Wick rotation for D-modules

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Abstract

We extend the classical Wick rotation to D-modules and higher codimensional submanifolds. [1] [2]

1 Introduction

Let M be a real analytic manifold of the type $N \times \mathbb{R}$ and let $X = Y \times \mathbb{C}$ be a complexification of M. Consider a differential operator P on X such that P is hyperbolic on M with respect to the direction $N \times \{0\}$, a typical example being the wave operator on a spacetime. Denote by L the real manifold $N \times \sqrt{-1}\mathbb{R}$. It may happen, and it happens for the wave operator, that P is elliptic on L. Passing from M to L is called the Wick rotation by physicists who deduce interesting properties of P on M from the study of P on L.

In the situation above, we had $codim_M N = codim_L N = 1$. In this paper, we treat the general case of two real analytic manifolds M and L in X, X being a complexification of both M and L, such that the intersection $N := M \cap L$ is clean, and we consider a coherent $\mathscr{D}_X$-module $\mathscr{M}$ which is hyperbolic with respect to M on N and elliptic on L. The main result is Theorem 3.10 which describes an isomorphism between the complex of hyperfunction solutions of $\mathscr{M}$ on L defined in a given cone $\gamma \subset T_N L$ and the complex of hyperfunction solutions of $\mathscr{M}$ on M (in a neighborhood of N), with wave front set in a cone $\lambda \subset T^*_M X$ associated with γ. It is also proved that this isomorphism is compatible to the boundary values morphism from M to N and from L to N.

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2 Sheaves, D-modules and wave front sets

2.1 Sheaves

We shall use the microlocal theory of sheaves of [KS90] and mainly follow its terminology. For the reader's convenience, we recall a few notations and results.

[1]Key words: Lorentzian manifolds, microlocal sheaf theory, hyperbolic $\mathscr{D}$-modules

[2]MSC: 35A27, 58J15, 58J45, 81T20

Geometry

Let X be a real manifold of class C^∞. For a subset $A \subset X$, we denote by $\overline{A}$ its closure and by $\mathrm{Int}(A)$ its interior. We denote by

$$\tau_X : TX \to X, \quad \pi_X : T^*X \to X$$

the tangent bundle and the cotangent bundle to X. For a closed submanifold M of X, we denote by $\tau_M : T_M X \to M$ and $\pi_M : T^*_M X \to M$ the normal bundle and the conormal bundle to M in X. In particular, $T^*_X X$ is the zero-section of T^*X, that we identify with X.

For a vector bundle $\pi : E \to X$, we identify X with the zero-section, we denote by E_x the fiber of E at $x \in X$, we set $\dot{E} = E \setminus X$ and we denote by $\dot{\pi} : \dot{E} \to X$ the projection. For a cone γ in a vector bundle $E \to X$, we set $\gamma_x = \gamma \cap E_x$, we denote by $\gamma^a = -\gamma$ the opposite cone and by γ° the polar cone in the dual vector bundle E^*,

$$\gamma^\circ = \{(x;\xi) \in E^*; \langle \xi, v \rangle \geq 0 \text{ for all } x \in M, v \in \gamma_x\}.$$

For $A \subset X$, the Whitney normal cone of A along M, $C_M(A) \subset T_M X$, is defined in [KS90, Def. 4.1.1].

To a morphism of manifolds $f : Y \to X$, one associates the maps:

(2.1)
$$\begin{array}{ccccc}
T^*Y & \xleftarrow{f_d} & Y \times_X T^*X & \xrightarrow{f_\pi} & T^*X \\
 & \searrow^{\pi_Y} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_X} \\
 & & Y & \xrightarrow{f} & X
\end{array}$$

where f_d is the transpose of the tangent map to $Tf : TY \to Y \times_X TX$.

Definition 2.1. Let Λ be a closed conic subset of T^*X. One says that f is non characteristic for Λ if the map f_d is proper on $f_\pi^{-1}(\Lambda)$.

Sheaves

Let $\mathbf{k}$ be a field. One denotes by $\mathrm{D^b}(\mathbf{k}_X)$ the bounded derived category of sheaves of $\mathbf{k}$-modules on X. We simply call an object of this category "a sheaf". For a closed subset A of a manifold we denote by $\mathbf{k}_A$ the constant sheaf on A with stalk $\mathbf{k}$ extended by 0 outside of A. More generally, we shall identify a sheaf on A and its extension by 0 outside of A. If A is locally closed, we keep the notation $\mathbf{k}_A$ as far as there is no risk of confusion. We denote by ω_X the dualizing complex on X. Recall that $\omega_X \simeq \mathrm{or}_X[\dim X]$ where or_X is the orientation sheaf and $\dim X$ is the dimension of X. More generally, we consider the relative dualizing complex associated with a morphism $f : Y \to X$, $\omega_{Y/X} = \omega_Y \otimes f^{-1}(\omega_X^{\otimes -1})$ and its inverse, $\omega_{X/Y} = \omega_{Y/X}^{\otimes -1}$. We denote by $\mathrm{D}'_X(\,\bullet\,) = \mathrm{R}\mathscr{H}om\,(\,\bullet\,, \mathbf{k}_X)$ the duality functor on X.

We shall use freely the six Grothendieck operations on sheaves.

