Tomography of constructible functions*

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Abstract

An explicit inversion formula for general integral transforms is given in the framework of constructible functions. It applies in particular to the real Radon transform in any dimension or the real X-rays transform in even dimension. For example, it allows us to reconstruct a body in a three dimensional vector space from the knowledge of the number of connected components and the number of holes of all its intersection by two dimensional affine slices.

Contents

1 Introduction

A constructible function on a manifold X is a $\mathbb{Z}$-valued function which is constant on a stratification. This stratification may be algebraic, complex analytic, P-L linear, etc. according to the situation. Here we shall work with real analytic manifolds and subanalytic stratifications. In [6, 7, 8], a "Euler calculus" for constructible functions is developed, and its applications to computational geometry ("the piano mover problem") is emphazised. In this paper, we treat general integral transforms. More precisely, consider a double morphism of real analytic manifolds:

$$\begin{array}{ccc} & S & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

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and let φ be a constructible function on X. Define its "Radon transform" $\mathscr{R}(\varphi)$ by the formula:

$$\mathscr{R}(\varphi) = \int_g f^*(\varphi).$$

Under suitable hypotheses, we give a reconstruction formula which allows us to recover φ from $\mathscr{R}(\varphi)$. Note that the integral should be understood in the sense of constructible functions, and for example, the integral of the characteristic function of a subanalytic compact set is nothing but its Euler-Poincaré index. This is a purely topological invariant. Our hypotheses are satisfied in some cases of real flag correspondences. This includes in particular the usual Radon transform in any dimension (which is nothing but real projective duality), or the so-called X-rays transform (also called the Penrose transform in the complex case) in even dimension. As an amazing application, consider a body (a compact subanalytic subset) in $\mathbb{R}^3$. Then, one can reconstruct it from the knowledge of the integrals of the characteristic functions of all its slices, that is, from the knowledge of the number of connected components minus the number of holes of all its slices.

Note that in the complex case, the same calculus as ours was independently and at the same period, introduced by Viro [9], who already gave an inversion formula for the complex Radon transform of algebraic constructible functions. Let us also mention the paper [5] which treats curved polygons, and the recent paper [2] which considers transformations of flag manifolds.

2 Review on constructible functions

In this section we recall without proofs the main constructions and results on constructible functions. For more details, we refer to [6, 8, 3].

Let X be a real analytic manifold. Recall that the family of subanalyitc subsets of X contains the family of semi-analytic subsets (those locally defined by analytic inequalities), and is stable by closure, complement, inverse images and proper direct images.

Definition 2.1. A function $\varphi : X \to \mathbb{Z}$ is constructible if:

(i) for all $m \in \mathbb{Z}, \varphi^{-1}(m)$ is subanalytic,

(ii) the family $\{\varphi^{-1}(m)\}_{m \in \mathbb{Z}}$ is locally finite.

It follows from Hardt triangulation theorem (a generalization of a theorem of Lojasiewicz) that φ is constructible if and only if there exists a locally finite family of compact subanalytic *contractible* subsets $\{K_i\}_i$, such that:

$$\varphi = \sum_i c_i \mathbf{1}_{K_i},$$

where $c_i \in \mathbb{Z}$ and $\mathbf{1}_A$ is the characteristic function of the subset A.

If φ has compact support, one may assume that the sum above is finite, and one checks that the integer $\sum_i c_i$ depends only on φ, not on its decomposition. One sets:

$$\int_X \varphi = \sum_i c_i.$$

Notice that if K is a compact subanalytic subset of X, then $\int_X \mathbf{1}_K = \chi(K)$, the Euler-Poincaré index of K. In other words, if

$$b_j(K) \quad := \quad \dim_{\mathbb{Q}} \mathrm{H}^j(K; \mathbb{Q}_K),$$

is the j-th Betti number of K, then:

$$\begin{aligned} \chi(K) &= \sum_j (-1)^j \mathrm{b}_j(K) \\ &= \int_X \mathbf{1}_K. \end{aligned}$$

(Recall that $\mathbb{Q}_K$ denotes the constant sheaf on K with stalk $\mathbb{Q}$.)

Let $CF(X)$ denote the group of constructible functions on X, and let CF_X denote the presheaf $U \mapsto CF(U)$. This presheaf is actually a sheaf on X.

