\mathrm{S}ets$ is the category of sets in a given universe.

In § 2 we generalize the extension lemma to the unbounded setting, that is, to objects of $\mathrm{D}(\mathbf{k}_M)$. Our proof is rather elementary and is based on the tools of [KS90]. This generalization being achieved, the reader can persuade himself that most of the results, such as the functorial behavior of the micro-support, of [KS90] extend to the unbounded case.

Next, we consider an higher categorical generalization of this result. In § 3 we generalize the constant functor criterion to the case where the 1-category $\mathrm{S}ets$ is replaced with the ∞-category $\mathcal{S}$ of spaces. Using this new tool, in § 4.1, we generalize the extension lemma for ∞-sheaves with values in any stable compactly generated ∞-category $\mathscr{D}$. When $\mathscr{D}$ is the ∞-category $\mathrm{Mod}^{\infty}(\mathbf{k}_M)$ of ∞-sheaves of unbounded complexes of $\mathbf{k}$-modules we recover the results of § 2.

Finally, in § 4.2 we define the micro-support of any ∞-sheaf F with general stable higher coefficient.

Remark 1.1. After this paper has been written, David Treumann informed us of the result of Dmitri Pavlov [Pav16] who generalizes Kashiwara's "constant functor criterion" to the case where the functor takes values in the ∞-category of spectra. Note that Corollary 3.2 below implies Pavlov's result on spectra.

2 Unbounded derived category of sheaves

Let $\mathrm{S}ets$ denote the category of sets, in a given universe $\mathscr{U}$. In the sequel, we consider $\mathbb{R}$ as a category with the morphisms being given by the natural order $\leq$.

We first recall a result due to M. Kashiwara (see [KS90, § 1.12]).

Lemma 2.1 (The constant functor criterion). *Consider a functor $X\colon \mathbb{R}^{\mathrm{op}} \to \mathrm{S}ets$. Assume that for each $s \in \mathbb{R}$*

$$(2.1) \qquad\qquad \varinjlim_{t>s} X_t \xrightarrow{\sim} X_s \xrightarrow{\sim} \varprojlim_{r<s} X_r.$$

Then the functor X is constant.

Let $\mathbf{k}$ denotes a unital ring and denote by $\mathrm{Mod}(\mathbf{k})$ the abelian Grothendieck category of $\mathbf{k}$-modules. Set for short

$$
\begin{aligned}
\mathrm{C}(\mathbf{k}) &:= \mathrm{C}(\mathrm{Mod}(\mathbf{k})), \text{ the category of chains complexes of } \mathrm{Mod}(\mathbf{k}), \\
\mathrm{D}(\mathbf{k}) &:= \mathrm{D}(\mathrm{Mod}(\mathbf{k})) \text{ the (unbounded) derived category of } \mathrm{Mod}(\mathbf{k}).
\end{aligned}
$$

We look at the ordered set $(\mathbb{R}, \leq)$ as a category and consider a functor $X\colon \mathbb{R}^{\mathrm{op}} \to \mathrm{C}(\mathbf{k})$. We write for short $X_s = X(s)$.

The next result is a variant on Lemma 2.1 and the results of [KS90, § 1.12].

Lemma 2.2. *Assume that*

(2.2) *for any $k \in \mathbb{Z}$, any $r \leq s$ in $\mathbb{R}$, the map $X_s^k \to X_r^k$ is surjective,*

(2.3) *for any $k \in \mathbb{Z}$, any $s \in \mathbb{R}$, $X_s^k \xrightarrow{\sim} \varprojlim\limits_{r<s} X_r^k$,*

(2.4) *for any $j \in \mathbb{Z}$, any $s \in \mathbb{R}$, $\varinjlim\limits_{t>s} H^j(X_t) \xrightarrow{\sim} H^j(X_s)$.*

Then for any $j \in \mathbb{Z}$, $r, s \in \mathbb{R}$ with $r \leq s$, one has the isomorphism $H^j(X_t) \xrightarrow{\sim} H^j(X_s)$. In other words, for all $j \in \mathbb{Z}$, the functor $H^j(X)$ is constant.

