Quantization of complex Lagrangian submanifolds

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Abstract

Let Λ be a smooth Lagrangian submanifold of a complex symplectic manifold $\mathfrak{X}$. We construct twisted simple holonomic modules along Λ in the stack of deformation-quantization modules on $\mathfrak{X}$.

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1 Introduction

Let $\mathfrak{Y}$ be a complex contact manifold. A local model for $\mathfrak{Y}$ is an open subset of the projective cotangent bundle P^*Y to a complex manifold Y. The manifold P^*Y is endowed with the sheaf $\mathcal{E}_Y$ of microdifferential operators of [20]. In [9], Kashiwara proves the existence of a canonical stack $\mathsf{Mod}(\mathsf{E}_{\mathfrak{Y}})$ on $\mathfrak{Y}$, locally equivalent to the stack of $\mathcal{E}_Y$-modules. Let $\Lambda \subset \mathfrak{Y}$ be a smooth Lagrangian submanifold. In the same paper, Kashiwara states that there exists a globally

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defined holonomic system simple along Λ in the stack $\mathsf{Mod}(\mathsf{E}_{\mathfrak{Y}}|_{\Lambda})$ twisted by half-forms on Λ.

Now, let $\mathfrak{X}$ be a complex symplectic manifold. A local model for $\mathfrak{X}$ is an open subset of the cotangent bundle T^*X to a complex manifold X. The manifold T^*X is endowed with the sheaf $\mathcal{W}_X$ of WKB-differential operators, similar to $\mathcal{E}_X$, but with an extra central parameter, a substitute to the lack of homogeneity. (Note that, in the literature, $\mathcal{W}_X$ is also called a deformation-quantization ring, or a ring of semi-classical differential operators. See [19] for a precise description of the ring $\mathcal{W}_X$ and its links with $\mathcal{E}_X$.) A stack $\mathsf{Mod}(\mathsf{W}_{\mathfrak{X}})$ on $\mathfrak{X}$ locally equivalent to the stack of $\mathcal{W}_X$-modules has been constructed in the formal case (in the general setting of Poisson manifolds) by [15] and in the analytic case (and by a different method, similar to [9]) by [19]. (See also [16,2,22] for papers closely related to this subject.)

In this paper, we prove that if Λ is a smooth Lagrangian submanifold of the complex symplectic manifold $\mathfrak{X}$, there exists a globally defined simple holonomic module along Λ in the stack $\mathsf{Mod}(\mathsf{W}_{\mathfrak{X}}|_{\Lambda})$ twisted by half-forms on Λ. As a by-product, we prove that there is an equivalence of stacks between that of twisted regular holonomic modules along Λ and that of local systems on Λ. The local model for our theorem is given by $\mathfrak{X} = T^*X$ and $\Lambda = T^*_X X$, the zero-section of T^*X. In this case, a simple module is the sheaf $\mathcal{O}_X^{\tau}$ whose sections are series $\sum_{-\infty < j \leq m} f_j \tau^j$ $(m \in \mathbb{Z})$, where the f_j's are sections of $\mathcal{O}_X$ and the family $\{f_j\}_j$ satisfies certain growth conditions on compact subsets of X. The problem we solve here is how to patch together these local models.

Our proof consists in showing that if $\mathfrak{X}$ is a complex symplectic manifold and Λ a Lagrangian submanifold, then there exists a "contactification" $\mathfrak{Y}$ of $\mathfrak{X}$ in a neighborhood of Λ. Local models for $\mathfrak{X}$ and $\mathfrak{Y}$ are an open subset of the cotangent bundle T^*X to a complex manifold X and an open subset of the projective cotangent bundle $P^*(X \times \mathbb{C})$, respectively. With the same techniques as in [9,19], we construct a stack $\mathsf{Mod}(\mathsf{E}_{\mathfrak{Y},\hat{t}})$ on $\mathfrak{Y}$ locally equivalent to the stack of modules over the ring $\mathcal{E}_{X \times \mathbb{C}, \hat{t}}$ of microdifferential operators commuting with ∂/∂_t, where t is the coordinate on $\mathbb{C}$. We then apply Kashiwara's existence theorem for simple modules along Lagrangian manifolds in the contact case to deduce the corresponding result in the symplectic case. In fact, the technical heart of this paper is devoted to giving a detailed proof, based on the theory of symbols of simple sections of holonomic modules, of Kashiwara's result stated in [9].

2 Stacks

Stacks were invented by Grothendieck and Giraud [6] and we refer to [14] for an exposition. Roughly speaking, a prestack (resp. a stack) is a presheaf (resp. a sheaf) of categories, as we shall see below.

In sections 2 and 3, we denote by X a topological space. However, all definitions and results easily extend when replacing X with a site, that is, a small category $\mathcal{C}_X$ endowed with a Grothendieck topology.

Definition 2.1. (a) A prestack S on X is the assignment of a category $\mathsf{S}(U)$ for every open subset $U \subset X$, a functor $\rho_{VU} \colon \mathsf{S}(U) \to \mathsf{S}(V)$ for every open inclusion $V \subset U$, and an isomorphism of functors $\lambda_{WVU} \colon \rho_{WV}\rho_{VU} \Rightarrow \rho_{WU}$ for every open inclusion $W \subset V \subset U$, such that $\rho_{UU} = \mathrm{id}_{\mathsf{S}(U)}$, $\lambda_{UUU} = \mathrm{id}_{\mathrm{id}_{\mathsf{S}(U)}}$, and the following diagram of isomorphisms of functors from $\mathsf{S}(U)$ to $\mathsf{S}(Y)$ commutes for every open inclusion $Y \subset W \subset V \subset U$

$$
\begin{array}{ccc}
\rho_{YW}\rho_{WV}\rho_{VU} & \xrightarrow{\lambda_{YWV}\,\mathrm{id}_{\rho_{VU}}} & \rho_{YV}\rho_{VU} \\
{\scriptstyle \mathrm{id}_{\rho_{YW}}\,\lambda_{WVU}} \big\Downarrow & & \big\Downarrow {\scriptstyle \lambda_{YVU}} \\
\rho_{YW}\rho_{WU} & \xrightarrow{\lambda_{YWU}} & \rho_{YU}.
\end{array}
$$

For $F \in \mathsf{S}(U)$ and $V \subset U$, we will write $F|_V$ for short instead of $\rho_{VU}(F)$.

(b) A separated prestack is a prestack S such that for any $F, G \in \mathsf{S}(U)$, the presheaf $\mathcal{H}om_{\mathsf{S}|_U}(F, G)$, defined by $V \mapsto \mathrm{Hom}_{\mathsf{S}(V)}(F|_V, G|_V)$, is a sheaf.

A stack is a separated prestack satisfying suitable glueing conditions, which may be expressed in terms of descent data.

Definition 2.2. Let U be an open subset of X, $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of U and S a separated prestack on X.

(a) A descent datum on $\mathcal{U}$ for S is a pair

$$
(\{\mathcal{F}_i\}_{i \in I}, \{\theta_{ij}\}_{i,j \in I}), \ \text{ with } \mathcal{F}_i \in \mathsf{S}(U_i), \quad \theta_{ij} \colon \mathcal{F}_j|_{U_{ij}} \xrightarrow{\ \sim\ } \mathcal{F}_i|_{U_{ij}} \qquad (2.1)
$$

such that the following diagram of isomorphisms in $\mathsf{S}(U_{ijk})$ commutes

(b) The descent datum (2.1) is called effective if there exist $\mathcal{F} \in \mathsf{S}(U)$ and isomorphisms $\theta_i \colon \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ in $\mathsf{S}(U_i)$ satisfying the natural compatibility conditions with the θ_{ij}'s and λ's.

Note that if the descent datum (2.1) is effective, then $\mathcal{F}$ is unique up to unique isomorphism.

Definition 2.3. A stack is a separated prestack such that for any open subset U of X and any open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of U, the descent datum is effective.

To end this section, let us go up one level, and recall the glueing conditions for stacks.

Definition 2.4. Let U be an open subset of X, $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of U.

(a) A descent datum for stacks on $\mathcal{U}$ is a triplet

$$(\{\mathsf{S}_i\}_{i \in I}, \{\varphi_{ij}\}_{i,j \in I}, \{\alpha_{ijk}\}_{i,j,k \in I}), \tag{2.2}$$

where the S_i's are stacks on U_i, $\varphi_{ij} \colon \mathsf{S}_j|_{U_{ij}} \to \mathsf{S}_i|_{U_{ij}}$ are equivalences of stacks, and $\alpha_{ijk} \colon \varphi_{ij}\varphi_{jk} \Rightarrow \varphi_{ik} \colon \mathsf{S}_k|_{U_{ijk}} \to \mathsf{S}_i|_{U_{ijk}}$ are isomorphisms of functors such that for any $i, j, k, l \in I$ the following diagram of isomorphisms of functors from $\mathsf{S}_l|_{U_{ijkl}}$ to $\mathsf{S}_i|_{U_{ijkl}}$ commutes

$$\begin{array}{ccc} \varphi_{ij}\varphi_{jk}\varphi_{kl} & \xrightarrow{\ \alpha_{ijk}\,\mathrm{id}_{\varphi_{kl}}\ } & \varphi_{ik}\varphi_{kl} \\ {\scriptstyle\mathrm{id}_{\varphi_{ij}}\,\alpha_{jkl}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle\alpha_{ikl}} \\ \varphi_{ij}\varphi_{jl} & \xrightarrow[\quad\alpha_{ijl}\quad]{} & \varphi_{il}. \end{array} \tag{2.3}$$

(b) The descent datum (2.2) is called effective if there exist a stack S on U, equivalences of stacks $\varphi_i \colon \mathsf{S}|_{U_i} \to \mathsf{S}_i$ and isomorphisms of functors $\alpha_{ij} \colon \varphi_{ij}\varphi_j \Rightarrow \varphi_i \colon \mathsf{S}|_{U_{ij}} \to \mathsf{S}_i|_{U_{ij}}$, satisfying the natural compatibility conditions.

Note that if the descent datum (2.3) is effective, then S is unique up to equivalence and such an equivalence is unique up to unique isomorphism.

In the language of 2-categories, the following theorem asserts that *the 2-prestack of stacks is a 2-stack.*

Theorem 2.5. (cf [6,3]) Descent data for stacks are effective.

Denote by S the stack associated with the descent datum (2.3). Objects of $\mathsf{S}(U)$ can be described by pairs

$$(\{\mathcal{F}_i\}_{i \in I}, \{\xi_{ij}\}_{i,j \in I}), \tag{2.4}$$

where $\mathcal{F}_i \in \mathsf{S}_i(U_i)$ and $\xi_{ij}\colon \varphi_{ij}(\mathcal{F}_j|_{U_{ij}}) \to \mathcal{F}_i|_{U_{ij}}$ are isomorphisms in $\mathsf{S}_i(U_{ij})$ such that for $i,j,k \in I$ the following diagram in $\mathsf{S}_i(U_{ijk})$ commutes

$$
\begin{array}{ccc}
\varphi_{ij}\varphi_{jk}(\mathcal{F}_k|_{U_{ijk}}) & \xrightarrow{\varphi_{ij}(\xi_{jk})} & \varphi_{ij}(\mathcal{F}_j|_{U_{ijk}}) \\
\Big\downarrow{\alpha_{ijk}(\mathcal{F}_k)} & & \Big\downarrow{\xi_{ij}} \\
\varphi_{ik}(\mathcal{F}_k|_{U_{ijk}}) & \xrightarrow{\ \xi_{ik}\ } & \mathcal{F}_i|_{U_{ijk}}.
\end{array}
\tag{2.5}
$$

3 Twisted modules

Let us now recall how stacks of twisted modules are constructed in [8,9].

