DUALITY FUNCTORS FOR QUANTUM GROUPOIDS

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Abstract

We present a formal algebraic language to deal with quantum deformations of Lie-Rinehart algebras — or Lie algebroids, in a geometrical setting. In particular, extending the ice-breaking ideas introduced by Xu in [34], we provide suitable notions of "quantum groupoids". For these objects, we detail somewhat in depth the formalism of linear duality; this yields several fundamental antiequivalences among (the categories of) the two basic kinds of "quantum groupoids". On the other hand, we develop a suitable version of a "quantum duality principle" for quantum groupoids, which extends the one for quantum groups — dealing with Hopf algebras — originally introduced by Drinfeld (cf. [9], §7) and later detailed in [13]. [1]

Contents

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1 Introduction

The classical theory of Lie groups, or alternatively algebraic groups, has a quantum counterpart in the theory of "quantum groups". This notion became of widespread interest after Drinfeld landmarking contribution in 1986 (see [9]), which opened the way for a long, fruitful investigation.

In particular, Drinfeld set a rigorous definition: quantum groups are suitable topological Hopf algebras which are formal deformations either of the algebra of functions on a formal group, or of the universal enveloping algebras of a Lie algebra. In this context, the deformation provides an additional structure on the classical object: namely, the formal group inherits a structure of *Poisson* formal group, and the Lie algebra a structure of *Lie bialgebra*. Both objects can be thought of as the infinitesimal counterpart of Poisson Lie groups (or of Poisson algebraic groups), whose importance in Poisson geometry is clear. Drinfeld himself also sketched the way to introduce linear duality among quantum groups, in a quite natural way. On the other hand, he also revealed a more surprising feature of quantum groups, later named "quantum duality". Namely, there exists an equivalence of categories which turns every quantized enveloping algebra into a quantized formal group, and viceversa; even more, in this equivalence one shifts from a quantization of a given Lie bialgebra, say L, to one of the *dual* Lie bialgebra L^*.

On the other hand, there exists another extension of Lie group theory which still moves within the realm of "classical" (vs. "quantum") Lie theory: this is the theory of Lie-Rinehart algebras (sometimes also called "Lie algebroids"), developed by Rinehart, Huebschmann and others. A Lie-Rinehart algebra (sometimes also called "Lie algebroid") over a commutative ring A is a structure $\big(L\,,[\ ,\ ]\,,\omega\big)$ between A–Lie algebras (this corresponds to the case where the anchor map ω is 0) and the k–Lie algebra of derivations of A, namely $Der(A)$. Global sections of Lie algebroids provide examples of Lie-Rinehart algebras that come from geometry: if X is a manifold, then $\big(\Gamma(TX)\,,\mathcal{C}^\infty(X)\,,id\big)$ is a Lie-Rinehart algebra whose enveloping algebra is the algebra of globally defined differential operators. When X is a Poisson manifold, it is known (cf. [8] and [16]) that $\Omega^1(X)$ is a Lie-Rinehart algebra, which is very used in Poisson geometry (see [11]).

The natural algebraic gadgets attached with a Lie-Rinehart algebra are its universal enveloping algebra $V(L)$ and its algebra of jets $J(L)$ — direct generalizations of the universal enveloping algebra of a Lie algebra and of the algebra of functions on a formal group.

As A is commutative, any Lie-Rinehart algebra L over A can also be seen as a *right* Lie-Rinehart algebra: this leads to define the less studied right enveloping algebra of L, call it $V^r(L)$. If L is a Lie algebra, then $V^\ell(L)$ and $V^r(L)$ coincide. In general, the algebra $V^r(L)$ as is anti-isomorphic to $V^\ell(L)$ via the map $\Xi : V^\ell(L) \longrightarrow V^r(L)\,,\ (L \ni) \, D \mapsto -D\,,\ (A \ni) \, a \mapsto a$. The existence of two enveloping algebras $V^\ell(L)$ and $V^r(L)$ will induce a phenomenon that does not appear in standard quantum group theory: namely, there will be two different notions of quantum enveloping algebroids and quantum formal series algebroids, the *left* ones and the *right* ones.

All these algebraic objects attached to L — $V^\ell(L)$, $V^r(L)$, $J^r(L)$ and $J^\ell(L)$ — bear a richer structure, namely of *(topological) bialgebroid* — a *left* for those with a superscript "ℓ", a *right* one for those with an "r". The notion of left/right bialgebroid is in turn a generalization — somewhat less rigid, and less symmetric — of that of Hopf algebra. It appears (in the left version, cf. [32] and [25]) when one wants to replace the commutative ground ring k of a bialgebra by a possibly noncommutative k–algebra A. Thus a left bialgebroid over a ground ring A is a sextuple $\mathcal{H} = \big(H, A, s^\ell, t^\ell, \Delta, \epsilon\big)$ where H is an algebra, the source map $s^\ell : A \longrightarrow H$ is an algebra morphism, the target map $t^\ell : A \longrightarrow H$ is an algebra anti-morphism, Δ is a coproduct (in a suitable sense) and $\epsilon : H \longrightarrow A$ is the counit for that coproduct. It is well known that $V^\ell(L)$ is naturally endowed with a standard left bialgebroid structure.

If U is a k–Hopf algebra, its opposite-coopposite U^{op}_{coop} is also a k–Hopf algebra. Instead, this is not true anymore if U is a left bialgebroid. This remark leads to the concept of right bialgebroid, that was introduced in [18]. Then one has that sextuple $\mathcal{W} = \big(W, A, s^r, t^r, \Delta, \partial\big)$ is a right bialgebroid if and only if $\mathcal{W}^{op}_{coop} = \big(W^{op}, A^{op}, s^r, t^r, \Delta^{coop}, \partial\big)$ is a left bialgebroid. If L is a Lie Rinehart algebra, its right enveloping algebra $V^r(L)$ is endowed with a standard right bialgebroid structure.

Any left bialgebroid U is naturally a left A–module and a right A–module. Then one may consider its *left dual* U_* — i.e. its dual as a left A–module — as well as its *right dual* U^* — i.e. its dual as a right A–module. Then (under mild conditions) U^* and U_* can be endowed with a right bialgebroid structure (see [18]). Similarly, a right bialgebroid W also has a left dual ${}_*W$ and a right dual ${}^*W$ which can both be endowed with a left bialgebroid structure.

In particular, $J^r(L) := V^\ell(L)^*$ is a right bialgebroid. It is well know that one recovers the left bialgebroid $V^\ell(L)$ from $J^r(L)$ by taking the continuous left dual $V^\ell(L) = {}_\star J(L)$. Similarly, $J^\ell(L) := {}_*V^r(L)$ is a left bialgebroid, and the right bialgebroid $V^r(L)$ can be recovered from $J^\ell(L)$ by taking the continuous right dual $V^r(L) = J^\ell(L)^\star$.

When looking for quantizations of Lie-Rinehart algebras, one should consider formal deformations of either $V^{\ell/r}(L)$ or $J^{r/\ell}(L)$, among the corresponding objects, i.e. left or right (topological) bialgebroids: loosely speaking, we shall call such objects "quantum groupoids".

The first, ice-breaking step in this direction was made by Ping Xu in [34]: in that paper he introduced a formal notion of quantization of $V^\ell(L)$, called quantum universal enveloping algebroid (LQUEAd in short). Then he noticed that any such quantization endows the Lie-Rinehart algebra L itself with an additional piece of structure which eventually makes it into a *Lie-Rinehart bialgebra*. Indeed, this notion of Lie-Rinehart bialgebra turns out to be a direct extension of that of Lie bialgebra, and can be studied in purely classical terms (i.e., without involving quantizations) much like it occurs for the notion of Lie bialgebra. In particular, this is a self-dual notion, so that if L is a Lie-Rinehart bialgebra then its dual space L^* is a Lie-Rinehart bialgebra as well ([21]).

Still in [34], Xu provides also a construction which provides examples of LQUEAd's, in the following way. Let X be a Poisson manifold $\mathcal{D}_X := V^\ell\big(\Gamma(TX)\big)$. Let $*$ be a star product which quantizes the Poisson bracket $\{\ ,\ \}$ of X, defined by a "twist" $\mathcal{F} \in \big(\mathcal{D}_X \otimes \mathcal{D}_X\big)[[h]]$. The twist (in the sense of [34]) of $\mathcal{D}_X[[h]]$ — which is the trivial deformation of $\mathcal{D}_X$ — by $\mathcal{F}$ provides a new, non-trivial LQUEAd, denoted by $\mathcal{D}_X[[h]]^{\mathcal{F}}$, with non trivial coproduct.

The purpose of the present paper is to move some further steps in the work of laying bases for a quantum theory of Lie-Rinehart algebras; in other words, we provide our contribution to the foundations of a theory of "quantum Lie algebroids" or, speaking globally, of "quantum groupoids".

After recalling some basics of the theory of Lie-Rinehart algebras and bialgebras (Sec. 2), we introduce also some basics of the theory of bialgebroids (Sec. 3): in particular, we dwell on the relevant examples, i.e. universal enveloping algebras and jet spaces for Lie-Rinehart algebras.

Then we introduce "quantum groupoids" (Sec. 4). Besides the original notion of LQUEAd given by Xu in [34] — yet with different conventions — we introduce its right counterpart (the notion of right quantum universal enveloping algebra, in short RQUEAd): a topological right bialgebroid which is a formal deformation of some $V^r(L)$. Similarly, we introduce quantizations of jet spaces; a topological right bialgebroid which is a formal deformation of some $J^r(L)$ will be called a *right quantum formal series algebroid* (RQFSAd in short); similarly, the left-handed version of this notion gives rise to the definition of *left quantum formal series algebroid* (LQFSAd in short). Altogether, this gives us four kinds of quantum groupoids; each one of these induces a Lie-Rinehart *bialgebra* structures on the original Lie-Rinehart algebra one deals with, extending what happens with LQUEAd's (after [34]).

As a next step, we discuss linear duality for quantum groupoids (Sec. 5). Here the natural language is that of linear duality for left and right bialgebroids, with some precisions. First, the bialgebroids we deal with have infinite rank, so one has to consider *topological* duals, with respect to suitably chosen topologies (which is harmless). Second, both left and right duals are available: thus a priori taking duals causes a proliferation of objects. Nevertheless, there is still enough symmetry to keep this phenomenon under control, so eventually we can bound ourselves to deal with only a handful of duality functors.

In the end, our main result on the subject claims the following: our duality functors provide (well-defined) anti-equivalences between the category of all LQUEAd's and the category of all RQFSAd's (on a same, fixed ring A_h), and similarly also anti-equivalences between the category of all RQUEAd's the category of all LQFSAd's (on A_h again). In addition, if one starts with a given quantum groupoid, which induces a specific (Lie-Rinehart) bialgebra structure on the underlying Lie-Rinehart algebra, then the dual quantization yields the same or the coopposite Lie-Rinehart bialgebra structure — see Theorems 5.5 and 5.7 for further precisions. To be precise, let us denote by LQUEAd_{A_h} the category of all LQUEAD over A_h (a quantization of the ground ring A), and by RQFSAd_{A_h} the category of all RQFSAd over A_h, and so on. Then denote by a $\star$ the continuous dual (with respect to a suitable topology), not to be confused with the full dual (denoted instead by a $*$). Our main result on linear duality for quantum groupoids then reads as follows:

Theorem 1. *(cf. Theorems 5.5, 5.7 and 5.8 in Sec. 5)*

Left and right duals yield pairs of well-defined contravariant functors

$$\begin{array}{ll}
(\mathrm{LQUEAd})_{A_h} \longrightarrow (\mathrm{RQFSAd})_{A_h}\,,\ \ H_h \mapsto H_h^{\star}\,, & (\mathrm{RQFSAd})_{A_h} \longrightarrow (\mathrm{LQUEAd})_{A_h}\,,\ \ K_h \mapsto {}_{\star}K_h \\
(\mathrm{LQUEAd})_{A_h} \longrightarrow (\mathrm{RQFSAd})_{A_h}\,,\ \ H_h \mapsto H_{h\,*}\,, & (\mathrm{RQFSAd})_{A_h} \longrightarrow (\mathrm{LQUEAd})_{A_h}\,,\ \ K_h \mapsto {}^{\star}K_h \\
(\mathrm{RQUEAd})_{A_h} \longrightarrow (\mathrm{LQFSAd})_{A_h}\,,\ \ H_h \mapsto {}^{*}H_h\,, & (\mathrm{LQFSAd})_{A_h} \longrightarrow (\mathrm{RQUEAd})_{A_h}\,,\ \ K_h \mapsto K_{h\,\star} \\
(\mathrm{RQUEAd})_{A_h} \longrightarrow (\mathrm{LQFSAd})_{A_h}\,,\ \ H_h \mapsto {}_{*}H_h\,, & (\mathrm{LQFSAd})_{A_h} \longrightarrow (\mathrm{RQUEAd})_{A_h}\,,\ \ K_h \mapsto K_h^{\star}
\end{array}$$

which are (pairwise) inverse to each other, hence yield pairs of antiequivalences of categories.

In addition, the Lie-Rinehart bialgebra structure induced by the dual of a given quantum groupoid is the same or the coopposite one as that induced by the initial quantum groupoid.

Finally (Sec. 6), we develop a suitable "Quantum Duality Principle" for quantum groupoids. Indeed, we introduce functors "à la Drinfeld", denoted by $(\ )^\vee$ and $(\ )'$, which turns (L/R)QFSAd's into (L/R)QUEAd's and viceversa, so to provide an equivalence between the category of all LQFSAd's and that of all LQUEAd's, and a similar equivalence between RQFSAd's and RQUEAd's. In addition, if one starts with a quantization of some Lie-Rinehart bialgebra L, then the (appropriate) Drinfeld functor gets out of it a quantization of the *dual* Lie-Rinehart bialgebra L^*.

Let us be more precise. For the functor $(\ )^\vee$, Drinfeld's original definition groups can be easily extended to RQFSAd and LQFSAd. We get then the following generalization of Drinfeld's result:

Theorem 2. *(cf. Theorem 6.4 in Subsec. 6.1)*

Let $J^r(L)_h \in (\mathrm{RQFSAd})_{A_h}$, where L is a finite projective Lie-Rinehart algebra. Then $J^r(L)_h^\vee \in (\mathrm{RQUEAd})_{A_h}$, with semiclassical limit isomorphic to $V^r(L^*)$, for which the structure of Lie-Rinehart bialgebra induced on L^* by the quantization $J^r(L)_h^\vee$ of $V^r(L^*)$ is dual to that induced on L by the quantization $J^r(L)_h$ of $J^r(L)$. Moreover, the definition of $J^r(L)_h \mapsto J^r(L)_h^\vee$ extends to morphisms in (RQFSAd), so to give a (covariant) functor $(\ )^\vee : (\mathrm{RQFSAd}) \longrightarrow (\mathrm{RQUEAd})$.

A similar result holds with the rôles of "left" and "right" reversed all over the place.

On the other hand, the original definition by Drinfeld of the functor $(\ )'$ — in terms of the coproduct — cannot be easily extended to the quantum groupoid case (because, in general, the source and the target differ). For that, we instead define the functor $(\ )'$ using linear duality. If $H_h = V(L)_h$ is a LQUEAd, we define $H_h' := {}_{*}\big(({H_h}^{*})^{\vee}\big)$, ${}'H_h := {}^{*}\big(((H_h)_{*})^{\vee}\big)$. Mimicking Drinfeld's construction, we also introduce an additional object $\delta_s(H_h)$. With a bit of work, we prove that $H_h' = \delta_s(H_h) = {}'H_h$. If $H_h = V^r(L)_h$ is a RQUEAd, we proceed in a similar way. In the end, we extend Drinfeld's result about $(\ )'$ in the following form:

Theorem 3. *(see Theorem 6.19 in Subsec. 6.2)*

(a) Let $V^\ell(L)_h \in (\mathrm{LQUEAd})_{A_h}$, where L is a finite projective Lie-Rinehart algebra. Then ${}'V^\ell(L)_h = V^\ell(L)_h' \in (\mathrm{LQFSAd})_{A_h}$, with semiclassical limit isomorphic to $J^\ell(L^*)$, for which the structure of Lie-Rinehart bialgebra induced on L^* by the quantization $V^\ell(L)_h'$ of $J^\ell(L^*)$ is dual to that on L by the quantization $V^\ell(L)_h$ of $V^\ell(L)$. Moreover, the definition of $V^\ell(L)_h \mapsto {}'V^\ell(L)_h =$

$V^\ell(L)_h'$ *extends to morphisms in* (LQUEAd) *, so that we have a well defined (covariant) functor* $'(\ ) = (\ )' : (\text{LQUEAd}) \longrightarrow (\text{LQFSAd})$ *.*

A similar result holds with the rôles of "left" and "right" reversed all over the place.

As a final outcome, we find a "quantum duality principle" in the context of quantum groupoids which is expressed by Theorems 2 and 3 above along with the following result:

Theorem 4. *(cf. Theorem 6.22 in Subsec. 6.3)*

The functors $(\ )^\vee : (\text{RQFSAd}) \to (\text{RQUEAd})$ *and* $(\ )' = '(\ ) : (\text{RQUEAd}) \to (\text{RQFSAd})$ *are inverse to each other, hence they are equivalences of categories. Similarly for the functors* $(\ )^\vee : (\text{LQFSAd}) \longrightarrow (\text{LQUEAd})$ *and* $(\ )' = '(\ ) : (\text{LQUEAd}) \longrightarrow (\text{LQFSAd})$ *.*

It is worth remarking that one could work with "quantum groupoids" *on one side only* (left or right, say). Then if we pick opposite sides for QUEAd's and QFSAd's — that is we take either LQUEAd's and RQFSAd's, or RQUEAd's and LQFSAd's — then we still have a good theory of linear duality, i.e. we have well-behaving antiequivalences given by the functors of (left/right) linear duality. Indeed, the key point here is that linear duality "reverses the orientation" of our quantum groupoids (and in general of bialgebroids), turning "left" into "right" and viceversa. On the other hand, Drinfeld functors (in whatever reasonable sense one introduces them) for quantum groupoids instead "do preserve the orientation", so to define them one keeps quantum groupoids "on the same side". Thus, if we want to deal with "quantum groupoids" which have *both* a nice theory of linear duality *and* a suitable quantum duality principle, then we must necessarily work with *four* kinds of "quantum groupoids", i.e. *left* and *right* QUEAd's and *left* and *right* QFSAd's.

At the end (Sec. 7) we present an example, just to illustrate some of our main results on a single — and simple, yet significant enough — toy model.

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2 Lie-Rinehart algebras and bialgebras

Throughout this paper, k will be a field and A will be a unital, associative k–algebra; we assume k to have characteristic zero (though for most definitions and constructions this is not necessary). Moreover, for all objects defined in *this* section we assume in addition that A is also commutative.

2.1 Lie-Rinehart algebras

To begin with, we introduce the notion of *(left) Lie-Rinehart algebra* (or "Lie algebroid").

Definition 2.1. *A (left)* Lie-Rinehart algebra *(see [30]) is a triple* (A, L, ω) *where*

(a) L *is a* k*–Lie algebra,*

(b) L *is an* A*–module,*

(c) ω *is an* A*–linear morphism of Lie* k*–algebras from* L *to* $\text{Der}(A)$*, called* anchor (map)*,*

such that the following compatibility relation holds:

$$\forall\ D, D' \in L\,,\ \ \forall\ f \in A\,, \qquad \big[D, f\, D'\big] \;=\; \omega(D)(f)\, D' + f\, \big[D, D'\big]$$

In particular, if L *is finitely generated projective as an* A*–module, then* (A, L, ω) *will be called a finite projective Lie-Rinehart algebra.*

Notation: *when there is no ambiguity, the Lie-Rinehart algebra* (A, L, ω) *will be written* L *.*

Examples 2.2.

(1) The simplest example of Lie-Rinehart algebras is obtained when the anchor map ω is 0. In this case, L is just an A–Lie algebra.

(2) $\big(A, Der(A), id\big)$ is a Lie-Rinehart algebra.

(3) Let A be a Poisson algebra and D_A be the A–module of Kähler differentials. Then D_A is naturally endowed with a Lie-Rinehart algebra structure (see [16]).

(4) *Crossed products:* Let $\mathfrak{g}$ be a k–Lie algebra and A be an associative commutative k–algebra. Assume moreover that a k–Lie algebra morphism $\sigma : \mathfrak{g} \longrightarrow Der(A)$ is given. Consider $L = A \otimes_k \mathfrak{g}$. Then L is a Lie-Rinehart algebra with respect to the Lie bracket and anchor map defined as follows: for all $X, Y \in \mathfrak{g}$ and $a, b \in A$,

$$\omega(a \otimes X)(b) \;:=\; a\,\sigma(X)(b) \quad , \qquad [a \otimes X, b \otimes Y] \;:=\; a\,\sigma(X)(b) \otimes Y \,-\, b\,\sigma(Y)(a) \otimes X \,+\, a\,b \otimes [X,Y]$$

This Lie-Rinehart algebra L is called *the crossed product of A with* $\mathfrak{g}$ and is denoted $A\#\mathfrak{g}$.

(5) If L is any A–module, we denote by L^{ab} the trivial Lie algebroid, that is L itself endowed with trivial Lie bracket and trivial anchor map.