Proof. Consider the assertions for all $j \in \mathbb{Z}$, all $r, s \in \mathbb{R}$ with $r \leq s$:

(2.5) for any $j \in \mathbb{Z}$, $s \in \mathbb{R}$, the map $H^j(X_s) \to \varprojlim\limits_{r<s} H^j(X_r)$ is surjective,

(2.6) for any $j \in \mathbb{Z}$, $r \leq s$, the map $H^j(X_s) \to H^j(X_r)$ is surjective,

(2.7) for any $j \in \mathbb{Z}$, $s \in \mathbb{R}$, the map $H^j(X_s) \to \varprojlim\limits_{r<s} H^j(X_r)$ is bijective.

Assertion (2.5) follows from hypotheses (2.2) and (2.3) by applying [KS90, Prop. 1.12.4 (a)].

Assertion (2.6) follows from (2.5) and hypothesis (2.4) in view of [KS90, Prop. 1.12.6].

It follows from (2.6) that for any $j \in \mathbb{Z}$ and $s \in \mathbb{R}$, the projective system $\{H^j(X_r)\}_{r<s}$ satisfies the Mittag-Leffler condition. We get (2.7) by using [KS90, Prop. 1.12.4 (b)], .

To conclude, apply [KS90, Prop. 1.12.6], using (2.4) and (2.7).   Q.E.D.

Theorem 2.3 (The non-characteristic deformation lemma). *Let[1] M be a Hausdorff space and let $F \in D(\mathbf{k}_M)$. Let $\{U_s\}_{s \in \mathbb{R}}$ be a family of open subsets of M. We assume*

(a) *for all $t \in \mathbb{R}$, $U_t = \bigcup_{s < t} U_s$,*

(b) *for all pairs (s, t) with $s \leq t$, the set $\overline{U_t \setminus U_s} \cap \operatorname{supp} F$ is compact,*

(c) *setting $Z_s = \bigcap_{t > s} \overline{(U_t \setminus U_s)}$, we have for all pairs (s, t) with $s \leq t$ and all $x \in Z_s$, $(R\Gamma_{X \setminus U_t} F)_x \simeq 0$.*

Then we have the isomorphism in $D(\mathbf{k})$, for all $t \in \mathbb{R}$

$$R\Gamma(\bigcup_s U_s; F) \xrightarrow{\sim} R\Gamma(U_t; F).$$

We shall adapt the proof of [KS90, Prop. 2.7.2], using Lemma 2.2.

Proof. (i) Following loc. cit., we shall first prove the isomorphism:

(2.8) $\qquad (a)^s \colon \varinjlim_{t > s} H^j(U_t; F) \xrightarrow{\sim} H^j(U_s; F), \text{ for all } j$

Replacing M with $\operatorname{supp} F$, we may assume from the beginning that $\overline{U_t \setminus U_s}$ is compact. For $s \leq t$, consider the distinguished triangle

$$(R\Gamma_{M \setminus U_t} F)|_{Z_s} \to (R\Gamma_{M \setminus U_s} F)|_{Z_s} \to (R\Gamma_{U_t \setminus U_s} F)|_{Z_s} \xrightarrow{+1} .$$

The two first terms are 0 by hypothesis (c). Therefore $(R\Gamma_{U_t \setminus U_s} F)|_{Z_s} \simeq 0$ and we get

$$0 \simeq H^j(Z_s; R\Gamma_{U_t \setminus U_s} F) \simeq \varinjlim_{U \supset Z_s} H^j(U \cap U_t; R\Gamma_{M \setminus U_s} F), \text{ for all } j$$

where U ranges over the family of open neighborhoods of Z_s.

For any such U there exists t' with $s < t' \leq t$ such that $U \cap U_t \supset U_{t'} \setminus U_s$. Therefore,

$$\varinjlim_{t, t > s} H^j(U_t; R\Gamma_{M \setminus U_s} F) \simeq 0 \text{ for all } j.$$

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[1]Stéphane Guillermou informed us that Claude Viterbo has obtained some time ago a similar result (unpublished).

By using the distinguished triangle $\mathrm{R}\Gamma_{M \setminus U_s} F \to F \to \mathrm{R}\Gamma_{U_s} F \xrightarrow{+1}$, we get (2.8).