Let $\mathbf{k}$ be a commutative unital ring and $\mathcal{A}$ a sheaf of $\mathbf{k}$-algebras on X. Denote by $\mathrm{Mod}(\mathcal{A})$ the category of left $\mathcal{A}$-modules, and by $\mathsf{Mod}(\mathcal{A})$ the associated stack $X \supset U \mapsto \mathrm{Mod}(\mathcal{A}|_U)$.

Consider an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X, a family of $\mathbf{k}$-algebras $\mathcal{A}_i$ on U_i and $\mathbf{k}$-algebra isomorphisms $f_{ij}\colon \mathcal{A}_j|_{U_{ij}} \to \mathcal{A}_i|_{U_{ij}}$. The existence of a sheaf of $\mathbf{k}$-algebras locally isomorphic to $\mathcal{A}_i$ requires the condition $f_{ij}f_{jk} = f_{ik}$ on triple intersections. The weaker conditions (3.2) and (3.3) below are needed for the existence of a $\mathbf{k}$-additive stack locally equivalent to $\mathsf{Mod}(\mathcal{A}_i)$.

Definition 3.1. A $\mathbf{k}$-algebroid descent datum A on $\mathcal{U}$ is a triplet

$$
\mathsf{A} = (\{\mathcal{A}_i\}_{i\in I}, \{f_{ij}\}_{i,j\in I}, \{a_{ijk}\}_{i,j,k\in I}),
\tag{3.1}
$$

where $\mathcal{A}_i$ is a $\mathbf{k}$-algebra on U_i, $f_{ij}\colon \mathcal{A}_j|_{U_{ij}} \to \mathcal{A}_i|_{U_{ij}}$ is a $\mathbf{k}$-algebra isomorphism, $a_{ijk} \in \mathcal{A}_i^\times(U_{ijk})$ is an invertible section, and (3.2) and (3.3) below are satisfied:

$$
f_{ij}f_{jk} = \mathrm{Ad}(a_{ijk})f_{ik} \text{ as } \mathbf{k}\text{-algebra isomorphisms } \mathcal{A}_k|_{U_{ijk}} \xrightarrow{\sim} \mathcal{A}_i|_{U_{ijk}}, \tag{3.2}
$$
$$
a_{ijk}a_{ikl} = f_{ij}(a_{jkl})a_{ijl} \text{ in } \mathcal{A}_i^\times(U_{ijkl}). \tag{3.3}
$$

(Here $\mathrm{Ad}(a_{ijk})$ denotes the automorphism of $\mathcal{A}_i|_{U_{ijk}}$ given by $a \mapsto a_{ijk}\, a\, a_{ijk}^{-1}$.)

Remark 3.2. The notion of an algebroid stack exists intrinsically, without using coverings or descent data. It has been introduced by [15] and developed in [4]. In this paper, we shall restrict ourselves to algebroids presented by descent data.

Remark 3.3. Let $\mathcal{B}_i \subset \mathcal{A}_i^\times$ be multiplicative subgroups, invariant by f_{ij}, and such that for any $b_i, b_i' \in \mathcal{B}_i$, the equality $\mathrm{Ad}(b_i) = \mathrm{Ad}(b_i')$ implies $b_i = b_i'$. Assume that $a_{ijk} \in \mathcal{B}_i$. Then, as noticed *e.g.*, in [9, pag. 2], condition (3.3) follows from (3.2).

Let us recall how to define the stack of "A-modules" in terms of local data.

A **k**-algebra morphism $f\colon \mathcal{B} \to \mathcal{A}$ induces a functor

$$\widetilde{f}\colon \mathsf{Mod}(\mathcal{A}) \to \mathsf{Mod}(\mathcal{B})$$

defined by $\mathcal{M} \mapsto {}_f\mathcal{M}$, where ${}_f\mathcal{M}$ denotes the sheaf of **k**-vector spaces $\mathcal{M}$, endowed with the $\mathcal{B}$-module structure given by $bm := f(b)m$ for $b \in \mathcal{B}$ and $m \in \mathcal{M}$.

For $a \in \mathcal{A}^\times$ an invertible section, the automorphism $\mathrm{Ad}(a)$ induces the functor $\widetilde{\mathrm{Ad}(a)}$ between $\mathsf{Mod}(\mathcal{A})$ and itself, and we denote by

$$\widetilde{a}\colon \widetilde{\mathrm{Ad}(a)} \Rightarrow \mathrm{id}_{\mathsf{Mod}(\mathcal{A})}$$

the isomorphism of functors given by $\widetilde{a}(\mathcal{M})\colon {}_{\mathrm{Ad}(a)}\mathcal{M} \to \mathcal{M}$, $u \mapsto a^{-1}u$, for $u \in \mathcal{M} \in \mathsf{Mod}(\mathcal{A})$. (Note that $\widetilde{a}(\mathcal{M})(a'u) = a^{-1}aa'a^{-1}u = a'\widetilde{a}(\mathcal{M})(u)$.)

Definition 3.4.(i) The stack of twisted modules associated to the **k**-algebroid descent datum A on $\mathcal{U}$ in (3.1) is the stack defined (using Theorem 2.5) by the descent datum

$$\mathsf{Mod}(\mathsf{A}) = (\{\mathsf{Mod}(\mathcal{A}_i)\}_{i \in I}, \{\widetilde{f}_{ji}\}_{i,j \in I}, \{\widetilde{a}_{kji}\}_{i,j,k \in I}). \tag{3.4}$$

(ii) One sets $\mathrm{Mod}(\mathsf{A}) := \mathsf{Mod}(\mathsf{A})(X)$. Objects of the category $\mathrm{Mod}(\mathsf{A})$ are called twisted modules.

According to (2.4), objects of $\mathrm{Mod}(\mathsf{A})$ are described by pairs

$$\mathcal{M} = (\{\mathcal{M}_i\}_{i \in I}, \{\xi_{ij}\}_{i,j \in I}),$$

where $\mathcal{M}_i$ are $\mathcal{A}_i$-modules and $\xi_{ij}\colon {}_{f_{ji}}\mathcal{M}_j|_{U_{ij}} \to \mathcal{M}_i|_{U_{ij}}$ are isomorphisms of $\mathcal{A}_i$-modules, such that for any $u_k \in \mathcal{M}_k$ one has

$$\xi_{ij}({}_{f_{ji}}\xi_{jk}(u_k)) = \xi_{ik}(a_{kji}^{-1}u_k) \tag{3.5}$$

as morphisms ${}_{f_{kj}f_{ji}}\mathcal{M}_k \to \mathcal{M}_i$. Indeed, (3.5) translates the commutativity of (2.5).

Example 3.5. Let X be a complex manifold, and denote by Ω_X the sheaf of holomorphic forms of maximal degree. Take an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that there are nowhere vanishing sections $\omega_i \in \Omega_{U_i}$. Let $t_{ij} \in \mathcal{O}_{U_{ij}}^\times$ be the transition functions given by $\omega_j|_{U_{ij}} = t_{ij}\omega_i|_{U_{ij}}$. Choose determinations $s_{ij} \in \mathcal{O}_{U_{ij}}^\times$ for the multivalued functions $t_{ij}^{1/2}$. Since $s_{ij}s_{jk}$ and s_{ik} are both determinations of $t_{ik}^{1/2}$, there exists $c_{ijk} \in \{-1, 1\}$ such that

$$s_{ij}s_{jk} = c_{ijk}s_{ik}. \tag{3.6}$$

We thus get a $\mathbb{C}$-algebroid descent datum

$$\mathbb{C}_{\sqrt{\Omega_X}} := (\{\mathbb{C}_{U_i}\}_{i \in I}, \{\mathrm{id}_{\mathbb{C}_{U_{ij}}}\}_{i,j \in I}, \{c_{ijk}\}_{i,j,k \in I}).$$

Note that, since $c_{ijk}^2 = 1$, there is an equivalence $\mathrm{Mod}(\mathbb{C}_{\sqrt{\Omega_X}}) \simeq \mathrm{Mod}(\mathbb{C}_{\sqrt{\Omega_X^{-1}}})$.

Recall from [8] (see also [5, §1]), that there is an equivalence

$$\mathrm{Mod}(\mathbb{C}_{\sqrt{\Omega_X}}) \simeq \mathrm{Mod}(\mathbb{C}_X) \tag{3.7}$$

if and only if the cohomology class $[c_{ijk}] \in H^2(X; \mathbb{C}_X^\times)$ is trivial. Consider the long exact cohomology sequence

$$H^1(X; \mathbb{C}_X^\times) \xrightarrow{\alpha} H^1(X; \mathcal{O}_X^\times) \xrightarrow{\beta} H^1(X; d\mathcal{O}_X) \xrightarrow{\gamma} H^2(X; \mathbb{C}_X^\times)$$

associated with the short exact sequence

$$1 \to \mathbb{C}_X^\times \to \mathcal{O}_X^\times \xrightarrow{d\log} d\mathcal{O}_X \to 0.$$

One has $[c_{ijk}] = \gamma(\frac{1}{2}\beta([\Omega_X]))$, so that $[c_{ijk}] = 1$ if and only if there exists a line bundle $\mathcal{L}$ such that $\frac{1}{2}\beta([\Omega_X]) = \beta([\mathcal{L}])$, i.e. such that $\beta([\Omega_X \otimes_\mathcal{O} \mathcal{L}^{-2}]) = 0$. This last condition holds if and only if there exists a local system of rank one L such that $[\Omega_X \otimes_\mathcal{O} \mathcal{L}^{-2}] = \alpha([L])$. Summarizing, (3.7) holds if and only if there exist $\mathcal{L}$ and L as above, such that

$$\Omega_X \simeq L \otimes \mathcal{L}^2.$$

The twisted sheaf of half-forms in $\mathrm{Mod}(\mathbb{C}_{\sqrt{\Omega_X}})$ is given by

$$\sqrt{\Omega_X} = (\{\mathcal{O}_{U_i}\}_{i \in I}, \{s_{ij}\}_{i,j \in I}).$$

We denote by $\sqrt{\omega_i}$ the section corresponding to $1 \in \mathcal{O}_{U_i}$. Hence, on U_{ij} we have

$$\sqrt{\omega_j} = s_{ij}\sqrt{\omega_i}.$$

Denote by $[s_{ij}]$ the equivalence class of s_{ij} in $\mathcal{O}_{U_{ij}}^\times / \mathbb{C}_{U_{ij}}^\times$. Since the s_{ij}'s satisfy (3.6), we notice that

$$\sqrt{\Omega_X^\times}/\mathbb{C}_X^\times = (\{\mathcal{O}_{U_i}^\times / \mathbb{C}_{U_i}^\times\}_{i \in I}, \{[s_{ij}]\}_{i,j \in I}) \in \mathrm{Mod}(\mathbb{Z}_X) \tag{3.8}$$

is a (usual, i.e. not twisted) sheaf.