(6) In the setup of differential geometry, natural examples of Lie-Rinehart algebras arise as spaces of global sections of Lie algebroids (see here below for the definition).

Definition 2.3. *Let P be a smooth (real) manifold. A Lie algebroid is a vector bundle $\mathcal{L}$ over P together with a Lie algebra structure on the space $\Gamma(\mathcal{L})$ of smooth global sections of $\mathcal{L}$ and a bundle map $\rho : \mathcal{L} \longrightarrow TP$ such that, for any $f \in \mathcal{C}^\infty(P)$ and $X, Y \in \Gamma(\mathcal{L})$, one has*

$$(i) \quad \rho\big([X,Y]\big) \;=\; \big[\rho(X), \rho(Y)\big] \quad , \qquad\qquad (ii) \quad [X, f\,Y] \;=\; f\,[X,Y] \,+\, \rho(X)(f)\,Y$$

2.4. Lie-Rinehart algebras from geometry. The first example of Lie-Rinehart algebras arising in (differential) geometry is the following, basic one. Let $\mathcal{L}$ be a Lie algebroid over the smooth (real) manifold P: then the triple $\big(\mathcal{C}^\infty(P), \Gamma(\mathcal{L}), \omega\big)$ is a Lie-Rinehart algebra, the anchor being $\omega := \Gamma(\rho)$ and the ground field being $k := \mathbb{R}$.

The simplest example of Lie algebroid over P is given by $\mathcal{L} := TP$, the tangent bundle of P, with $\rho := id_{TP}$: by the above, this implies that $\big(\mathcal{C}^\infty(P), \Gamma(TP), \Gamma\big(id_{TP}\big)\big)$ has a canonical structure of Lie-Rinehart algebra.

Let us also say a word about the geometric counterpart of Example 2.2*(3)* ([8]). Let P be a Poisson manifold: write $\{\ ,\ \}$ for the Poisson bracket over $A := \mathcal{C}^\infty(P)$, then $\Pi \in \bigwedge_A^2 TP$ for the corresponding Poisson bivector on P and $\Pi^\# : \Omega_P^1 \longrightarrow TP$ for the map defined by $\Pi^\#(\omega_1) := \Pi(\omega_1, -)$. The vector bundle of differential forms of degree 1 on P, i.e. Ω_P^1, is endowed with a natural Lie algebroid structure. Indeed, the Lie bracket over $\Gamma\big(\Omega_P^1\big)$ is the standard one given by

$$[\omega_1, \omega_2] \;:=\; \mathcal{L}_{\Pi^\#\omega_1}\omega_2 - \mathcal{L}_{\Pi^\#\omega_2}\omega_1 \,-\, d\big(\Pi(\omega_1, \omega_2)\big)$$

where $\mathcal{L}_{\Pi^\#\omega_i}$ is the Lie derivative with respect to $\Pi^\#\omega_i$ and d is the de Rham differential. On the other hand, the anchor map is given by

$$\Pi^\# : \Omega_P^1 \longrightarrow TP \;, \quad f\,dg \mapsto f\,\{g, -\}$$

2.5. Differentials for Lie-Rinehart algebras. If (A, L, ω) is a Lie-Rinehart algebra, it is well known that $\bigwedge_A L$, endowed with the exterior product and the generalized Schouten bracket, is a Gerstenhaber algebra (cf. [22]). The generalized Schouten bracket is the unique extension of the Lie bracket on L such that

$$\begin{aligned}
&\forall\, Q \in {\textstyle\bigwedge_A^{q+1}} L\,,\ Q' \in {\textstyle\bigwedge_A^{q'+1}} L\,, \qquad [Q, Q'] \;=\; -(-1)^{qq'}\,[Q', Q] \\
&\forall\, f \in {\textstyle\bigwedge_A^{0}} L = A\,,\ \forall\, X \in L\,, \qquad [X, f] \;=\; \omega(X)(f) \\
&\forall\, Q \in {\textstyle\bigwedge_A^{q+1}} L\,, \qquad [Q, -] \ \text{ is a derivation of degree } q \text{ of } \ {\textstyle\bigwedge_A} L \;=\; \oplus_{n \geq 0} {\textstyle\bigwedge_A^n} L
\end{aligned}$$

Given a finite projective Lie-Rinehart algebra (A, L, ω), it is known that $\bigwedge_A L^* = \oplus_n \bigwedge_A^n L^*$ admits a differential d_L that makes it into a differential algebra. Here $d_L : \bigwedge_A^n L^* \longrightarrow \bigwedge_A^{n+1} L^*$ is defined as follows: for all $\lambda \in \bigwedge_A^n L^*$ and for all $(X_1, X_2, \ldots, X_{n+1}) \in L^{n+1}$, one has

$$\begin{aligned}(d_L\lambda)(X_1, \ldots, X_{n+1}) \;&=\; \textstyle\sum_{i=1}^{n+1} (-1)^{i+1}\omega(X_i)\big(\lambda(X_1, \ldots, \widehat{X_i}, \ldots, X_{n+1}\big) \;+ \\ &+\; \textstyle\sum_{i<j} (-1)^{i+j}\lambda\big([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{n+1}\big)\end{aligned}$$

In the case where $L = TX$, the differential d_L coincides with the de Rham differential.

Lie-Rinehart algebra structures on L are in bijective correspondence with differential graded A–algebra structures on $\bigwedge_A L^*$ ([33], lemma 2.2).

Remark 2.6. If L is not finite projective, then the differential d_L still exists but just as a map from $Hom_A\big(\bigwedge_A^n L, A\big)$ to $Hom_A\big(\bigwedge_A^{n+1} L, A\big)$.

Definition 2.7. *Let (A, L, ω) be a (left) Lie-Rinehart algebra. The (left) universal enveloping algebra of L is the k–algebra*

$$V^\ell(L) \;:=\; T_k^+(A \oplus L)\Big/ I$$

where $T_k^+(A \oplus L)$ is the positive part of the the tensor k–algebra over $A \oplus L$ and I is the two sided ideal in $T_k^+(A \oplus L)$ generated by the elements

$$a \otimes b - a\,b \;, \qquad a \otimes \xi - a\,\xi \;, \qquad \xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] \;, \qquad \xi \otimes a - a \otimes \xi - \omega(\xi)(a)$$

for all $a, b \in A$, $\xi, \eta \in L$.

Remark 2.8. Note that $V^\ell(L)$ is a filtered ring, its (increasing) filtration $\big\{V_n^\ell(L)\big\}_{n\in\mathbb{N}}$ being defined as follows: $V_0^\ell(L) := A$, $V_{n+1}^\ell(L) := V_n^\ell(L) + V_n^\ell(L) \cdot L$ for all $n \in \mathbb{N}$. In the following we shall denote by $Gr\big(V^\ell(L)\big)$ the graduate algebra associated with this filtration.

The following basic result is proved in [30]:

Theorem 2.9. *Assume that L is projective as an A–module. Then $Gr\big(V^\ell(L)\big) \cong S_A(L)$.*
Moreover, the natural maps $\iota_A : A \longrightarrow V^\ell(L)$ and $\iota_L : L \longrightarrow V^\ell(L)$ are monomorphisms.

Remarks 2.10.

(a) Let S be a multiplicative system. We know (cf. [12]) that $L_S = A_S \otimes_A L$ is naturally endowed with a natural Lie-Rinehart algebra structure over A_S, extending that of L.

(b) The Lie-Rinehart algebras $L = \big(L, A, [\;,\;], \omega\big)$ and $L^{op} := \big(L, A, -[\;,\;], -\omega\big)$ are isomorphic via the isomorphims F defined by $F(D) := -D$ for all $D \in L$ and $F(a) := a$ for all $a \in A$.

Examples 2.11.

(1) If L is simply a Lie algebra over k, then its universal enveloping algebra is just the usual one of Lie algebra theory.

(2) Let X be a smooth manifold and set $A = \mathcal{C}^\infty(X)$. Then the universal enveloping algebra $U\big(Der(A)\big)$ associated to the Lie-Rinehart algebra $\big(A, Der(A), id\big)$ is the ring of global differential operators on X.

2.12. From a finite projective Lie-Rinehart algebra to a free Lie-Rinehart algebra. Most of the time, we will work with finite projective Lie-Rinehart algebras. This is a reasonable hypothesis as Lie-Rinehart algebras coming from the geometry are finite projective. Several times in this article, we will prove results for (finite) free Lie-Rinehart algebra and then extend them to finite projective Lie-Rinehart algebras. We now explain the key step for this.

Let L be a finite projective Lie-Rinehart algebra. There exist a finite projective A–module Q such that $F = L \oplus Q$ is a finite rank free A–module. We can endow F with the following Lie-Rinehart algebra structure:

$$\begin{aligned} \forall\ D \in L\,,\ \forall\ E \in Q\,, \qquad & \omega_F(D+E) := \omega_L(D) \\ \forall\ D_1\,, D_2 \in L\,,\ \forall\ E_1\,, E_2 \in Q\,, \qquad & [D_1 + E_1\,, D_2 + E_2] := [D_1\,, D_2] \end{aligned}$$

(in other words, the Lie-Rinehart structure of L is extended trivially to $F = L \oplus Q$). Then $V^\ell(F) = V^\ell(L) \otimes_A S(Q)$. We shall not make use of the Lie-Rinehart algebra F because, as Q is not free, a quantization of $V^\ell(F)$ is not easy to construct from a quantization of $V^\ell(L)$.

The A–module $L_Q := L \oplus Q \oplus L \oplus Q \oplus L \oplus Q \oplus \cdots = F \oplus F \oplus F \oplus \cdots$ is a free A–module. Set $R = Q \oplus L \oplus Q \oplus L \oplus Q \oplus \cdots$; then R is a free A–module such that $L_Q = L \oplus R$ is a free A–module (cf. [15]). We endow L_Q with the structure of a Lie-Rinehart algebra as follows:

$$\begin{aligned} \forall\ D \in L\,,\ \forall\ B \in R\,, \qquad & \omega_{L_Q}(D+B) := \omega_L(D) \\ \forall\ D_1\,, D_2 \in L\,,\ \forall\ B_1\,, B_2 \in R\,, \qquad & [D_1 + B_1\,, D_2 + B_2] := [D_1\,, D_2] \end{aligned}$$

(in other words, the Lie-Rinehart structure of L is extended trivially to $L_Q = L \oplus R$).

Let $\{b_1, \dots, b_n\}$ be an A–basis of F, and set $T := \mathbb{N} \times \{1, \dots, n\}$, $Z := \bigoplus_{t \in T} k\, v_t$, $Y := \bigoplus_{i=1}^{n} k\, b_i$. Then $F = \bigoplus_{i=1}^{n} A b_i = A \otimes_k Y$ and $R \simeq F \oplus F \oplus \cdots = A \otimes_k (Y \oplus Y \oplus \cdots) = A \otimes_k Z$; also, $V^\ell(L_Q) = V^\ell(L) \otimes_k S(Z)$.

Definition 2.13. *Let an A–basis $\{b_1, \dots, b_n\}$ of F be given. Then one can construct a basis $\{v_t\}_{t \in T}$ of R and an A–basis $\{e_t\}_{t \in T}$ of L_Q both indexed by $T := \mathbb{N} \times \{1, \dots, n\}$. For L_Q such a basis will be called a good basis. For later use, if $i = (i_1, i_2) \in T$ we set $\varpi(e_i) := i_1$.*

2.14. Right Lie-Rinehart algebras. For the sake of completeness, we have to mention that one can also, in a symmetric way, consider the notion of *right* Lie-Rinehart algebra, as follows:

Definition 2.15. *A right Lie-Rinehart algebra is a triple (A, L, ω) where*

(a) L is a k–Lie algebra,

(b) L is a right A–module,

(c) ω is an A–linear morphism of Lie k–algebras from L to $\mathrm{Der}(A)$, called anchor (map)*,*

such that the following compatibility relation holds:

$$\forall\ D, D' \in L\,,\ \forall\, f \in A\,, \qquad \big[D, D' \cdot f\big] = D' \cdot \omega(D)(f) + \big[D, D'\big] \cdot f$$

Remark 2.16. As A is commutative, a Lie-Rinehart algebra can be considered as a right Lie-Rinehart algebra and viceversa. However, the enveloping algebra defined by the notion of right Lie-Rinehart algebra is different from that defined by a Lie-Rinehart algebra.

Definition 2.17. *Let (A, L, ω) be a right Lie-Rinehart algebra. The (right)* universal enveloping algebra *of L is the k–algebra*

$$V^r(L) := T_k^+(A \oplus L) \Big/ I$$

where $T_k^+(A \oplus L)$ is the positive part of the tensor k–algebra over $A \oplus L$ and I is the two sided ideal in $T_k^+(A \oplus L)$ generated by the elements

$$a \otimes b - a\, b \ , \qquad \xi \otimes a - \xi \cdot a \ , \qquad \xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] \ , \qquad \xi \otimes a - a \otimes \xi - \omega(\xi)(a)$$

for all $a, b \in A$, $\xi, \eta \in L$.

Remark 2.18. Just like $V^\ell(L)$, also $V^r(L)$ is a filtered ring, with increasing filtration $\big\{V_n^r(L)\big\}_{n\in\mathbb{N}}$ being defined by $V_0^r(L) := A$, $V_{n+1}^r(L) := V_n^r(L) + V_n^r(L)\cdot L$ for all $n\in\mathbb{N}$. In the following then $Gr\big(V^r(L)\big)$ will denote the graduate algebra associated with this filtration.

Next result clarifies the link between left and right enveloping algebras of a single Lie-Rinehart algebra L. Hereafter, L^{op} denotes the "opposite" Lie-Rinehart algebra to L — cf. Remarks 2.10*(b)* — while $\mathfrak{A}^{op}$ denotes the opposite of any (associative) algebra $\mathfrak{A}$.

Proposition 2.19. *Let L be a Lie-Rinehart algebra. Then the following holds:*

(a) The algebras $V^r(L)^{op}$ and $V^l(L^{op})$ are equal.

(b) There exists an algebra isomorphism $\Xi : V^\ell(L) \to V^r(L)^{op}$ given, for all $a\in A$, $D\in L$, by

$$a \mapsto \Xi(a) := a \quad , \quad D \mapsto \Xi(D) := -D \ .$$

2.2 Lie-Rinehart bialgebras

We are now ready to introduce the notion of *Lie-Rinehart bialgebra* (cf. [26], [21], [17]).

Definition 2.20. *A Lie-Rinehart bialgebra is a pair $\big(L_1, L_2\big)$ of finitely generated projective A–modules in duality — that is, $L_1 \cong L_2^*$ and $L_2 \cong L_1^*$ — each of them being endowed with Lie-Rinehart algebra structures such that the differential d_1 on $\bigwedge_A L_1$ arising from the Lie-Rinehart structure on $L_2 \cong L_1^*$ is a derivation of the Schouten bracket on $\bigwedge_A L_1$. Equivalently, d_1 is a derivation of the Lie bracket of L_1, that is to say*

$$d_1\big([X,Y]\big) \ = \ \big[d_1(X)\,,Y\big] \, + \, \big[X, d_1(Y)\big] \qquad\qquad \forall \;\; X,Y\in L_1$$

In general, if L is a finitely generated projective A–module, then its linear dual L^* (as an A–module) is finitely generated projective as well: in this case, in the following we shall say that "L is a Lie-Rinehart bialgebra" to mean that $\big(L, L^*\big)$ has a structure of Lie-Rinehart bialgebra, and we shall denote the differential of $\bigwedge_A L$ mentioned above by d_{L^*} or δ_L.

Remarks 2.21.

(a) The conditions of the theorems are equivalent to those with the rôles of L_1 and L_2 interchanged (cf. [21]).

(b) It follows at once from the very definition that the differential δ_L of L in a Lie-Rinehart bialgebra $\big(L, L^*\big)$ is uniquely determined by its restriction to A and L — the degree 0 and degree 1 pieces of $\bigwedge_A L$.

(c) If $\big(L, L^*\big)$ is a Lie-Rinehart bialgebra, we can read off the explicit relation between the Lie-Rinehart algebra structure of L^* — that is, its anchor map ω_{L^*} and its Lie bracket $[\ ,\ ]_{L^*}$ — and the differential δ_L of L through the following formulas: if $D^*, E^*\in L^*$, $X\in L$ and $a\in A$,

$$\omega_{L^*}(D^*)(a) \ = \ \big\langle\, \delta_L(a)\,, D^* \big\rangle$$
$$\big\langle\, X\,, [D^*, E^*]_{L^*} \big\rangle \, + \, \big\langle\, \delta_L(X)\,,\, D^*\wedge E^* \big\rangle \ = \ \omega_{L^*}(D^*)\big(\langle\, X\,, E^*\rangle\big) \, - \, \omega_{L^*}(E^*)\big(\langle\, X\,, D^*\rangle\big)$$

where $\langle\ ,\ \rangle$ denote the natural pairing between L and L^*. Indeed, one can use these formulas either to deduce ω_{L^*} and $[\ ,\ ]_{L^*}$ from δ_L, or to deduce the latter from ω_{L^*} and $[\ ,\ ]_{L^*}$

(d) Lie-Rinehart bialgebras structures on a given (finitely generated projective) A–module L correspond to strict differential Gerstenhaber algebra structures on $\bigwedge_A L$ (cf. [21], [33]).

(e) Let $\big(L\,, L^*\big)$ be a Lie-Rinehart bialgebra. Denote by d the differential on $\bigwedge_A L^*$ arising from the Lie-Rinehart structure on L and $d_*\,(=\delta_L)$ the differential on $\bigwedge_A L$ coming from the Lie-Rinehart structure on L^*. The base A inherits a natural Poisson structure as follows:

$$\{f,g\} \ := \ \big\langle\, df\,, d_* g \big\rangle \qquad\qquad \forall \;\; f\,,g\in A$$

(see [21], [33]); moreover, one has the identities $[\,df, dg\,] = d\{f,g\}$ and $d_*\{f,g\} = -[\,d_*f, d_*g\,]$.

(f) Let $\big(L\,,L^*\big)$ be a Lie-Rinehart bialgebra. Then $\big(L^{op},L^*\big)$, $\big(L\,,(L^*)^{op}\big)$ and $\big(L^{op},(L^*)^{op}\big)$ are Lie-Rinehart bialgebras as well. If we identify any Lie-Rinehart bialgebra, written as a pair, with the left-hand of the pair, say $L \equiv \big(L\,,L^*\big)$, then we shall also write $L^{op} \equiv \big(L^{op},L^*\big)$ — the Lie-Rinehart bialgebra "opposite" to L — $L_{coop} \equiv \big(L\,,(L^*)^{op}\big)$ — the "coopposite" — and $L^{op}_{coop} \equiv \big(L^{op},(L^*)^{op}\big)$ — the "opposite-coopposite".

Example 2.22. *r–matrices for Lie-Rinehart bialgebras.*

Let L be a Lie-Rinehart algebra, which is finitely generated projective as an A–module. An r–matrix of L is, by definition, any section $\Lambda \in \bigwedge_A^2 L$ such that $\big[X,[\Lambda,\Lambda]\big] = 0$ for all $X \in L$.

Any r–matrix of L defines a Lie-Rinehart bialgebra structure on it: the Lie bracket on L^* is given by

$$[\xi,\eta] \,=\, \mathcal{L}_{\Lambda^{\#}\xi}\eta \,-\, \mathcal{L}_{\Lambda^{\#}\eta}\xi \,-\, d\big(\Lambda(\xi,\eta)\big) \qquad\qquad \forall\;\xi,\eta \in L^*$$

and the anchor is the composition $\rho \circ \Lambda^{\#} : L^* \longrightarrow \mathit{Der}(A)$ where $\Lambda^{\#} : L^* \longrightarrow L$ is defined by $\Lambda^{\#}(\xi)(\eta) := \Lambda(\xi,\eta)$ for all $\xi,\eta \in L^*$. The differential $d_{L^*} : \bigwedge_A^{\bullet} L \longrightarrow \bigwedge_A^{\bullet+1} L$ is given by $d_{L^*} := [-\,,\Lambda]$. Any such Lie-Rinehart bialgebra is called *coboundary Lie-Rinehart bialgebra*, in analogy to the Lie algebra case. In the special case where $[\Lambda,\Lambda] = 0$ we call it a *triangular Lie-Rinehart bialgebra*. In the case where A reduces to k , i.e. L is just a Lie k–algebra, Λ is an ordinary r–matrix in the sense of Lie (bi)algebra theory.

When L is the space of global sections of the tangent bundle TP on a smooth manifold P, with the standard Lie algebroid structure, triangular Lie-Rinehart bialgebra structures on L correspond to Poisson structure on P ; thus we recover the geometric counterpart of Example 2.2*(3)* — cf. Example 2.2*(6)*. More in general, the space of global sections of a Lie bialgebroid is a Lie-Rinehart bialgebra (cf. [26]).

3 Left and right bialgebroids

Let again k be a field, and A a unital, associative k–algebra. We define $A^e := A \otimes_k A^{op}$.

3.1 A–rings, A–corings

We begin this section introducing the notions of A*–ring* and A*–coring*, which are direct generalizations of the notions of algebra and coalgebra over a commutative ring.