(ii) We shall follow [KS06, Prop. 14.1.6, Th. 14.1.7] and recall that if $\mathscr{C}$ is a Grothendieck category, then any object of $\mathrm{C}(\mathscr{C})$ is qis to a homotopically injective object whose components are injective. Hence, given $F \in \mathrm{D}(\mathbf{k}_M)$, we may represent it by a homotopically injective object $F^{\bullet} \in \mathrm{C}(\mathbf{k}_M)$ whose components F^k are injective. Then $\mathrm{R}\Gamma(U_s; F)$ is represented by $\Gamma(U_s; F^{\bullet}) \in \mathrm{C}(\mathbf{k})$. Set

$$ X_s^k = \Gamma(U_s; F^k), \quad X_s = \Gamma(U_s; F^{\bullet}). $$

Then (2.2) is satisfied since F^k is flabby, (2.3) is satisfied since F^k is a sheaf and (2.4) is nothing but (2.8).

(iii) To conclude, apply Lemma 2.2.   Q.E.D.

3 The constant functor criterion for $\mathcal{S}$

3.1 On ∞-categories

The aim of this subsection is essentially notational and references are made to [Lur09, Lur16]. We use Joyal's quasi-categories to model $(\infty, 1)$-categories. If not necessary we will simply use the terminology ∞-categories.

Denote by ∞–Cat the $(\infty, 1)$-category of all $(\infty, 1)$-categories in a given universe $\mathscr{U}$ and by 1–Cat the 1-category of all 1-categories in $\mathscr{U}$.

To $\mathscr{C} \in 1$–Cat, one associates its nerve, $N(\mathscr{C}) \in \infty$–C$at$. Denoting by $N(1$–C$at)$ the image of 1–Cat by N, the embedding $\iota \colon N(1$–C$at) \hookrightarrow \infty$–C$at$ admits a left adjoint h. We get the functors :

$$ \mathrm{h} \colon \quad \infty\text{--C}at \rightleftarrows N(1\text{--C}at) \quad \colon \iota $$

Hence, $\mathrm{h} \circ \iota \simeq \mathrm{id}_1$ and there exists a natural morphism of ∞-functors $\mathrm{id}_\infty \to \iota \circ \mathrm{h}$, where id_1 and id_∞ denote the identity functors of the categories 1–Cat and ∞–Cat, respectively.

Looking at ∞–Cat as a simplicial set, its degree 0 elements are the $(\infty, 1)$-categories, its degree 1 elements are the ∞-functors, etc. Hence the functor h sends a $(\infty, 1)$-category to a usual category, an ∞-functor to a usual functor, etc. Its sends a stable $(\infty, 1)$-category to a triangulated category where the

distinguished triangles are induced by the cofiber-fiber sequences. Moreover, it sends an ∞-functor to a triangulated functor, etc. See [Lur16, 1.1.2.15].

Let $\mathcal{S}$ (resp. $\mathcal{S}_*$) denote the $(\infty, 1)$-category of spaces (resp. pointed spaces) [Lur09, 1.2.16.1]. Informally, one can think of $\mathcal{S}$ as a simplicial set whose vertices are CW-complexes, 1-cells are continuous maps, 2-cells are homotopies between continuous maps, etc. Recall that $\mathcal{S}$ admits small limits and colimits in the sense of [Lur09, 1.2.13]. Moreover, by Whitehead's theorem, a map $f : X \to Y$ in $\mathcal{S}$ is an equivalence if and only if the induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is an isomorphism of sets and for every base point $x \in X$, the induced maps $\pi_n(X, x) \to \pi_n(Y, f(x))$ are isomorphisms for all $n \geq 1$.