Microlocalization

For a closed submanifold M of X, we have the functors

$$\begin{aligned}
&\nu_M : \mathrm{D^b}(\mathbf{k}_X) \to \mathrm{D^b_{\mathbb{R}^+}}(\mathbf{k}_{T_M X}) \text{ specialization along } M,\\
&\mu_M : \mathrm{D^b}(\mathbf{k}_X) \to \mathrm{D^b_{\mathbb{R}^+}}(\mathbf{k}_{T^*_M X}) \text{ microlocalization along } M,\\
&\mu hom : \mathrm{D^b}(\mathbf{k}_X) \times \mathrm{D^b}(\mathbf{k}_X)^{\mathrm{op}} \to \mathrm{D^b_{\mathbb{R}^+}}(\mathbf{k}_{T^*X}).
\end{aligned}$$

Here, for a vector bundle $E \to M$ or $E \to X$, $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$ is the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_E)$ consisting of conic sheaves, that is, sheaves locally constant under the $\mathbb{R}^+$-action.

The functor μ_M, called Sato's microlocalization functor, is the Fourier–Sato transform of the specialization functor ν_M. The bifunctor μhom of [KS90] is a slight generalization of μ_M. Recall that $\mu_M(\,\bullet\,) = \mu hom(\mathbf{k}_M, \,\bullet\,)$.

Let λ be a closed convex proper cone of T^*_MX containing the zero-section M. For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, we have an isomorphism (see [KS90, Th. 4.3.2]):

$$\mathrm{R}\pi_{M*}\mathrm{R}\Gamma_\lambda(\mu_M(F)) \otimes \omega_{X/M} \simeq \mathrm{R}\tau_{M*}\mathrm{R}\Gamma_{\mathrm{Int}(\lambda^{\circ a})}(\nu_M(F)). \tag{2.2}$$

Microsupport

To a sheaf F is associated (see [KS90]) its microsupport $\mu\mathrm{supp}(F)$[3], a closed $\mathbb{R}^+$-conic *co-isotropic* subset of T^*X.

Let us recall some results that we shall use.

Theorem 2.2. *Let $f\colon Y \to X$ be a morphism of real manifolds and let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. Assume that f is non characteristic for F, that is, for $\mu\mathrm{supp}(F)$. Then the morphism $f^{-1}F \otimes \omega_{Y/X} \to f^!F$ is an isomorphim.*

As a particular case of this result, we get a kind of Petrowski theorem for sheaves (see Theorem 2.11 below):

Corollary 2.3. *Let M be a closed submanifold of X and let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. Assume that $T^*_MX \cap \mu\mathrm{supp}(F) \subset T^*_XX$. Then $F \otimes \mathbf{k}_M \simeq \mathrm{R}\Gamma_M F \otimes \mathrm{or}_{M/X}\,[\mathrm{codim}_X M]$.*

Let M be a closed submanifold of X. If $\Lambda \subset T^*X$ is a closed conic subset, its Whitney normal cone along T^*_MX is a closed biconic subset of $T_{T^*_MX}T^*X \simeq T^*T^*_MX$. Moreover, there exists a natural embedding

$$T^*M \hookrightarrow T^*T^*_MX \simeq T_{T^*_MX}T^*X. \tag{2.3}$$

Now we consider a morphism of manifolds $g\colon Z \to X$ and let $M \subset X$ and $N \subset Z$ be two closed submanifolds with $g(N) \subset M$. One gets the maps

$$\begin{array}{ccccc} T^*Z & \xleftarrow{\;g_d\;} & Z \times_X T^*X & \xrightarrow{\;g_\pi\;} & T^*X \\ \uparrow & & \uparrow & & \uparrow \\ T^*_NZ & \xleftarrow{\;g_{Nd}\;} & N \times_M T^*_MX & \xrightarrow{\;g_{N\pi}\;} & T^*_MX. \end{array} \tag{2.4}$$

The next result is a particular case of [KS90, Th. 6.7.1] in which we choose $V = T^*_NZ$ and write $g\colon Z \to X$ instead of $f\colon Y \to X$. (The reason of this change of notations is that we need to consider the complexification of the embedding $N \hookrightarrow M$ that we shall denote by $f\colon Y \hookrightarrow X$.)

Theorem 2.4. *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ and assume*

(a) *g is non characteristic for $\mu\mathrm{supp}(F)$,*

(b) *the map $N \times_M T^*_MX \to T^*_MX$ is non characteristic for $C_{T^*_MX}(\mu\mathrm{supp}(F))$,*

[3] $\mu\mathrm{supp}(F)$ was denoted $\mathrm{SS}(F)$ in loc. cit., a shortcut for "singular support".

(c) $g_d^{-1}T_N^*Z \cap g_\pi^{-1}\,\mu\mathrm{supp}(F) \subset N \times_M T_M^*X$.

Then one has the commutative diagram of natural isomorphisms on T_Z^*X*:*

$$
\begin{array}{ccc}
\mathrm{R}g_{Nd!}(\omega_{N/M} \otimes g_{N\pi}^{-1}\mu_M(F)) & \xrightarrow{\sim} & \mu_N(\omega_{Z/X} \otimes g^{-1}F) \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{R}g_{Nd*}(g_{N\pi}^{!}\mu_M(F)) & \xleftarrow[\sim]{} & \mu_N(g^!F).
\end{array}
\tag{2.5}
$$

Notation 2.5. As usual, we have simply writen ω_M instead of $\pi^{-1}\omega_M$ and similarly with other locally constant sheaves.