Definition 3.1. *Let A be a k–algebra as above. An A–ring is a triple (H, m_H, ι) where H is an A^e–module, $m_H : H \otimes_A H \longrightarrow H$ and $\iota : A \longrightarrow H$ are A^e–module morphisms such that*

- $m_H \circ (m_H \otimes id_H) \,=\, m_H \circ (id_H \otimes m_H)$
- *making the identifications $H \otimes_A A \,\simeq\, H$ and $A \otimes_A H \,\simeq\, H$, one has the identities* $m_H \circ (\iota \otimes id_H) \,=\, m_H \circ (id_H \otimes \iota) = id_H$

It is well known (see [4]) that A–rings H correspond bijectively to k–algebra homomorphism $\iota : A \longrightarrow H$. With this characterization, the A^e–module structure on H can be expressed as follows: $a \cdot h \cdot b \,=\, \iota(a)\, h\, \iota(b)$ for all $a, b \in A\,,\; h \in H$.

The dual notion (of an A–ring) is the following notion of A–coring:

Definition 3.2. *An A–coring is a triple (C, Δ, ϵ) where C is an A^e–module (with left action L_A and right action R_A), $\Delta : C \longrightarrow C \otimes_A C$ and $\epsilon : C \longrightarrow A$ are A^e–module morphisms such that*

- $(\Delta \otimes id_C) \circ \Delta \,=\, (id_C \otimes \Delta) \circ \Delta$
- $L_A \circ (\epsilon \otimes id_C) \circ \Delta \,=\, R_A \circ (id_C \otimes \epsilon) \circ \Delta \,=\, id_C$

3.3. **A^e–module structures and special products.** Let A be as above, and consider now A^e as base k–algebra. An A^e–ring H can be described by a k–algebra morphism $\iota : A^e \longrightarrow H$. Let us consider its restrictions

$$s := \iota(-\otimes_k 1_H) : A \longrightarrow H \quad , \qquad t := \iota(1_H \otimes_k -) : A \longrightarrow H$$

which we call respectively *source* and *target* maps. Thus an A^e–ring H carries two A–module structures and two A^{op}–module structures: for all $a, a' \in A$, $h \in H$, we write

$$a \triangleright h \triangleleft \tilde{a} := s(a)\, t(\tilde{a})\, h \quad , \qquad a \blacktriangleright h \blacktriangleleft \tilde{a} := h\, t(a)\, s(\tilde{a})$$

As usual, the tensor product of H with itself (as an A–bimodule, i.e. an A^e–module) is defined as

$$\begin{aligned} H_{\triangleleft} \underset{A}{\otimes}{}_{\triangleright} H &:= H \otimes_k H \Big/ \big\{ (u \triangleleft a) \otimes u' - u \otimes (a \triangleright u') \big\}_{a \in A,\, u, u' \in H} \\ &= H \otimes_k H \Big/ \big\{ \big(t(a)\, u\big) \otimes u' - u \otimes \big(s(a)\, u'\big) \big\}_{a \in A,\, u, u' \in H} \end{aligned}$$

Now we *define* $H_{\triangleleft} \underset{A}{\times}{}_{\triangleright} H \subseteq H_{\triangleleft} \underset{A}{\otimes}{}_{\triangleright} H$ as follows:

$$\begin{aligned} H_{\triangleleft} \underset{A}{\times}{}_{\triangleright} H &:= \Big\{ \textstyle\sum_i u_i \otimes u'_i \in H_{\triangleleft} \underset{A}{\otimes}{}_{\triangleright} H \,\Big|\, \sum_i (a \blacktriangleright u_i) \otimes u'_i = \sum_i u_i \otimes_A (u'_i \blacktriangleleft a) \Big\} \\ &= \Big\{ \textstyle\sum_i u_i \otimes u'_i \in H_{\triangleleft} \underset{A}{\otimes}{}_{\triangleright} H \,\Big|\, \sum_i \big(u_i t(a)\big) \otimes_A u'_i = \sum_i u_i \otimes \big(u'_i s(a)\big) \ , \ \forall\, a \in A \Big\} \end{aligned}$$

This $H_{\triangleleft} \underset{A}{\times}{}_{\triangleright} H$ is called the *left Takeuchi product* of the A^e–ring H with itself.

By construction, the Takeuchi product $H_{\triangleleft} \underset{A}{\times}{}_{\triangleright} H$ has a natural structure of A^e–module, induced by that of $H_{\triangleleft} \underset{A}{\otimes}{}_{\triangleright} H$. Even more, $H_{\triangleleft} \underset{A}{\times}{}_{\triangleright} H$ is also an A^e–ring, via factorwise multiplication, with unit element $1_H \otimes 1_H$ and $\iota_{H_{\triangleleft} \underset{A}{\times}{}_{\triangleright} H}$ defined by $\iota_{H_{\triangleleft} \underset{A}{\times}{}_{\triangleright} H}(a \otimes \tilde{a}) := s(a) \otimes t(\tilde{a})$. Note that this instead is *not* the case for $H_{\triangleleft} \underset{A}{\otimes}{}_{\triangleright} H$.

In a similar way, we can consider a second type of "Takeuchi product". In order to easily distinguish it from the previous one, we shall now denote the base k–algebra by B instead of A .

Let B be a (unital, associative) k–algebra; let H be a B^e–ring given by a k–algebra morphism $\eta_r : B^e \longrightarrow H$, a source map $s^r := \eta^r(-\otimes 1)$ and a target map $t^r := \eta^r(1 \otimes -)$. We consider now the right B^e–module structure on H given by $h \cdot (b \otimes \tilde{b}) := h \cdot \eta^r(b \otimes \tilde{b})$, for $b, \tilde{b} \in B$, $h \in H$. Then the tensor product of H with itself (as a B–bimodule, i.e. a B^e–module) is defined as

$$\begin{aligned} H_{\blacktriangleleft} \overset{B}{\otimes}{}_{\blacktriangleright} H &:= H \otimes_k H \Big/ \big\{ (u \blacktriangleleft b) \otimes u' - u \otimes (b \blacktriangleright u') \big\}_{b \in B,\, u, u' \in H} = \\ &= H \otimes_k H \Big/ \big\{ \big(u\, s^r(b)\big) \otimes u' - u \otimes \big(u'\, t^r(b)\big) \big\}_{b \in B,\, u, u' \in H} \end{aligned}$$

Now we *define* $H_{\blacktriangleleft} \overset{B}{\times}{}_{\blacktriangleright} H \subseteq H_{\blacktriangleleft} \overset{B}{\otimes}{}_{\blacktriangleright} H$ as follows:

$$\begin{aligned} H_{\blacktriangleleft} \overset{B}{\times}{}_{\blacktriangleright} H &:= \Big\{ \textstyle\sum_i u_i \otimes u'_i \in H_{\blacktriangleleft} \overset{B}{\otimes}{}_{\blacktriangleright} H \,\Big|\, \sum_i (a \triangleright u_i) \otimes u'_i = \sum_i u_i \otimes (u'_i \triangleleft a) \Big\} = \\ &= \Big\{ \textstyle\sum_i u_i \otimes u'_i \in H_{\triangleleft} \underset{A}{\otimes}{}_{\triangleright} H \,\Big|\, \sum_i \big(s^r(b)\, u_i\big) \otimes u'_i = \sum_i u_i \otimes \big(t^r(b)\, u'_i\big) \ , \ \forall\, b \in B \Big\} \end{aligned}$$

This $H_{\blacktriangleleft} \overset{B}{\times}{}_{\blacktriangleright} H$ is called the *right Takeuchi product* of the B^e–ring H with itself.

3.2 Left bialgebroids

We introduce now the notion of *left bialgebroid*, as well as some related items (see [32], [25], [34] and [23], Chapter 2, for a detailed history of this notion). We begin with the very definition:

Definition 3.4. *A left A–bialgebroid is a k–module H that carries simultaneously a structure of an A^e–ring (H, s^ℓ, t^ℓ) and of an A–coring $(H, \Delta_\ell, \epsilon)$ subject to the following compatibility relations:*

(i) The A^e–module structure on the A-coring $(H, \Delta_\ell, \epsilon)$ is that of ${}_\triangleright H_\triangleleft$, namely (for all $a, \tilde{a} \in A$, $h \in H$) $a \triangleright h \triangleleft \tilde{a} := s^\ell(a)\, t^\ell(\tilde{a})\, h$.

(ii) The coproduct map Δ^ℓ is a unital k–algebra morphism taking values in $H_\triangleleft \underset{A}{\times}{}_\triangleright H$.

(iii) The (left) counit map ϵ has the following property: for $a, \tilde{a} \in A$, $u, u' \in H$, one has

$$\epsilon\big(s^\ell(a)\, t^\ell(\tilde{a})\, u\big) \;=\; a\, \epsilon(u)\, \tilde{a} \quad , \qquad \epsilon(u\, u') \;=\; \epsilon\big(u\, s^\ell\big(\epsilon(u')\big)\big) \;=\; \epsilon\big(u\, t^\ell\big(\epsilon(u')\big)\big) \quad , \qquad \epsilon(1) = 1$$

Remarks 3.5. A left bialgebroid has the following properties :

(a) $$\Delta_\ell\big(s^\ell(a)\big) \;=\; s^\ell(a) \otimes 1 \quad , \qquad \Delta_\ell\big(t^\ell(a)\big) \;=\; 1 \otimes t^\ell(a)$$

(b) If $\Delta_l(u) = u_{(1)} \otimes u_{(2)}$, then

$$\Delta_\ell(a \triangleright u \triangleleft \tilde{a}) \;=\; (a \triangleright u_{(1)}) \otimes (u_{(2)} \triangleleft \tilde{a}) \quad , \qquad \Delta_\ell\big(a \blacktriangleright u \blacktriangleleft \tilde{a}\big) \;=\; (u_{(1)} \blacktriangleleft \tilde{a}) \otimes (a \blacktriangleright u_{(2)})$$

(c) $$\epsilon(s^\ell(a)) = a \quad , \qquad \epsilon(t^\ell(a)) = a$$

(d) H acts on its base algebra A on the left as follows (cf. [23]):

$$u.a \;:=\; \epsilon\big(u\, s^\ell(a)\big) \;=\; \epsilon\big(u\, t^\ell(a)\big) \qquad\qquad \forall\ u \in H\,,\ a \in A$$

in fact, in the following we shall also use the notation $u(a) := u.a$ (for all $u \in H$, $a \in A$). This may be called the left anchor of the left bialgebroid H ([34]).

(e) $$t^\ell\big(\epsilon(x)\big) \otimes 1 \;=\; t^\ell\big(\epsilon\big(x_{(1)}\big)\big) \otimes s^\ell\big(\epsilon\big(x_{(2)}\big)\big) \;=\; 1 \otimes s^\ell\big(\epsilon(x)\big) \qquad \text{for all } \ x \in H \,.$$

(f) As a matter of notation, if $\big(H, A, s^\ell, t^\ell, \Delta, \epsilon\big)$ is a left bialgebroid, we set $H^+ := Ker(\epsilon)$.

Definition 3.6. *Let $\mathcal{H} = \big(H, A, s^\ell, t^\ell, \Delta, \epsilon\big)$ and $\hat{\mathcal{H}} = \big(\hat{H}, \hat{A}, \hat{s}^\ell, \hat{t}^\ell, \hat{\Delta}, \hat{\epsilon}\big)$ be two left bialgebroids. A morphism of left bialgebroids Φ from $\mathcal{H}$ to $\hat{\mathcal{H}}$ is a pair (f, F) where*

(a) $f : A \longrightarrow \hat{A}$ is a morphism of algebras;

(b) $F : H \longrightarrow \hat{H}$ is a morphism of algebras and of coalgebras — that is, we have

$$\hat{\Delta} \circ F = (F \otimes F) \circ \Delta \ , \qquad \hat{\epsilon} \circ F = f \circ \epsilon \ ;$$

(c) $$F \circ s^\ell \;=\; \hat{s}^\ell \circ f \ , \qquad F \circ t^\ell \;=\; \hat{t}^\ell \circ f \ .$$

We denote by (LBialg) *the category of left bialgebroids, whose objects are left bialgebroids and morphisms are defined as above. Inside it,* (LBialg_A) *is the subcategory whose objects are all the left bialgebroids over A, and whose morphisms are all the morphisms in* (LBialg) *of the form (id, F) .*

N.B.: the notion of left bialgebroid generalizes that of Hopf algebra.

3.7. Twists of left bialgebroids. Let H be a left bialgebroid. Given $\mathfrak{F} = \sum_i x_i \otimes y_i \in H {}_\triangleleft\!\underset{A}{\otimes}\!{}_\triangleright H$ (with x_i, $y_i \in H$), define $s^\ell_\mathfrak{F} : A \longrightarrow H$ and $t^\ell_\mathfrak{F} : A \longrightarrow H$ by

$$s^\ell_\mathfrak{F}(a) \;=\; \textstyle\sum_i s^\ell\big(x_i(a)\big)\, y_i \qquad , \qquad t^\ell_\mathfrak{F}(a) \;=\; \textstyle\sum_i t^\ell\big(y_i(a)\big)\, x_i$$

Moreover, for any $a, b \in A$ set

$$a *_\mathfrak{F} b \;:=\; s^\ell_\mathfrak{F}(a)(b) \;=\; \textstyle\sum_i x_i(a)\, y_i(b)$$

Proposition 3.8. *(cf. [34]) Assume that* $\mathfrak{F} \in H \underset{A}{\otimes} H$ *satisfies the following conditions:*

$$(i) \qquad (\Delta \otimes id)(\mathfrak{F}) \cdot \mathfrak{F}_{1,2} \; = \; (id \otimes \Delta)(\mathfrak{F}) \cdot \mathfrak{F}_{2,3} \qquad \textit{inside} \;\; H \underset{A}{\otimes} H \underset{A}{\otimes} H$$

$$(ii) \qquad m\big((\epsilon \otimes id)(\mathfrak{F})\big) \; = \; 1_H \quad , \qquad m\big((id \otimes \epsilon)\mathfrak{F}\big) \; = \; 1_H$$

where $\mathfrak{F}_{1,2} \; = \; \mathfrak{F} \otimes 1_H \; \in \; H \underset{A}{\otimes} H \underset{A}{\otimes} H$ *and* $\mathfrak{F}_{2,3} \; = \; 1_H \otimes \mathfrak{F} \; \in \; H \underset{A}{\otimes} H \underset{A}{\otimes} H$ *. Then one has*

$$\mathfrak{F} \cdot \big(t^\ell_{\mathfrak{F}}(a) \otimes 1_H - 1_H \otimes s^\ell_{\mathfrak{F}}(a)\big) \; = \; 0 \qquad \textit{inside} \quad H \underset{A}{\otimes} H \qquad\qquad \forall \;\, a \in A$$

Moreover, if $\mathfrak{F}$ *satisfies (i) and (ii) above, then*

(a) $(A, *_{\mathfrak{F}})$ *is an associative algebra, denoted* $A_{\mathfrak{F}}$ *, and* $a *_{\mathfrak{F}} 1 \; = \; a \; = \; 1 *_{\mathfrak{F}} a$ *for all* $a \in A$ *;*

(b) $s^\ell_{\mathfrak{F}} : A_{\mathfrak{F}} \longrightarrow H$ *is an algebra homomorphism and* $t^\ell_{\mathfrak{F}} : A_{\mathfrak{F}} \longrightarrow H$ *is an algebra antihomomorphism.*

Now let M be a module over H (as an algebra): then M has also a natural A^e–module structure. If $\mathfrak{F}$ is a twist of H, then M has also a natural $A^e_{\mathfrak{F}}$–module structure. Consequently, if M_1 and M_2 are two H–modules, the tensor products $M_{1\triangleleft}\underset{A}{\otimes}{}_{\triangleright}M_2$ and $M_{1\triangleleft}\underset{A_{\mathfrak{F}}}{\otimes}{}_{\triangleright}M_2$ are well defined.

Corollary 3.9. *(cf. [34]) Let* M_1 *and* M_2 *be two left* H*–modules. Then*

$$\mathfrak{F}^\# : M_{1\triangleleft}\underset{A_{\mathfrak{F}}}{\otimes}{}_{\triangleright}M_2 \longrightarrow M_{1\triangleleft}\underset{A}{\otimes}{}_{\triangleright}M_2 \quad , \qquad m_1 \otimes m_2 \mapsto \mathfrak{F} \cdot (m_1 \otimes m_2)$$

is a well defined k-linear map.

We say that $\mathfrak{F}$ *is invertible* if $\mathfrak{F}^\#$ is a k–vector space isomorphism for any pair of left H–modules M_1 and M_2. In this case, in particular, we can take $M_1 = M_2 = H$ so that we have an isomorphism of k–vector spaces $\mathfrak{F}^\# : H \underset{A_{\mathfrak{F}}}{\otimes} H \longrightarrow H \underset{A}{\otimes} H$.

Definition 3.10. *An element* $\mathfrak{F} \in H \underset{A}{\otimes} H$ *is called a twistor (of* H*) if it satisfies equations (i) and (ii) in Proposition 3.8 and it is invertible.*

Assume now that $\mathfrak{F}$ is a twistor of H. Then we may define a new coproduct $\Delta_{\mathfrak{F}} : H \longrightarrow H \underset{A_{\mathfrak{F}}}{\otimes} H$ of H by the formula $\Delta_{\mathfrak{F}}(x) := \big(\mathfrak{F}^\#\big)^{-1}\big(\Delta(x) \cdot \mathfrak{F}\big)$. The key result is then the following (see [34]):

Theorem 3.11. *Let* $\big(H, A, s^\ell, t^\ell, m, \Delta, \epsilon\big)$ *be a left bialgebroid. Then* $\big(H, A, s^\ell_{\mathcal{F}}, t^\ell_{\mathcal{F}}, m, \Delta_{\mathcal{F}}, \epsilon\big)$ *is a left bialgebroid too.*

3.12. Left bialgebroid structures on universal enveloping algebras $V^\ell(L)$. Given a Lie-Rinehart algebra L, there is a standard left bialgebroid structure on $V^\ell(L)$.

Source and target maps are equal and given by $\iota_A : A \longrightarrow V^\ell(L)$. Then the A^e–module structure ${}_{\triangleright}V^\ell(L)_{\triangleleft}$ is given by $a{\triangleright}u{\triangleleft}\tilde{a} := a\,\tilde{a}\,u$. The coproduct $\Delta_\ell : V^\ell(L) \longrightarrow V^\ell(L)_{\triangleleft}\underset{A}{\otimes}{}_{\triangleright}V^\ell(L)$ and the counit map $\epsilon : V^\ell(L) \longrightarrow A$ are determined by

$$\Delta_\ell(a) \, = \, a \otimes 1 \; , \quad \Delta_\ell(X) \, = \, X \otimes 1 + 1 \otimes X \; , \qquad \epsilon(a) \, = \, a \; , \quad \epsilon(X) \, = \, 0 \qquad \forall \, a \in A \; , \; X \in L$$

Remark that the anchor map ω endows A with an obvious left $V^\ell(L)$–module structure, given by $u.a := \omega(u)(a)$ for all $u \in V^\ell(L)$, $a \in A$, that coincides with the anchor of the left bialgebroid $V^\ell(L)$. More in general (see [23]), left $V^\ell(L)$–module structures on A correspond to left bialgebroid structures on $V^\ell(L)$ over A. Indeed, one has $\tilde{\epsilon}(u) = u \cdot 1$ for all $u \in V^\ell(L)$ and

$$\Delta_\ell(a) \, = \, a \otimes 1 \quad , \qquad \Delta_\ell(X) \, = \, X \otimes 1 \, + \, 1 \otimes X \, - \, \tilde{\epsilon}(X) \qquad\qquad \forall \;\, a \in A \, , \; X \in L$$

Finally, one can recover the anchor of L from the left bialgebroid structure of $V^\ell(L)$ as follows:

$$\omega_L(X)(a) \; = \; \epsilon_{V^\ell(L)}(X\,a) \qquad\qquad \text{for all } \; X \in L \, , \; a \in A$$

Remark 3.13. Let (A, L) and (A', L') be two Lie-Rinehart algebras. Endow $V^\ell(L)$ and $V^\ell(L')$ with their standard left bialgebroid structure. A Lie-Rinehart algebra morphism from (A, L) to (A', L') as it was defined in [16] — see also "morphisms of Lie pseudo-algebras" as they are defined in [14] — gives rise to a left bialgebroid morphism from $V^\ell(L)$ to $V^\ell(L')$.

Our next theorem is a suitable version for left bialgebroids of the well-known Cartier-Milnor-Moore theorem (for Hopf algebras). A similar result is given in [28], yet in this paper we do need (later on) that kind of result exactly as stated here below.

Theorem 3.14. *Assume that A is a unital commutative algebra over the field k.*

(a) Let $(U, A, s^\ell, t^\ell, \Delta_\ell, \epsilon)$ be a left bialgebroid such that $s^\ell = t^\ell$. Set

$$P^\ell(U) \;:=\; \big\{ u \in U \,\big|\, \Delta_\ell(u) = u \otimes 1 + 1 \otimes u \big\}$$

(the set of "left primitive elements" of U). Then the pair $\big(A, P^\ell(U)\big)$ is a Lie-Rinehart algebra.

(b) Assume in addition that s_ℓ is injective, $P^\ell(U)$ is projective as an A–module, and $P^\ell(U)$ and $s^\ell(A)$ generate U as an algebra. Then U is isomorphic to $V^\ell\big(P^\ell(U)\big)$ as a left bialgebroid.