It is also convenient to recall the existence of a Grothendieck construction for $(\infty, 1)$-categories. Namely, for any $(\infty, 1)$-category $\mathscr{C}$ we have an equivalence of $(\infty, 1)$-categories

$$(3.1) \qquad \mathrm{St} : \infty\text{--}\mathrm{C}at^{\mathrm{cart}}/\mathscr{C} \simeq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \infty\text{--}\mathrm{C}at) : \mathrm{Un}$$

where on the l.h.s we have the $(\infty, 1)$-category of ∞-functors $\mathscr{D} \to \mathscr{C}$ that are cartesian fibrations and functors that preserve cartesian morphisms (see [Lur09, Def. 2.4.1.1]), and on the r.h.s. we have the $(\infty, 1)$-category of ∞-functors from $\mathscr{C}^{\mathrm{op}}$ to $\infty\text{--}\mathrm{C}at$. See [Lur09, 3.2.0.1]. The same holds for diagrams in $\mathcal{S}$, where we find

$$(3.2) \qquad \mathrm{St} : \infty\text{--}\mathrm{C}at^{\mathrm{Right-fib}}/\mathscr{C} \simeq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathcal{S}) : \mathrm{Un}$$

where this time on the l.h.s. we have the $(\infty, 1)$-category of ∞-functors $\mathscr{D} \to \mathscr{C}$ that are right fibrations. See [Lur09, 2.2.1.2]. The equivalence (3.2) will be useful for the following reason: for any diagram $X : \mathscr{C}^{\mathrm{op}} \to \mathcal{S}$, its limit in $\mathcal{S}$ can be identified with the space of sections of $\mathrm{Un}(X)$ [Lur09, 3.3.3.4]:

$$(3.3) \qquad \lim X \simeq \mathrm{Map}_{\mathscr{C}}(\mathscr{C}, \mathrm{Un}(X)).$$

3.2 A criterion for a functor to be constant

In this subsection, we generalize [KS90, Prop. 1.12.6] to the case of an ∞-functor. Let $X \colon \mathbb{R}^{\mathrm{op}} \to \mathcal{S}$ be an ∞-functor. We set

$$(3.4) \qquad X_s = X(s), \quad \rho_{s,t} \colon X_t \to X_s \, (s \leq t).$$

Lemma 3.1. *Let $X \colon \mathbb{R}^{\mathrm{op}} \to \mathcal{S}$ be an ∞-functor. Assume that for each $s \in \mathbb{R}$, the natural morphisms in $\mathcal{S}$*

$$(3.5) \qquad \operatorname*{colim}_{s<t} X_t \to X_s \to \lim_{r<s} X_r$$

both are equivalences. Then for every $t \geq s$, the morphism $X_t \to X_s$ in $\mathcal{S}$ is an equivalence.

The proof adapts to the case of $\mathcal{S}$ that of [KS90, Prop. 1.12.6] and will also use this result.

Proof.

(Step I) It is enough to prove that for each $c \in \mathbb{R}$, the restriction of X to $\mathbb{R}_{<c}$ is constant.

(Step II: Choosing base points) Let $c \in \mathbb{R}$ and let again X denote the restriction of X to $\mathbb{R}_{<c}$. The hypothesis

$$\lim_{s<c} X_s \simeq X_c$$

ensures that the choice of a base point in X_c determines a compatible system of base points up to homotopy at every X_s with $s < c$, i.e. the choice of a 2-simplex $\sigma : \Delta^2 \to \infty\text{–Cat}$

$$(3.6)$$

$$
\begin{array}{ccc}
 & & \mathcal{S}_* \\
 & \overline{X} \nearrow & \downarrow \\
 & \sigma & \\
\mathrm{N}(\mathbb{R}^{\mathrm{op}}_{<c}) & \xrightarrow[X]{} & \mathcal{S}.
\end{array}
$$

For the reader's convenience we explain how to construct the 2-simplex σ. Thanks to (3.3), the limit $\lim_{s<c} X_s$ can be identified with the category of sections of the right fibration $\mathrm{Un}(X) \to \mathrm{N}(\mathbb{R}_{<c})$. Therefore, the choice of a base point in X_c provides a section of $\mathrm{Un}(X)$, which we can see as a map from the trivial cartesian fibration $\mathrm{Id} : \mathrm{N}(\mathbb{R}_{<c}) \to \mathrm{N}(\mathbb{R}_{<c})$ to $\mathrm{Un}(X)$. Its image via the functor St of (3.2) provides the lifting (3.6).