Let $\mathsf{A} = (\{\mathcal{A}_i\}_{i \in I}, \{f_{ij}\}_{i,j \in I}, \{a_{ijk}\}_{i,j,k \in I})$ be a $\mathbf{k}$-algebroid descent datum on $\mathcal{U}$ as in (3.1), and consider a pair

$$\mathcal{M} = (\{\mathcal{M}_i\}_{i \in I}, \{\xi_{ij}\}_{i,j \in I}),$$

where $\mathcal{M}_i$ are $\mathcal{A}_i$-modules and $\xi_{ij}\colon {}_{f_{ji}}\mathcal{M}_j|_{U_{ij}} \to \mathcal{M}_i|_{U_{ij}}$ are isomorphisms of $\mathcal{A}_i$-modules which do not necessarily satisfy (3.5). Assume instead that there are isomorphisms

$$\mathcal{H}om_{\mathcal{A}_i}({}_{f_{ji}}\mathcal{M}_j|_{U_{ij}}, \mathcal{M}_i|_{U_{ij}}) \simeq \mathbf{k}_{U_{ij}}. \tag{3.9}$$

Under this assumption, we will show in Proposition 3.8 below that $\mathcal{M}$ makes sense as a global object of $\mathsf{Mod}(\mathsf{A} \otimes_{\mathbf{k}} \mathsf{S})$, for a suitable twist S.

Consider the isomorphisms

$$\phi_{ij}\colon \mathcal{H}om_{\mathcal{A}_i}({}_{f_{ji}}\mathcal{M}_j|_{U_{ij}}, \mathcal{M}_i|_{U_{ij}}) \xrightarrow{\sim} \mathbf{k}_{U_{ij}}, \tag{3.10}$$

defined by $\xi_{ij} \mapsto 1$. The dotted arrow defined by the following commutative diagram of isomorphisms is the multiplication by a section c_{ijk} of $\mathbf{k}_{U_{ijk}}^{\times}$.

$$\begin{array}{ccc}
\mathcal{H}om_{\mathcal{A}_j}({}_{f_{ij}}\mathcal{M}_i, \mathcal{M}_j) \otimes_{\mathbf{k}} \mathcal{H}om_{\mathcal{A}_k}({}_{f_{jk}}\mathcal{M}_j, \mathcal{M}_k) & \xrightarrow[\sim]{\varphi_{ji} \otimes \varphi_{kj}} & \mathbf{k}_{U_{ijk}} \otimes_{\mathbf{k}} \mathbf{k}_{U_{ijk}} \quad (3.11) \\
\wr \downarrow {\widetilde{f_{jk}} \otimes \mathrm{id}} & & \wr \downarrow \\
\mathcal{H}om_{\mathcal{A}_k}({}_{f_{ij}f_{jk}}\mathcal{M}_i, {}_{f_{jk}}\mathcal{M}_j) \otimes_{\mathbf{k}} \mathcal{H}om_{\mathcal{A}_k}({}_{f_{jk}}\mathcal{M}_j, \mathcal{M}_k) & & \mathbf{k}_{U_{ijk}} \\
\wr \uparrow {\widetilde{a_{ijk}} \otimes \mathrm{id}} & & \uparrow \\
\mathcal{H}om_{\mathcal{A}_k}({}_{f_{ik}}\mathcal{M}_i, {}_{f_{jk}}\mathcal{M}_j) \otimes_{\mathbf{k}} \mathcal{H}om_{\mathcal{A}_k}({}_{f_{jk}}\mathcal{M}_j, \mathcal{M}_k) & & c_{ijk} \wr \vdots \\
\wr \downarrow {\circ} & & \\
\mathcal{H}om_{\mathcal{A}_k}({}_{f_{ik}}\mathcal{M}_i, \mathcal{M}_k) & \xrightarrow[\sim]{\varphi_{ki}} & \mathbf{k}_{U_{ijk}}.
\end{array}$$

Here, the first vertical arrow on the left follows from the equality ${}_{f_{ij}f_{jk}}\mathcal{M}_i = {}_{f_{jk}}({}_{f_{ij}}\mathcal{M}_i)$, and the second one from ${}_{f_{ij}f_{jk}}\mathcal{M}_i = {}_{\mathrm{Ad}(a_{ijk})f_{ik}}\mathcal{M}_i$.

Lemma 3.6. The constants c_{ijk} defined above satisfy the cocycle condition

$$c_{ijk}c_{ikl} = c_{jkl}c_{ijl}.$$

Proof. Consider morphisms

$$\eta_{ji}\colon {}_{f_{ij}}\mathcal{M}_i \to \mathcal{M}_j.$$

These induce morphisms

$$f_{jk}\eta_{ji}\colon {}_{f_{jk}}({}_{f_{ij}}\mathcal{M}_i) \to {}_{f_{jk}}\mathcal{M}_j.$$

The composition

$${}_{f_{ik}}\mathcal{M}_i \xrightarrow{\widetilde{a_{ijk}}^{-1}} {}_{\mathrm{Ad}(a_{ijk})f_{ik}}\mathcal{M}_i = {}_{f_{jk}}({}_{f_{ij}}\mathcal{M}_i) \xrightarrow{f_{jk}\eta_{ji}} {}_{f_{jk}}\mathcal{M}_j$$

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is given by
$$u_i \mapsto a_{ijk}u_i \mapsto {}_{f_{jk}}\eta_{ji}(a_{ijk}u_i).$$
Hence, the composition of the vertical isomorphisms in the left column of (3.11) is given by
$$\eta_{ji} \otimes \eta_{kj} \mapsto \eta_{kj}({}_{f_{jk}}\eta_{ji}(a_{ijk} \cdot \bullet)).$$
One then has

$$\varphi_{ji}(\eta_{ji})\varphi_{kj}(\eta_{kj}) = \varphi_{ki}(\eta_{kj}({}_{f_{jk}}\eta_{ji}(a_{ijk} \cdot \bullet)))c_{ijk}. \qquad (3.12)$$

Using (3.12), we have, on one hand

$$\begin{aligned}
\varphi_{ji}(\eta_{ji})\varphi_{kj}(\eta_{kj})\varphi_{lk}(\eta_{lk}) &= \varphi_{ji}(\eta_{ji})\varphi_{lj}(\eta_{lk}({}_{f_{kl}}\eta_{kj}(a_{jkl} \cdot \bullet)))c_{jkl} \\
&= \varphi_{li}(\eta_{lk}({}_{f_{kl}}\eta_{kj}(a_{jkl}{}_{f_{jl}}\eta_{ji}(a_{ijl} \cdot \bullet))))c_{ijl}c_{jkl} \\
&= \varphi_{li}(\eta_{lk}({}_{f_{kl}}\eta_{kj}({}_{f_{jk}f_{kl}}\eta_{ji}(f_{ij}(a_{jkl})a_{ijl} \cdot \bullet))))c_{ijl}c_{jkl},
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\varphi_{ji}(\eta_{ji})\varphi_{kj}(\eta_{kj})\varphi_{lk}(\eta_{lk}) &= \varphi_{ki}(\eta_{kj}({}_{f_{jk}}\eta_{ji}(a_{ijk} \cdot \bullet)))\varphi_{lk}(\eta_{lk})c_{ijk} \\
&= \varphi_{li}(\eta_{lk}({}_{f_{kl}}\eta_{kj}({}_{f_{jk}f_{kl}}\eta_{ji}(a_{ikl}a_{ijk} \cdot \bullet))))c_{ikl}c_{ijk}.
\end{aligned}$$

The conclusion follows using (3.3). q.e.d.

Remark 3.7. Lemma 3.6 is a particular case of a general result which asserts that equivalence classes of locally trivial **k**-algebroids on X are in one-to-one correspondence with $H^2(X; \mathbf{k}^\times)$. The analogue result for gerbes is discussed in [6], and we refer to [4] for the formulation in terms of algebroids.

Let us recall our setting. Given a **k**-algebroid descent datum

$$\mathsf{A} = (\{\mathcal{A}_i\}_{i \in I}, \{f_{ij}\}_{i,j \in I}, \{a_{ijk}\}_{i,j,k \in I}),$$

consider a pair

$$\mathcal{M} = (\{\mathcal{M}_i\}_{i \in I}, \{\xi_{ij}\}_{i,j \in I}),$$

where $\mathcal{M}_i$ are $\mathcal{A}_i$-modules and $\xi_{ij} \colon {}_{f_{ji}}\mathcal{M}_j|_{U_{ij}} \to \mathcal{M}_i|_{U_{ij}}$ are isomorphisms of $\mathcal{A}_i$-modules. Assuming (3.9), Lemma 3.6 guarantees that the constants c_{ijk} defined by (3.11) satisfy the cocycle condition. We can thus consider the **k**-algebroid descent datum

$$\mathsf{S} = (\{\mathbf{k}_{U_i}\}_{i \in I}, \{\mathrm{id}_{\mathbf{k}_{U_{ij}}}\}_{i,j \in I}, \{c_{ijk}\}_{i,j,k \in I}).$$

Set

$$\mathsf{A} \otimes_\mathbf{k} \mathsf{S} = (\{\mathcal{A}_i\}_{i \in I}, \{f_{ij}\}_{i,j \in I}, \{a_{ijk}c_{ijk}\}_{i,j,k \in I}).$$

Proposition 3.8. Let $\mathcal{M} = (\{\mathcal{M}_i\}_{i \in I}, \{\xi_{ij}\}_{i,j \in I})$ be as above, and assume (3.9), then

$$\mathcal{M} = (\mathcal{M}_i, \xi_{ij}) \in \mathrm{Mod}(\mathsf{A} \otimes_\mathbf{k} \mathsf{S}).$$

Proof. By (3.5), it is enough to show that

$$\xi_{ik} = \xi_{ij}\big(_{f_{ji}}\xi_{jk}(a_{kji}c_{kji}\cdot\bullet)\big).$$

We have $\varphi_{ik}(\xi_{ik}) = 1 = \varphi_{jk}(\xi_{jk})\varphi_{ij}(\xi_{ij}) = \varphi_{ik}(\xi_{ij}(_{f_{ji}}\xi_{jk}(a_{kji}c_{kji}\cdot\bullet)))$ by (3.12). The conclusion follows since φ_{ik} is an isomorphism. q.e.d.

4 Microdifferential modules

Here we review a few notions from the theory of microdifferential modules. References are made to [20] and also to [12] for complementary results. See [10,21] for an exposition.