Proof. (a) On $P^\ell(U)$ we set the following A–module structure: $a \cdot D := s^\ell(a) D$ for all $a \in A$, $D \in P^\ell(U)$. Moreover, if $D, D' \in P^\ell(U)$ then $\big[D, D'\big] := D \cdot D' - D' \cdot D \in P^\ell(U)$, by direct check: this defines a Lie bracket on $P^\ell(U)$. Finally, we define $\omega : P^\ell(U) \longrightarrow Der(A)$ by $D \mapsto \big(b \xrightarrow{\omega(D)} \epsilon(D\, s^\ell(b))\big)$.

It is proved in [23] (Proposition 4.2.1) that $\big(A, P^\ell(U), \omega\big)$ is a Lie-Rinehart algebra.

(b) By assumption, the natural algebra morphism from $T_k\big(A \oplus P^\ell(U)\big)$ to U is surjective and it induces a surjective algebra morphism $f : V^\ell\big(P^\ell(U)\big) \to U$. As $P^\ell(U)$ and $s^\ell(A)$ generates $V^\ell\big(P^\ell(U)\big)$ as an algebra, this map is also a morphism of corings. By the same argument as in [27], Lemma 5.3.3, we will show that f is also injective because $f\big|_{P^\ell(U)}$ is injective (which is obvious).

Consider the following (increasing) filtration $\big\{C_n\big\}_{n \in \mathbb{N}}$ of $V^\ell\big(P^\ell(U)\big)$: $C_0 := A$, $C_1 := A + P^\ell(U)$, $\dots$, $C_n := A + P^\ell(U) + \cdots + \big(P^\ell(U)\big)^n$, $\dots$; then one has $\Delta(C_n) = \sum_{p=0}^{n} C_p \otimes C_{n-p}$, and if $c \in C_n$ then $\Delta(c) = c \otimes 1 + 1 \otimes c + y$ for some $y \in \sum_{p=1}^{n-1} C_p \otimes C_{n-p}$.

We will show that $f\big|_{C_n}$ is injective for all $n \in \mathbb{N}$, by the same arguments as in [27].

We know that $f\big|_{C_1}$ is injective by hypothesis. So now assume that $f\big|_{C_n}$ is injective, for some $n > 0$, and choose $x \in C_{n+1}$. One has $\Delta(x) = x \otimes 1 + 1 \otimes x + y$ with $y \in C_n \otimes C_n$: thus

$$\Delta\big(f(x)\big) \;=\; (f \otimes f)\big(\Delta(x)\big) \;=\; f(x) \otimes f(1) + f(1) \otimes f(x) + (f \otimes f)(y)$$

If $x \in Ker(f)$, then $(f \otimes f)(y) = 0$. But $f \otimes f$ is injective on $C_n \otimes C_n$ (as C_n is projective) so $y = 0$. Then $x \in P^\ell(U)$, and of course f is injective on $P^\ell(U)$: thus $x = 0$ and f is injective. □

Remarks 3.15. *(a)* We can improve a little the previous result as follows : Assume that A is commutative and let $\big(U, A, s^\ell, t^\ell, \Delta_\ell, \partial\big)$ be a left bialgebroid such that $s^\ell = t^\ell$ and s^ℓ is injective. Let $Q \subseteq P^\ell(U)$ be a left Lie-Rinehart subalgebra of $P^\ell(U)$ such that

[i] Q is a projective A–module; *[ii]* Q and A generate U as an algebra.

Then U is isomorphic to $V^\ell(Q)$ as a left bialgebroid, and $Q = P^\ell(U)$.

(b) Under the conditions in Theorem 3.14*(b)*, any left bialgebroid structure on $V^\ell(L)$ can be seen as the standard left bialgebroid structure of some well chosen Lie-Rinehart algebra.

3.3 Right bialgebroids

Just like for left bialgebroids, one can consider the notion of *right bialgebroids*. We shall now introduce this theme, first considered in [18]. Hereafter B is a (unital, associative) k–algebra, and we use notations as in § 3.1.

Definition 3.16. *A right B–bialgebroid is a k–module H that carries simultaneously a structure of a B^e–ring (H, s^r, t^r) and of a B–coring (H, Δ_r, ∂) subject to the following compatibility relations:*

(i) The B^e–module structure on the B–coring (H, s^r, t^r) is that of ${}_{\blacktriangleright}H_{\blacktriangleleft}$, namely (for all $b, \tilde{b} \in B$, $h \in H$,) $b \blacktriangleright h \blacktriangleleft \tilde{b} := h\, s^r(\tilde{b})\, t^r(b) = h\, \eta(\tilde{b} \otimes b)$.

(ii) The coproduct map Δ_r is a unital k–algebra morphism taking values in $H_{\blacktriangleleft} \overset{B}{\times}{}_{\blacktriangleright} H$.

(iii) The (right) counit map ∂ has the following property: for all $b, \tilde{b} \in B$, $u, u' \in H$, one has

$$\partial\big(u\, s^r\big(\tilde{b}\big)\, t^r(b)\big) \;=\; b\, \partial(u)\, \tilde{b} \quad , \qquad \partial(u\, u') \;=\; \partial\big(s^r\big(\partial(u)\big)\, u'\big) \;=\; \partial\big(t^r\big(\partial(u)\big)\, u'\big) \quad , \qquad \partial(1) \;=\; 1$$

Remarks 3.17. A right bialgebroid has the following properties :

(a) $$\Delta_r\big(s^r(b)\big) \;=\; 1 \otimes s^r(b) \quad , \qquad \Delta_r\big(t^r(b)\big) \;=\; t^r(b) \otimes 1$$

(b) If $\Delta_r(u) = u^{(1)} \otimes u^{(2)}$ then

$$\Delta_r\big(b \triangleright u \triangleleft \tilde{b}\,\big) \;=\; \big(u^{(1)} \triangleleft \tilde{b}\,\big) \otimes \big(b \triangleright u^{(2)}\big) \quad , \qquad \Delta_r\big(b \blacktriangleright u \blacktriangleleft \tilde{b}\,\big) \;=\; \big(b \blacktriangleright u^{(1)}\,\big) \otimes \big(u^{(2)} \blacktriangleleft \tilde{b}\big)$$

(c) $$\partial(s^r(b)) \;=\; b \quad , \qquad \partial(t^r(b)) \;=\; b$$

(d) H acts on its base algebra B on the right as follows:

$$b.u \;:=\; \partial\big(s^r(b)\, u\big) \;=\; \partial\big(t^r(b)\, u\big) \qquad\qquad \forall\; b \in B\,,\; u \in H$$

(e) $$s^r\big(\partial(x)\big) \otimes 1 \;=\; s^r\big(\partial\big(x_{(1)}\big)\big) \otimes t^r\big(\partial\big(x_{(2)}\big)\big) \;=\; 1 \otimes t^r\big(\partial(x)\big) \qquad \text{for all } \; x \in H\,.$$

(f) As a matter of notation, if $\big(H, A, s^r, t^r, \Delta, \partial\big)$ is a right bialgebroid, we set $H^+ := \mathit{Ker}(\partial)$.

N.B.: in literature, right bialgebroids are also called $\times^B$*–bialgebras*.

Definition 3.18. *Let $\mathcal{H} = \big(H, B, s^r, t^r, \Delta, \partial\big)$ and $\hat{\mathcal{H}} = \big(\hat{H}, \hat{B}, \hat{s}^r, \hat{t}^r, \hat{\Delta}, \hat{\partial}\big)$ be two right bialgebroids. A morphism of right bialgebroids Φ from $\mathcal{H}$ to $\hat{\mathcal{H}}$ is a pair (f, F) where*

(a) $f : B \longrightarrow \hat{B}$ is a morphism of algebras;

(b) $F : H \longrightarrow \hat{H}$ is a morphism of algebras and of coalgebras — that is, we have

$$\hat{\Delta} \circ F = (F \otimes F) \circ \Delta \;\; , \qquad \hat{\partial} \circ F = f \circ \partial \;\; ;$$

(c) $$F \circ s^r \;=\; \hat{s}^r \circ f \;\; , \qquad F \circ t^r \;=\; \hat{t}^r \circ f \;\; .$$

We denote (RBialg) *the category of right bialgebroids, whose objects are right bialgebroids and morphisms are defined as above. Inside it,* (RBialg_A) *is the subcategory whose objects are all the right bialgebroids over A and whose morphisms are all those in* (RBialg) *of the form* $(\mathrm{id}\,, F)$.

Remarks 3.19. (cf. [23])

The "opposite" of a left bialgebroid $U = \big(U, A, s^\ell, t^\ell, \Delta_\ell, \epsilon\big)$ is given, by definition, by $U^{op} := \big(U^{op}, A, t^\ell, s^\ell, \Delta_\ell, \epsilon\big)$. This can be shown to be a right bialgebroid whereas its "coopposite", given by $U^{coop} := \big(U, A^{op}, t^\ell, s^\ell, \Delta_\ell^{coop}, \epsilon\big)$ with Δ_ℓ^{coop} thought of as a map

$$U \longrightarrow {}_{\triangleright}U \otimes_{A^{op}} U_{\triangleleft}\,, \quad u \mapsto u_{(2)} \otimes u_{(1)} \quad \text{if } \Delta(u) = u_{(1)} \otimes u_{(2)}$$

is still a left bialgebroid. As a consequence, U^{op}_{coop} is a right bialgebroid.

3.20. Right bialgebroid structures on universal enveloping algebras $V^r(L)$. Given a Lie-Rinehart algebra L, now considered as a *right* one, its right universal enveloping algebra $V^r(L)$ bears a natural structure of right bialgebroid over A. In order to describe it, one can mimick stepwise the construction of the canonical left bialgebroid structure in $V^\ell(L)$. Alternatively, one can proceed as follows. First, we have an algebra identification $\Xi : V^r(L) \cong V^\ell(L)^{op}$ by Proposition 2.19. As $V^\ell(L)$ is a left bialgebroid, its opposite $V^\ell(L)^{op}$ is a right bialgebroid. Then we pull-back — via the previous algebra isomorphism — this right bialgebroid structure onto $V^r(L)$, so that $V^r(L) \cong V^\ell(L)^{op}$ as right bialgebroids over A.

More explicitly, let us use notation $\big(V^\ell(L), A, s_\ell, t_\ell, \Delta_\ell, \epsilon\big)$ for the given left bialgebroid structure on $V^\ell(L)$, and similarly $\big(V^r(L), A, s^r, t^r, \Delta_r, \partial\big)$ for the right bialgebroid structure we have to fix on $V^r(L)$; moreover, let Ξ be the suitable algebra anti-isomorphism as in Proposition 2.19. Then

$$s^r := \Xi \circ t_\ell \ , \qquad t^r := \Xi \circ s^\ell \ , \qquad \partial := \epsilon \circ \Xi^{-1} \ , \qquad \Delta_r := (\Xi \otimes \Xi) \circ \Delta \circ \Xi^{-1}$$

In another language, the A^e–module structure ${}_\blacktriangleright V^r(L)_\blacktriangleleft$ is given by $a \blacktriangleright u \blacktriangleleft \tilde{a} := u\, a\, \tilde{a}$. The coproduct $\Delta_r : V^r(L) \longrightarrow V^r(L)_\blacktriangleleft \underset{A}{\otimes} {}_\blacktriangleright V^r(L)$ and the counit $\partial : V^r(L) \longrightarrow A$ are determined by

$$\Delta_r(a) \,=\, a \otimes 1 \ , \quad \Delta_r(X) \,=\, X \otimes 1 + 1 \otimes X \ , \qquad \partial(a) \,=\, a \ , \quad \partial(X) \,=\, 0 \qquad \forall\, a \in A \ , \ X \in L$$

Note also that the right $V^r(L)$–module structure on A determined by this structure is given by $u.a := \big(\big(\omega \circ \Xi^{-1}\big)(u)\big)(a)$ for all $u \in V^r(L)$, $a \in A$.

Finally, one can recover the anchor of L from the left bialgebroid structure of $V^r(L)$ as follows:

$$\omega_L(X)(a) \;=\; -\partial_{V^r(L)}(a\, X) \qquad\qquad \text{for all } X \in L \,,\ a \in A$$

We also have an analog for right bialgebroids of Theorem 3.14:

Proposition 3.21. *Assume that A is a unital commutative algebra over the field k.*

(a) Let $(W, A, s^r, t^r, \Delta_r, \partial)$ be a right bialgebroid such that $s^r = t^r$. Set

$$P^r(W) \;:=\; \big\{\, w \in W \,\big|\, \Delta_r(w) = w \otimes 1 + 1 \otimes w \,\big\}$$

(the set of "right primitive elements" of W). Then the pair $\big(A, P^r(W)\big)$ is a right Lie-Rinehart algebra for the following right action and anchor map

$$w.a \,:=\, w\, s^r(a) \ , \quad \omega(D)(a) \,:=\, -\partial\big(s^r(a)\, D\big) \ , \qquad \forall\ w \in W \,, \ \forall\ D \in P^r(W) \,, \ \forall\ a \in A$$

(b) Assume in addition that s^r is injective, $P^r(W)$ is projective as an A–module, and $P^r(W)$ and $s^r(A)$ generate W as an algebra. Then W is isomorphic to $V^r\big(P^r(W)\big)$ as a right bialgebroid.

The proof of this theorem is a variation of the proof of Theorems 3.14.

Remark 3.22. We can improve a little the previous result as follows. Assume that A is commutative and let $\big(W, A, s^r, t^r, \Delta_r, \partial\big)$ be a right bialgebroid such that $s^r = t^r$ and s^r is injective. Let $Q \subseteq P^r(U)$ be a right Lie-Rinehart subalgebra of $P^r(U)$ such that

[i] Q is a projective A–module; *[ii]* Q and A generate W as an algebra.

Then W is isomorphic to $V^r(Q)$ as a right bialgebroid, and $Q = P^r(W)$.

3.4 Hopf algebroids

The following definition is due to Bőhm-Szlachànyi ([5],[3], see [23] for a survey).

Definition 3.23. *Let A and B be two k–algebras and H a k–module. A Hopf algebroid structure on H consists of*

(1) a left bialgebroid structure $H^\ell = \big(H, A, s^\ell, t^\ell, \Delta_\ell\,, \epsilon\big)$ on H over A,

(2) a right bialgebroid structure $H^r = \big(H, B, s^r, t^r, \Delta_r\,, \partial\big)$ on H over B,

(3) a k–module map $S : H \longrightarrow H$,

and these structures are subject to the following compatibility axioms $\big(\forall\, a, a' \in A,\ b, b' \in B,\ h \in H\big)$ *:*

(i) the underlying k–algebra structure on H in (1) and (2) are the same,

(ii) $\quad s^\ell \circ \epsilon \circ t^r = t^r \quad , \quad t^\ell \circ \epsilon \circ s^r = s^r \quad , \quad s^r \circ \partial \circ t^\ell = t^\ell \quad , \quad t^r \circ \partial \circ s^\ell = s^\ell$

(iii) twisted coassociativity holds, that is to say

$$\big(\Delta_\ell \otimes id_H\big) \circ \Delta_r = \big(id_H \otimes \Delta_r\big) \circ \Delta_\ell \quad , \qquad \big(\Delta_r \otimes id_H\big)\Delta_\ell = \big(id_H \otimes \Delta_\ell\big) \circ \Delta_r$$

(iv) $\quad S\big(t^\ell(a)\, h\, t^\ell(b)\big) = s^\ell(b)\, S(h)\, s^\ell(a) \quad , \qquad S\big(t^r(a)\, h\, t^r(b)\big) = s^r(b)\, S(h)\, s^r(a)$

(v) $\quad m_H \circ \big(S \otimes id_H\big) \circ \Delta_\ell = s^r \circ \partial \quad , \qquad m_H \circ \big(id_H \otimes S\,\big) \circ \Delta_r = s^\ell \circ \epsilon$

The map $S : H \longrightarrow H$ is called the antipode of H .

Remarks 3.24.

(a) When using Σ–notation, we shall adopt lower indices for the left coproduct Δ_ℓ and upper ones for the right coproduct Δ_r : in other words, for any $h \in H$ we shall write $\Delta_\ell(h) = h_{(1)} \otimes h_{(2)}$ and $\Delta_r(h) = h^{(1)} \otimes h^{(2)}$. In these terms, the twisted coassociativity is expressed, for $h \in H$, by

$$h^{(1)}{}_{(1)} \otimes h^{(1)}{}_{(2)} \otimes h^{(2)} = h_{(1)} \otimes h_{(2)}{}^{(1)} \otimes h_{(2)}{}^{(2)} \quad \text{and} \quad h_{(1)}{}^{(1)} \otimes h_{(1)}{}^{(2)} \otimes h_{(2)} = h^{(1)} \otimes h^{(2)}{}_{(1)} \otimes h^{(2)}{}_{(2)}$$

and similarly the identities in *(v)* read $S\big(h_{(1)}\big)\, h_{(2)} = s^r\big(\partial(h)\big)$ and $h^{(1)}\, S\big(h^{(2)}\big) = s^\ell\big(\epsilon(h)\big)$.

(b) The images of s^ℓ and t^r , as well as those of s^r and t^ℓ , respectively, are coinciding subalgebras inside H .

(c) It follows from definitions that $\mu := \partial \circ s^\ell : A^{op} \longrightarrow B$ *and* $\nu := \partial \circ t^\ell : A \longrightarrow B^{op}$ *are isomorphisms of algebras. Moreover, one has* $\mu^{-1} = \epsilon \circ t^r : B \longrightarrow A^{op}$ *and* $\nu^{-1} = \epsilon \circ s^r : B^{op} \longrightarrow A$ *.*

(c) If (H^l, H^r, S) is a Hopf algebroid, so is $\big((H^r)^{op}_{coop}, (H^l)^{op}_{coop}, S\big)$.

Theorem 3.25. *(cf. [3]) We keep the notation of the previous definition.*

(a) The pair $\big(S, \epsilon \circ s_r\big)$ is a left bialgebroid homomorphism from H^ℓ to $\big(H^r\big)^{op}_{coop}$.

(b) The pair $\big(S, \partial \circ s_\ell\big)$ is a left bialgebroid homomorphism from $\big(H^r\big)^{op}_{coop}$ to H^ℓ .

For alternative definitions of a Hopf algebroid, we refer the reader to [25] and [19].

3.26. Left Hopf algebroids. Left Hopf algebroids (called "$\times_A$–Hopf algebras" in [31]) are a generalization of the notion of Hopf algebra. Given a left bialgebroid U over A, define the so-called (Hopf-)Galois map of U as ${}_\blacktriangleright U \underset{A^{op}}{\otimes} U_\triangleleft \longrightarrow U_\triangleleft \underset{A}{\otimes} {}_\triangleright U$, $u \otimes v \mapsto u_{(1)} \otimes u_{(2)}\, v$, where

$${}_\blacktriangleright U \underset{A^{op}}{\otimes} U_\triangleleft := U \underset{k}{\otimes} U \Big/ Span\big(\big\{\, (a \blacktriangleright u) \otimes v - u \otimes (v \triangleleft a) \;\big|\; u, v \in U\,,\ a \in A \,\big\}\big)$$

For *bialgebras* over *fields*, it is easily seen that β is bijective if and only if U is a Hopf algebra, with $\beta^{-1}(u \otimes v) = u_{(1)} \otimes S\big(u_{(2)}\big)\, v$ where S is the antipode of U . This motivates next definition:

Definition 3.27. *A left A-bialgebroid U is called a left Hopf algebroid (or a $\times_A$–Hopf algebroid) if the map β considered above is a bijection. As a matter of notation we shall then adopt the following (Σ–like) notation:* $u_+ \otimes u_- := \beta^{-1}(u \otimes 1)$ *for all* $u \in U$ *.*

N.B.: the following property then holds: $(uv)_+ \otimes (uv)_- = u_+ v_+ \otimes v_- u_-$ for any $u, v \in U$.

The notion of left Hopf algebroid was introduced in [31], under the name of $\times_A$*–Hopf algebras*; here instead we adopt the conventions and terminology of [23], Ch. 2 (where a detailed history of this notion can be found).

Example 3.28. If L is any Lie-Rinehart algebra, then its enveloping algebra $V(L)$, endowed with its standard bialgebroid structure, is a left Hopf bialgebroid, whose map β^{-1} is given on generators by $a_+ \otimes a_- = a \otimes 1 \ \big(\forall\, a \in A\big)$, $X_+ \otimes X_- = X \otimes 1 - 1 \otimes X \ \big(\forall\, X \in L\big)$.

The link between Hopf algebroids and left Hopf algebroids is the following (cf. [23], §2.6.14):

Proposition 3.29. *Every Hopf algebroid is a left Hopf algebroid.*

3.5 Duals of bialgebroids

We shall now consider left and right duals of (left and right) bialgebroids, and investigate their main properties. To begin with, we introduce the notions of left dual and right dual of a left bialgebroid; later on, we shall give the parallel definitions for right bialgebroids.

Definition 3.30.

(a) Let U be a left A–bialgebroid, with structure maps as before.