Recall that the forgetful functor $\mathcal{S}_* \to \mathcal{S}$ preserves filtrant colimits and all small limits and is conservative. Therefore the hypothesis are also valid for $\overline{X}$.

(Step III: working with a fixed choice of base points) Choose any lifting $\overline{X}$ of X. We have for each $n \in \mathbb{N}$, $s \in \mathbb{R}_{<c}$, a short exact sequence[2], called the Milnor exact sequence (see for instance [MP12, Prop. 2.2.9][3]):

$$(3.7) \qquad 0 \to R^1 \lim_{r<s} \pi_{n+1}(\overline{X}_r) \to \pi_n(\lim_{r<s} \overline{X}_r) \to \lim_{r<s} \pi_n(\overline{X}_r) \to 0.$$

Under the hypothesis of the lemma, we get short exact sequences:

$$(3.8) \qquad 0 \to R^1 \lim_{r<s} \pi_{n+1}(\overline{X}_r) \to \pi_n(\overline{X}_s) \to \lim_{r<s} \pi_n(\overline{X}_r) \to 0.$$

For each $n \geq 0$, each $s, t \in \mathbb{R}_{\leq x}$ with $t \geq s$, we shall prove:

$$(3.9) \qquad \text{the map } \operatorname*{colim}_{c>t>s} \pi_n(\overline{X}_t) \to \pi_n(\overline{X}_s) \text{ is bijective,}$$

$$(3.10) \qquad \text{the map } \pi_n(\overline{X}_s) \to \lim_{r<s} \pi_n(\overline{X}_r) \text{ is surjective,}$$

$$(3.11) \qquad \text{the map } \pi_n(\overline{X}_t) \to \pi_n(\overline{X}_s) \text{ is surjective,}$$

$$(3.12) \qquad \text{the map } \pi_n(\overline{X}_s) \to \lim_{r<s} \pi_n(\overline{X}_r) \text{ is bijective.}$$

Assertion (3.9) follows from the hypothesis, the fact that the system $\{t : c > t > s\}$ is cofinal in $\{t : t > s\}$ and the fact that π_n commutes with filtrant colimits for $n \geq 0$.

Assertion (3.10) follows from (3.8).

Let us prove (3.11). By the surjectivity result in [KS90, Prop. 1.12.6], it is enough to prove the surjectivity of $\operatorname*{colim}_{c>t>s} \pi_n(\overline{X}_t) \to \pi_n(\overline{X}_s)$ and $\pi_n(\overline{X}_s) \to \lim_{r<s} \pi_n(\overline{X}_r)$ for all $s \in \mathbb{R}_{<c}$, which follows from (3.9) and (3.10).

By (3.11) we know that the projective systems $\{\pi_n(\overline{X}_r)\}_{r<s}$ satisfy the Mittag-Leffler condition for all $n \geq 0$, $s < c$. Therefore, $R^1 \lim_{r<s} \pi_{n+1}(\overline{X}_r) \simeq 0$ for all n, all $s \in \mathbb{R}_{<c}$ and (3.12) follows from (3.8). Therefore, we have isomorphisms for every $n \geq 0$

$$\operatorname*{colim}_{s<t<c} \pi_n(\overline{X}_t) \simeq \pi_n(\overline{X}_s) \simeq \lim_{r<s} \pi_n(\overline{X}_r).$$

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[2]of groups when $n \geq 1$ and pointed sets when $n = 0$.

[3]The Milnor exact sequence is usually define for $\mathbb{N}^{op}$-towers. However, the argument works for $\mathbb{R}^{op}$-towers as the inclusion $\mathbb{N}^{op} \subseteq \mathbb{R}^{op}$ is cofinal.

Applying [KS90, Prop. 1.12.6], we get that the diagram of sets $s \mapsto \pi_n(\overline{X}_s)$ is constant for every n.

(Step IV: End of the Proof) The conclusion of Step III holds for any lifting $\overline{X}$ of the restriction of X to $\mathbb{R}_{<c}$. As the result holds for $n = 0$, the diagram $s < c \mapsto \pi_0(X_s)$ is also constant, seen as a diagram of sets rather than pointed sets.