Let Y be a complex analytic manifold, and $\pi\colon T^*Y \to Y$ its cotangent bundle. The sheaf $\mathcal{E}_Y$ of microdifferential operators on T^*Y is a $\mathbb{C}$-central algebra endowed with a $\mathbb{Z}$-filtration by the order. Denote by $\mathcal{E}_Y(m)$ its subsheaf of operators of order at most m, and by $\mathcal{O}_{T^*Y}(m)$ the sheaf of functions homogeneous of degree m in the fiber of π. Denote by eu the Euler vector field, i.e. the infinitesimal generator of the action of $\mathbb{C}^\times$ on T^*Y. Then $f \in \mathcal{O}_{T^*Y}(m)$ if and only if eu $f = mf$. In a homogeneous symplectic local coordinate system $(x;\xi)$ on $U \subset T^*Y$, a section $P \in \Gamma(U;\mathcal{E}_Y(m))$ is written as a formal series

$$P = \sum_{j\leq m} p_j(x;\xi), \quad p_j \in \Gamma(U;\mathcal{O}_{T^*Y}(j)), \tag{4.1}$$

with the condition that for any compact subset K of U there exists a constant $C_K > 0$ such that $\sup_K |p_j| \leq C_K^{-j}(-j)!$ for all $j < 0$.

If $Q = \sum_{j\leq n} q_j$ is another section, the product $PQ = R = \sum_{j\leq m+n} r_j$ is given by the Leibniz rule

$$r_k = \sum_{k=i+j-|\alpha|} \frac{1}{\alpha!}(\partial_\xi^\alpha p_i)(\partial_x^\alpha q_j).$$

The symbol map

$$\sigma_m\colon \mathcal{E}_Y(m) \to \mathcal{O}_{T^*Y}(m), \qquad P \mapsto p_m$$

does not depend on the choice of coordinates and induces the symbol map

$$\sigma\colon \mathcal{E}_Y \to \operatorname{gr}\mathcal{E}_Y \xrightarrow{\sim} \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_{T^*Y}(m).$$

The formal adjoint of $P = \sum_{j\leq m} p_j$ is defined by

$$P^* = \sum_{j\leq m} p_j^*, \quad p_j^*(x;\xi) = \sum_{j=k-|\alpha|} \frac{(-1)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\partial_x^\alpha p_k(x;-\xi).$$

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It depends on the choice of coordinates, and more precisely on the choice of the top degree form $dx_i \wedge \cdots \wedge dx_n \in \Omega_Y$. One thus considers the twist of $\mathcal{E}_Y$ by half-forms

$$\mathcal{E}_Y^{\sqrt{v}} := \pi^{-1}\sqrt{\Omega_Y} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{E}_Y \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}\sqrt{\Omega_Y^{-1}}.$$

This is a sheaf of filtered $\mathbb{C}$-algebras endowed with a canonical anti-isomorphism

$$*\colon \mathcal{E}_Y^{\sqrt{v}} \xrightarrow{\sim} a_*\mathcal{E}_Y^{\sqrt{v}},$$

where a denotes the antipodal map on T^*Y. There is a subprincipal symbol

$$\sigma'_{m-1}\colon \mathcal{E}_Y^{\sqrt{v}}(m) \to \mathcal{O}_{T^*Y}(m-1), \quad P \mapsto \frac{1}{2}\sigma_{m-1}(P - (-1)^m P^*).$$

In local coordinates, $\sigma'_{m-1}(P) = p_{m-1} - \frac{1}{2}\sum_i \partial_{x_i}\partial_{\xi_i}p_m$ for P as in (4.1).

The following definition is adapted from [12].

Definition 4.1. Let Λ be a smooth, locally closed, $\mathbb{C}^\times$-conic submanifold of T^*Y. Let $\mathcal{M}$ be a coherent $\mathcal{E}_Y$-module supported by Λ.

(a) One says that $\mathcal{M}$ is regular (resp. simple) along Λ if there locally exists a coherent sub-$\mathcal{E}_Y(0)$-module $\mathcal{M}_0$ of $\mathcal{M}$ which generates it over $\mathcal{E}_Y$, and such that $\mathcal{M}_0/\mathcal{E}_Y(-1)\mathcal{M}_0$ is an $\mathcal{O}_\Lambda(0)$-module (resp. a locally free $\mathcal{O}_\Lambda(0)$-module of rank one).

(b) Let $\mathcal{M}$ be simple along Λ. A section $u \in \mathcal{M}$ is called a simple generator if $\mathcal{M}_0 = \mathcal{E}_Y(0)u$ satisfies the conditions in (a), and the image of u in $\mathcal{M}_0/\mathcal{E}_Y(-1)\mathcal{M}_0$ generates this module over $\mathcal{O}_\Lambda(0)$.

Set
$$\mathcal{I}_\Lambda = \{P \in \mathcal{E}_Y(1)|_\Lambda;\ \sigma_1(P)|_\Lambda = 0\}, \tag{4.2}$$
and denote by $\mathcal{E}_\Lambda$ the sub-algebra of $\mathcal{E}_Y|_\Lambda$ generated by $\mathcal{I}_\Lambda$.

Remark 4.2. Let u be a generator of a coherent $\mathcal{E}_Y$-module $\mathcal{M}$. Then $\mathcal{M} \simeq \mathcal{E}_Y/\mathcal{I}$, where $\mathcal{I} = \{P \in \mathcal{E}_Y\colon Pu = 0\}$. Set

$$\overline{\mathcal{I}} = \{\sigma_m(P); m \in \mathbb{Z},\ P \in \mathcal{I} \cap \mathcal{E}_Y(m)\},$$

and note that $\operatorname{supp}\mathcal{M} = \operatorname{supp}(\mathcal{O}_{T^*Y}/\overline{\mathcal{I}})$. Then $\mathcal{M}$ is simple if and only if there locally exists a generator u such that the ideal $\overline{\mathcal{I}}$ is reduced. Moreover, such a section u is a simple generator and the sub-$\mathcal{E}_Y(0)$-module $\mathcal{M}_0$ generated by u satisfies

$$\mathcal{E}_\Lambda \mathcal{M}_0 \subset \mathcal{M}_0.$$

Indeed, in a homogeneous symplectic local coordinate system we may write $P \in \mathcal{I}_\Lambda$ as $P = P' + Q$ with Q of order ≤ 0 and $P'u = 0$.

Let us recall the notion of symbol for simple generators. References are made to [20,7].

For a vector field $v \in \Theta_\Lambda$ on Λ, denote by $\mathcal{L}_v^{1/2}$ its Lie derivative action on the twisted sheaf $\sqrt{\Omega_\Lambda}$. Then $\mathcal{L}_v^{1/2}$ is an operator of order one in the ring
$$\mathcal{D}_\Lambda^{\sqrt{v}} = \sqrt{\Omega_\Lambda} \otimes_{\mathcal{O}_\Lambda} \mathcal{D}_\Lambda \otimes_{\mathcal{O}_\Lambda} \sqrt{\Omega_\Lambda^{-1}}$$

of differential operators acting on $\sqrt{\Omega_\Lambda}$.

Define $\mathcal{I}_\Lambda^{\sqrt{v}}$ and $\mathcal{E}_\Lambda^{\sqrt{v}}$ as in (4.2), replacing $\mathcal{E}_Y$ with $\mathcal{E}_Y^{\sqrt{v}}$, and denote by H_f the Hamiltonian vector field of $f \in \mathcal{O}_{T^*Y}$. Note that $H_f \in T\Lambda$ if $f|_\Lambda = 0$. For $P \in \mathcal{I}_\Lambda^{\sqrt{v}}$, consider the transport operator

$$\mathrm{L}(P) = \mathcal{L}_{H_{\sigma_1(P)}|_\Lambda}^{1/2} + \sigma_0'(P)|_\Lambda.$$

One checks that L satisfies the relations $\mathrm{L}(AP) = \sigma_0(A)\mathrm{L}(P)$, $\mathrm{L}(PA) = \mathrm{L}(P)\sigma_0(A)$, and $\mathrm{L}([P,Q]) = [\mathrm{L}(P), \mathrm{L}(Q)]$, for $P, Q \in \mathcal{I}_\Lambda^{\sqrt{v}}$ and $A \in \mathcal{E}_\Lambda^{\sqrt{v}}(0)$ (see *e.g.* [10, §8.3]). It follows that L extends as a $\mathbb{C}$-algebra morphism

$$\mathrm{L} \colon \mathcal{E}_\Lambda^{\sqrt{v}} \to \mathcal{D}_\Lambda^{\sqrt{v}} \tag{4.3}$$

by setting $\mathrm{L}(P_1 \cdots P_r) = \mathrm{L}(P_1) \cdots \mathrm{L}(P_r)$, for $P_i \in \mathcal{I}_\Lambda^{\sqrt{v}}$.

Let $\mathcal{M}$ be a simple $\mathcal{E}_Y^{\sqrt{v}}$-module along Λ, and $u \in \mathcal{M}$ a simple generator. The twisted subsheaf of $\sqrt{\Omega_\Lambda}$ defined by

$$\{\sigma \in \sqrt{\Omega_\Lambda};\ \mathrm{L}(P)\sigma = 0\ \forall P \in \mathcal{E}_\Lambda^{\sqrt{v}},\ Pu = 0\} \tag{4.4}$$

is locally a free sheaf of rank one over $\mathbb{C}$.

Definition 4.3. Let $\mathcal{M}$ be a simple $\mathcal{E}_Y^{\sqrt{v}}$-module along Λ and let $u \in \mathcal{M}$ be a simple generator. The symbol of u is defined by

$$\sigma_\Lambda(u) = [\sigma] \in \sqrt{\Omega_\Lambda^\times}/\mathbb{C}_\Lambda^\times.$$

for σ as in (4.4).

The Euler vector field eu acts on $\sqrt{\Omega_\Lambda}$ by $\mathcal{L}_{\mathrm{eu}}^{1/2}$, and one says that $\sigma \in \sqrt{\Omega_\Lambda}$ is homogeneous of degree λ if $\mathrm{eu}\,\sigma = \lambda\sigma$. Hence, the notion of homogeneous section of $\sqrt{\Omega_\Lambda^\times}/\mathbb{C}_\Lambda^\times$ makes sense. One calls *order of* u the homogeneous degree of $\sigma_\Lambda(u)$. Then the equivalence class of λ in $\mathbb{C}/\mathbb{Z}$ does not depend on u, and is called the *order of* $\mathcal{M}$.

If $P \in \mathcal{E}_Y^{\sqrt{v}}(m)$ is such that $\sigma_m(P)|_\Lambda$ never vanishes, then

$$\sigma_\Lambda(Pu) = \sigma_m(P)\sigma_\Lambda(u). \tag{4.5}$$

Also recall from *loc.cit.* that simple modules of the same order are locally isomorphic.

5 Quantization of contact manifolds

Here we review Kashiwara's construction [9] of the stack of microdifferential modules on a contact manifold.

Let Y be a complex analytic manifold, $\pi\colon T^*Y \to Y$ its cotangent bundle, $\dot{T}^*Y = T^*Y \setminus Y$ the complementary of the zero-section, $\varpi\colon P^*Y \to Y$ the projective cotangent bundle, and $\gamma\colon \dot{T}^*Y \to P^*Y$ the projection. The sheaf of microdifferential operators on P^*Y is given by $\gamma_*(\mathcal{E}_Y|_{\dot{T}^*X})$, and we still denote it by $\mathcal{E}_Y$ for short. Since the antipodal map induces the identity on P^*Y, the anti-involution $*$ is well defined on the sheaf $\mathcal{E}_Y^{\sqrt{v}}$.