(a.1) The left dual of U is the space

$$U_* := \left\{ \phi : U \longrightarrow A \,\middle|\, \phi(u'+u'') = \phi(u') + \phi(u'') \,,\ \phi\big(s^\ell(a)\,u\big) = a\,\phi(u) \right\} = Hom_A\big({}_{\triangleright}U \,,\, {}_A A\big)$$

(a.2) The right dual of U is the space

$$U^* := \left\{ \phi : U \longrightarrow A \,\middle|\, \phi(u'+u'') = \phi(u') + \phi(u'') \,,\ \phi\big(t^\ell(a)\,u\big) = \phi(u)\,a \right\} = Hom_A\big(U_{\triangleleft} \,,\, A_A\big)$$

(b) Let W be a right A–bialgebroid, with structure maps as before.

(b.1) The left dual of W is the space

$${}_*W := \left\{ \psi : W \longrightarrow A \,\middle|\, \psi(w'+w'') = \psi(w')+\psi(w'') \,,\ \psi\big(w\,t^r(a)\big) = a\,\psi(w) \right\} = Hom_A\big({}_{\blacktriangleright}W \,,\, {}_A A\big)$$

(b.2) The right dual of W is the space

$${}^*W := \left\{ \psi : W \longrightarrow A \,\middle|\, \psi(w'+w'') = \psi(w')+\psi(w'') \,,\ \psi\big(w\,s^r(a)\big) = \psi(w)\,a \right\} = Hom_A\big(W_{\blacktriangleleft} \,,\, A_A\big)$$

3.31. Bialgebroid structures on dual spaces. Let U be a *left* A–bialgebroid as above. We shall now introduce on its dual spaces U_* and U^* a structure of right A–bialgebroid; most of the structure is well-defined in general, but for the coproduct we need an additional assumption, namely U as an A–module (on the left, or the right, see below) has to be *projective*.

<u>*Product structure*</u>*:* First we recall (see [18], and also [23] for a nice exposition) that U_* and U^* can be equipped with a product, for which the counit map ϵ is a two-sided unit. For any $\phi, \phi' \in U_*$ and $\psi, \psi' \in U^*$ set

$$\big(\phi\,\phi'\big)(u) \;\equiv\; m_{U_*}\big(\phi \otimes \phi'\big)(u) \;:=\; \phi'\Big(m_{U^{op}}\big(id \otimes (t^\ell \circ \phi)\big)\big(\Delta_\ell(u)\big)\Big) \;=\; \phi'\big(t^\ell(\phi(u_{(2)})) \cdot u_{(1)}\big)$$

$$\big(\psi\,\psi'\big)(u) \;\equiv\; m_{U^*}\big(\psi \otimes \psi'\big)(u) \;:=\; \psi'\Big(m_U\big((s^\ell \circ \psi) \otimes id\big)\big(\Delta_\ell(u)\big)\Big) \;=\; \psi'\big(s^\ell(\psi(u_{(1)})) \cdot u_{(2)}\big)$$

for every $u \in U$, with $\Delta_\ell(u) = u_{(1)} \otimes u_{(2)}$.

<u>*A–module structures*</u>*:* For the left dual space U_* , the left dual source map $s_*^r : A \longrightarrow U_*$ and the right dual target map $t_*^r : A \longrightarrow U_*$ are defined as follows:

$$\big(s_*^r(a)\big)(u) := \epsilon\big(t^\ell(a)\,u\big) = \epsilon(u)\,a \;, \qquad \big(t_*^r(a)\big)(u) := \epsilon\big(u\,t^\ell(a)\big) \qquad \forall\, a \in A \,,\ u \in U \,.$$

Then one has, in the usual way, two left and two right actions of A on U_* , given by

$$(a \triangleright \phi)(u) := \big(s_*^r(a)\,\phi\big)(u) = \phi\big(t^\ell(a)\,u\big) \;, \qquad (\phi \triangleleft a)(u) := \big(t_*^r(a)\,\phi\big)(u) = \phi\big(u\,t^\ell(a)\big)$$
$$(a \blacktriangleright \phi)(u) := \big(\phi\,t_*^r(a)\big)(u) = \phi\big(u\,s^\ell(a)\big) \;, \qquad (\phi \blacktriangleleft a)(u) := \big(\phi\,s_*^r(a)\big)(u) = \phi(u)\,a$$

Similarly, for the right dual space U^* the left dual source map $s_r^* : A \longrightarrow U^*$ and the right dual target map $t_r^* : A \longrightarrow U^*$ are defined as follows:

$$\big(s_r^*(a)\big)(u) := \epsilon\big(u\,s^\ell(a)\big) \;, \qquad \big(t_r^*(a)\big)(u) := \epsilon\big(s^\ell(a)\,u\big) = a\,\epsilon(u) \qquad \forall\, a \in A \,,\ u \in U \,.$$

Then one has, like before, two left and two right A–actions on U^* , given by

$$(a \triangleright \psi)(u) := \big(s_r^*(a)\,\psi\big)(u) = \psi\big(u\,s^\ell(a)\big) \;, \qquad (\psi \triangleleft a)(u) := \big(t_r^*(a)\,\psi\big)(u) = \psi\big(s^\ell(a)\,u\big)$$
$$(a \blacktriangleright \psi)(u) := \big(\psi\,t_r^*(a)\big)(u) = a\,\psi(u) \;, \qquad (\psi \blacktriangleleft a)(u) := \big(\psi\,s_r^*(a)\big)(u) = \psi\big(u\,t^\ell(a)\big)$$

Coproduct structure: Now *assume that ${}_{\triangleright}U$ as an A–module be projective*, Then we shall now endow the left dual U_* with a coproduct Δ_r which — together with the previously introduced structures — makes it into a right bialgebroid.

Consider the injective map $\chi : U_{*\,\blacktriangleleft} \otimes {}_{\blacktriangleright}U_* \longrightarrow \mathit{Hom}_{(A,-)}\big({}_{\triangleright}(U_{\blacktriangleleft} \otimes {}_{\triangleright}U\,)\,,\,{}_AA\big)$ given by

$$\phi \otimes \phi' \;\mapsto\; \chi\big(\phi \otimes \phi'\big)\Big(\, u \otimes u' \;\mapsto\; \chi\big(\phi \otimes \phi'\big)\big(u \otimes u'\big) \;:=\; \phi'\big(u\, s^\ell(\phi(u'))\big)\Big)$$

Now, if U is finite projective (as an A–module) then the previous map is even an isomorphism. If instead U is projective but not finite, one can endow $U_{*\,\blacktriangleleft} \otimes {}_{\blacktriangleright}U_*$ with a suitable topology (typically, the "weak" one), and denote by $U_{*\,\blacktriangleleft} \widetilde{\otimes}\, {}_{\blacktriangleright}U_*$ the corresponding completion: then the above map extends — by continuity, using the notion of basis for a projective module (cf. [2]) — to an isomorphism from $U_{*\,\blacktriangleleft} \widetilde{\otimes}\, {}_{\blacktriangleright}U_*$ to $\mathit{Hom}_{(A,-)}\big({}_{\triangleright}(U_{\blacktriangleleft} \otimes {}_{\triangleright}U\,)\,,\,{}_AA\big)$. This allows us to define a coproduct Δ_r on U_* as the transpose of the multiplication on U, namely

$$\begin{aligned} \Delta_r \,:\, U_* \quad &\longrightarrow \quad \mathit{Hom}_{(A,-)}\big({}_{\triangleright}(U_{\blacktriangleleft} \otimes {}_{\triangleright}U\,)\,,\,{}_AA\big) \underset{\chi^{-1}}{\cong} U_{*\,\blacktriangleleft} \widetilde{\otimes}\, {}_{\blacktriangleright}U_* \\ \phi \quad &\mapsto \quad \Delta_r(\phi)\,\big(\, u \otimes u' \;\mapsto\; \phi\big(u\,u'\big)\big) \end{aligned}$$

This coproduct makes U_* into an A–coring, with counit $\eta_* : U_* \longrightarrow A$ given by $\eta_*(\phi) := \phi(1)$.

Similarly, *if $U_{\triangleleft}$ as an A–module is projective*. Then for its right dual U^* a coproduct is defined as follows. Consider the injective map $\vartheta : U^*_{\blacktriangleleft} \otimes {}_{\blacktriangleright}U^* \longrightarrow \mathit{Hom}_{(-,A)}\big(\,(U_{\triangleleft} \otimes {}_{\blacktriangleright}U\,)_{\triangleleft}\,,\,A_A\,\big)$ given by

$$\psi \otimes \psi' \;\mapsto\; \vartheta\big(\psi \otimes \psi'\big)\Big(\, u \otimes u' \;\mapsto\; \vartheta\big(\psi \otimes \psi'\big)\big(u \otimes u'\big) \;:=\; \psi\big(u'\, t^\ell(\psi'(u))\big)\Big)$$

Again, if U is finite projective (as an A–module) then this map is an isomorphism. If instead U is projective but not finite, one can endow $U^*_{\blacktriangleleft} \otimes {}_{\blacktriangleright}U^*$ with a suitable topology (like the weak one), and denote by $U^*_{\blacktriangleleft} \widetilde{\otimes}\, {}_{\blacktriangleright}U^*$ the corresponding completion: then the above map extends — by continuity — to an isomorphism $\widetilde{\vartheta}$ from $U^*_{\blacktriangleleft} \widetilde{\otimes}\, {}_{\blacktriangleright}U^*$ to $\mathit{Hom}_{(-,A)}\big(\,(U_{\triangleleft} \otimes {}_{\blacktriangleright}U\,)_{\triangleleft}\,,\,A_A\,\big)$. Thus we can define a coproduct Δ_r on U^* as the transpose of the *opposite* multiplication on U, namely

$$\begin{aligned} \Delta_r \,:\, U^* \quad &\longrightarrow \quad \mathit{Hom}_{(-,A)}\big(\,(U_{\triangleleft} \otimes {}_{\blacktriangleright}U\,)_{\triangleleft}\,,\,A_A\,\big) \underset{\widetilde{\vartheta}^{-1}}{\cong} U^*_{\blacktriangleleft} \widetilde{\otimes}\, {}_{\blacktriangleright}U^* \\ \psi \quad &\mapsto \quad \Delta_r(\psi)\,\big(\, u \otimes u' \;\mapsto\; \psi\big(u'\,u\big)\big) \end{aligned}$$

This makes U^* into an A–coring, which has counit $\partial_* : U^* \longrightarrow A$ given by $\partial_*(\psi) := \psi(1)$.

Conclusion: If U is any *left* bialgebroid over A, projective as an A–module, then U_* and U^* with the structures introduced above are both *right* bialgebroids over A: indeed, to be precise they are *topological* bialgebroids, in that their coproduct take values in a (suitably) *topological* tensor product, rather than in the "plain" one. Clearly, one can formalize all this by saying that they are bialgebroids in a suitable tensor category; we leave this task to the interested reader.

Similarly, we consider the case of a *right* A–bialgebroid W, and we introduce canonical structures of *left* A–bialgebroids on its left and right dual spaces ${}_*W$ and ${}^*W$: indeed, everything is strictly similar to what occurs in the previous case for U, so we skip details.

Notation: in the following, we shall also use the following, standard notation: if v is an element of some (left or right) A–module, and ϕ is an element of the (left or right) dual of that module, then we shall write $\big\langle \phi\,,\, v \big\rangle := \phi(v)$ or $\big\langle v\,,\, \phi \big\rangle := \phi(v)$.

Remarks 3.32.

(a) If U is a left bialgebroid which is projective of finite type as an A–module, then it is isomorphic to ${}^*(U_*)$ and to ${}_*(U^*)$ — as a left bialgebroid. This follows from the following equalities:

$$\begin{aligned} \big\langle\, u\,,\, \phi\, s^r_*(a) \,\big\rangle \;&=\; \big\langle\, u\,,\, \phi \,\big\rangle\, a \qquad \text{which shows that} \quad U \subseteq {}^*(U_*) \\ \big\langle\, u\,,\, \phi\, t^*_r(a) \,\big\rangle \;&=\; a\, \big\langle\, u\,,\, \phi \,\big\rangle \qquad \text{which shows that} \quad U \subseteq {}_*(U^*) \end{aligned}$$

(b) If ${}_{\triangleright}U$ as an A–module is free, of finite type, with A–basis $\{e_1\,,\,\dots\,,\,e_n\}$, then we define $e^i_* \in U_*$ by $\big\langle\, e_j\,,\, e^i_* \,\big\rangle = \delta_{i,j}$. Then $\{e^1_*\,,\,\dots\,,\,e^r_*\}$ is an A–basis of $(U_*)_{\blacktriangleleft}$ and $\{e_1\,,\,\dots\,,\,e_n\}$ is

an A–basis of ${}^*(U_*)$. This shows that $U \cong {}^*(U_*)$ whenever ${}_\triangleright U$ is a finite free A–module. This can be extended to the case where ${}_\triangleright U$ is only a finite type projective A–module (using the notion of basis for a projective A–module, cf. [2]).

We introduce now the vocabulary of "pairings", which provides us with another tool to deal with linear duality among bialgebroids. The natural pattern is the familiar one of "pairings" or "matchings" in Hopf algebra theory.

Definition 3.33. *(a) Let $\big(U, s_\ell, t_\ell\big)$ and $\big(W, s^r, t^r\big)$ be two A^e–modules. An A^e–left pairing is a k–bilinear map $\langle\ ,\ \rangle : U \times W \longrightarrow A$ such that, for any $u \in U$, $w \in W$ and $a \in A$, one has*

$$\begin{aligned}
\big\langle\, u \,, a \triangleright w \,\big\rangle \;&=\; \big\langle\, u, s^r(a)\, w \,\big\rangle \;=\; \big\langle\, t_\ell(a)\, u \,, w \,\big\rangle \;=\; \big\langle\, u \triangleleft a \,, w \,\big\rangle \\
\big\langle\, u \,, w \triangleleft a \,\big\rangle \;&=\; \big\langle\, u \,, t^r(a)\, w \,\big\rangle \;=\; \big\langle\, u\, t_\ell(a) \,, w \,\big\rangle \;=\; \big\langle\, a \blacktriangleright u \,, w \,\big\rangle \\
\big\langle\, u \,, a \blacktriangleright w \,\big\rangle \;&=\; \big\langle\, u \,, w\, t^r(a) \,\big\rangle \;=\; \big\langle\, u\, s_\ell(a) \,, w \,\big\rangle \;=\; \big\langle\, u \blacktriangleleft a \,, w \,\big\rangle \\
\big\langle\, u \,, w \blacktriangleleft a \big\rangle \;&=\; \big\langle u \,, w\, s^r(a) \,\big\rangle \;=\; \big\langle\, u \,, w \,\big\rangle\, a \\
\big\langle\, a \triangleright u \,, w \,\big\rangle \;&=\; \big\langle\, s^\ell(a)\, u \,, w \,\big\rangle \;=\; a \big\langle\, u \,, w \,\big\rangle
\end{aligned}$$

Then there exist natural morphisms of A^e–modules $W \longrightarrow U_*$ and $U \longrightarrow {}^*W$. The pairing is non degenerate if the left and right kernels of this pairing are trivial, that is to say

$$\begin{aligned}
\big\langle\, u \,, w \,\big\rangle = 0 \;\;, \quad \forall\; w \in W \qquad &\Longrightarrow \qquad u = 0 \\
\big\langle\, u \,, w \,\big\rangle = 0 \;\;, \quad \forall\; u \in U \qquad &\Longrightarrow \qquad w = 0
\end{aligned}$$

In other words, the pairing is non degenerate if and only if the above maps $W \longrightarrow U_*$ and $U \longrightarrow {}^*W$ (which are morphisms of A^e–modules) are injective.

(b) Let $\big(U, s^\ell, t^\ell\big)$ and $\big(W, s_r, t_r\big)$ be two A^e–modules. An A^e–right pairing is a k–bilinear map $\langle\ ,\ \rangle : U \times W \longrightarrow A$ such that, for any $u \in U$, $w \in W$ and $a \in A$, one has

$$\begin{aligned}
\big\langle\, u \,, w \triangleleft a \,\big\rangle \;&=\; \big\langle\, u \,, t_r(a)\, w \,\big\rangle \;=\; \big\langle\, s^\ell(a)\, u \,, w \,\big\rangle \;=\; \big\langle\, a \triangleright u \,, w \,\big\rangle \\
\big\langle\, u \,, a \triangleright w \,\big\rangle \;&=\; \big\langle\, u, s_r(a)\, w \,\big\rangle \;=\; \big\langle\, u\, s^\ell(a) \,, w \,\big\rangle \;=\; \big\langle\, u \blacktriangleleft a \,, w \,\big\rangle \\
\big\langle\, u \,, w \blacktriangleleft a \big\rangle \;&=\; \big\langle u \,, w\, s_r(a) \,\big\rangle \;=\; \big\langle\, u\, t^\ell(a) \,, w \,\big\rangle \;=\; \big\langle u \triangleleft a \,, w \,\big\rangle \\
\big\langle\, u \,, a \blacktriangleright w \,\big\rangle \;&=\; \big\langle\, u \,, w\, t_r(a) \,\big\rangle \;=\; a \big\langle\, u \,, w \,\big\rangle \\
\big\langle\, u \triangleleft a \,, w \,\big\rangle \;&=\; \big\langle\, t^\ell(a)\, u \,, w \,\big\rangle \;=\; \big\langle\, u \,, w \,\big\rangle\, a
\end{aligned}$$

Then there exist natural morphisms of A^e–modules $W \longrightarrow U^*$ and $U \longrightarrow {}_*W$. The pairing is non degenerate if the left and right kernels of this pairing are trivial, that is to say

$$\begin{aligned}
\big\langle\, u \,, w \,\big\rangle = 0 \;\;, \quad \forall\; w \in W \qquad &\Longrightarrow \qquad u = 0 \\
\big\langle\, u \,, w \,\big\rangle = 0 \;\;, \quad \forall\; u \in U \qquad &\Longrightarrow \qquad w = 0
\end{aligned}$$

In other words, the pairing is non degenerate if and only if the above maps $W \longrightarrow U^*$ and $U \longrightarrow {}_*W$ (which are morphisms of A^e–modules) are injective.

Definition 3.34.

(a) Let $\big(U, s_\ell, t_\ell, \Delta, \epsilon\big)$ be a left A–bialgebroid and $\big(W, s^r, t^r, \Delta, \eta\big)$ be a right A–bialgebroid. A bialgebroid left pairing is a non degenerate A^e–left pairing $\langle\ ,\ \rangle : U \times W \longrightarrow A$ such that, for any $u, u' \in U$ and any $w, w' \in W$, one has

$$\begin{aligned}
\big\langle\, u\,u' \,,\, w \,\big\rangle \;&=\; \Big\langle\, u \,,\, w_{(2)}\, t^r\big(\big\langle\, u' \,,\, w_{(1)} \,\big\rangle\big) \Big\rangle \;=\; \Big\langle\, u\, s_\ell\big(\big\langle\, u' \,,\, w_{(1)} \,\big\rangle\big) \,,\, w_{(2)} \Big\rangle \\
\big\langle\, u \,,\, w\,w' \,\big\rangle \;&=\; \Big\langle\, t^\ell\big(\big\langle\, u_{(2)} \,,\, w \,\big\rangle\big)\, u_{(1)} \,,\, w' \,\Big\rangle \;=\; \Big\langle\, u_{(1)} \,,\, s_r\big(\big\langle\, u_{(2)} \,,\, w \,\big\rangle\big)\, w' \,\Big\rangle \\
\big\langle\, u \,,\, 1 \,\big\rangle \;&=\; \epsilon(u) \quad , \qquad \big\langle\, 1 \,,\, w \,\big\rangle \;=\; \eta(w)
\end{aligned}$$

In other words, the natural maps $W \longrightarrow U_*$ and $U \longrightarrow {}^*W$ are (injective) morphisms of right and left bialgebroids respectively.

(b) Let $\big(U, s^\ell, t^\ell, \Delta, \epsilon\big)$ be a left A–bialgebroid and $\big(W, s_r, t_r, \Delta, \eta\big)$ be a right A–bialgebroid. A bialgebroid right pairing is a non degenerate A^e–right pairing $\langle\ ,\ \rangle : U \times W \longrightarrow A$ such that, for any $u, u' \in U$ and any $w, w' \in W$, one has

$$\langle\, u\, u' \,,\, w \,\rangle \;=\; \Big\langle\, u\, t_\ell\big(\langle\, u' \,,\, w_{(2)} \,\rangle\big) \,,\, w_{(1)} \Big\rangle \;=\; \Big\langle\, u \,,\, w_{(1)}\, s^r\big(\langle\, u' \,,\, w_{(2)} \,\rangle\big)\Big\rangle$$

$$\langle\, u \,,\, w\, w' \,\rangle \;=\; \Big\langle\, s^\ell\big(\langle\, u_{(1)} \,,\, w \,\rangle\big)\, u_{(2)} \,,\, w' \Big\rangle \;=\; \Big\langle\, u_{(2)} \,,\, t_r\big(\langle\, u_{(1)} \,,\, w \,\rangle\big)\, w' \Big\rangle$$

$$\langle\, u \,,\, 1 \,\rangle \;=\; \epsilon(u) \quad , \qquad \langle\, 1 \,,\, w \,\rangle \;=\; \eta(w)$$

In other words, the natural maps $W \longrightarrow U^*$ and $U \longrightarrow {}_*W$ are (injective) morphisms of right and left bialgebroids respectively.

Next result should come as no surprise:

Proposition 3.35.