Consider the projections

$$
\begin{array}{ccccc}
T_N^*Z & \xleftarrow{g_{Nd}} & N \times_M T_M^*X & \xrightarrow{g_{N\pi}} & T_M^*X \\
 & \searrow^{\pi_N} & \downarrow \pi & & \downarrow \pi_M \\
 & & N & \xrightarrow{g} & M
\end{array}
\tag{2.6}
$$

One has the isomorphisms

$$
\begin{aligned}
\mathrm{R}\pi_{N*}\mathrm{R}g_{Nd*}(g_{N\pi}^{!}\mu_M(F)) &\simeq \mathrm{R}\pi_*(g_{N\pi}^{!}\mu_M(F)) \\
&\simeq g^!\mathrm{R}\pi_{M*}\mu_M(F) \simeq \mathrm{R}\Gamma_N F,
\end{aligned}
\tag{2.7}
$$

and

$$
\mathrm{R}\pi_{N*}\mu_N(g^!F) \simeq \mathrm{R}\Gamma_N g^!F \simeq \mathrm{R}\Gamma_N F.
\tag{2.8}
$$

Moreover, one easily proves:

Lemma 2.6. *The isomorphisms* (2.7) *and* (2.8) *are compatible with the morphisms obtained by applying* $\mathrm{R}\pi_{Z*}$ *to* (2.5).

Lemma 2.7. *In the situation of* Theorem 2.4 *assume moreover that* $g\colon Z \to X$ *is a closed embedding,* $N = Z \cap M$ *and the intersection is clean* (*that is,* $TN = N \times_M TM \cap N \times_Z TZ$). *Then condition* (c) *follows from* (b).

Proof. Let us choose a local coordinate system (x', x'', y', y'') on X such that $M = \{y' = y'' = 0)\}$ and $Z = \{x'' = y'' = 0\}$. Denote by $(x', x'', y', y''; \xi', \xi'', \eta', \eta'')$ the coordinates on T^*X and by $(x', x''; \xi', \xi'')$ the coordinates on T^*M. Then

$$
\begin{array}{ll}
M = \{y' = y'' = 0)\}, & T_M^*X = \{y' = y'' = \xi' = \xi'' = 0\}, \\
Z = \{x'' = y'' = 0\}, & T_Z^*X = \{x'' = y'' = \xi' = \eta' = 0\}, \\
N = \{x'' = y' = y'' = 0)\}, & T_N^*X = \{x'' = y' = y'' = \xi' = 0\}, \\
g_d\colon (x', y'; \xi', \xi'', \eta', \eta'') \mapsto (x', y'; \xi', \eta'), & .
\end{array}
$$

Therefore $g_d^{-1}T_N^*Z = \{(x', y'; \xi', \xi'', \eta', \eta'') \in Z \times_X T^*X; y' = \xi' = 0\} = T_N^*X$. Let $\theta \in T_{T_M^*X}T^*X$ with $\theta \notin C_{T_M^*X}\,\mu\mathrm{supp}(F)$. Then $(x', x''; \eta', \eta'') + \theta \notin \mu\mathrm{supp}(F)$. Choosing $\theta \in T_N^*M$, $\theta \neq 0$, we get that $(x', 0; 0, \xi'', \eta', \eta'') \in \mu\mathrm{supp}(F)$ implies $\xi'' = 0$. Q.E.D.

2.2 Analytic wave front set

From now on and until the end of this paper, unless otherwise specified, all manifolds are (real or complex) analytic and the base field $\mathbf{k}$ is $\mathbb{C}$.

Let M be a real manifold of dimension n and let X be a complexification of M. One denotes by $\mathscr{A}_M$ the sheaf of complex valued real analytic functions on M, that is, $\mathscr{A}_M = \mathscr{O}_X|_M$.

One denotes by $\mathscr{B}_M$ and $\mathscr{C}_M$ the sheaves on M and $T^*_M X$ of Sato's hyperfunctions and microfunctions, respectively. Recall that these sheaves are defined by

$$\mathscr{A}_M := \mathscr{O}_X \otimes \mathbb{C}_M, \quad \mathscr{B}_M := \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X), \quad \mathscr{C}_M := \mu hom(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X).$$

In particular, $\mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X)$ and $\mu hom(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X)$ are concentrated in degree 0. Since $\mathrm{D}'_X\mathbb{C}_M \simeq \mathrm{or}_M\,[-n] \simeq \omega_{M/X} \simeq \omega_M^{\otimes -1}$, we get that

$$\begin{aligned}\mathscr{B}_M &\simeq \mathrm{R}\Gamma_M(\mathscr{O}_X) \otimes \omega_M \simeq H^n_M(\mathscr{O}_X) \otimes \mathrm{or}_M,\\ \mathscr{C}_M &\simeq \mu_M(\mathscr{O}_X) \otimes \omega_M \simeq H^n(\mu_M(\mathscr{O}_X)) \otimes \mathrm{or}_M\,.\end{aligned}$$

The sheaf $\mathscr{B}_M$ is flabby and the sheaf $\mathscr{C}_M$ is conically flabby.

Moreover, since $\mathrm{R}\pi_* \circ \mu hom \simeq \mathrm{R}\mathscr{H}om$, we have the isomorphism $\mathscr{B}_M \xrightarrow{\sim} \pi_*\mathscr{C}_M$. One deduces the isomorphism:

$$\mathrm{spec}\colon \Gamma(\mathrm{M}; \mathscr{B}_\mathrm{M}) \xrightarrow{\sim} \Gamma(\mathrm{T}^*_\mathrm{M}\mathrm{X}; \mathscr{C}_\mathrm{M}).$$

Definition 2.8 ([Sat70])**.** The analytic wave front set of a hyperfunction $u \in \Gamma(M; \mathscr{B}_M)$, denoted WF(u), is the support of spec(u), a closed conic subset of $T^*_M X$.