Let $\chi_{ij}\colon P^*Y_i \supset V_i \to V_j \subset P^*Y_j$ be a contact transformation. Recall that there locally exists a quantized contact transformation (QCT for short) above χ_{ij}. This is an isomorphism of filtered $\mathbb{C}$-algebras $\Phi_{ij}\colon \chi_{ij}^{-1}\mathcal{E}_{Y_j}^{\sqrt{v}}|_{V_j} \xrightarrow{\sim} \mathcal{E}_{Y_i}^{\sqrt{v}}|_{V_i}$. Moreover, one can ask that Φ_{ij} is $*$-preserving. Such a quantization is not unique, but a key remark by Kashiwara is that $*$-preserving filtered automorphisms of $\mathcal{E}_Y^{\sqrt{v}}$ are of the form $\mathrm{Ad}(Q)$ for a *unique* operator $Q \in \mathcal{E}_Y^{\sqrt{v}}(0)$ satisfying

$$QQ^* = 1, \quad \sigma_0(Q) = 1. \tag{5.1}$$

Definition 5.1. A complex contact manifold $\mathfrak{Y} = (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}(1), \alpha)$ is a complex manifold $\mathfrak{Y}$ of dimension $2n+1$ endowed with a line bundle $\mathcal{O}_{\mathfrak{Y}}(1)$ and a 1-form $\alpha \in \Gamma(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}} \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{Y}}(1))$, such that $\alpha \wedge (d\alpha)^{\wedge n}$ is a non-degenerate section of $\Omega^{2n+1}_{\mathfrak{Y}} \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{Y}}(n+1)$.

(Here we set $\mathcal{O}_{\mathfrak{Y}}(k) = \mathcal{O}_{\mathfrak{Y}}(1)^{\otimes k}$, and we use the fact that $\alpha \wedge (d\alpha)^{\wedge r}$ is a well-defined section of $\Omega^{2r+1}_{\mathfrak{Y}} \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{Y}}(r+1)$ for $0 \leq r \leq n$.)

There is an open covering $\mathcal{V} = \{V_i\}_{i \in I}$ of $\mathfrak{Y}$ and contact embeddings $\chi_i\colon V_i \hookrightarrow P^*Y_i$. Up to refining the covering (we still denote it by $\mathcal{V}$), the induced contact transformations $\chi_{ij}\colon \chi_i(V_{ij}) \to \chi_j(V_{ij})$ can be quantized to a $*$-preserving filtered $\mathbb{C}$-algebra isomorphism

$$\Phi_{ij}\colon \chi_{ij}^{-1}(\mathcal{E}_{Y_j}^{\sqrt{v}}|_{\chi_j(V_{ij})}) \to \mathcal{E}_{Y_i}^{\sqrt{v}}|_{\chi_i(V_{ij})}.$$

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The composition $\Phi_{ij}\Phi_{jk}\Phi_{ik}^{-1}$ is a $*$-preserving automorphism of $\mathcal{E}_{Y_i}^{\sqrt{v}}|_{\chi_i(V_{ijk})}$, and hence is equal to $\mathrm{Ad}(Q_{ijk})$ for a unique $Q_{ijk} \in \mathcal{E}_{Y_i}^{\sqrt{v}}(V_{ijk})$ satisfying (5.1). This proves the theorem below, thanks to Remark 3.3.

Theorem 5.2. (cf [9, Theorem 2]) The triplet

$$\mathsf{E}_{\mathfrak{Y}} = (\{\chi_i^{-1}\mathcal{E}_{Y_i}^{\sqrt{v}}|_{V_i}\}_{i \in I}, \{\chi_i^{-1}(\Phi_{ij})\}_{i,j \in I}, \{\chi_i^{-1}(Q_{ijk})\}_{i,j,k \in I}) \qquad (5.2)$$

is a $\mathbb{C}$-algebroid descent datum over $\mathfrak{Y}$. In particular, there is an associated stack $\mathsf{Mod}(\mathsf{E}_{\mathfrak{Y}})$ on $\mathfrak{Y}$ locally equivalent to the stack $\mathsf{Mod}(\mathcal{E}_{Y_i}^{\sqrt{v}})$ of microdifferential modules.

Good modules

Any local notion, such as that of being coherent, simple, or regular, immediately extends to the category $\mathrm{Mod}(\mathsf{E}_{\mathfrak{Y}})$. Here, we will discuss the non local notion of being good (cf [7]).

Remark first that the construction in Theorem 5.2 also applies when replacing $\mathcal{E}_{Y_i}$ with $\mathcal{E}_{Y_i}(0)$. One thus gets a $\mathbb{C}$-algebroid descent datum $\mathsf{E}_{\mathfrak{Y}}(0)$ as well as a $\mathbb{C}$-linear functor $\mathsf{E}_{\mathfrak{Y}}(0) \to \mathsf{E}_{\mathfrak{Y}}$. This induces a forgetful functor

$$\mathrm{Mod}(\mathsf{E}_{\mathfrak{Y}}) \xrightarrow{for} \mathrm{Mod}(\mathsf{E}_{\mathfrak{Y}}(0)).$$

This functor locally admits a left adjoint, hence it has a left adjoint

$$\mathrm{Mod}(\mathsf{E}_{\mathfrak{Y}}(0)) \xrightarrow{ext} \mathrm{Mod}(\mathsf{E}_{\mathfrak{Y}}).$$

Definition 5.3. A coherent module $\mathcal{M} \in \mathrm{Mod}(\mathsf{E}_{\mathfrak{Y}})$ is good if for any relatively compact open subset $U \subset \mathfrak{Y}$ there exists $\mathcal{M}_0 \in \mathrm{Mod}(\mathsf{E}_{\mathfrak{Y}}(0)|_U)$ such that $\mathcal{M}|_U \simeq ext(\mathcal{M}_0)$.

Quantization with parameters

Assume now that the bundle $\mathcal{O}_{\mathfrak{Y}}(1)$ is trivial, *i.e.* that there exists a nowhere vanishing section $\tau \in \Gamma(\mathfrak{Y}; \mathcal{O}_{\mathfrak{Y}}(1))$. Consider an open covering $\{V_i\}_{i \in I}$ of $\mathfrak{Y}$ and contact embeddings $\chi_i \colon V_i \hookrightarrow P^*Y_i$ to which is attached the $\mathbb{C}$-algebroid descent datum (5.2).

Lemma 5.4. Up to refining the covering, there exist $*$-preserving QCT's Φ_{ij} above χ_{ij}, and sections $T_i \in \Gamma(\chi_i(V_i); \mathcal{E}_{Y_i}^{\sqrt{v}}(1))$, such that

$$\sigma(T_i) \circ \chi_i = \tau, \quad T_i^* = -T_i, \quad \Phi_{ij}(T_j) = T_i.$$

Proof. Let $(t, x_1, \ldots, x_n, u_1, \ldots, u_n)$ be a local coordinate system on V_i such that the contact form is given by $dt + \sum_i u_i dx_i$ and $\tau = \sigma(\partial_t) \circ \chi_i$. Then set $T_i = \partial_t$.

Let $\Phi_{ij}^\dagger$ be a QCT above χ_{ij} such that $\Phi_{ij}^\dagger(T_j) = T_i$. Then the proof goes as that of [19, Lemma 5.3 (iii)]. Consider the QCT above the identity given by

$$\Phi_{ij}^\ddagger := \Phi_{ij}^{\dagger -1} \circ * \circ \Phi_{ij}^\dagger \circ *.$$

There exists $Q \in \mathcal{E}_{Y_j}^{\sqrt{v}}$ such that $\Phi_{ij}^\ddagger = \mathrm{Ad}(QQ^*)$. Hence $\Phi_{ij} := \Phi_{ij}^\dagger \circ \mathrm{Ad}(Q)$ is $*$-preserving QCT above χ_{ij}, and $\Phi_{ij}(T_j) = T_i$.   q.e.d.

Denote by $\mathcal{E}_{Y_i, T_i}^{\sqrt{v}}$ the subalgebra of $\mathcal{E}_{Y_i}^{\sqrt{v}}$ of operators commuting with T_i. As above, denote by $Q_{ijk} \in \mathcal{E}_{Y_i}^{\sqrt{v}}(\chi_i(V_{ijk}))$ the unique operator satisfying (5.1) such that $\Phi_{ij}\Phi_{jk}\Phi_{ik}^{-1} = \mathrm{Ad}(Q_{ijk})$. Since $\mathrm{Ad}(Q_{ijk})(T_i) = T_i$, it follows that Q_{ijk} is a section of $\mathcal{E}_{Y_i, T_i}^{\sqrt{v}}$.

Proposition 5.5. Let $\mathfrak{Y}$ be a complex contact manifold. Assume that there exists a nowhere vanishing section $\tau \in \Gamma(\mathfrak{Y}; \mathcal{O}_{\mathfrak{Y}}(1))$. Then the triplet

$$\mathsf{E}_{\mathfrak{Y}, \tau} = (\{\chi_i^{-1}\mathcal{E}_{Y_i, T_i}^{\sqrt{v}}|_{\chi_i(V_i)}\}_{i \in I}, \{\chi_i^{-1}(\Phi_{ij})\}_{i,j \in I}, \{\chi_i^{-1}(Q_{ijk})\}_{i,j,k \in I}) \qquad (5.3)$$

is a $\mathbb{C}$-algebroid descent datum on $\mathcal{V}$. In particular, there is an associated stack $\mathsf{Mod}(\mathsf{E}_{\mathfrak{Y}, \tau})$ on $\mathfrak{Y}$ locally equivalent to the stack $\mathsf{Mod}(\mathcal{E}_{Y_i, T_i}^{\sqrt{v}})$.

6   Simple holonomic modules on contact manifolds

Here, we give a proof of a result of Kashiwara [9] on the existence of twisted simple holonomic modules along smooth Lagrangian submanifolds (also called Legendrian in the literature) of complex contact manifolds.

Let $\mathfrak{Y}$ be a complex contact manifold. Recall from Theorem 5.2 that there is an algebroid descent datum $\mathsf{E}_{\mathfrak{Y}}$ of microdifferential operators on $\mathfrak{Y}$. Let $\Lambda \subset \mathfrak{Y}$ be a Lagrangian submanifold. With notations as in Example 3.5, consider the stack $\mathsf{Mod}(\mathsf{E}_{\mathfrak{Y}}|_\Lambda \otimes_{\mathbb{C}} \mathbb{C}_{\sqrt{\Omega_\Lambda}})$ of twisted microdifferential modules on Λ.

Let us recall Kashiwara's theorem announced in [9].

Theorem 6.1. Let $\mathfrak{Y}$ be a complex contact manifold and let $\Lambda \subset \mathfrak{Y}$ be a Lagrangian submanifold. There exists $\mathcal{L} \in \mathrm{Mod}(\mathsf{E}_{\mathfrak{Y}}|_\Lambda \otimes_{\mathbb{C}} \mathbb{C}_{\sqrt{\Omega_\Lambda}})$ which is simple along Λ.