(a) Let $\big(U, s_\ell, t_\ell\big)$ and $\big(W, s^r, t^r\big)$ be two A^e–rings and $\langle\ ,\ \rangle : U \times W \longrightarrow A$ be a non degenerate A^e–left pairing. If $\big(U, s_\ell, t_\ell\big)$ is a left bialgebroid, then there is at most one right bialgebroid structure on $\big(W, s^r, t^r\big)$ such that $\langle\ ,\ \rangle$ is a bialgebroid left pairing. Similarly, if $\big(W, s^r, t^r\big)$ is a right bialgebroid, then there is at most one left bialgebroid structure on $\big(U, s_\ell, t_\ell\big)$ such that $\langle\ ,\ \rangle$ is a bialgebroid left pairing.

(b) Let $\big(U, s^\ell, t^\ell\big)$ and $\big(W, s_r, t_r\big)$ be two A^e–rings and $\langle\ ,\ \rangle : U \times W \longrightarrow A$ be a non degenerate A^e–right pairing. If $\big(U, s^\ell, t^\ell\big)$ is a left bialgebroid, then there is at most one right bialgebroid structure on $\big(W, s_r, t_r\big)$ such that $\langle\ ,\ \rangle$ is a bialgebroid right pairing. Similarly, if $\big(W, s_r, t_r\big)$ is a right bialgebroid, then there is at most one left bialgebroid structure on $\big(U, s^\ell, t^\ell\big)$ such that $\langle\ ,\ \rangle$ is a bialgebroid right pairing.

Remark 3.36. Let U be a left bialgebroid. Then the left bialgebroids $(U^*)^{op}_{coop}$ and ${}_*(U^{op}_{coop})$ are isomorphic: indeed, the right A^e–pairings between U and U^* and between ${}_*\big(U^{op}_{coop}\big)$ and U^{op}_{coop} give rise to the same formulas. Similarly, the left bialgebroids $(U_*)^{op}_{coop}$ and ${}^*\big(U^{op}_{coop}\big)$ are isomorphic.

3.6 The jet space(s) of a Lie-Rinehart algebra

3.37. Bialgebroids of jets: the right version. Let (L, A) be a Lie-Rinehart algebra, projective as an A–module. Consider its enveloping algebra $V^\ell(L)$ endowed with its trivial left bialgebroid structure and define the *right jet space* of the Lie-Rinehart algebra L as

$$J^r(L) \;:=\; V^\ell(L)^* \;=\; Hom_{(-,A)}\big(V^\ell(L)_\triangleleft \,,\, A_A\big)$$

From §3.31 we know that a multiplication in $J^r(L)$ can be given as follows:

$$\big(\phi\, \phi'\big)(u) \;=\; \phi\big(u_{(1)}\big)\, \phi'\big(u_{(2)}\big) \qquad \forall\ \phi, \phi' \in J^r(L)\,,\ \ u \in V^\ell(L) \quad , \quad \text{where} \quad \Delta(u) = u_{(1)} \otimes u_{(2)}$$

In particular, this multiplication is commutative in $J^r(L)$, and the unit element of $J^r(L)$ is the counit map of $V^\ell(L)$. Moreover, the map $\partial = \partial_{J^r(L)} : J^r(L) \longrightarrow A\,,\ \phi \mapsto \partial(\phi) := \phi\big(1_{V^\ell(L)}\big)$, will play the role of the counit map of $J^r(L)$; hereafter, we write $\mathfrak{J}_{J^r(L)} := Ker\big(\partial_{J^r(L)}\big)$. Moreover, we have a structure of A^e–ring on $J^r(L)$, whose source and target maps are given — for all $a \in A$, $u \in V^\ell(L)$ — by the formulas $\big(s^r(a)\big)(u) := \epsilon\big(u\, s^\ell(a)\big)$, $\big(t^r(a)\big)(u) := \epsilon\big(s^\ell(a)\, u\big) = a\, \epsilon(u)$.

In order to define a coproduct on $J^r(L) := V^\ell(L)^*$ we implement the construction in §3.31 again. Indeed, consider the (increasing) filtration $\big\{V^\ell_n(L)\big\}_{n\in\mathbb{N}}$ of $V^\ell(L)$ introduced in Remark 2.8; dually, define

$$J^r_n(L) \;:=\; Hom_{(-,A)}\big(V^\ell_n(L)_\triangleleft \,,\, A_A\big)\ , \quad J^r_{(n)}(L) \;:=\; \Big\{\, \phi \in Hom_{(-,A)}\big(V^\ell(L)_\triangleleft \,,\, A_A\big) \,\Big|\, \phi\big|_{V^\ell_n(L)} = 0 \,\Big\}$$

for all $n \in \mathbb{N}$, so that restriction yields natural monomorphisms $\rho_n : J^r(L) \Big/ J^r_{(n)}(L) \lhook\joinrel\longrightarrow J^r_n(L)$, and $\big\{J^r_{(n)}(L)\big\}_{n\in\mathbb{N}}$ is a decreasing filtration in $J^r(L)$.

By our overall assumptions, L is finite projective (as an A–module); it follows that every $V_n^\ell(L)$ and $J_n^r(L)$ are finite projective too, and each monomorphism ρ_n as above is actually an isomorphism. As $V^\ell(L) = \varinjlim_n V_n^\ell(L)$, one has

$$J^r(L) := Hom_A\big(V_n^\ell(L)_\triangleleft\,, A_A\big) = Hom_A\Big(\varinjlim_n V_n^\ell(L)_\triangleleft\,, A_A\Big) = \varprojlim_n \Big(Hom_A\big(V_n^\ell(L)_\triangleleft\,, A_A\big)\Big) = \varprojlim_n J_n^r(L)$$

i.e. $J^r(L)$ is the inverse limit of the $J_n^r(L)$'s. The decreasing filtration $\big\{J_{(n)}^r(L)\big\}_{n\in\mathbb{N}}$ defines a topology in $J^r(L)$, for which $J^r(L)$ itself is automatically complete, as $J^r(L) = \varprojlim_n J_n^r(L)$. It is worth remarking that $J_{(n)}^r(L) = \mathfrak{J}_{J^r(L)}^{n+1}\ \big(\forall\ n \in \mathbb{N}\big)$, so that $\big\{J_{(n)}^r(L)\big\}_{n\in\mathbb{N}}$ is — up to a shift — nothing but the $\mathfrak{J}_{J^r(L)}$–adic filtration of $J^r(L)$ — see, e.g., [6], §4.2.5.

Similarly, we can consider in $V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)$ the increasing filtration

$$\Big\{ \big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)_n := \textstyle\sum_{r+s=n} V_r^\ell(L)_\triangleleft \otimes_\blacktriangleright V_s^\ell(L) \Big\}_{n\in\mathbb{N}}$$

and dually define $Hom_{(-,A)}^{\otimes,n} := \Big\{ \psi \in Hom_{(-,A)}\Big(\big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)_\triangleleft\,, A_A\Big) \,\Big|\, \psi\big|_{(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L))_n} = 0 \Big\}$ for all $n \in \mathbb{N}$: this yields a decreasing filtration $\big\{Hom_{(-,A)}^{\otimes,n}\big\}_{n\in\mathbb{N}}$ inside $\big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)^* := Hom_{(-,A)}\Big(\big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)_\triangleleft\,, A_A\Big)$, and this filtration defines a topology in $\big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)^*$. By construction, together with the finite projectivity assumption on L, one has

$$\begin{aligned}\big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)^* &= \Big(\varinjlim_n \big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)_n\Big)^* = \\ &= \varprojlim_n \Big(\big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)_n\Big)^* = \varprojlim_n \big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)^* \Big/ Hom_{(-,A)}^{\otimes,n}\end{aligned}$$

so that $\big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)^*$ is complete for that topology.

Now consider the injective map $\vartheta : J^r(L)_\blacktriangleleft \otimes_\blacktriangleright J^r(L) = V^\ell(L)^*{}_\blacktriangleleft \otimes_\blacktriangleright V^\ell(L)^* \longrightarrow \big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)^*$ given (as in §3.31) by $\psi \otimes \psi' \mapsto \vartheta\big(\psi \otimes \psi'\big)\Big(u \otimes u' \mapsto \vartheta\big(\psi \otimes \psi'\big)\big(u \otimes u'\big) := \psi\big(u'\, t^\ell(\psi'(u))\big)\Big)$.

Consider in $J^r(L)_\blacktriangleleft \otimes_\blacktriangleright J^r(L)$ the $\mathfrak{J}_\otimes$–adic filtration, with $\mathfrak{J}_\otimes := \mathfrak{J}_{J^r(L)} \otimes J^r(L) + J^r(L) \otimes \mathfrak{J}_{J^r(L)} = Ker\big(\partial_{J^r(L)} \otimes \partial_{J^r(L)}\big)$, and the corresponding topology defined by it in $J^r(L)_\blacktriangleleft \otimes_\blacktriangleright J^r(L)$; then denote by $J^r(L)_\blacktriangleleft \widetilde{\otimes}_\blacktriangleright J^r(L)$ the $\mathfrak{J}_\otimes$–adic completion of $J^r(L)_\blacktriangleleft \otimes_\blacktriangleright J^r(L)$. One has $\vartheta\big(\mathfrak{J}_\otimes^{n+1}\big) \subseteq Hom_{(-,A)}^{\otimes,n}$ for all n, directly by definitions, so that ϑ is *continuous* (for the topologies considered above), hence it extends to a continuous monomorphism $\widetilde{\vartheta} : J^r(L)_\blacktriangleleft \widetilde{\otimes}_\blacktriangleright J^r(L) \longrightarrow \big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)^*$. By the finite projective assumption on L, one sees — through the Poincaré-Birkhoff-Witt Theorem, etc. — that all the natural maps $\vartheta_n : \big(J^r(L)_\blacktriangleleft \otimes_\blacktriangleright J^r(L)\big) \Big/ \mathfrak{J}_\otimes^{n+1} \longrightarrow \big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)^* \Big/ Hom_{(-,A)}^{\otimes,n}$ induced by ϑ (for each $n \in \mathbb{N}$) are all isomorphisms; as a direct consequence, the completion $\widetilde{\vartheta} : J^r(L)_\blacktriangleleft \widetilde{\otimes}_\blacktriangleright J^r(L) \longrightarrow \big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)^*$ is an isomorphism as well. Therefore, we can complete the procedure explained in §3.31 and define a coproduct $\Delta : J^r(L) \longrightarrow J^r(L)_\blacktriangleleft \widetilde{\otimes}_\blacktriangleright J^r(L)$ on $J^r(L) := V^\ell(L)^*$ as $\Delta := \widetilde{\vartheta}^{-1} \circ \nabla$ where $\nabla : J^r(L) := V^\ell(L)^* \longrightarrow \big(V^\ell(L)_\triangleleft \otimes_\blacktriangleright V^\ell(L)\big)^*$ is given by $\nabla : \psi \mapsto \nabla(\psi)\,\big(u \otimes u' \mapsto \psi\big(u'\,u\big)\big)$ for all $\psi \in J^r(L)$, $u, u' \in V^\ell(L)$. As an outcome, using Σ–notation to denote the coproduct of an element, we have (for all $\psi \in J^r(L)$, $u, u' \in V^\ell(L)$)

$$\Delta(\psi) = \psi_{(1)} \otimes \psi_{(2)} \in J^r(L)_\blacktriangleleft \widetilde{\otimes}_\blacktriangleright J^r(L) \qquad \text{with} \qquad \psi_{(1)}\big(u\, t^\ell\big(\psi_{(2)}(u')\big)\big) = \psi\big(u\, u'\big)$$

This Δ makes $J^r(L)$ into a (topological) A–coring, with counit map $\partial = \partial_{J^r(L)} : J^r(L) \longrightarrow A$ given as above by $\phi \mapsto \partial(\phi) := \phi(1_{V^\ell(L)})$. All in all, this makes $J^r(L)$ into a right bialgebroid over A.

Finally, we stress the point that the natural pairing (given by evaluation) between the left bialgebroid $V^\ell(L)$ and the right bialgebroid $J^r(L) := V^\ell(L)^*$ is a bialgebroid *right* pairing, in the sense of Definition 3.34.

As $J^r(L)$ is commutative, it is equal to $J^r(L)^{op}$ and can be considered as a left bialgebroid. Now introduce the map $S : J^r(L) \longrightarrow J^r(L)$ defined (with notation of §3.27) by

$$S(\phi)(u) := \epsilon_{V^\ell(L)}\big(u_+\, \phi(u_-)\big) \qquad\qquad \forall\ \phi \in J^r(L)\,, \quad \forall\ u \in V^\ell(L)$$

Theorem 3.38. *(see [23], [6], [29]) Let L be a finite projective Lie-Rinehart algebra. Then $\big(J^r(L)\,, J^r(L)^{op}\,, S\,\big)$ is a Hopf algebroid, whose antipode S is involutive.*

As a simple application of Theorem 3.25 we then find the following corollary:

Corollary 3.39. $\big(id_A\,, S\,\big)$ *is a right bialgebroid isomorphism from $J^r(L)$ to $J^r(L)_{coop}$.*

3.40. Bialgebroids of jets: the left version. Let again L be a Lie-Rinehart algebra over A, again projective as an A–module. Considering now L as a *right* A–module, we look at its *right* enveloping algebra $V^r(L)$, endowed with its natural structure of right bialgebroid (cf. §3.20).

We define the *left jet space* of the Lie-Rinehart algebra L as the left dual space

$$J^\ell(L) \;:=\; {}_*V^r(L) \;=\; Hom_{(-,A)}\big(V^r(L)_{\blacktriangleleft}\,, A_A\big)$$

Again from §3.31 we know that a multiplication in $J^\ell(L)$ is given by

$$\big(\psi\,\psi'\big)(u) \;=\; \psi\big(u_{(1)}\big)\,\psi'\big(u_{(2)}\big) \qquad \forall\; \phi, \phi' \in J^\ell(L)\;,\;\; u \in V^r(L) \quad , \quad \text{with} \quad \Delta(u) = u_{(1)} \otimes u_{(2)}$$

hence in particular this multiplication is commutative in $J^\ell(L)$, and the unit element of $J^\ell(L)$ is the counit map of $V^r(L)$. Moreover, the map $\epsilon = \epsilon_{J^\ell(L)} : J^\ell(L) \longrightarrow A\,,\;\psi \mapsto \epsilon(\psi) := \psi(1_{V^r(L)})$, works as counit map of $J^\ell(L)$; in the sequel we write $\mathfrak{J}_{J^\ell(L)} := Ker\,(\epsilon_{J^\ell(L)})$.

Still from §3.31 we get a structure of A^e–ring on $J^\ell(L)$, with source and target maps given by $\big(s^\ell(a)\big)(u) := \partial\big(\,a\,u\big)\;,\;\;\big(t^\ell(a)\,\big)(u) := \partial(u)\,a\;$, — for all $a \in A\,,\;u \in V^r(L)$.

Finally, mimicking the analysis in §3.37 — realizing $J^\ell(L)$ as an inverse limit, since $V^r(L)$ is a direct limit — we can also endow $J^\ell(L)$ with a suitable coproduct (following the recipe given in §3.31). Eventually, all this makes $J^\ell(L)$ into a (topological) left bialgebroid.

At last, notice that the natural (evaluation) pairing between the right bialgebroid $V^r(L)$ and the left bialgebroid $J^\ell(L) := {}_*V^r(L)$ is a bialgebroid *right* pairing, in the sense of Definition 3.34.

Remark 3.41. As $J^\ell(L) = J^r(L^{op})^{op}_{coop}$, the triple $\big(J^\ell(L)\,, J^\ell(L)^{op}\,, S_{J^r(L^{op})}\big)$ is a Hopf algebroid and $S_{J^r(L^{op})}$ induces an isomorphism of left bialgebroids from $J^\ell(L)$ to $J^\ell(L)_{coop}$.

3.42. Further jet spaces, and comparison. Besides the jet spaces $J^r(L)$ and $J^\ell(L)$, we can consider also further possibilities.

First, we can consider the *left dual* ${}^rJ(L) := V^\ell(L)_*$ of $V^\ell(L)$, which again is a right bialgebroid. Indeed, it is equal to $\big(V^\ell(L)^*\big)_{coop}$, i.e. to $J^r(L)_{coop}$. Furthermore, the natural pairing (given by evaluation) between the left bialgebroid $V^\ell(L)$ and the right bialgebroid $V^\ell(L)_*$ is a bialgebroid *left* pairing, in the sense of Definition 3.34 — in contrast with the case of $V^\ell(L)^* =: J^r(L)$.

Second, we can consider the *left dual* ${}^\ell J(L) := {}^*V^r(L)$ of $V^r(L)$, which again is a left bialgebroid. Indeed, it is equal to $\big({}_*V^r(L)\big)_{coop}$, i.e. to $J^\ell(L)_{coop}$. Moreover, the natural pairing (given by evaluation) between the right bialgebroid $V^r(L)$ and the left bialgebroid ${}^*V^r(L)$ is a bialgebroid *left* pairing, in the sense of Definition 3.34 — in contrast with the case of ${}_*V^r(L) =: J^\ell(L)$.

One can also establish some relevant links among all these bialgebroids of jets.

We have already noticed that that $J^\ell(L) \cong J^r(L^{op})^{op}_{coop} \cong J^r(L^{op})_{coop}$ (cf. also Remark 3.36).

Finally, note that all in all we considered *four* different types of "jet bialgebroids", namely

$$V^\ell(L)^* =: J^r(L) \quad , \qquad {}_*V^r(L) =: J^\ell(L) \quad , \qquad V^\ell(L)_* =: {}^rJ(L) \quad , \qquad {}^*V^r(L) =: {}^\ell J(L)$$

However, we saw above that $V^\ell(L)_* = \big(V^\ell(L)^*\big)_{coop} = J^r(L)_{coop} \cong J^r\big(L\big)$ (corollary 3.39) and ${}^*V^r(L) = \big({}_*V^r(L)\big)_{coop} = J^\ell(L)_{coop} \cong J^\ell\big(L\big)$ (remark 3.41) . Therefore, in the end, jet bialgebroids of type $J^r(L)$ or $J^\ell(L)$ are enough to consider all possible situations, for every possible L .

We introduce now suitable "topological duals" for jet bialgebroids $J^r(L)$ and $J^\ell(L)$:

Definition 3.43. *Let $K = J^r(L)$ be a right jet bialgebroid, for some Lie-Rinehart algebra L. Set $I := \big\{ \lambda \in J^r(L) \,\big|\, \langle 1 \,, \lambda \rangle = 0 \big\} = \mathrm{Ker}\big(\partial_{J^r(L)}\big)$ — which is a (two-sided) ideal in $J^r(L)$, as one easily sees. Then we introduce the following subsets of ${}^*K$ and ${}_*K$:*

$$\begin{aligned} {}^\star K \; &:= \; \big\{ u \in {}^*K \,\big|\, u\big(I^n\big) = 0 \;\; \forall \;\; n \gg 0 \big\} \\ &= \; \big\{ u : K \longrightarrow A \,\big|\, u\big(\lambda' + \lambda''\big) = u\big(\lambda'\big) + u\big(\lambda''\big) \,,\; u\big(\lambda \, s^r_*(a)\big) = u(\lambda)\, a \,,\; u\big(I^n\big) = 0 \;\; \forall \;\; n \gg 0 \big\} \end{aligned}$$

$$\begin{aligned} {}_\star K \; &:= \; \big\{ u \in {}_*K \,\big|\, u\big(I^n\big) = 0 \;\; \forall \;\; n \gg 0 \big\} \\ &= \; \big\{ u : K \longrightarrow A \,\big|\, u\big(\lambda' + \lambda''\big) = u\big(\lambda'\big) + u\big(\lambda''\big) \,,\; u\big(\lambda \, t^*_r(a)\big) = a \, u(\lambda) \,,\; u\big(I^n\big) = 0 \;\; \forall \;\; n \gg 0 \big\} \end{aligned}$$

Similarly, if $K := J^\ell(L)$ is a left jet bialgebroid, and $I := \mathrm{Ker}\big(\partial_{J^\ell(L)}\big)$, we define

$$K^\star := \big\{ u \in K^* \,\big|\, u\big(I^n\big) = 0 \;\; \forall \;\; n \gg 0 \big\} \quad , \qquad K_\star := \big\{ u \in K_* \,\big|\, u\big(I^n\big) = 0 \;\; \forall \;\; n \gg 0 \big\}$$

It should be clear by the very definition that, in the first case, ${}^\star K$, resp. ${}_\star K$, is nothing but the subset of those functions in ${}^*K$, resp. in ${}_*K$, which are *continuous* with respect to the I–adic topology in ${}^*K$, resp. in ${}_*K$, and the discrete topology in A. Similarly for $K^\star$ and $K_\star$ in the second case. The key reason of interest for these objects lies in the following, well-known result:

Theorem 3.44. *Let L be a Lie-Rinehart algebra which, as an A–module, is finite projective.*

(a) Consider the right bialgebroid $J^r(L) := V^\ell(L)^*$. Then ${}_\star J^r(L)$, as a left bialgebroid, is isomorphic to $V^\ell(L)$: more precisely, the canonical map $V^\ell(L) \longrightarrow {}_\star\big(V^\ell(L)^*\big) = {}_\star J^r(L)$ given by evaluation is an isomorphism of left bialgebroids.