The next result is well-known to the specialists. Let M be a real analytic manifold, X a complexification of M and let λ be a closed convex proper cone in $T^*_M X$.

Theorem 2.9. *Let $u \in \Gamma(M; \mathscr{B}_M)$ with $\mathrm{WF(u)} \subset \lambda$. Assume that M is connected and that $u \equiv 0$ on an open subset $U \subset M$, $U \neq \varnothing$. Then $u \equiv 0$ on M.*

Proof. Let $S = \mathrm{supp}(u)$ and let $x \in \partial S$. Choosing a local chart in a neighborhood of x, we may assume from the beginning that M is open in $\mathbb{R}^n$ and that $\lambda \subset M \times \sqrt{-1}\gamma^\circ$ where γ is a non empty open convex cone of $\mathbb{R}^n$. Then there exists a holomorphic function $f \in \Gamma((M \times \sqrt{-1}\gamma) \cap W; \mathscr{O}_X)$, where W is a connected open neighborhood of M in X, such that $u = b(f)$, that is, u is the boundary value of f. If $b(f)$ is analytic on U, then f extends holomorphically in a neighborhood of U in X. If moreover $f = 0$ on U, then $f \equiv 0$ on $M \times \sqrt{-1}\gamma) \cap W$ and thus $u \equiv 0$. Q.E.D.

2.3 D-modules

Let $(X, \mathscr{O}_X)$ be a complex manifold. One denotes by $\mathscr{D}_X$ the sheaf of rings of finite order holomorphic differential operators on X. In the sequel, a $\mathscr{D}_X$-module means a left $\mathscr{D}_X$-module. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Locally on X, $\mathscr{M}$ may be represented as the cokernel of a matrix $\cdot P_0$ of differential operators acting on the right:

$$\mathscr{M} \simeq \mathscr{D}_X^{N_0}/\mathscr{D}_X^{N_1} \cdot P_0$$

and one shows that $\mathscr{M}$ is locally isomorphic to the cohomology of a bounded complex

$$\mathscr{M}^\bullet := 0 \to \mathscr{D}_X^{N_r} \to \cdots \to \mathscr{D}_X^{N_1} \xrightarrow{\cdot P_0} \mathscr{D}_X^{N_0} \to 0. \tag{2.9}$$

Clearly, $\mathscr{O}_X$ is a left $\mathscr{D}_X$-module. It is indeed coherent since $\mathscr{O}_X \simeq \mathscr{D}_X/\mathscr{I}$ where $\mathscr{I}$ is the left ideal generated by the vector fields. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, one sets for short

$$\mathscr{S}ol(\mathscr{M}) := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X).$$

Representing (locally) $\mathscr{M}$ by a bounded complex $\mathscr{M}^\bullet$ as above, we get

$$\mathscr{S}ol(\mathscr{M}) \simeq 0 \to \mathscr{O}_X^{N_0} \xrightarrow{P_0\cdot} \mathscr{O}_X^{N_1} \to \cdots \mathscr{O}_X^{N_r} \to 0, \tag{2.10}$$

where now $P_0\cdot$ operates on the left.

Hence a coherent $\mathscr{D}_X$-module is nothing but a system of linear partial differential equations.

To a coherent $\mathscr{D}_X$-module $\mathscr{M}$ is associated its characteristic variety, a closed analytic $\mathbb{C}^\times$-conic co-isotropic subset of T^*X.

Theorem 2.10 (see [KS90, Th. 11.3.3])**.** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then* $\mu\mathrm{supp}(\mathscr{S}ol(\mathscr{M})) = \mathrm{char}(\mathscr{M})$.

Let $f\colon Y \to X$ be a morphism of complex manifolds. One can define the inverse image $f^D\mathscr{M}$, an object of $\mathrm{D^b}(\mathscr{D}_Y)$. The Cauchy-Kowalevska theorem has been extended to D-modules in Kashiwara's thesis of 1970.

Theorem 2.11 (see [Kas95, Kas03])**.** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module and assume that f is non characteristic for $\mathscr{M}$, that is, for* $\mathrm{char}(\mathscr{M})$. *Then*

(i) *$f^D(\mathscr{M})$ is concentrated in degree 0 and is a coherent $\mathscr{D}_Y$-module,*

(ii) $\mathrm{char}(f^D(\mathscr{M})) = f_d f_\pi^{-1}\,\mathrm{char}(\mathscr{M})$,

(iii) *one has a natural isomorphism* $f^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(f^D\mathscr{M}, \mathscr{O}_Y)$.

Example 2.12. Assume $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X \cdot P$ for a differential operator P of order m and Y is a hypersurface, non characteristic for P. Let $s = 0$ be a reduced equation of Y. Then, $f^D(\mathscr{M}) \simeq \mathscr{D}_Y/(s \cdot \mathscr{D}_Y + \mathscr{D}_X \cdot P)$ and it follows from the Weierstrass division theorem that, locally, $f^D\mathscr{M} \simeq \mathscr{D}_Y^m$. In this case, isomorphism (iii) in the above theorem is nothing but the Cauchy-Kowalevska theorem.

Definition 2.13. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module and let $L \subset X$ be a real submanifold. One says that the pair $(L, \mathscr{M})$ is elliptic if $\mathrm{char}(\mathscr{M}) \cap T^*_L X \subset T^*_X X$.