To conclude one must show that for any $n \in \mathbb{N}$, $t \geq s \in \mathbb{R}_{<c}$ and for every choice of a base point y in X_t, the induce maps

$$(3.13) \qquad \rho_{s,t}^n \colon \pi_n(X_t, y) \to \pi_n(X_s, \rho_{s,t}^n(y)).$$

are bijective. Since, for $\alpha < c$, $\alpha \mapsto \pi_0(X_\alpha) \in \mathrm{S}ets$ is constant, choosing $l \in \mathbb{R}$ with $t < l < c$, y determines a unique element $\bar{y}$ in $\pi_0(X_l)$ and again using the hypothesis $X_l \simeq \lim_{r<l} X_r$, the choice of a representative for $\bar{y}$ determines an homotopy compatible system of base points at every X_r for $r < l$ and therefore a new lifting $\overline{X}$ of the restriction of X to $\mathbb{R}_{<l}$ whose associated base point at X_t is a representative of y and the composition with π_n provides the maps (3.13). By (3.9), (3.10), (3.11), (3.12) and [KS90, Prop. 1.12.6] the maps (3.13) are isomorphisms. This conclusion holds for any $c \in \mathbb{R}$ and thus for any $t \geq s$ in $\mathbb{R}^{\mathrm{op}}$. Q.E.D.

We refer to [Lur09, 5.5.7.1] for the notion of presentable compactly generated $(\infty, 1)$-category.

Corollary 3.2. *Let $\mathscr{C}$ be a presentable compactly generated $(\infty, 1)$-category and let $X \colon \mathbb{R}^{\mathrm{op}} \to \mathscr{C}$ be an ∞-functor. Assume that for each $s \in \mathbb{R}$, the natural morphisms*

$$\operatorname*{colim}_{s<t} X_t \to X_s \to \lim_{r<s} X_r$$

both are equivalences. Then for any $t \geq s$ the induced map $X_t \to X_s$ is an equivalence.

Proof. Apply the Lemma 3.1 to all mapping spaces $\mathrm{Map}(Z, X_t)$ for each compact object Z. Q.E.D.

Remark 3.3. This result does not apply to $\mathscr{C} = \mathbb{R}^{\mathrm{op}}$ and X the identity functor. Indeed, $\mathbb{R}^{\mathrm{op}}$ is not compactly generated in the sense of [Lur09, 5.5.7.1].

Remark 3.4. As noticed by M.Porta, the category $\mathbb{R}^{\mathrm{op}}$ being contractible, the condition that for any $t \geq s$ the induced map $X_t \to X_s$ is an equivalence, is equivalent to X being a constant functor.

4 Micro-support

4.1 The non-characteristic deformation lemma with stable coefficients

In this subsection, we generalize [KS90, Prop. 2.7.2] and Theorem 2.3 to more general coefficients. Let $\mathscr{D}$ be a presentable compactly generated stable $(\infty, 1)$-category. Given a topological space M we denote by Op_M its category of open subsets. One defines an higher categorical version of sheaves on M as follows. Let $\mathrm{Psh}(M, \mathscr{D})$ denote the $(\infty, 1)$-category of ∞-functors

$$\mathrm{N}(\mathrm{Op}_M^{\mathrm{op}}) \to \mathscr{D}$$

See [Lur09, 1.2.7.2, 1.2.7.3]. The category Op_M is equipped with a Grothendieck topology whose covering of U are the families $\{U_i\}_i$ such that $U_i \subseteq U$ and $\bigcup_\alpha U_\alpha = U$. We let $\mathrm{Sh}(M, \mathscr{D})^\wedge$ denote the full subcategory of $\mathrm{Psh}(M, \mathscr{D})$ spanned by those functors that satisfy the sheaf condition and are hypercomplete. See [Lur09, 6.2.2] and [Lur11a, Section 1.1] for the theory of ∞-sheaves and [Lur09, 6.5.2, 6.5.3, 6.5.4] for the notion of hypercomplete. The $(\infty, 1)$-category $\mathrm{Sh}(M, \mathscr{D})^\wedge$ is again a stable compactly generated $(\infty, 1)$-category and when $M = \mathrm{pt}$, one recovers $\mathrm{Sh}(M, \mathscr{D})^\wedge \simeq \mathscr{D}$.