Similarly, replacing $J^r(L) := V^\ell(L)^*$ with the right bialgebroid $V^\ell(L)_*$ one has a corresponding isomorphism of left bialgebroids $V^\ell(L) \longrightarrow {}^\star\big(V^\ell(L)_*\big)$ still given by evaluation.

(b) Consider the left bialgebroid $J^\ell(L) := {}_*V^r(L)$. Then $J^\ell(L)^\star$, as a right bialgebroid, is isomorphic to $V^r(L)$: more precisely, the canonical map $V^r(L) \longrightarrow \big({}_*V^r(L)\big)^\star = J^\ell(L)^\star$ given by evaluation is an isomorphism of right bialgebroids.

Similarly, replacing $J^\ell(L) := {}_*V^r(L)$ with the left bialgebroid ${}^*V^r(L)$ one has a corresponding isomorphism of right bialgebroids $V^r(L) \longrightarrow \big({}^*V^r(L)\big)_\star$ still given by evaluation.

Remark 3.45. As the antipode $S_{J^r(L)}$ provides an isomorphism" between $V^\ell(L)_*$ and $V^\ell(L)^*$, we have an isomorphism ${}^\star\big(\big(V^\ell(L)^*\big) \cong V^\ell(L)$. Similarly, we have an analogous isomorphism $\big({}_*V^r(L)\big)_\star \cong V^r(L)$.

Remark 3.46. Let L be a finite projective Lie-Rinehart algebra and Q be a (finite projective) A–module such that $L \oplus Q = F$ is a finite rank free A–module. We resume notation of §2.12: so we take an A–basis $\{b_1, \dots, b_n\}$ of F, and we set $Y = k\, b_1 \oplus \cdots \oplus k\, b_n$, so that $F = A \otimes_k Y$; moreover, $L_Q = L \oplus (A \otimes_k Z)$ is a Lie-Rinehart algebra with $Z = Y \oplus Y \oplus Y \oplus \cdots$. One has $S(Y)^{\otimes \infty} := S(Z) = S(Y) \otimes S(Y) \otimes \cdots$ (recall that elements of an infinite tensor product of algebras are sum of tensor products with only finitely many factors different from 1). For $T \in \{Y, Z\}$, we let $\epsilon : S(T) \longrightarrow k$ — the counit map of $S(T)$ — be the unique k-algebra morphism given by $S(t) := 0$ for $t \in T$, and we set $S(T)^+ := Ker(\epsilon)$.

For any n, denote by $J^r_{f,n}(L_Q) \equiv V^\ell(L_Q)^*_{f,n}$ the subset of $V^\ell(L_Q)^*$ whose elements are all the $\lambda \in V^\ell(L_Q)^*$ such that $\lambda\big|_{V^\ell(L) \otimes S(Y)^{\otimes n} \otimes S(Z)^+} = 0$ and set $J^r_f(L_Q) \equiv V^\ell(L_Q)^*_f := \bigcup_{n \in \mathbb{N}} J^r_{f,n}(L_Q)$. Then one can describe $J^r_{f,n}(L_Q)$ as $J^r_{f,n}(L_Q) = J^r(L) \,\widetilde{\otimes}\, \widetilde{S}(Y^*)^{\widetilde{\otimes}^n} \,\widetilde{\otimes}\, 1 \,\widetilde{\otimes}\, 1 \,\widetilde{\otimes} \cdots$, where $\widetilde{S}\big(Y^*\big)$ denotes the completion of $S\big(Y^*\big)$ with respect to the weak topology; so we have also

$$J^r_f(L_Q) \;\cong\; \textstyle\sum_{n \in \mathbb{N}} J^r_{f,n}(L_Q) \;=\; \sum_{n \in \mathbb{N}} J^r(L) \,\widetilde{\otimes}\, \widetilde{S}(Y^*)^{\widetilde{\otimes}^n} \,\widetilde{\otimes}\, 1 \,\widetilde{\otimes}\, 1 \,\widetilde{\otimes} \cdots$$

This $J^r_f(L_Q)$ is a sub-bialgebroid of $J^r(L_Q)$: indeed, its right bialgebroid structure is described by

$$s_r : A \longrightarrow J^r_f(L_Q), \quad a \mapsto s_r(a) \otimes 1, \qquad t_r : A \longrightarrow J^r_f(L_Q)\,, \quad a \mapsto t_r(a) \otimes 1$$
$$\Delta(\phi \otimes s) := \big(\phi_{(1)} \otimes s_{(1)}\big) \otimes \big(\phi_{(2)} \otimes s_{(2)}\big) \quad \text{if} \quad \Delta_{J^r(L)}(\phi) = \phi_{(1)} \otimes \phi_{(2)}\,, \ \ \Delta(s) = s_{(1)} \otimes s_{(2)}$$
$$\partial(\phi \otimes s) := \partial(\phi)\epsilon(s)\,, \qquad (\phi \otimes s)\big(\phi' \otimes s'\big) := \phi\,\phi' \otimes s\,s'$$

for all $a \in A$, $\phi, \phi' \in J^r(L)$, $s, s' \in \sum_{n\in\mathbb{N}} \widetilde{S}(Y^*)^{\widetilde{\otimes}^n} \,\widetilde{\otimes}\, 1 \,\widetilde{\otimes}\, 1 \,\widetilde{\otimes} \cdots \ (n \in \mathbb{N})$.

Finally, denote by ${}^{\star_f}J^r_f(L_Q)$ the subset of ${}^\star J^r_f(L_Q)$ consisting of all $\delta \in {}^\star J^r_f(L_Q)$ such that there exists $n \in \mathbb{N}$ such that $\delta\big|_{J^r(L)\,\widetilde{\otimes}\, S(Y^*)^{\widetilde{\otimes} n}\,\widetilde{\otimes}\, S(Z^*)^+} = 0$. It is easy to check that ${}^{\star_f}J^r_f(L_Q)$ is a left sub-bialgebroid of ${}^\star J^r_f(L_Q)$, isomorphic to $V^\ell(L_Q)$.

4 Quantum groupoids

In this section we introduce quantum groupoids — i.e. topological bialgebroids which are formal deformations of those attached to Lie-Rinehart algebras. Then we show that taking suitable "(linear) duals" we get an antiequivalence among the categories of objects of these two types.

4.1. The h–adic topology. If V is any $k[[h]]$–module, it is endowed with the following decreasing filtration: $V \supseteq h\,V \supseteq h^2V \supseteq \cdots \supseteq h^nV \supseteq h^{n+1}V \supseteq \cdots \supseteq$. Then V is also endowed with the h–adic topology, which is the unique one for which V is a topological $k[[h]]$–module in which $\big\{ h^mV \big\}_{m\in\mathbb{N}}$ is a basis of neighborhoods of 0. Indeed, V is then a pseudo-metric space, as the h–adic topology is the one induced by the following pseudo-metric:

$$d(x,y) := \|x-y\| = 2^{-m} \quad \text{with} \quad m := \sup\big\{ s \in \mathbb{N} \,\big|\, (x-y) \in h^s\,V \big\} \qquad \forall\ x,y \in V$$

The topological space V is Hausdorff if and only if the pseudo-metric d is a metric: in turn, this occurs if and only if $\bigcap_{m\in\mathbb{N}} h^mV = \{0\}$, which means that each point in V forms a closed subset.

4.1 Quantum groupoids

In this subsection we introduce the notion of "quantum groupoids": these are special "quantum bialgebroids", namely (topological) bialgebroids which are formal deformations of those of type $V^\ell(L)$, $V^r(L)$, $J^r(L)$ or $J^\ell(L)$. We begin with the ones associated with the first two cases:

Definition 4.2. *A left quantum universal enveloping algebroid (=LQUEAd) is a topological left bialgebroid $\big(H_h\,, A_h\,, s^\ell_h\,, t^\ell_h\,, m_h\,, \Delta_h\,, \epsilon_h\big)$ over a topological $k[[h]]$–algebra A_h such that:*

(i) H_h is isomorphic to $V^\ell(L)[[h]]$ as a topological $k[[h]]$–module, with identity $1_{V^\ell(L)}$, and A_h is isomorphic to $A[[h]]$ as a topological $k[[h]]$–module, with identity 1_A, where $V^\ell(L)$ is the left bialgebroid associated with some Lie-Rinehart A–algebra L, as in §3.12;

(ii) $H_h\big/h\,H_h \cong V^\ell(L)[[h]]\big/h\,V^\ell(L)[[h]]$ is isomorphic to $V^\ell(L)$ as a left A–bialgebroid via the isomorphism $A_h\big/h\,A_h \cong A[[h]]\big/h\,A[[h]] \cong A$ induced from the isomorphism $A_h \cong A[[h]]$ mentioned in (i);

(iii) denote by $H_{h\,\triangleleft}\underset{A_h}{\widehat{\otimes}}{}_{\triangleright}H_h$ the completion of $H_{h\,\triangleleft}\underset{A_h}{\otimes}{}_{\triangleright}H_h$ with respect to the h–adic topology, and define the (h–adically completed) Takeuchi product *as*

$$H_{h\,\triangleleft}\underset{A_h}{\widehat{\times}}{}_{\triangleright}H_h := \Big\{ \textstyle\sum_i u_i \otimes u'_i \in H_{h\,\triangleleft}\underset{A_h}{\widehat{\otimes}}{}_{\triangleright}H_h \,\Big|\, \sum_i (a \blacktriangleright u_i) \otimes u'_i = \sum_i u_i \otimes \big(u'_i \blacktriangleleft a\big) \Big\}$$
$$= \Big\{ \textstyle\sum_i u_i \otimes u'_i \in H_{h\,\triangleleft}\underset{A_h}{\widehat{\otimes}}{}_{\triangleright}H_h \,\Big|\, \sum_i \big(u_i\,t^\ell_h(a)\big) \otimes u'_i = \sum_i u_i \otimes \big(u'_i\,s^\ell_h(a)\big) \Big\}$$

then the coproduct Δ_h of H_h takes values in $H_{h\,\triangleleft}\underset{A_h}{\widehat{\otimes}}{}_{\triangleright}H_h$.

In this setting, we shall say that H_h is a quantization, or a quantum deformation, of $V^\ell(L)$; we shall resume it in short using notation $V^\ell(L)_h := H_h$.

In a similar, parallel way, we define the notion of right quantum universal enveloping algebroid (=RQUEAd) as well, just replacing "left" with "right" and $V^\ell(L)$ with $V^r(L)$, cf. §3.20.

We define morphisms among left, resp. right, quantum universal enveloping algebroids like in Definitions 3.6 and 3.18; moreover, we use notation (LQUEAd), *resp.* RQUEAd, *to denote the category of all left, resp. right, quantum universal enveloping algebroids. If A_h is a fixed ground $k[[h]]$–algebra, then we write* $(\text{LQUEAd})_{A_h}$, *resp.* $(\text{RQUEAd})_{A_h}$, *to denote the subcategory — in* (LQUEAd), *resp.* (RQUEAd) *— whose objects are all the left, resp. right, quantum universal enveloping algebroids over A_h , and whose morphisms are selected as in Definitions 3.6 and 3.18.*

Remarks 4.3.

(a) U is a LQUEAd if and only if U^{op}_{coop} is a RQUEAd.

(b) U is a LQUEAd if and only if U^{op} is a RQUEAd.

(c) If $\big(V^\ell(L)_h, A_h, s^\ell_h, t^\ell_h, m_h, \Delta_h, \epsilon_h\big)$ is any LQUEAd, then A_h is a deformation of the algebra A : then, as usual, one can define a Poisson structure on the base algebra A as follows:

$$\{f,g\} \;:=\; \frac{f' *_h g' - g' *_h f'}{h} \quad \bmod h\, A_h \qquad\qquad \forall\;\, f,g \in A$$

where $f' \in A_h$ and $g' \in A_h$ are such that $f' \bmod h\, A_h = f$ and $g' \bmod h\, A_h = g$. The same observations makes sense if one has to do with a RQUEAd $V^r(L)_h$.

(d) The definitions given so far make sense for any Lie-Rinehart algebra L . However, *from now on we shall assume in addition that L , as an A–module, is finitely generated projective.*

The following theorem is proved in [34] (Theorem 5.16) :

Theorem 4.4. *Let $\big(V^\ell(L)_h, A_h, s^\ell_h, t^\ell_h, m_h, \Delta_h, \epsilon_h\big)$ be a LQUEAd. Define*

$$\delta(a) \;:=\; \frac{t^\ell_h(a') - s^\ell_h(a')}{h} \quad \bmod h\, V^\ell(L)_h \qquad\qquad \forall\;\, a \in A$$

$$\delta(X) \;:=\; \Delta^{[1]}(X)_{2,1} - \Delta^{[1]}(X) \;\in\; V^\ell(L) \underset{A}{\otimes} V^\ell(L) \qquad\qquad \forall\;\, X \in L$$

$$\textit{with} \qquad \Delta^{[1]}(X) \;:=\; \frac{\Delta_h(X') - X' \otimes 1 - 1 \otimes X'}{h} \quad \bmod h \Big(V^\ell(L)_h \underset{A_h}{\widehat{\otimes}} V^\ell(L)_h\Big)$$

$$\textit{and} \qquad \Delta^{[1]}(X)_{2,1} \;:=\; \textstyle\sum_{[X]} X_{[2]} \otimes X_{[1]} \qquad \textit{if} \qquad \Delta^{[1]}(X) \;=\; \textstyle\sum_{[X]} X_{[1]} \otimes X_{[2]}$$

where $X' \in V^\ell(L)_h$ is any lift of X (i.e. $X' \bmod h\, V^\ell(L)_h = X$) and $a' \in A_h$ is any lift of a .

Then $\delta(a) \in L$ and $\delta(X) \in \bigwedge^2_A L$; this gives to L the structure of a Lie-Rinehart bialgebra. Moreover, the Poisson structure on A induced by this Lie-Rinehart bialgebra (cf. Remarks 2.21(c)) coincides with the one obtained as the classical limit of the base $*$–algebra A_h (cf. Remarks 4.3(c)).

Remark 4.5. In the above statement, we took formulas opposite to those in [34]: indeed, this allows us to deduce the very last claim.

Example 4.6. (cf. [34]) Let P be a smooth manifold, D the algebra of global differential operators on P and $A := \mathcal{C}^\infty(P)$. Let $D[[h]]$ be the trivial deformation of D . Let

$$\mathcal{F} \;=\; 1 \otimes 1 + h\, B_1 + \cdots \in \big(D \otimes_A D\big)[[h]] \;\cong\; D[[h]]\, \widehat{\otimes}_{A[[h]]} D[[h]]$$

be a formal series of bidifferential operators. It is easy to see that $\mathcal{F}$ is a twistor — cf. Definition 3.10 — iff the multiplication on $A[[h]]$ defined by

$$f *_h g \;=\; \mathcal{F}(f,g) \qquad\qquad \forall\;\, f,g \in A[[h]]$$

is associative with identity being the constant function 1, i. e. iff $*_h$ is a *star product* on P. The twisted bialgebroid structure on $D_h := D[[h]]$ can be easily described: $A_h = A[[h]]$ has the star product defined above, $s_h^\ell : A_h \longrightarrow D_h$ and $t_h^\ell : A_h \longrightarrow D_h$ are given by

$$s_h^\ell(f)\,g \,=\, f *_h g, \qquad t_h^\ell(f)\,g \,=\, g *_h f \qquad\qquad \forall\ f,g \in A$$

The coproduct $\Delta_h : D_h \longrightarrow D_h \widehat{\otimes}_{A_h} D_h$ is $\Delta_h(x) := \mathcal{F}^{\#-1}\big(\Delta(x)\cdot\mathcal{F}\big)$ for all $x \in D_h$.

In Section 7 later on we shall explicitly provide a specific example of this kind.

Theorem 4.4 has a natural counterpart for RQUEAd's as follows:

Theorem 4.7. *Let $\big(V^r(L)_h\,, A_h\,, s_h^r\,, t_h^r\,, m_h\,, \Delta_h\,, \epsilon_h\big)$ be a RQUEAd. Define*

$$\begin{aligned} \delta(a) \;&:=\; \frac{s_h^r(a) - t_h^r(a)}{h} \quad \mod h\,V^r(L)_h && \forall\ a \in A \\ \delta(X) \;&:=\; \Delta^{[1]}(X)_{2,1} - \Delta^{[1]}(X) \;\in\; V^r(L) \underset{A}{\otimes} V^r(L) && \forall\ X \in L \\ \textit{with} \qquad \Delta^{[1]}(X) \;&:=\; \frac{\Delta_h\big(X'\big) - X' \otimes 1 - 1 \otimes X'}{h} \quad \mod h\,\Big(V^r(L)_h \underset{A_h}{\widehat{\otimes}} V^r(L)_h\Big) \\ \textit{and} \qquad \Delta^{[1]}(X)_{2,1} \;&:=\; \textstyle\sum_{[X]} X_{[2]} \otimes X_{[1]} \qquad \textit{if} \qquad \Delta^{[1]}(X) \,=\, \textstyle\sum_{[X]} X_{[1]} \otimes X_{[2]} \end{aligned}$$

where $X' \in V^r(L)_h$ is any lift of X (i.e. $X' \mod h\,V^r(L)_h = X$) and $a' \in A_h$ is any lift of a.

Then $\delta(a) \in L$ and $\delta(X) \in \bigwedge_A^2 L$; this gives to L the structure of a Lie-Rinehart bialgebra. Moreover, the induced Poisson structure of this Lie bialgebroid on the base algebra A coincides with the opposite to the one obtained as the classical limit of the base $*$–algebra A_h (cf. Remarks 4.3(c)).

Remark 4.8. The previous result can be proved duplicating that of Theorem 4.4 in [34]. Otherwise, one can deduce Theorem 4.7 from Theorem 4.4 applied to $V^\ell(L)_h := V^r(L)_h^{op}$, which is a LQUEAd — cf. Remarks 4.3*(b)*. In particular, the Lie-Rinehart bialgebra structure induced on L by the RQUEAd $V^r(L)_h$ is opposite to that induced by the LQUEAd $V^\ell(L)_h := V^r(L)_h^{op}$.

We introduce now the second pair of "quantum bialgebroids" we shall deal with, namely quantizations of jet bialgebroids:

Definition 4.9. *A right quantum formal series algebroid (=RQFSAd) is a topological right bialgebroid $\big(K_h\,, A_h\,, s_h^r\,, t_h^r\,, m_h\,, \Delta_h\,, \partial_h\big)$ over a topological $k[[h]]$–algebra A_h such that:*

(i) K_h is isomorphic to $J^r(L)[[h]]$ as a topological $k[[h]]$–module, with identity $1_{J^r(L)}$, and A_h is isomorphic to $A[[h]]$ as a topological $k[[h]]$–module, with identity 1_A, where $J^r(L)$ is the right bialgebroid of jets associated with some finite projective Lie-Rinehart A–algebra L as in Subs. 3.6;

(ii) $K_h\big/h\,K_h$ is isomorphic to $J^r(L)$ as a right A–bialgebroid via the isomorphism $A_h\big/h\,A_h \cong A[[h]]\big/h\,A[[h]] \cong A$ induced from the isomorphism $A_h \cong A[[h]]$ mentioned in (i);

(iii) letting $I_h := \big\{\psi \in K_h \,\big|\, \partial(\psi) \in h\,A_h\big\}$ — which is easily seen to be a two-sided ideal in K_h — we have that K_h is complete in the I_h–adic topology.

(iv) let $I_h^\otimes := \big(I_h \otimes_{A_h} K_h + K_h \otimes_{A_h} I_h\big)$, denote by $K_h {}_\blacktriangleleft \underset{A_h}{\widetilde{\otimes}} {}_\blacktriangleright K_h$ the $I_h^\otimes$–adic completion of $K_h {}_\blacktriangleleft \underset{A_h}{\otimes} {}_\blacktriangleright K_h$; also, define the ($I_h^\otimes$–adically completed) Takeuchi product *as*

$$\begin{aligned} K_h {}_\blacktriangleleft \underset{A_h}{\widetilde{\times}} {}_\blacktriangleright K_h \;&:=\; \Big\{ \textstyle\sum_i u_i \otimes u_i' \in K_h {}_\blacktriangleleft \underset{A_h}{\widetilde{\otimes}} {}_\blacktriangleright K_h \,\Big|\, \textstyle\sum_i (a \triangleright u_i) \otimes u_i' = \sum_i u_i \otimes \big(u_i' \triangleleft a\big) \Big\} \\ &=\; \Big\{ \textstyle\sum_i u_i \otimes u_i' \in K_h {}_\blacktriangleleft \underset{A_h}{\widetilde{\otimes}} {}_\blacktriangleright K_h \,\Big|\, \textstyle\sum_i \big(s_r(a) u_i\big) \otimes u_i' = \sum_i u_i \otimes \big(t^r(a) u_i'\big) \Big\} \end{aligned}$$

Then the coproduct Δ_h of K_h takes values in $K_h {}_\blacktriangleleft \underset{A_h}{\widetilde{\times}} {}_\blacktriangleright K_h$.

In this setting, we shall say that K_h is a quantization, or a quantum deformation, of $J^r(L)$; we shall resume it in short using notation $J^r(L)_h := K_h$.

In a similar, parallel way, we define the notion of left formal series algebroid (=LQFSAd) as well, just replacing "left" with "right" and $J^r(L)$ with $J^\ell(L)$.