If X is a complexification of a real manifold M, the pair $(M, \mathscr{M})$ is elliptic if and only if $\mathscr{M}$ is elliptic in the usual sense and Corollary 2.3 gives the isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_M) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M). \tag{2.11}$$

In particular, the hyperfunction solutions of the system $\mathscr{M}$ are real analytic. More generally, we have

Theorem 2.14 ([Sat70])**.** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module and let $u \in \Gamma(M; \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M))$. Then* $\mathrm{WF(u)} \subset \mathrm{T^*_M X} \cap \mathrm{char}(\mathscr{M})$.

When $L = Y$ is a complex submanifold of complex codimension d, $(Y, \mathscr{M})$ is elliptic if and only if the embedding $Y \hookrightarrow X$ is non-characteristic for $\mathscr{M}$. In this case, Corollary 2.3 gives the isomorphism

$$f^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{R}\Gamma_Y\mathscr{O}_X)\,[2d]. \tag{2.12}$$

3 Wick rotation for D-modules

3.1 Hyperbolic D-modules

Let M be a real manifold and let X be a complexification of M. Recall the embedding $T^*M \hookrightarrow T^*T^*_MX$ of (2.3) and recall that for $S \subset T^*X$, the Whitney cone $C_{T^*_MX}(S)$ is contained in $T_{T^*_MX}T^*X \simeq T^*T^*_MX$. The next definition is extracted form [KS90]. See [Sch13] for details.

Definition 3.1. Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module.

(a) We set

$$\mathrm{hypchar}_M(\mathscr{M}) = T^*M \cap C_{T^*_MX}(\mathrm{char}(\mathscr{M})) \tag{3.1}$$

and call $\mathrm{hypchar}_M(\mathscr{M})$ the *hyperbolic characteristic variety of $\mathscr{M}$ along M.*

(b) *A vector $\theta \in T^*M$ such that $\theta \notin \mathrm{hypchar}_M(\mathscr{M})$ is called* hyperbolic with respect to $\mathscr{M}$.

(c) A submanifold N of M is called *hyperbolic for $\mathscr{M}$ if*

$$T^*_NM \cap \mathrm{hypchar}_M(\mathscr{M}) \subset T^*_MM, \tag{3.2}$$

that is, any nonzero vector of T^*_NM is hyperbolic for $\mathscr{M}$.

(d) *For a differential operator P, we set* $\mathrm{hypchar}(P) = \mathrm{hypchar}_M(\mathscr{D}_X/\mathscr{D}_X \cdot P)$.

Example 3.2. Assume we have a local coordinate system $(x+\sqrt{-1}y)$ on X with $M = \{y = 0\}$ and let $(x + \sqrt{-1}y; \xi + \sqrt{-1}\eta)$ be the coordinates on T^*X so that $T^*_MX = \{y = \xi = 0\}$. Let $(x_0; \theta_0) \in T^*M$ with $\theta_0 \neq 0$. Let P be a differential operator with principal symbol $\sigma(P)$. Then $(x_0; \theta_0)$ is hyperbolic for P if and only if

$$\begin{cases} \text{there exist an open neighborhood } U \text{ of } x_0 \text{ in } M \text{ and an open conic} \\ \text{neighborhood } \gamma \text{ of } \theta_0 \in \mathbb{R}^n \text{ such that } \sigma(P)(x; \theta + \sqrt{-1}\eta) \neq 0 \text{ for} \\ \text{all } \eta \in \mathbb{R}^n,\ x \in U \text{ and } \theta \in \gamma. \end{cases} \tag{3.3}$$

As noticed by M. Kashiwara, it follows from the local Bochner's tube theorem that Condition (3.3) can be simplified: $(x_0; \theta_0)$ is hyperbolic for P if and only if

$$\begin{cases} \text{there exists an open neighborhood } U \text{ of } x_0 \text{ in } M \text{ such that} \\ \sigma(P)(x; \theta_0 + \sqrt{-1}\eta) \neq 0 \text{ for all } \eta \in \mathbb{R}^n, \text{ and } x \in U. \end{cases} \tag{3.4}$$

Hence, one recovers the classical notion of a (weakly) hyperbolic operator.

Notation 3.3. As usual, we shall write $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_M)$ instead of $\mathrm{R}\mathscr{H}om_{\pi^{-1}\mathscr{D}_X}(\pi^{-1}\mathscr{M}, \mathscr{C}_M)$ and similarly with other sheaves on cotangent bundles.

3.2 Main tool

Consider as above a real manifold M and a complexification X of M, a closed submanifold N of M, and Y a complexification of N in X. Denote as above by $f\colon Y \hookrightarrow X$ the embedding. Consider also another closed real submanifold $L \subset X$ such that $L \cap M = N$ and the intersection is clean. Denote by $g\colon L \hookrightarrow X$ the embedding and consider the Diagram 2.4 with $Z = L$.

(We prefer to use the notation L better than L since now it is a real manifold, playing a role similar to that of M.)

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module and consider the hypotheses:

(3.5) the pair $(L, \mathscr{M})$ is elliptic,

(3.6) the submanifold N is hyperbolic for $\mathscr{M}$ on M,

(3.7) Y is non characteristic for $\mathscr{M}$.

Set $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$. Then hypothesis (a) of Theorem 2.4 is translated as hypothesis (3.5) and hypothesis (b) is translated as hypothesis (3.6).

We shall constantly use the next result.