Let X and Y be two real analytic manifolds. One defines the external product of two constructible functions by the formula:

$$(\varphi \boxtimes \psi)(x, y) = \varphi(x)\psi(y).$$

By this formula, one gets a morphism of sheaves:

$$CF_X \boxtimes CF_Y \to CF_{X \times Y}.$$

Now, let $f : Y \to X$ be a morphism of manifolds. One defines the inverse image of a constructible function φ on X by the formula:

$$f^*\varphi(y) = \varphi(f(y)).$$

By this formula, one gets a morphism of sheaves:

$$f^* : f^{-1}CF_X \to CF_Y.$$

One defines the direct image, or integral, of a constructible function ψ on Y whose support is proper over X by the formula:

$$(\int_f \psi)(x) = \int_Y (\psi \cdot \mathbf{1}_{f^{-1}(x)}).$$

By this formula, one gets a morphism of sheaves:

$$\int_f : f_! CF_Y \to CF_X.$$

(Recall that a section of $f_! CF_Y$ on an open subset $U \subset X$ is a section of CF_Y on $f^{-1}(U)$ such that f is proper on its support. Hence the integral makes sense.)

Finally, one defines the dual of a constructible function φ as follows. Let $x_0 \in X$, and choose a local chart in a neighborhood of x_0. Let $B(x_0; \varepsilon)$ denote the open ball with center x_0 and radius $\varepsilon > 0$ in this chart. Then the integral $\int_X \varphi . \mathbf{1}_{B(x_0;\varepsilon)}$ neither depends on the local chart nor on ε, for $\varepsilon << 1$, and defines $(D_X\varphi)(x_0)$. One gets a morphism of sheaves:

$$D_X : CF_X \to CF_X.$$

Theorem 2.2. *(i) The operations:*

$CF_X \boxtimes CF_Y \to CF_{X\times Y}$,
$f^* : f^{-1}CF_X \to CF_Y$,
$\int_f : f_! CF_Y \to CF_X$,
$D_X : CF_X \to CF_X$
are well defined morphisms of sheaves.

(ii) Duality is an involution ($D_X \circ D_X = id_X$) and commutes to integration:

$$D_X\left(\int_f \psi\right) = \int_f D_Y(\psi).$$

(iii) Inverse and direct images are functorial, that is, if $f : Y \to X$ and $g : Z \to Y$ are morphisms of manifolds, then:
$g^* \circ f^* = (f \circ g)^*$,
$\int_{f\circ g} = \int_f \circ \int_g$.

(iv) Consider a Cartesian square of morphisms of real analytic manifolds:

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X. \end{array}$$

Then, if ψ is a constructible function on Y such that f is proper on its support, one has:

$$g^* \int_f \psi = \int_{f'} (h^*\psi).$$

Recall that the square in (iv) is Cartesian means that Y' is isomorphic to the submanifold $\{(x', y) \in X' \times Y; g(x') = f(y)\}$.

The duality morphism is an important tool closely related to the notion of "link" in algebraic topology (see [4]), but we shall not make use of it here.

3 Inversion formula

Let X and Y be two real analytic manifolds and let $S \subset X \times Y$ be a locally closed subanalytic subset of $X \times Y$. Denote by q_1 and q_2 the first and second projection defined on $X \times Y$ and by f and g the restriction of q_1 and q_2 to S:

$$\begin{array}{ccccc} & & S & & \\ & \swarrow & & \searrow & \\ X & & & & Y \end{array}$$

We shall assume:

(3.1) $\qquad q_2$ is proper on $\overline{S}$, the closure of S in $X \times Y$.

Let φ be a constructible function on X. We define its Radon transform:

$$\begin{aligned}\mathscr{R}_S(\varphi) &= \int_g f^*\varphi \\ &= \int_{q_2} (q_1^*\varphi)\mathbf{1}_S.\end{aligned}$$

This is a constructible function on Y. The aim of this section is to give an inversion formula under general hypotheses.

Let $S' \subset Y \times X$ be another locally closed subanalytic subset, and denote again by q_1 and q_2 the first and second projection defined on $Y \times X$. Denote by f' and g' the restriction of q_1 and q_2 to S', and by r the projection $S \times_Y S' \to X \times X$.

We shall make the hypotheses:

(3.2) $$q_2 \text{ is proper on } \overline{S'}, \text{ the closure of } S' \text{ in } Y \times X,$$

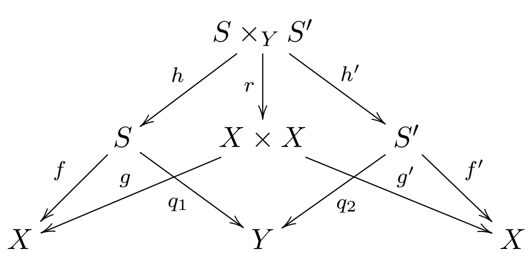
(3.3) $$\text{there exists } \lambda \neq \mu \in \mathbb{Z} \text{ such that} : \chi(r^{-1}(x,x')) = \begin{cases} \lambda & \text{if } x \neq x' \\ \mu & \text{if } x = x'. \end{cases}$$

Notice that $\lambda \neq 0$ implies $\overline{r(S \times_Y S')} = X \times X$, and q_2 being proper on this set, this implies that X is compact.