Remark 6.2. It will follow from the proof that moreover $\mathcal{L}$ is good.

Here, we give a proof of this result using the notion of symbol for simple sections of holonomic modules recalled in Section 4.

Proof. Let us denote for short by

$$\mathsf{E}_{\mathfrak{Y}} = (\{\mathcal{A}_i\}_{i \in I}, \{f_{ij}\}_{i,j \in I}, \{a_{ijk}\}_{i,j,k \in I})$$

the $\mathbb{C}$-algebroid descent datum (5.2) attached to an open covering $\mathfrak{Y} = \bigcup_{i \in I} V_i$. Up to a refinement, we may assume that the $\mathbb{C}$-algebroid descent datum $\mathbb{C}_{\sqrt{\Omega_\Lambda}}$ is attached to the same covering. More precisely, using notations as in Example 3.5 for $X = \Lambda$ and $U_i = \Lambda_i = \Lambda \cap V_i$, we assume that there are nowhere vanishing sections $\omega_i \in \Omega_{\Lambda_i}$, and $s_{ij} \in \mathcal{O}_{\Lambda_{ij}}^\times$ with $\sqrt{\omega_j} = s_{ij}\sqrt{\omega_i}$, such that

$$\mathbb{C}_{\sqrt{\Omega_X}} = (\{\mathbb{C}_{\Lambda_i}\}_{i \in I}, \{\mathrm{id}_{\mathbb{C}_{\Lambda_{ij}}}\}_{i,j \in I}, \{c_{ijk}\}_{i,j,k \in I}),$$

where the c_{ijk}'s are defined by

$$s_{ij} s_{jk} = c_{ijk} s_{ik}.$$

Up to a further refinement of the covering, we may assume that there exist simple $\mathcal{A}_i$-modules $\mathcal{L}_i$ of order 0 along Λ_i and simple generators u_i of $\mathcal{L}_i$ such that

$$\sigma_{\Lambda_i}(u_i) = [\sqrt{\omega_i}] \in (\sqrt{\Omega_\Lambda^\times}/\mathbb{C}_\Lambda^\times)|_{\Lambda_i}.$$

Since $\mathcal{L}_i$ and $_{f_{ji}}\mathcal{L}_j$ are simple $\mathcal{A}_i$-modules along Λ_{ij} of the same order, they are isomorphic, and one has

$$\mathcal{H}om_{\mathcal{A}_i}(_{f_{ji}}\mathcal{L}_j, \mathcal{L}_i) \simeq \mathbb{C}_{\Lambda_{ij}}.$$

Let us describe explicitly such an isomorphism. For $\widetilde{\xi}_{ij} \in \mathcal{H}om_{\mathcal{A}_i}(_{f_{ji}}\mathcal{L}_j, \mathcal{L}_i)$, let $\widetilde{b}_{ij}$ is a section of $\mathcal{A}_i$ satisfying $\widetilde{\xi}_{ij}(u_j) = \widetilde{b}_{ij} u_i$. One has

$$\sigma_\Lambda(\widetilde{\xi}_{ij}(u_j)) = \sigma_\Lambda(u_j) = [\sqrt{\omega_j}] = [s_{ij}\sqrt{\omega_i}],$$
$$\sigma_\Lambda(\widetilde{b}_{ij} u_i) = \sigma(\widetilde{b}_{ij})\sigma_\Lambda(u_i) = [\sigma(\widetilde{b}_{ij})\sqrt{\omega_i}],$$

where the fourth equality follows from (4.5). In particular, $s_{ij}^{-1}\sigma(\widetilde{b}_{ij}) \in \mathbb{C}_{\Lambda_{ij}}^\times$. We can thus consider the isomorphism

$$\varphi_{ij} \colon \mathcal{H}om_{\mathcal{A}_i}(_{f_{ji}}\mathcal{L}_j, \mathcal{L}_i) \xrightarrow{\sim} \mathbb{C}_{\Lambda_{ij}}$$
$$\widetilde{\xi}_{ij} \mapsto s_{ij}^{-1}\sigma(\widetilde{b}_{ij}).$$

Set

$$\xi_{ij} = \varphi_{ij}^{-1}(1) \colon {}_{f_{ji}}\mathcal{L}_j \xrightarrow{\sim} \mathcal{L}_i,$$

and let $b_{ij} \in \mathcal{A}_i$ satisfy $\xi_{ij}(u_j) = b_{ij} u_i$. As $\varphi_{ij}(\xi_{ij}) = 1$, one has

$$\sigma(b_{ij}) = s_{ij}. \tag{6.1}$$

By Proposition 3.8, we get a twisted module

$$\mathcal{L} = (\mathcal{L}_i, \xi_{ij}) \in \mathrm{Mod}(\mathsf{E}_\mathfrak{Y}|_\Lambda \otimes_\mathbb{C} \mathsf{S}),$$

where

$$\mathsf{S} = (\{\mathbb{C}_{\Lambda_i}\}_{i \in I}, \{\mathrm{id}_{\mathbb{C}_{\Lambda_{ij}}}\}_{i,j \in I}, \{\widetilde{c}_{ijk}\}_{i,j,k \in I}),$$

the $\widetilde{c}_{ijk}$'s being given by (3.11). Since $\varphi_{ij}(\xi_{ij}) = 1 = \varphi_{jk}(\xi_{jk})$, by (3.12) we have

$$\widetilde{c}_{ijk} = \varphi_{ik}(\xi_{ij}({}_{f_{ji}}\xi_{jk}(a_{kji} \cdot \bullet))).$$

To show that $\mathsf{S} = \mathbb{C}_{\sqrt{\Omega_\Lambda}}$, it thus remains to prove that $\widetilde{c}_{ijk} = c_{ijk}$, i.e. that

$$s_{ij}s_{jk} = \widetilde{c}_{ijk}s_{ik}.$$

One has

$$\begin{aligned}
\xi_{ij}({}_{f_{ji}}\xi_{jk}(a_{kji}u_k)) &= f_{ij}f_{jk}(a_{kji})\,\xi_{ij}({}_{f_{ji}}\xi_{jk}(u_k)) \\
&= f_{ij}f_{jk}(a_{kji})\,\xi_{ij}(b_{jk}u_j) \\
&= f_{ij}f_{jk}(a_{kji})f_{ij}(b_{jk})\,\xi_{ij}(u_j) \\
&= f_{ij}f_{jk}(a_{kji})f_{ij}(b_{jk})b_{ij}\,u_i.
\end{aligned}$$

Then

$$\begin{aligned}
\widetilde{c}_{ijk} &= \varphi_{ik}(\xi_{ij}({}_{f_{ji}}\xi_{jk}(a_{kji} \cdot \bullet))) \\
&= s_{ik}^{-1}\sigma(f_{ij}f_{jk}(a_{kji})f_{ij}(b_{jk})b_{ij}) \\
&= s_{ik}^{-1}\sigma(b_{jk})\sigma(b_{ij}) \\
&= s_{ik}^{-1}s_{jk}s_{ij},
\end{aligned}$$

where the third equality follows from the fact that $\sigma(a_{kji}) = 1$, and the fourth one from (6.1). q.e.d.

Definition 6.3. A complex homogeneous symplectic manifold $\mathfrak{Z} = (\mathfrak{Z}, \omega, v)$ is a complex symplectic manifold $(\mathfrak{Z}, \omega)$ endowed with a vector field v satisfying $\mathcal{L}_v \omega = \omega$.

Corollary 6.4 below was announced in [9]. Although we shall not use it, we give a proof for the reader's convenience. Also note that our statement corrects that in loc. cit., following a private communication with M. Kashiwara.

Let $\mathfrak{Y} = (\mathfrak{Y}, \mathcal{O}_\mathfrak{Y}(1), \alpha)$ be a contact manifold and denote by $\gamma \colon \widetilde{\mathfrak{Y}} \to \mathfrak{Y}$ the total space of the $\mathbb{C}^\times$-principal bundle associated with the dual of the line bundle $\mathcal{O}_\mathfrak{Y}(1)$. Then $\widetilde{\mathfrak{Y}}$ is a homogeneous symplectic manifold and there exists a covering $\{V_i\}_{i \in I}$ of $\mathfrak{Y}$ by contact charts $\chi_i \colon V_i \hookrightarrow P^*Y_i$ such that, setting

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$\widetilde{V}_i = \gamma^{-1} V_i$, $\{\widetilde{V}_i\}_{i \in I}$ is a covering of $\widetilde{\mathfrak{Y}}$ and there are commutative diagrams

$$
\begin{array}{ccc}
\widetilde{V}_i & \overset{\widetilde{\chi}_i}{\hookrightarrow} & \dot{T}^*Y_i \\
{\scriptstyle \gamma|_{\widetilde{V}_i}} \downarrow & & \downarrow {\scriptstyle \gamma_i} \\
V_i & \overset{\chi_i}{\hookrightarrow} & P^*Y_i.
\end{array}
$$

where γ_i is the the projection $\dot{T}^*Y_i \to P^*Y_i$ and $\widetilde{\chi}_i \colon \widetilde{V}_i \hookrightarrow \dot{T}^*Y_i$ are homogeneous symplectic maps.

We denote by $\mathsf{Mod}_{\text{loc-sys}}(\mathbb{C}_{\gamma^{-1}\Lambda})$ the full substack of $\mathsf{Mod}(\mathbb{C}_{\gamma^{-1}\Lambda})$ consisting of local systems, and by $\mathsf{Mod}_{\text{reg-}\Lambda}(\mathsf{E}_{\mathfrak{Y}}|_\Lambda)$ the full substack of $\mathsf{Mod}(\mathsf{E}_{\mathfrak{Y}}|_\Lambda)$ consisting of modules regular along Λ

Corollary 6.4. (cf [9]) There is an equivalence of stacks

$$
\mathsf{Mod}_{\text{reg-}\Lambda}(\mathsf{E}_{\mathfrak{Y}}|_\Lambda \otimes_{\mathbb{C}} \mathbb{C}_{\sqrt{\Omega_\Lambda}}) \simeq \gamma_* \mathsf{Mod}_{\text{loc-sys}}(\mathbb{C}_{\gamma^{-1}\Lambda}).
$$

Proof. The sheaf of rings $\mathcal{E}_Y$ on T^*Y is a subsheaf of the sheaf of rings $\mathcal{E}_Y^{\mathbb{R}}$ of [20], this last sheaf being the microlocalization along the diagonal of the sheaf $\mathcal{O}_{Y \times Y}$ (up to a shift and tensorizing by holomorphic forms), see [13, chapter 11] for a detailed construction. One defines the sheaf $\mathcal{E}_Y^{\sqrt{v}, \mathbb{R}}$ similarly as we have defined $\mathcal{E}_Y^{\sqrt{v}}$.