Lemma 3.4 (see [JS16, Lem. 3.5])**.** *Hypothesis* (3.6) *implies hypothesis* (3.7)*.*

Theorem 3.5. *Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module. Assume* (3.5) *and* (3.6)*. Then one has the natural isomorphism*

$$\mathrm{R}g_{Nd!}g_{N\pi}^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_M) \xrightarrow{\sim} \mu_N(\omega_{L/N} \otimes g^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)).$$

Proof. Apply Theorem 2.4 together with Lemma 2.7 to the sheaf $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$. We get:

$$\mathrm{R}g_{Nd!}(\omega_{N/M} \otimes g_{N\pi}^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_M(\mathscr{O}_X))) \simeq \mu_N(\omega_{L/X} \otimes g^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)).$$

Equivalently, we have

$$\mathrm{R}g_{Nd!}g_{N\pi}^{-1}(\omega_{X/M} \otimes \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_M(\mathscr{O}_X))) \simeq \mu_N(\omega_{L/N} \otimes g^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)).$$

Finally $\omega_{X/M} \otimes \mu_M(\mathscr{O}_X) \simeq \mathscr{C}_M$. Q.E.D.

Example 1: Cauchy problem for microfunctions

Let M, X, L, N and f be as above and assume that $L = Y$, hence $f = g$.

Corollary 3.6. *Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module. Assume* (3.6)*. Then one has the natural isomorphism*

$$f_{Nd!}f_{N\pi}^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_M) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(f^D\mathscr{M}, \mathscr{C}_N).$$

Proof. Applying Theorem 2.11, we get $f^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(f^D\mathscr{M}, \mathscr{O}_Y)$. (Recall that (3.6) implies (3.7).) Moreover, $\omega_{Y/N} \otimes \mu_N(\mathscr{O}_Y) \simeq \mathscr{C}_N$. Finally, since f_{Nd} is finite on $\mathrm{char}(\mathscr{M})$, we may replace $\mathrm{R}f_{Nd!}$ with $f_{Nd!}$. Q.E.D.

3.3 Boundary values

Let M be a real n-dimensional manifold, N a closed submanifod of codimesnion d, X a complexification of M and Y a complexification of N in X. We denote by $f\colon Y \hookrightarrow X$ the embedding.

Notation 3.7. We set

$$\widetilde{\mathscr{B}}_N = \mathrm{R}\Gamma_N(\mathscr{O}_X) \otimes \mathrm{or}_N [n] \simeq H^n_N(\mathscr{O}_X) \otimes \mathrm{or}_N .$$

We shall not confuse the sheaf $\widetilde{\mathscr{B}}_N$ with the sheaf $\mathscr{B}_N$ of hyperfunctions on N. We have an isomorphism

$$\widetilde{\mathscr{B}}_N \simeq \Gamma_N \mathscr{B}_M \otimes \mathrm{or}_{N/M}\,.$$

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Applying the functor $\mathrm{R}\Gamma_N(\,\bullet\,) \otimes \mathrm{or}_N\,[n-d]$ to the isomorphism (iii) in Theorem 2.11 together with isomorphism (2.12) one recovers a well known result:

Lemma 3.8. *Assume* (3.7)*. One has a natural isomorphism*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \widetilde{\mathscr{B}}_N)\,[d] \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(f^D\mathscr{M}, \mathscr{B}_N).$$

Appying the functor D'_X to the morphism $\mathbb{C}_M \to \mathbb{C}_N$, we get the morphism $\mathrm{D}'_X(\mathbb{C}_N) \to \mathrm{D}'_X(\mathbb{C}_M)$, that is, the morphism $\mathrm{or}_N\,[d+n] \to \mathrm{or}_M\,[n]$. Applying the functor $\mathrm{R}\mathscr{H}om\,(\,\bullet\,, \mathscr{O}_X)$ we get the "restriction" morphism

$$\rho_{MN}\colon \mathscr{B}_M \to \widetilde{\mathscr{B}}_N\,[d] \simeq \Gamma_N\mathscr{B}_M \otimes \omega_{M/N}. \tag{3.8}$$

For a closed cone $\lambda \subset T^*_M X$, we set for short

$$\mathscr{B}_{M,\lambda} := \pi_{M*}\Gamma_\lambda \mathscr{C}_M. \tag{3.9}$$

For an open cone $\gamma \subset T_N M$, we set for short :

$$\Gamma_\gamma \mathscr{B}_{NM} := \tau_{N*}\Gamma_\gamma(\nu_N(\mathscr{B}_M)). \tag{3.10}$$

(In the sequel, we shall use this notation for another real manifold Z instead of M.)

Hence, for a closed convex proper cone λ with $\lambda \supset N$, setting $\gamma = \mathrm{Int}(\lambda^{\circ a})$, we have by (2.2):

$$\pi_{N*}\Gamma_\lambda(\mu_N\mathscr{B}_M) \otimes \omega_{M/N} \simeq \Gamma_\gamma \mathscr{B}_M. \tag{3.11}$$

One can use (3.11) and the morphism $\pi_{N*}\Gamma_\lambda(\mu_N\mathscr{B}_M) \to \pi_{N*}\mu_N\mathscr{B}_M \simeq \Gamma_N\mathscr{B}_M$ to obtain the morphism

$$b_{\gamma,N}\colon \Gamma_\gamma\mathscr{B}_M \to \Gamma_N\mathscr{B}_M \otimes \omega_{M/N}. \tag{3.12}$$

One can also construct (3.12) directly as follows. Let U be an open subset of M such that $\overline{U} \supset N$, U is locally cohomologically trivial (see [KS90, Exe. III.4]). Then the morphism $\mathbb{C}_{\overline{U}} \to \mathbb{C}_N$ gives by duality the morphism $\mathrm{or}_N\,[d+n] \to \mathrm{or}_U\,[n]$ and one gets the morphism $\Gamma_U\mathscr{B}_M \to \Gamma_N\mathscr{B}_M \otimes \omega_{M/N}$ by applying $\mathrm{R}\mathscr{H}om\,(\,\bullet\,, \mathscr{O}_X)$ similarly as for ρ_{MN}. Taking the inductive limit with respect to the family of open sets U such that $C_M(X \setminus U) \cap \gamma = \varnothing$ (see [KS90, Th. 4.2.3]), we recover the morphism (3.12).