The usual pullback and push-forward functorialites can be lifted to the higher categorical setting and are given by exact functors. See for instance the discussion in [PYY16, Section 2.4]. Let $j_U \colon U \hookrightarrow M$ be an open embedding and let $a_M \colon M \to \mathrm{pt}$ be the map from M to one point. We introduce the notations

$$\Gamma^\infty(U; \bullet) := a_{M*}^\infty \circ j_{U*}^\infty \circ j_U^{\infty -1} \colon \mathrm{Sh}(M, \mathscr{D})^\wedge \to \mathscr{D},$$

where a_{M*}^∞, j_{U*}^∞, $j_U^{\infty -1}$ are the direct and inverse image functors for $(\infty, 1)$-categories of sheaves. If Z is a closed subset of U, using the cofiber-fiber sequence associated to $\Gamma^\infty(U; \bullet) \to \Gamma^\infty(U \setminus Z; \bullet)$, we define

$$\Gamma_Z^\infty(U; \bullet) \colon \mathrm{Sh}(M, \mathscr{D})^\wedge \to \mathscr{D}.$$

The following result generalizes [KS90, Prop. 2.7.2] and Theorem 2.3 to any context of sheaves with stable coefficients:

Theorem 4.1 (The non-characteristic deformation lemma for stable coefficients). *Let M be a Hausdorff space and let $F \in \mathrm{Sh}(M, \mathscr{D})^{\wedge}$. Let $\{U_s\}_{s \in \mathbb{R}}$ be a family of open subsets of M. We assume*

(a) *for all $t \in \mathbb{R}$, $U_t = \bigcup_{s < t} U_s$,*

(b) *for all pairs (s,t) with $s \leq t$, the set $\overline{U_t \setminus U_s} \cap \mathrm{supp}\, F$ is compact,*

(c) *setting $Z_s = \bigcap_{t > s} \overline{(U_t \setminus U_s)}$, we have for all pairs (s,t) with $s \leq t$ and all $x \in Z_s$, $(\Gamma^{\infty}_{X \setminus U_t} F)_x \simeq 0$.*

Then we have the equivalences in $\mathscr{D}$, for all $s,t \in \mathbb{R}$

$$\Gamma^{\infty}(\bigcup_s U_s; F) \xrightarrow{\sim} \Gamma^{\infty}(U_t; F).$$

We shall almost mimic the proof of [KS90, Prop. 2.7.2].

Proof. (i) We shall prove the equivalences

$$(a)^t : \lim_{s < t} \Gamma^{\infty}(U_s; F) \xleftarrow{\sim} \Gamma^{\infty}(U_t; F),$$
$$(b)^s : \operatorname*{colim}_{t > s} \Gamma^{\infty}(U_t; F) \xrightarrow{\sim} \Gamma^{\infty}(U_s; F).$$

(ii) Equivalence $(a)^t$ is always true by hypothesis (a). Indeed one has $\mathbf{k}_{U_s} \simeq \varprojlim_{r < s} \mathbf{k}_{U_r}$, which implies $\varprojlim_{s < t} \Gamma^{\infty}_{U_s} F \xleftarrow{\sim} \Gamma^{\infty}_{U_t} F$, and the result follows since the direct image functor commutes with $\lim$ (because it is a right adjoint).

(iii) The proof of the equivalence $(b)^s$ for all s is formally the same as the proof of (2.8) which itself mimics that of [KS90, Prop. 2.7.2] and we shall not repeat it.

To conclude, apply Corollary 3.2 to $\mathscr{D}$.   Q.E.D.