With notations as in (5.2), the isomorphisms Φ_{ij} are induced by sections of a microdifferential module supported by the graph of χ_{ij}, and hence extend to isomorphisms $\Phi_{ij}^{\mathbb{R}}$. Considering Q_{ijk} as sections of $\mathcal{E}_{Y_i}^{\sqrt{v}, \mathbb{R}}$, we get a $\mathbb{C}$-algebroid descent datum

$$
\mathsf{E}_{\mathfrak{Y}}^{\mathbb{R}} = (\{\widetilde{\chi}_i^{-1} \mathcal{E}_{Y_i}^{\sqrt{v}, \mathbb{R}}|_{\widetilde{\chi}_i(\widetilde{V}_i)}\}_{i \in I}, \{\widetilde{\chi}_i^{-1}(\Phi_{ij}^{\mathbb{R}})\}_{i,j \in I}, \{\widetilde{\chi}_i^{-1}(Q_{ijk})\}_{i,j,k \in I}),
$$

and the associated stack $\mathsf{Mod}(\mathsf{E}_{\mathfrak{Y}}^{\mathbb{R}})$ on $\widetilde{\mathfrak{Y}}$. The inclusion $\gamma^{-1} \mathcal{E}_Y^{\sqrt{v}} \subset \mathcal{E}_Y^{\mathbb{R}}$ induces a functor of extension of scalars $\mathsf{Mod}(\mathsf{E}_{\mathfrak{Y}}) \to \gamma_* \mathsf{Mod}(\mathsf{E}_{\mathfrak{Y}}^{\mathbb{R}})$. Let us denote by $\mathcal{M} \mapsto \mathcal{M}^{\mathbb{R}}$ this functor.

By Theorem 6.1 there is a simple system $\mathcal{L}$ in $\mathrm{Mod}(\mathsf{E}_{\mathfrak{Y}}|_\Lambda \otimes_{\mathbb{C}} \mathbb{C}_{\sqrt{\Omega_\Lambda}})$. Consider the functor

$$
\Psi \colon \mathsf{Mod}_{\text{reg-}\Lambda}(\mathsf{E}_{\mathfrak{Y}}|_\Lambda \otimes_{\mathbb{C}} \mathbb{C}_{\sqrt{\Omega_\Lambda}}) \to \gamma_* \mathsf{Mod}_{\text{loc-sys}}(\mathbb{C}_{\gamma^{-1}\Lambda})
$$

given by $\mathcal{M} \mapsto \mathcal{H}om_{\mathsf{E}_{\mathfrak{Y}}^{\mathbb{R}}}(\mathcal{L}^{\mathbb{R}}, \mathcal{M}^{\mathbb{R}})$. The functor Ψ is locally an equivalence by the results of [11], and hence is an equivalence. q.e.d.

Example 6.5. Let $\mathfrak{Y} = P^*\mathbb{C}$, $\Lambda = P_{\{0\}}^*\mathbb{C} = \{\text{pt}\}$, $\gamma^{-1}\Lambda = \mathbb{C}^\times$. In this case, simple objects of rank one are classified by $\mathbb{C}^\times \simeq \mathbb{C}/\mathbb{Z}$.

7 WKB-modules

The relationship between microdifferential operators on a complex contact manifold and WKB-differential operators on a complex symplectic manifold is classic, and is discussed *e.g.*, in [1,18] in the case of cotangent bundles. This study (including the analysis of the action of quantized contact transformations) is systematically performed in [19], and here we follow their presentation.

Let X be a complex manifold, $t \in \mathbb{C}$ the coordinate, and set

$$\mathcal{E}_{X \times \mathbb{C}, \hat{t}} = \mathcal{E}_{X \times \mathbb{C}, \partial_t} = \{P \in \mathcal{E}_{X \times \mathbb{C}}; \ [P, \partial/\partial_t] = 0\}.$$

Set $\dot{P}^*(X \times \mathbb{C}) = \{\tau \neq 0\}$, and consider the projection

$$\rho \colon \dot{P}^*(X \times \mathbb{C}) \to T^*X, \tag{7.1}$$

given in local coordinates by $\rho(x, t; \xi, \tau) = (x; \xi/\tau)$. The ring of WKB-operators on T^*X is defined by

$$\mathcal{W}_X := \rho_*(\mathcal{E}_{X \times \mathbb{C}, \hat{t}}).$$

We similarly set $\mathcal{W}_X^{\sqrt{v}} := \rho_*(\mathcal{E}_{X \times \mathbb{C}, \hat{t}}^{\sqrt{v}})$. In a local symplectic coordinate system $(x; \xi)$ on T^*X, a section $P \in \mathcal{W}_X(U)$ is written as a formal series

$$P = \sum_{j \leq m} p_j(x; \xi)\tau^j, \quad p_j \in \mathcal{O}_{T^*X}(U), \quad m \in \mathbb{Z},$$

with the condition that for any compact subset K of V there exists a constant $C_K > 0$ such that $\sup_K |p_j| \leq C_K^{-j}(-j)!$ for all $j < 0$.

One sets

$$\mathbf{k} := \mathcal{W}_{\{\mathrm{pt}\}}. \tag{7.2}$$

Hence, an element $a \in \mathbf{k}$ is written as a formal series

$$a = \sum_{j \leq m} a_j \tau^j, \quad a_j \in \mathbb{C}, \quad m \in \mathbb{Z},$$

with the condition that there exist $C > 0$ with $|a_j| \leq C^{-j}(-j)!$ for all $j < 0$.

Note that $\mathcal{W}_X$ is $\mathbb{Z}$-filtered $\mathbf{k}$-central algebra, and the principal symbol map $\sigma_m \colon \mathcal{W}_X(m) \to \mathcal{O}_{T^*X}$ induces an isomorphism of graded algebras $\operatorname{gr}\mathcal{W}_X \to \mathcal{O}_{T^*X}[\tau^{-1}, \tau]$. Note also that $\pi^{-1}\mathcal{D}_X$ is a subring of $\mathcal{W}_X$.

We can now mimic Definition 4.1 for $\mathcal{W}_X$-modules.

Definition 7.1. Let Λ be a smooth Lagrangian submanifold of T^*X. Let $\mathcal{M}$ be a coherent $\mathcal{W}_X$-module supported by Λ. One says that $\mathcal{M}$ is regular

(resp. simple) along Λ if there locally exists a coherent sub-$\mathcal{W}_X(0)$-module $\mathcal{M}_0$ of $\mathcal{M}$ which generates it over $\mathcal{W}_X$, and such that $\mathcal{M}_0/\mathcal{W}_X(-1)\mathcal{M}_0$ is an $\mathcal{O}_\Lambda$-module (resp. a locally free $\mathcal{O}_\Lambda$-module of rank one).

Note that, as follows e.g. from Corollary 9.2 below, if $\mathcal{M}$ is regular then it is locally a finite direct sum of simple modules.

Notation 7.2. Let X be a complex manifold. We denote by $\mathcal{O}_X^\tau$ the simple $\mathcal{W}_X$-module along the zero-section T_X^*X defined by $\mathcal{O}_X^\tau = \mathcal{W}_X/\mathcal{I}$, where $\mathcal{I}$ is the left ideal generated by the vector fields on X.

Proposition 7.3. (i) Any two $\mathcal{W}_X$-modules simple along Λ are locally isomorphic. In particular, any simple module along T_X^*X is locally isomorphic to $\mathcal{O}_X^\tau$.
  (ii) If $\mathcal{M}$, $\mathcal{N}$ are simple $\mathcal{W}_X$-modules along Λ, then $R\mathcal{H}om_{\mathcal{W}_X}(\mathcal{M},\mathcal{N})$ is a $\mathbf{k}$-local system of rank one on Λ.

Proof. Since both statements are local on Λ, we may assume that X is endowed with a local coordinate system $(x_1,\ldots,x_n)$ and that Λ is the zero-section T_X^*X of T^*X.

(i) Any $\mathcal{W}_X$-module simple along T_X^*X is locally isomorphic to $\mathcal{O}_X^\tau$. Indeed, the proof of the theorem (due to [20]) which asserts that if Y is a complex manifold, then simple $\mathcal{E}_Y$-modules along smooth regular involutive submanifolds of P^*Y are locally isomorphic applies when replacing $\mathcal{E}_{X\times\mathbb{C}}$ with $\mathcal{E}_{X\times\mathbb{C},\hat{t}}$ (for an exposition, see [21, Ch 1, Th 6.2.1]).

(ii) By (i) we may assume that $\mathcal{M} = \mathcal{N} = \mathcal{O}_X^\tau$. The result easily follows, representing $R\mathcal{H}om_{\mathcal{W}_X}(\mathcal{O}_X^\tau,\mathcal{O}_X^\tau)$ by the Koszul complex $K^\bullet(\mathcal{O}_X^\tau,(\partial_{x_1},\ldots,\partial_{x_n}))$ associated with the sequence $(\partial_{x_1},\ldots,\partial_{x_n})$ acting on $\mathcal{O}_X^\tau$.   q.e.d.

Recall the projection ρ in (7.1) and note that $\rho^{-1}\Lambda$ is an involutive submanifold of $\dot{P}^*(X\times\mathbb{C})$.

Proposition 7.4. Let Λ be a smooth Lagrangian submanifold of T^*X.

  (i) Locally, there exists a Lagrangian submanifold $\Lambda^0 \subset \rho^{-1}(\Lambda)$ on which ρ induces an isomorphism $\Lambda^0 \simeq \Lambda$.
  (ii) Let $\mathcal{L}$ be a simple $\mathcal{W}_X$-module along Λ. Then, locally there exists a simple $\mathcal{E}_{X\times\mathbb{C}}$-module $\mathcal{L}^0$ along Λ^0 such that $\mathcal{L} \simeq for(\mathcal{L}^0)$, where *for* denotes the forgetful functor $\mathrm{Mod}(\mathcal{E}_{X\times\mathbb{C}}) \to \mathrm{Mod}(\mathcal{E}_{X\times\mathbb{C},\hat{t}})$.

Proof. Since the problem is local, we may assume that $\Lambda = T_X^*X$ in a system of local symplectic coordinates $(x;u) \in T^*X$. In the corresponding system of

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homogeneous coordinates $(x, t; \xi, \tau) \in P^*(X \times \mathbb{C})$, one has $\rho^{-1}\Lambda = \{\xi = 0\}$. This set is foliated by the Lagrangian submanifolds $\Lambda^c = \{\xi = 0, t = c\}$, for $c \in \mathbb{C}$. For $Z \subset X$ a closed submanifold, denote by $\mathcal{C}_{Z|X}$ the sheaf of finite order holomorphic microfunctions on $P_Z^* X$. A simple $\mathcal{E}_X$-module along Λ^0 is given by $\mathcal{L}^0 = \mathcal{C}_{X \times \{0\}|X \times \mathbb{C}}$. One then immediately checks that $\mathcal{L}^0 \simeq \mathcal{O}_X^\tau$. q.e.d.

8   Contactification of symplectic manifolds

In this section, we recall some well-known facts from the specialists on contact and symplectic geometry.