In particular, for a coherent $\mathscr{D}_X$-module $\mathscr{M}$ we get the morphisms

$$\begin{aligned}
\gamma_{MN} \ &:\ \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{M,\lambda}) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \widetilde{\mathscr{B}}_N)\,[d],\\
b_{\gamma,N} \ &:\ \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_\gamma\mathscr{B}_{NM}) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \widetilde{\mathscr{B}}_N)\,[d].
\end{aligned}$$

3.4 Wick rotation

Let M, X, Y, N, L, f and g be as above. Now, we also assume that L is a real manifold of the same dimension than M and X is a complexification of L. We still consider diagram (2.4).

Consider the hypothesis

$$\text{(3.13)} \qquad \begin{cases} \text{in a neighborhood of } N, \operatorname{char}(\mathscr{M}) \cap T^*_M X \text{ is contained in the union of} \\ \text{two closed cones } \lambda^+ \text{ and } \lambda^- \text{ such that } \lambda^+ \cap \lambda^- \subset T^*_X X \text{ and } \lambda^\pm \supset T^*_M X. \end{cases}$$

Lemma 3.9. *Assume* (3.13)*. Then we have the natural isomorphism*

$$\text{(3.14)} \qquad g_{N\pi}^{-1} \mathrm{R}\Gamma_{\lambda^+} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_M) \xrightarrow{\sim} \mathrm{R}\Gamma_{g_{N\pi}^{-1}(\lambda^+)} g_{N\pi}^{-1} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_M).$$

Proof. (i) Set for short $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_M)$, $j = g_{N\pi}$, $A = \lambda^+$, $B = j^{-1}A$. With these new notations, we have to prove the morphism

$$\text{(3.15)} \qquad j^{-1}\mathrm{R}\Gamma_A F \xrightarrow{\sim} \mathrm{R}\Gamma_B j^{-1} F$$

is an isomorphism.

(ii) The morphism (3.15) is an isomorphism outside of the zero-section of $T^*_M X$ since $\operatorname{supp}(F) = A \sqcup C$ with A and C closed and $A \cap C = \varnothing$.

(iii) Consider the diagram in which s_N and s_M denote the embeddings of the zero-sections:

$$\text{(3.16)} \qquad \begin{array}{ccc} N \times_M T^*_M X & \xrightarrow{j} & T^*_M X \\ {\scriptstyle \pi_N} \downarrow \uparrow {\scriptstyle s_N} & & {\scriptstyle \pi_M} \downarrow \uparrow {\scriptstyle s_M} \\ N & \xrightarrow{j} & M. \end{array}$$

Since $\mathrm{R}\pi_{N*} \simeq s_N^{-1}$, when applied to conic sheaves, it remains to show that (3.15) is an isomorphism after applying the functor $\mathrm{R}\pi_{N*}$.

(iv) Consider the morphism of Sato's distinguished triangles:

$$\begin{array}{ccccccc} \mathrm{R}\pi_{N!} j^{-1} \mathrm{R}\Gamma_A F & \longrightarrow & \mathrm{R}\pi_{N*} j^{-1} \mathrm{R}\Gamma_A F & \longrightarrow & \dot{\pi}_{N*} j^{-1} \mathrm{R}\Gamma_A F & \xrightarrow{+1} & \\ \downarrow {\scriptstyle u} & & \downarrow {\scriptstyle v} & & \downarrow {\scriptstyle w} & & \\ \mathrm{R}\pi_{N!} \mathrm{R}\Gamma_B j^{-1} F & \longrightarrow & \mathrm{R}\pi_{N*} \mathrm{R}\Gamma_B j^{-1} F & \longrightarrow & \dot{\pi}_{N*} \mathrm{R}\Gamma_B j^{-1} F & \xrightarrow{+1} & \end{array}$$

It follows from (i) that the vertical arrow w on the right is an isomorphism. We are thus reduced to prove the isomorphism

$$\text{(3.17)} \qquad \mathrm{R}\pi_{N!} j^{-1} \mathrm{R}\Gamma_A F \xrightarrow{\sim} \mathrm{R}\pi_{N!} \mathrm{R}\Gamma_B j^{-1} F.$$

(v) Using the fact that $A \supset M$ and $B \supset N$ and that Diagram (3.16) (with the arrows going down) is Cartesian, we get

$$\begin{aligned} \mathrm{R}\pi_{N!} j^{-1} \mathrm{R}\Gamma_A F &\simeq j^{-1} \mathrm{R}\pi_{M!} \mathrm{R}\Gamma_A F \simeq j^{-1} s_M^! \mathrm{R}\Gamma_A F \\ &\simeq j^{-1} s_M^! F \simeq j^{-1} \mathrm{R}\pi_{M!} F \simeq \mathrm{R}\pi_{N!} j^{-1} F \\ &\simeq s_N^! j^{-1} F \simeq s_M^! \mathrm{R}\Gamma_B j^{-1} F \\ &\simeq \mathrm{R}\pi_{N!} \mathrm{R}\Gamma_B j^{-1} F. \end{aligned}$$

Q.E.D.