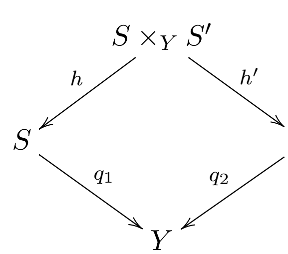
Theorem 3.1. *Assume (3.1), (3.2), (3.3) and let $\varphi \in CF(X)$. Then:*

$$\mathscr{R}_{S'} \circ \mathscr{R}_S(\varphi) = (\mu - \lambda)\varphi + [\int_X \lambda\varphi]\, \mathbf{1}_X.$$

Proof. Denote by h and h' the projections from $S \times_Y S'$ to S and S' respectively. We get the diagram:



Since the square



is Cartesian, we have:

$$
\begin{aligned}
\mathscr{R}_{S'} \circ \mathscr{R}_S(\varphi) &= \int_{f'} (g'^* \int_g (f^* \varphi)) \\
&= \int_{f' \circ h'} (h \circ f)^* \varphi \\
&= \int_{q_2} \int_r r^* q_1^* \varphi \\
&= \int_{q_2} k(x, x') q_1^* \varphi,
\end{aligned}
$$

where

$$
\begin{aligned}
k(x, x') &= \int_r r^* \mathbf{1}_{X \times X} \\
&= \int_r \mathbf{1}_{S \times_Y S'}.
\end{aligned}
$$

Hence, it is enough to notice that, by the hypothesis,

$$\int_r \mathbf{1}_{S \times_Y S'} = (\mu - \lambda)\delta_\Delta + \lambda \mathbf{1}_{X \times X},$$

where δ_Δ is the Dirac function (i.e. the characteristic function) of the diagonal. □

4 Application: correspondences of real flag manifolds

Let E be a real $(n+1)$-dimensional vector space, and denote by $F_{n+1}(p,q), (1 \leq p \leq q \leq n)$ the set of pairs $\{(l,h)\}$ of linear subspaces of E with $l \subset h, \dim l = p, \dim h = q$. This is a real compact submanifold of $F_{n+1}(p) \times F_{n+1}(q)$. We denote by $F_{n+1}(q,p)$ its image by the map $F_{n+1}(p) \times F_{n+1}(q) \to F_{n+1}(q) \times F_{n+1}(p), (x,y) \mapsto (y,x)$. One sets for short, $F_{n+1}(p,p) = F_{n+1}(p)$. Denoting by f and g the natural projections, we get the diagram:

$$
\begin{array}{ccc}
 & F_{n+1}(p,q) & \\
 \swarrow^{f} & & \searrow^{g} \\
F_{n+1}(p) & & F_{n+1}(q)
\end{array}
$$

We shall write for short $\mathscr{R}_{(n+1;p,q)}(\varphi)$ instead of $\mathscr{R}_{F_{n+1}(p,q)}(\varphi)$, the transform of a constructible function φ on $F_{n+1}(p)$ associated to this flag correspondence. In order to apply Theorem 3.1, it is enough to calculate the Euler-Poincaré index of $r^{-1}(x,x')$, where r is the projection:

$$r : F_{n+1}(p,q) \times_{F_{n+1}(q)} F_{n+1}(q,p) \to F_{n+1}(p) \times F_{n+1}(p).$$

Now we shall assume $p = 1, q > 1$. Notice that $F_{n+1}(1) = P_n$, the n-dimensional real projective space associated to E, and $F_{n+1}(n) = P_n^*$, the dual n-dimensional projective space. For $x \neq x', r^{-1}(x,x') \simeq F_{n-1}(q-2)$ and for $x = x', r^{-1}(x,x') \simeq F_n(q-1)$. Set

$$\mu_n(q) = \chi(F_n(q))$$

.