Definition 8.1. A contactification of a symplectic manifold $(\mathfrak{X}, \omega)$ is a complex contact manifold $(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}(1), \alpha)$, and a morphism of complex manifolds $\rho \colon \mathfrak{Y} \to \mathfrak{X}$ such that the following conditions are satisfied:

(a) the line bundle $\mathcal{O}_{\mathfrak{Y}}(1)$ has a nowhere vanishing section τ,
(b) there exists an open covering $\mathfrak{X} = \bigcup_{i \in I} U_i$, holomorphic functions t_i on $\rho^{-1}(U_i)$ and primitives $\sigma_i \in \Omega^1_{U_i}$ of $\omega|_{U_i}$ such that $\chi_i := (\rho, t_i)$ gives an isomorphism $\chi_i \colon \rho^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}$, and $dt_i + \rho^*(\sigma_i) = (\alpha/\tau)|_{\rho^{-1}(U_i)}$.

Lemma 8.2. A contactification $\rho \colon \mathfrak{Y} \to \mathfrak{X}$ has a structure of $\mathbb{C}$-principal bundle.

Proof. With notations as in Definition 8.1, set $\chi_{ij} = \chi_j \chi_i^{-1}|_{U_{ij}} \colon U_{ij} \times \mathbb{C} \to U_{ij} \times \mathbb{C}$. We have $\chi_{ij} = (\mathrm{id}_{U_{ij}}, \theta_{ij})$ for some transition functions $\theta_{ij} \colon U_{ij} \times \mathbb{C} \to \mathbb{C}$. Since $d(t_i - t_j) = \rho^*(\sigma_i - \sigma_j)$, up to shrinking the covering we may assume that $\sigma_i - \sigma_j = df_{ij}$ for some functions $f_{ij} \in \mathcal{O}_{U_{ij}}$. The transition functions are the translations $\theta_{ij}(p, t) = t + f_{ij}(p) + c_{ij}$ for some $c_{ij} \in \mathbb{C}$.   q.e.d.

Lemma 8.3. A symplectic manifold $(\mathfrak{X}, \omega)$ admits a contactification if and only if the de Rham cohomology class $[\omega] \in H^2(\mathfrak{X}; \mathbb{C}_{\mathfrak{X}})$ vanishes.

Proof. (a) Let $\mathfrak{X} = \bigcup_{i \in I} U_i$ be a covering such that $\omega|_{U_i}$ has a primitive $\sigma_i \in \Omega^1_{U_i}$. Up to shrinking the covering we may assume that $\sigma_i - \sigma_j = df_{ij}$ for some functions $f_{ij} \in \mathcal{O}_{U_{ij}}$. Then $d(f_{ij} + f_{jk} - f_{ik}) = 0$, and $\mathbb{C}_{U_{ijk}} \ni c_{ijk} = f_{ij} + f_{jk} - f_{ik}$ is a Čech cocycle representing $[\omega] \in H^2(\mathfrak{X}; \mathbb{C}_{\mathfrak{X}})$. Since $[\omega] = 0$, we have $c_{ijk} = c_{ij} + c_{jk} - c_{ik}$. Setting $\tilde{f}_{ij} = f_{ij} - c_{ij}$, we have $\tilde{f}_{ij} + \tilde{f}_{jk} - \tilde{f}_{ik} = 0$. Endow $U_i \times \mathbb{C}$ with the contact form $dt + \sigma_i$. A contactification of $\mathfrak{X}$ is thus given by the $\mathbb{C}$-principal bundle $\mathfrak{Y} \to \mathfrak{X}$ with local charts $U_i \times \mathbb{C}$, and transition functions over U_{ij} given by $(p, t) \mapsto (p, t + \tilde{f}_{ij}(p))$.

(b) Let $\rho\colon \mathfrak{Y} \to \mathfrak{X}$ be a contactification, and use notations as in Definition 8.1. We have $d(\alpha/\tau) = \rho^*\omega$ and hence $[\rho^*\omega] = 0$ in $H^2(\mathfrak{Y}; \mathbb{C}_{\mathfrak{Y}})$. Then $[\omega] = 0$ in $H^2(\mathfrak{X}; \mathbb{C}_{\mathfrak{X}}) \xrightarrow{\sim} H^2(\mathfrak{Y}; \mathbb{C}_{\mathfrak{Y}})$. q.e.d.

Lemma 8.4. Let $(\mathfrak{X}, \omega)$ be a symplectic manifold and Λ a Lagrangian submanifold. After replacing $\mathfrak{X}$ with a neighborhood of Λ there exists a contactification $\rho\colon \mathfrak{Y} \to \mathfrak{X}$ and a Lagrangian submanifold $\Lambda^0 \subset \mathfrak{Y}$ on which ρ induces an isomorphism $\Lambda^0 \simeq \Lambda$.

Proof. The restriction map $H^2(\mathfrak{X}, \mathbb{C}_{\mathfrak{X}}) \to H^2(\Lambda; \mathbb{C}_\Lambda)$ is given by $[\omega] \mapsto [j^*\omega]$, where $j\colon \Lambda \hookrightarrow \mathfrak{X}$ is the embedding. Since $\omega|_\Lambda = 0$, there exists an open neighborhood $U \supset \Lambda$ such that $[\omega|_U] = 0$. Thus, up to replacing $\mathfrak{X}$ with U, we can assume that $[\omega] = 0$.

Let us adapt the arguments in part (a) of the proof of Lemma 8.3 above. Let $\mathfrak{X} = \bigcup_{i \in I} U_i$ be a covering such that $\omega|_{U_i}$ has a primitive $\sigma_i \in \Omega^1_{U_i}$. We may assume that $\sigma_i|_\Lambda = 0$. Let $f_{ij} \in \mathcal{O}_{U_{ij}}$ satisfy $\sigma_i - \sigma_j = df_{ij}$. We may assume moreover that $f_{ij}|_\Lambda = 0$. Since $f_{ij} + f_{jk} - f_{ik}$ is locally constant on Λ, it must vanish on Λ. Thus, the $\mathbb{C}$-principal bundle $\rho\colon \mathfrak{Y} \to \mathfrak{X}$ is described by the local charts $U_i \times \mathbb{C}$, with transition functions $(p, t) \mapsto (p, t + f_{ij}(p))$ and contact form $dt + \sigma_i$. We define Λ^0 by $\Lambda^0|_{U_i \times \mathbb{C}} = \Lambda|_{U_i} \times \{0\}$. q.e.d.

9 Simple holonomic modules on symplectic manifolds

Let us start by recalling the construction of [19] of the stack of WKB-modules on a symplectic manifold, in the special case where there exists a contactification.

Let $\rho\colon \mathfrak{Y} \to \mathfrak{X}$ be a contactification of a complex symplectic manifold. Denote by
$$\mathsf{E}_{\mathfrak{Y}, \tau} = (\{\mathcal{A}_i\}_{i \in I}, \{f_{ij}\}_{i,j \in I}, \{a_{ijk}\}_{i,j,k \in I})$$
the $\mathbb{C}$-algebroid descent datum on $\mathfrak{Y}$ given by Proposition 5.5. Consider the $\mathbb{C}$-algebroid descent datum on $\mathfrak{X}$
$$\mathsf{W}_{\mathfrak{X}} = (\{\rho_*\mathcal{A}_i\}_{i \in I}, \{\rho_*f_{ij}\}_{i,j \in I}, \{\rho_*a_{ijk}\}_{i,j,k \in I}).$$

The stack $\mathsf{Mod}(\mathsf{W}_{\mathfrak{X}})$ associated with this $\mathbb{C}$-algebroid descent datum is the stack of WKB-modules in [19].

One defines the notion of good $\mathsf{W}_{\mathfrak{X}}$-module similarly as for $\mathsf{E}_{\mathfrak{Y}}$-modules.

Here, we prove the existence of twisted simple holonomic modules along smooth Lagrangian submanifolds of complex symplectic manifolds.

Theorem 9.1. Let $\mathfrak{X}$ be a complex symplectic manifold and let $\Lambda \subset \mathfrak{X}$ be a Lagrangian submanifold. There exists a module $\mathcal{L} \in \mathrm{Mod}(\mathsf{W}_{\mathfrak{X}}|_\Lambda \otimes_{\mathbb{C}} \mathbb{C}_{\sqrt{\Omega_\Lambda}})$ which is simple along Λ. Moreover, $\mathcal{L}$ is good.

Proof. By Lemma 8.3 there exist a contactification $\rho \colon \mathfrak{Y} \to \mathfrak{X}$ and a Lagrangian submanifold $\Lambda^0 \subset \mathfrak{Y}$ on which ρ induces an isomorphism $\Lambda^0 \simeq \Lambda$. By Theorem 6.1 there exists $\mathcal{L}^0 \in \mathrm{Mod}(\mathsf{E}_{\mathfrak{Y}}|_{\Lambda^0} \otimes_{\mathbb{C}} \mathbb{C}_{\sqrt{\Omega_{\Lambda^0}}})$ which is simple along Λ^0. As in Proposition 7.4 we then set $\mathcal{L} = for(\mathcal{L}^0)$, where

$$for \colon \mathrm{Mod}(\mathsf{E}_{\mathfrak{Y}}|_{\Lambda^0} \otimes_{\mathbb{C}} \mathbb{C}_{\sqrt{\Omega_{\Lambda^0}}}) \to \mathrm{Mod}(\mathsf{E}_{\mathfrak{Y},\hat{t}}|_{\Lambda^0} \otimes_{\mathbb{C}} \mathbb{C}_{\sqrt{\Omega_{\Lambda^0}}})$$
$$\simeq \mathrm{Mod}(\mathsf{W}_{\mathfrak{X}}|_\Lambda \otimes_{\mathbb{C}} \mathbb{C}_{\sqrt{\Omega_\Lambda}})$$

is the natural forgetful functor. $\hspace{3cm}$ q.e.d.

Corollary 9.2. There is a **k**-equivalence of stacks

$$\mathsf{Mod}_{\mathrm{reg}\text{-}\Lambda}(\mathsf{W}_{\mathfrak{X}}|_\Lambda \otimes_{\mathbb{C}} \mathbb{C}_{\sqrt{\Omega_\Lambda}}) \simeq \mathsf{Mod}_{\mathrm{loc\text{-}sys}}(\mathbf{k}_\Lambda). \tag{9.1}$$

Proof. By Theorem 9.1, there exists a simple module $\mathcal{L}$ in $\mathrm{Mod}(\mathsf{W}_{\mathfrak{X}}|_\Lambda \otimes_{\mathbb{C}} \mathbb{C}_{\sqrt{\Omega_\Lambda}})$. By Proposition 7.3, a functor (9.1) is given by $\mathcal{M} \mapsto \mathcal{H}om_{\mathsf{W}_{\mathfrak{X}}}(\mathcal{L}, \mathcal{M})$. Proving that it is an equivalence is a local problem, and so we may assume $\mathfrak{X} = T^*X$, $\Lambda = T^*_X X$. Then a quasi inverse is given by $F \mapsto F \otimes_{\mathbf{k}} \mathcal{O}^\tau_X$. q.e.d.

Remark 9.3. Note that in the real setting, a link between simple holonomic modules on real Lagrangian submanifolds and Fourier distributions, Maslov index, etc. is investigated in [17].

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