Consider

(3.18) $\qquad \gamma \subset T_N L$ an open convex cone such that $\overline{\gamma}$ contains the zero-section N

and recall notations (3.9) and (3.10).

Theorem 3.10 (Wick isomorphism Theorem)**.** *Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module and let γ be as in* (3.18)*. Assume* (3.5)*,* (3.6)*,* (3.13) *and also*

$$g_{N\pi}^{-1}(\lambda^+) = g_{Nd}^{-1}(\gamma^{\circ a}). \tag{3.19}$$

Then one has the commutative diagram in which the horizontal arrow is an isomorphism:

$$\begin{array}{ccc} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_{M,\lambda^+})|_N & \xrightarrow{\ \sim\ } & \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\Gamma_\gamma\mathscr{B}_{NL}) \\ & {\scriptstyle\rho_{MN}}\searrow \qquad \swarrow{\scriptstyle b_{\gamma,N}} & \\ & \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\widetilde{\mathscr{B}}_N)\,[d] & \end{array} \tag{3.20}$$

Proof. (i) As a particular case of Theorem 3.5 and using the fact that $g^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_L)$, we get the isomorphism

$$\mathrm{R}g_{Nd!}g_{N\pi}^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}_M) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mu_N\mathscr{B}_L)\otimes\omega_{L/N}.$$

(ii) Set for short $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}_M)$. Using Lemma 3.9 and the fact that g_{Nd} is proper on $\operatorname{supp} F$, we have the isomorphism

$$\begin{aligned} \mathrm{R}g_{Nd!}g_{N\pi}^{-1}\mathrm{R}\Gamma_{\lambda^+}F &\simeq \mathrm{R}g_{Nd!}\mathrm{R}\Gamma_{g_{N\pi}^{-1}(\lambda^+)}g_{N\pi}^{-1}F \\ &\simeq \mathrm{R}\Gamma_{\gamma^{\circ a}}\mathrm{R}g_{Nd!}g_{N\pi}^{-1}F. \end{aligned}$$

Therefore, we have proved the isomorphism

$$\mathrm{R}g_{Nd!}g_{N\pi}^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\Gamma_{\lambda^+}\mathscr{C}_M) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\Gamma_{\gamma^{\circ a}}\mu_N\mathscr{B}_L)\otimes\omega_{L/N}. \tag{3.21}$$

(iii) Let us apply the functor $\mathrm{R}\pi_{N*}$ to (3.21). Since g_{Nd} is proper on $\operatorname{supp} F$, setting $G = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\Gamma_{\lambda^+}\mathscr{C}_M)$, we have (see Diagram 2.6)

$$\begin{aligned} \mathrm{R}\pi_{N*}\mathrm{R}g_{Nd!}g_{N\pi}^{-1}G &\simeq \mathrm{R}\pi_* g_{N\pi}^{-1}G \\ &\simeq (\mathrm{R}\pi_{M*}G)|_N. \end{aligned}$$

Hence, we have proved the isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_{M,\lambda^+})|_N \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\pi_{N*}\Gamma_{\gamma^{\circ a}}\mu_N\mathscr{B}_L)\otimes\omega_{L/N}$$

and the result follows from (3.11). Q.E.D.

3.5 The classical Wick rotation

Let us treat the classical Wick rotation. Hence, we assume that $M = N \times \mathbb{R}$ and $L = N \times \sqrt{-1}\mathbb{R}$. As usual, Y is a complexification of N and $X = Y \times \mathbb{C}$. We denote by $t + is$ the holomorphic coordinate on $\mathbb{C}$, by $(t+is; \tau+i\sigma)$ the symplectic coordinates on $T^*\mathbb{C}$ and by $(x; i\eta)$ a point of $T^*_N Y$. We identify N and $N \times \{0\} \subset X$.

Let P is a differential operator of order m, elliptic on L and (weakly) hyperbolic on M in the $\pm dt$ codirections. A typical example is the wave operator on a globally hyperbolic spacetime $N \times \mathbb{R}_t$. Set

$$L^+ = N \times \{t + is; t = 0, s > 0\}, \quad \lambda^+ = T^*_N Y \times \{(t + is; \tau + i\sigma); s = 0, \tau = 0, \sigma \leq 0\}.$$

The map $g_{Nd} \colon N \times_M T^*_M X \to T^*_N L$ is given by

$$(x, 0; i\eta, i\sigma) \mapsto (x; \sigma).$$

We shall apply the preceding result with $\gamma = L^+$. In that case, $\gamma^{\circ a} = \lambda^+$ and (3.19) is satisfied.

Let $\mathscr{M} = \mathscr{D}_X / \mathscr{D}_X \cdot P$. In the sequel we write for short $\mathscr{B}^P_M$ instead of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)$ and similarly with other sheaves. Note that $\mathscr{E}xt^1_{\mathscr{D}_X}(\mathscr{M}, \widetilde{\mathscr{B}}_N) \simeq \widetilde{\mathscr{B}}_N / P \cdot \widetilde{\mathscr{B}}_N$.

As a particular case of Theorem 3.10, we get:

Corollary 3.11. *We have a commutative diagram in which the horizontal arrow is an isomorphism:*

$$\begin{array}{ccccc} \mathscr{B}^P_{M,\lambda^+}|_N & & \xrightarrow{\ \sim\ } & & \mathscr{B}^P_{L^+}|_N \\ & \searrow^{\rho} & & \swarrow^{b} & \\ & & \widetilde{\mathscr{B}}_N / P \cdot \widetilde{\mathscr{B}}_N & \xrightarrow[\sim]{} & \mathscr{B}^m_N. \end{array}$$

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