Remark 4.2. Let $\mathbf{k}$ denote a commutative unital ring. Theorem 4.1 recovers the result of Theorem 2.3 in the particular case where $\mathscr{D}$ is the ∞-version of the derived category of $\mathbf{k}$, which we will denote as $\mathrm{Mod}^{\infty}(\mathbf{k})$. We define it as follows: let $C(\mathbf{k})$ denote the 1-category of (unbounded) chain complexes over $\mathbf{k}$. One considers the nerve $N(C(\mathbf{k}))$ and settles $\mathrm{Mod}^{\infty}(\mathbf{k})$ as the localization

$\mathrm{N}(\mathrm{C}(\mathbf{k}))[\mathcal{W}^{-1}]$ along the class of edges $\mathcal{W}$ given by quasi-isomorphisms of complexes. This localization is taken inside the theory of $(\infty, 1)$-categories. See [Lur16, 4.1.3.1]. The homotopy category $\mathrm{h}(\mathrm{Mod}^{\infty}(\mathbf{k}))$ is canonically equivalent to $\mathrm{D}(\mathbf{k})$ by the universal properties of the higher and the classical localizations. In this case we settle the notation

$$\mathrm{Mod}^{\infty}(\mathbf{k}_M) := \mathrm{Sh}(M, \mathrm{Mod}^{\infty}(\mathbf{k}))^{\wedge}.$$

The homotopy category of $\mathrm{Mod}^{\infty}(\mathbf{k}_M)$ recovers the usual derived category of (unbounded) complexes of sheaves of $\mathbf{k}$-modules, $\mathrm{D}(\mathbf{k}_M)$. This follows from [Lur11b, Prop. 2.1.8] and the definition of hypercomplete sheaves. When $M = \mathrm{pt}$, one recovers $\mathrm{Mod}^{\infty}(\mathbf{k})$ and $\mathrm{D}(\mathbf{k})$, respectively.

Remark 4.3. If we assume that M is a topological manifold (therefore homotopy equivalent to a CW-complex), then $\mathrm{Sh}(M, \mathscr{D})^{\wedge}$ is equivalent to $\mathrm{Sh}(M, \mathscr{D})$. In other words, sheaves on topological manifolds are automatically hypercomplete. In particular, $\mathrm{Mod}^{\infty}(\mathbf{k}_M)$ is equivalent to the higher category $\mathrm{Sh}(M, \mathrm{Mod}^{\infty}(\mathbf{k}))$ of ∞-sheaves obtained without imposing hyperdescent. To see this we use the fact that every CW-complex can be obtained as a filtered colimit of finite CW-complexes. Then we combine [Lur16, 7.2.1.12, 7.2.3.6, 7.1.5.8, 6.5.2.13].

Remark 4.4. In [KS90, Prop. 2.7.2], Z_s was defined as $Z_s = \overline{\bigcap_{t>s} (U_t \setminus U_s)}$, which was a mistake. This mistake is already corrected in the "Errata" of: https://webusers.imj-prg.fr/ pierre.schapira/books/.

4.2 Micro-support

The definition [KS90, Def. 5.1.2] of the micro-support of sheaves immediately extends to ∞-sheaves with stable coefficients.

Let M be a real manifold of class C^1 and denote by T^*M its cotangent bundle.

Definition 4.5. Let $F \in \mathrm{Sh}(M, \mathscr{D})$. The micro-support of F, denoted $\mu supp(F)$, is the closed $\mathbb{R}^+$-conic subset of T^*M defined as follows. For U open in T^*M, $U \cap \mu supp(F) = \emptyset$ if for any $x_0 \in M$ and any real C^1-function φ on M defined in a neighborhood of x_0 satisfying $d\varphi(x_0) \in U$ and $\varphi(x_0) = 0$, one has $(\Gamma^{\infty}_{\{x; \varphi(x) \geq 0\}}(F))_{x_0} \simeq 0$.

When $\mathscr{D}$ is $\mathrm{Mod}^{\infty}(\mathbf{k})$, one recovers the classical definition of the micro-support.

Remark 4.6. As already mentioned in the introduction, Theorem 2.3 is the main tool to develop microlocal sheaf theory in the framework of classical derived categories. We hope that similarly Theorem 4.1 will be the main tool to develop microlocal sheaf theory in the new framework of sheaves with stable coefficients.

Remark 4.7. In [KS90], the micro-support of F was denoted $\mathrm{SS}(F)$, a short-cut for "singular support". Some people made the remark that this notation had very bad historical reminiscences and that is the reason of this new terminology, $\mu supp$.

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