Proposition 4.1. *Let $\varphi \in CF(P_n)$. Then:*

$$\mathscr{R}_{(n+1;q,1)} \circ \mathscr{R}_{(n+1;1,q)}(\varphi) = (\mu_n(q-1) - \mu_{n-1}(q-2))\varphi + [\mu_{n-1}(q-2)\int_{P_n} \varphi]\,\mathbf{1}_{P_n}.$$

Remark 4.2. [1] Using a cellular decomposition of the flag manifold $F_n(p)$, (see for example [1]), one can prove that:

$$\begin{aligned} \mu_n(p) &= 0 \text{if } p(n-p) \text{ is odd} \\ &= \begin{pmatrix} E(n/2) \\ 2\,E(p/2) \end{pmatrix} \text{if } p(n-p) \text{ is even} \end{aligned}$$

where $E(n/2)$ denotes the integral part of $n/2$, $\binom{a}{b}$ is the binomial coefficient, and we have assumed $p \leq E(n/2)$, which is not restrictive since $\mu_n(p) = \mu_n(n-p)$.

5 Example: the Radon transform

Let V be a n-dimensional real vector space, P_n its projective compactification, $P_n = V \sqcup h_\infty$, where h_∞ is the hyperplane at infinity. Let P_n^* be the dual projective space. Then $P_n^* \setminus \{h_\infty\}$ is nothing but the set of affine hyperplanes of V. Let φ be a constructible function on V with compact support, and denote by K its support. We set:

$$K^* = \{\xi \in P_n^*; \xi \cap K \neq \varnothing\}.$$

Then, clearly, K^* is a compact subset of P_n^* which does not contains h_∞. The Radon transform of φ is defined by:

$$\mathscr{R}_(n+1;1,n)(\varphi)(\xi) = \int_V \varphi.\mathbf{1}_\xi$$

and this function on P_n^* is supported by K^*. Hence, to calculate the Radon transform of φ, it is enough to restrict to those hyperplanes ξ of K^*.

Recall that the Euler-Poincaré index of the n-dimensional real projective space P_n is given by the formula:

(5.1)
$$\chi(P_n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Corollary 5.1. *Let $\varphi \in CF(P_n)$. Then:*

$$\mathscr{R}_{(n+1;n,1)} \circ \mathscr{R}_{(n+1;1,n)}(\varphi) = \begin{cases} \varphi & \textit{if } n \textit{ is odd,} \\ -\varphi + [\int_{P_n} \varphi]\mathbf{1}_{P_n} & \textit{if } n \textit{ is even and } n > 0. \end{cases}$$

Now assume $\dim V = 3$ and let us calculate the Radon transform of the characteristic function $\mathbf{1}_K$ of a compact subanalytic subset K of V. First, consider a compact subanalytic subset L of a two dimensional affine vector space W. By Poincaré's duality, there is an isomorphism $\mathrm{H}^1_L(W;\mathbb{Q}_W) \simeq \mathrm{H}^1(L;\mathbb{Q}_L)$ and moreover there is a short exact sequence:

$$0 \to \mathrm{H}^0(W;\mathbb{Q}_W) \to \mathrm{H}^0(W \setminus L;\mathbb{Q}_W) \to \mathrm{H}^1_L(W;\mathbb{Q}_W) \to 0,$$

[1] We thank P. Polo for useful comments on the topology of real flags manifolds.

from which one deduces that:

$$\mathrm{b}_1(L) = \mathrm{b}_0(W \setminus L) - 1.$$

Note that $\mathrm{b}_0(W \setminus L)$ is the number of connected components of $W \setminus L$, hence $\mathrm{b}_1(L)$ is the "number of holes" of the compact set L. We may summarize:

Proposition 5.2. *The value at ξ of the Radon transform of $\mathbf{1}_K$ is the number of connected components of $K \cap \xi$ minus the number of its holes.*

The inversion formula of the Radon transform tells us how to reconstruct the set K from the knowledge of the number of connected components and holes of all its affine slices.

6 Example: the X-rays transform

Again, let V be a real n-dimensional vector space, $P_n \simeq F_{n+1}(1)$ its projective compactification, and consider the correspondence:

$$\begin{array}{ccc} & F_{n+1}(1,2) & \\ {}^{f}\swarrow & & \searrow^{g} \\ F_{n+1}(1) & & F_{n+1}(2) \end{array}$$

Since $\mu_n(1) = 1$ or 0 according whether n is odd or even, and $\mu_{n-1}(0) = 1$, we can only apply Proposition 4.1 (with $q = 2$) when n is even.

Corollary 6.1. *Assume n is even and let $\varphi \in CF(P_n)$. Then:*

$$\mathscr{R}_{(n+1;2,1)} \circ \mathscr{R}_{(n+1;1,2)}(\varphi) = -\varphi + [\int_{P_n} \varphi]\, \mathbf{1}_{P_n}.$$

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