We define morphisms among right, resp. left, quantum formal series algebroids like in Definitions 3.18 and 3.6; moreover, we use notation (RQFSAd)*, resp.* LQFSAd*, to denote the category of all right, resp. left, quantum formal series algebroids. If A_h is a fixed ground (topological) $k[[h]]$–algebra, then we write* $(\mathrm{RQFSAd})_{A_h}$*, resp.* $(\mathrm{LQFSAd})_{A_h}$*, to denote the subcategory — in* (RQFSAd)*, resp.* (LQFSAd) *— whose objects are all the right, resp. left, quantum formal series algebroids over A_h, and whose morphisms are selected as in Definitions 3.18 and 3.6.*

Remarks 4.10.

(a) From the analysis in §3.42 we can argue that one could define a RQFSAd also as a deformation of the right bialgebroid $V^\ell(L)_*$, and a LQFSAd as a deformation of the left bialgebroid ${}^*V^r(L)$. On the other hand, the very conclusion of §3.42 itself also tells us that it is enough to consider the notions of RQFSAd and LQFSAd introduced in Definition 4.9 above.

(b) K_h is a LQFSAd if and only if $\big(K_h^{\,op}\big)_{coop}$ is a RQFSAd. Similarly, K_h is a LQFSAd if and only if $K_h^{\,op}$ is a RQFSAd.

4.11. Liftings in a (R/L)QFSAd. Let L be a Lie-Rinehart algebra which is finite projective (as an A–module). Set $\mathfrak{J}_{J^r(L)} := Ker\big(\partial_{J^r(L)}\big)$, a two-sided ideal — the "augmentation ideal" — in $J^r(L)$: then $\mathfrak{J}_{J^r(L)} \big/ \mathfrak{J}_{J^r(L)}^2 \cong L^*$ as A–modules, by definitions. For any $\Phi \in L^*$, we shall call *lift of Φ in $J^r(L)$* any $\phi \in \mathfrak{J}_{J^r(L)}$ such that $\phi \mod \mathfrak{J}_{J^r(L)}^2 = \Phi$ via the above isomorphism.

Now let $K_h = J^r(L)_h$ be a RQFSAd, namely a deformation of $J^r(L)$. For any $\Phi \in L^*$ as above, we shall call *lift of Φ in $J^r(L)_h$* any $\phi' \in J^r(L)_h$ such that $\phi' \mod h\, J^r(L)_h$ is a lift of Φ in $J^r(L)$. In short, this amounts to saying that $\Phi \equiv \big(\phi' \mod h\, J^r(L)_h\big) \mod \mathfrak{J}_{J^r(L)}^2$.

Also, if $a \in A$ we call *lift of a in A_h* any $a' \in A_h$ such that $a' \mod h\, A_h = a$.

Changing "right" into "left", similar remarks and terminology apply for defining "lifts" of elements of $J^\ell(L)$ in some associated LQFSAd, say $J^\ell(L)_h$.

Next result introduces semiclassical structures induced on a Lie-Rinehart algebra L by quantizations of the form $J^r(L)$ or $J^\ell(L)$. Indeed, this is the dual counterpart of Theorem 4.4.

Theorem 4.12. *Let $J^r(L)_h$ be a RQFSAd, namely a deformation of $J^r(L)$ as above. Then L inherits from this quantization a structure of Lie-Rinehart bialgebra, namely the unique one such that the Lie bracket and the anchor map of L^* are given (notation as above) by*

$$[\Phi\,,\Psi\,] \;:=\; \left(\frac{\phi'\,\psi' - \psi'\,\phi'}{h} \mod h\, J^r(L)_h \right) \mod \mathfrak{J}_{J^r(L)}^2$$

$$\omega(\Phi)(a) \;:=\; \left(\frac{\phi'\, r\big(a'\big) - r\big(a'\big)\, \phi'}{h} \mod h\, J^r(L)_h \right) \mod \mathfrak{J}_{J^r(L)} = \partial \left(\frac{\phi'\, r\big(a'\big) - r\big(a'\big)\, \phi'}{h} \right) \mod hA_h$$

for all $\Phi, \Psi \in L^*$ and $a \in A$, where ϕ' and ψ' are liftings in $J^r(L)_h$ of Φ and Ψ respectively, a' is a lifting in A_h of $a \in A$, and finally $r\big(a'\big)$ stands for either $s_h^r\big(a'\big)$ or $t_h^r\big(a'\big)$.

Proof. First of all, it is easy to see that the maps $[\;,\;]$ and ω as given in the statement are well-defined indeed, i.e. they do not depend on the choice of liftings, nor of the choice of either of $s_h^r\big(a'\big)$ or $t_h^r\big(a'\big)$ as playing the role of $r\big(a'\big)$. Moreover, by construction we have $[\Phi\,,\Psi\,] \in \mathfrak{J}_{J^r(L)} \big/ \mathfrak{J}_{J^r(L)}^2 \cong L^*$. Also, again by construction we have $\omega(\Phi)(a) \in J^r(L) \big/ \mathfrak{J}_{J^r(L)}$; now the latter space identifies with $\epsilon\big(J^r(L)\big) = A$, thus $\omega(\Phi)(a) \in A$ via these identifications, so that $\omega(\Phi)$ is a k–linear endomorphism of A.

Now, the definition of both $[\;,\;]$ and ω is made via a commutator in $J^r(L)_h$. As the commutator — in any associative k–algebra — is a k–bilinear Lie bracket and satisfies the Leibniz identity (involving the associative product), one can easily argue at once from definitions that L^* with the given bracket and anchor map is indeed a Lie-Rinehart algebra (over A).

What is more demanding is to prove that with this structure the pair $\big(L\,,L^*\big)$ of Lie-Rinehart A–algebras fulfills all constraints to be a Lie-Rinehart bialgebra. Indeed, we shall *not* provide a *direct* proof for that: instead, we have recourse to a duality argument, using the notions and results of Subsec. 5.1 later on. Indeed, there we shall see that ${}_\star J^r(L)_h$ is a LQUEAd, hence by Theorem 4.4 we know that $\big(L\,,L^*\big)$ is a Lie-Rinehart bialgebra. □

The analog of Theorem 4.12 for LQFSAd's (with essentially the same proof) is the following:

Theorem 4.13. *Let $J^\ell(L)_h$ be a LQFSAd, namely a deformation of $J^\ell(L)$. Then L inherits from this quantization a structure of Lie-Rinehart bialgebra, namely the unique one for which the Lie bracket and the anchor map of L^* are given (notation as above) by*

$$[\Phi\,,\Psi\,] \;:=\; \left(\frac{\phi'\,\psi' - \psi'\,\phi'}{h} \quad \mathrm{mod}\; h\,J^\ell(L)_h\right) \quad \mathrm{mod}\; \mathfrak{J}^{\,2}_{J^r(L)}$$

$$\omega(\Phi)(a) \;:=\; \left(\frac{\phi'\,r\big(a'\big) - r\big(a'\big)\,\phi'}{h} \quad \mathrm{mod}\; h\,J^\ell(L)_h\right) \quad \mathrm{mod}\; \mathfrak{J}_{J^r(L)}$$

for all $\Phi\,,\Psi \in L^*$ and $a \in A$, where ϕ' and ψ' are liftings in $J^\ell(L)_h$ of Φ and Ψ respectively, a' is a lifting in A_h of $a \in A$, and finally $r\big(a'\big)$ stands for either $s^r_h\big(a'\big)$ or $t^r_h\big(a'\big)$.

Remark 4.14. The result above can be proved like its analogue for RQFSAd's, i.e. Theorem 4.12. Otherwise, one can get the former from the latter applied to $J^r(L)_h := J^\ell(L)_h^{op}$, which is a RQFSAd — cf. Remarks 4.10*(c)*. In particular, the Lie-Rinehart bialgebra structure induced on L by the LQFSAd $J^\ell(L)_h$ is opposite-coopposite to that induced by the RQFSAd $J^r(L)_h := J^\ell(L)_h^{op}$.

4.2 Extending quantizations: from the finite projective to the free case

Let L be a Lie-Rinehart algebra over A which is finite projective as an A–module. With the procedure presented in §2.12, we can find a projective A–module Q (a complement of L in a finite free A–module) and use it to build a new Lie-Rinehart algebra $L_Q := L \oplus \big(Q \oplus L \oplus Q \oplus L \oplus \cdots\big)$, which as an A–module is free. Then we fix an A–basis $\{b_1,\dots,b_n\}$ of F and set $Y := \overset{n}{\underset{i=1}{\oplus}}\, k\,b_i$, so $F = A\otimes_k Y$. Set $T := \mathbb{N}\times\{1,\dots,n\}$, $Z := \underset{t\in T}{\oplus}\, k\,v_t$, $R = A\otimes_k (Y\oplus Y\oplus\cdots) = A\otimes_k Z$. One has also $V^\ell(L_Q) = V^\ell(L)\otimes_k S(Z)$, and $S(Z) = S(Y)\otimes S(Y)\otimes\cdots$ (recall that elements of an infinite tensor product are sum of tensor products with only finitely many elements unequal to 1).

From the basis $\{b_1,\dots,b_n\}$ of F we may construct a good basis $\{e_i\}_{i\in T}$ of L_Q (repeating the basis $\{b_1\,,b_2\,,\dots\,,b_n\}$.

4.15. Extending QUEAd's. Let L be a finite projective Lie-Rinehart algebra, for which we consider for it all the objects and constructions mentioned just above.

Let $V^\ell(L)_h \in (\mathrm{LQUEAd})_{A_h}$ be a (left) quantization of the left bialgebroid $V^\ell(L)$. Consider

$$V^\ell(L)_{h,Y} \;:=\; h\text{–adic completion of } V^\ell(L)_h\otimes_k S(Y)\otimes_k S(Y)\otimes_k S(Y)\otimes\cdots \;=:\; V^\ell(L)_h\widehat{\otimes}_k S(Z)$$

In order to describe it, for $\underline{d}\in T^{(\mathbb{N})}$ we set $e^{\underline{d}} := \prod_{t\in T} e_t^{\underline{d}(t)}$ and $\varpi\big(\underline{d}\big) := \max\big\{\varpi(e_t)\,\big|\,\underline{d}(t)\neq 0\,\big\}$ (cf. Definition 2.13);

Proposition 4.16. *Any element of $V^\ell(L)_{h,Y}$ can be written in a unique way as*

$$\sum_{\underline{d}\in T^{(\mathbb{N})}} t^\ell\big(a_{\underline{d}}\big)\,e^{\underline{d}} \;=\; \lim_{n\to+\infty} \sum_{\substack{\underline{d}\in T^{(\mathbb{N})}\\ |\underline{d}|+\varpi(\underline{d})\leq n}} t^\ell\big(a_{\underline{d}}\big)\,e^{\underline{d}} \qquad \textit{with} \qquad \lim_{|\underline{d}|+\varpi(\underline{d})\to+\infty} \big\|a_{\underline{d}}\big\| = 0 \qquad \textit{(notation of §4.1)}$$

Proof. It is obvious that any element of the given form belongs to $V(L)_{hY}$. Conversely, let $u \in V(L)_{hY}$. Write $u = u_0 + h\,u_1 + \cdots + h^n u_n + \cdots$ with $u_i \in V(L_Q)$ for all i. Now, for all $i \in \mathbb{N}$, each u_i can be written as $u_i = \sum_{\underline{\alpha}\in\mathbb{N}^{(T)}} t^l\big(u_i^{\underline{\alpha}}\big)\, e^{\underline{\alpha}}$ where all but a finite number of the $u_i^{\underline{\alpha}}$'s are zero. Set $u_{\underline{\alpha}} := \sum_i h^i u_i^{\underline{\alpha}}$, so that $u = \sum_{\underline{\alpha}} u_{\underline{\alpha}}\, e^{\underline{\alpha}}$. Let us show that $\lim\limits_{|\alpha|+\varpi(\underline{\alpha})\to+\infty} \big\|u_{\underline{\alpha}}\big\| = 0$. Pick $n_0 \in \mathbb{N}$; if we choose $A > \max\big\{\, |\,\underline{\alpha}\,| + \omega(\underline{\alpha}) \,\big|\, \exists\, i \leq n_0 : u_i^{\underline{\alpha}} \neq 0 \,\big\}$, then for $|\,\underline{\alpha}\,| + \omega(\underline{\alpha}) > A$ we have $\big\|u_{\underline{\alpha}}\big\| < 2^{-n_0}$. $\square$

Now, there exists a unique left bialgebroid structure on $V^\ell(L)_{h,Y}$ given as follows:

$$t_\ell^{Y} : A_h \longrightarrow V^\ell(L)_{h,Y}\ , \quad a \mapsto t_\ell(a)\otimes 1\ , \qquad s_\ell^{Y} : A_h \longrightarrow V^\ell(L)_{h,Y}\ , \quad a \mapsto s_\ell(a)\otimes 1$$
$$\Delta_{V^\ell(L)_{h,Y}}(a\otimes s) := \big(a_{(1)}\otimes s_{(1)}\big)\otimes\big(a_{(2)}\otimes s_{(2)}\big) \quad \text{if} \quad \Delta_h(a) = a_{(1)}\otimes a_{(2)}\ ,\ \ \Delta_{S(Z)}(s) = s_{(1)}\otimes s_{(2)}$$
$$\epsilon_h(a\otimes s) := \epsilon_h(a)\epsilon(s)\ , \qquad (a\otimes s)\big(a'\otimes s'\big) := a\,a'\otimes s\,s'$$

where the right-hand side factor map ϵ above is just the standard counit map $\epsilon : S(Z) \longrightarrow k$ of the Hopf k–algebra $S(Z)$, uniquely determined by $\epsilon(z) = 0$ for every $z \in Z$. It is easy to see that

(a) $V^\ell(L)_{h,Y}$ *is a quantization of the left bialgebroid* $V^\ell(L_Q)$;

(b) $\pi_Y := id_{V^\ell(L)_{h,Y}} \widehat{\otimes}_k\, \epsilon : V^\ell(L)_{h,Y} := V^\ell(L)_h \widehat{\otimes}_k S(Z) \to V^\ell(L)_h$ *is an epimorphism of left bialgebroids.*

An entirely similar construction is possible if we consider any RQUEAd $V^r(L)_h$ *instead of the LQUEAd* $V^\ell(L)_h$

4.17. Extending QFSAd's. Let L be a finite projective Lie-Rinehart algebra, and adopt again notations as before. Recall also that in Remark 3.46 we have introduced $J^r_f(L_Q) := V^\ell(L_Q)^{*_f}$.

Let $J^r(L)_h \in (\text{RQFSAd})_{A_h}$ be a quantization of $J^r(L)$. Keeping notation as in §4.15, consider

$$J^r(L)_{h,Y} \ := \ h\text{–adic completion of } \ \textstyle\sum_{n\in\mathbb{N}} J^r(L)_h \widetilde{\otimes}_k S(Y^*)^{\widetilde{\otimes}\, n} \otimes 1 \otimes 1 \otimes 1 \cdots$$

where $J^r(L)_h \widetilde{\otimes} S(Y^*)^{\widetilde{\otimes}\, n} \otimes 1 \otimes 1 \otimes \ldots$ is the $\big(\big(S(Y^*)^{\otimes n}\big)^+ \otimes 1 \otimes 1 \otimes 1 \cdots\big)$–adic completion of $J^r(L)_h \otimes S(Y^*)^{\otimes n} \otimes 1 \otimes 1 \otimes 1 \cdots$. There exists a unique right bialgebroid structure on $J^r(L)_{h,Y}$ described as follows:

$$t_r^{Y} : A \longrightarrow J^r(L)_{h,Y}\ , \quad a \mapsto t_r(a)\otimes 1\ , \qquad s_r^{Y} : A_h \longrightarrow J^r(L)_{h,Y}, \quad a \mapsto s_r(a)\otimes 1$$
$$\Delta(a\otimes s) := \big(a_{(1)}\otimes s_{(1)}\big)\otimes\big(a_{(2)}\otimes s_{(2)}\big) \quad \text{if} \quad \Delta_{J^r(L)_h}(a) = a_{(1)}\otimes a_{(2)}\ ,\ \ \Delta(s) = s_{(1)}\otimes s_{(2)}$$
$$\partial_h(a\otimes s) := \partial_h(a)\epsilon^*(s)\ , \qquad (a\otimes s)(a'\otimes s') := a\,a'\otimes s\,s'$$

Then one easily sees that

(a) $J^r(L)_{h,Y}$ *is a quantization of the right bialgebroid* $J^r_f(L_Q)$;

(b) $\pi^Y := id_{J^r(L)_{h,Y}} \otimes_k \epsilon^* : J^r(L)_{h,Y} \to J^r(L)_h$ *is an epimorphism of right bialgebroids.*

An entirely similar construction is possible if $J^r(L)_h$ *is replaced with a LQFSAd* $J^\ell(L)_h$.

Remark: Let $V^\ell(L)_h \in (\text{LQUEAd})_{A_h}$ be a quantization of $V^\ell(L)$. We have seen in §4.15 that $V^\ell(L)_{h,Y} := V^\ell(L)_h \widehat{\otimes}_k S(Z)$ is a LQUEAd which quantizes $V^\ell(L_Q)$. If $n \in \mathbb{N}$, let $S(Z)^+ := Ker\big(\epsilon : S(Z) \longrightarrow k\big)$ be the kernel of the counit of $S(Z)$, and let $V^\ell(L)_{h,Y}^{*_{f,n}}$ be the subspace of $V^\ell(L)_{h,Y}^{*}$ given by $V^\ell(L)_{h,Y}^{*_{f,n}} := \big\{\, \lambda \in V^\ell(L)_{h,Y}^{*} \,\big|\, \lambda\big(V^\ell(L)_h \otimes_k S(Y)^{\otimes n} \otimes_k S(Z)^+\big) = 0 \,\big\}$. Then set

$$V^\ell(L)_{h,Y}^{*_f} \ := \ h\text{–adic completion of } \textstyle\sum_{n\in\mathbb{N}} V^\ell(L)_{h,Y}^{*_{f,n}}$$

Then $V^\ell(L)_{h,Y}^{*_f}$ is a right subbialgebroid of $V^\ell(L)_{h,Y}^{*}$, which is isomorphic to the right bialgebroid $\big(V^\ell(L)_h^{*}\big)_Y$. Note also that $V^\ell(L)_{h,Y}^{*_{f,n}}$ is isomorphic to $V^\ell(L)_h^{*} \widetilde{\otimes}_k S(Y^*)^{\widetilde{\otimes} n} \otimes_k 1 \otimes_k 1 \otimes_k 1 \cdots$.

In a similar way, one can define also the right bialgebroid $\big(V^\ell(L)_{h,Y}\big)_{*_f}$: this is a right subbialgebroid of $\big(V^\ell(L)_{h,Y}\big)_{*}$ isomorphic to the right bialgebroid $\big(\big(V^\ell(L)_h\big)_*\big)_Y$.

Parallel "right-handed versions" of the previous constructions and results also make sense if one starts with some $V^r(L)_h \in (\text{RQUEAd})_{A_h}$ instead of $V^\ell(L)_h \in (\text{LQUEAd})_{A_h}$: in a nutshell, one still finds that "extension commutes with dualization". Details are left to the reader.

5 Linear duality for quantum groupoids

In this section we explore the relationship among quantum groupoids ruled by linear duality (i.e., by taking left or right duals). We shall see that the "(left/right) full dual" and the "(left/right) continuous dual" altogether provide category antiequivalences between $(\text{LQUEAd})_{A_h}$ and $(\text{RQFSAd})_{A_h}$ and between $(\text{RQUEAd})_{A_h}$ and $(\text{LQFSAd})_{A_h}$.

Essentially, we implement the construction of "dual bialgebroids" presented in Subsection 3.5, but still we need to make sure that several technical aspects do turn round.

5.1 Linear duality for quantum QUEAd's

We begin with the construction of duals for (L/R)QUEAD's. In this case, we consider "full duals" (versus topological ones, cf. Subsection 5.2 later on. Before giving the main result, we need a couple of auxiliary, technical lemmas.

Lemma 5.1. *Let $V^\ell(L)_h \in (\text{LQUEAd})_{A_h}$ and $u \in V^\ell(L)_h$. For any $r \in \mathbb{N}$, there exists $t_r \in \mathbb{N}$ such that*

(a) $\quad \Delta^{t_r}(u) \,=\, \delta_0 + h\,\delta_1 + h^2\,\delta_2 + \cdots + h^{r-1}\,\delta_{r-1} + h^r\,\delta_r \quad \Big(\in V^\ell(L)_h^{\widehat{\otimes}\, t_r}\Big)$

(b) for every $i = 0, \dots, r-1$, the element $\delta_i \in V^\ell(L)_h^{\widehat{\otimes}\, t_r}$ contains at least r terms equal to 1 .

Proof. Let us expand u as $u \,=\, u_0 + h\,u_1 + h^2\,u_2 + \cdots + h^r\,u_r + \cdots$. Then there exists $t'_0 \in \mathbb{N}$ such that $\overline{\Delta^{t'_0}(u_0)}$ contains at least r terms equal to 1 . We lift $\overline{\Delta^{t'_0}(u_0)}$ to some element $\delta^0_0 \in V^\ell(L)_h^{\widehat{\otimes}\, t'_0}$ containing at least r terms equal to 1 . Then $\Delta^{t'_0}(u) \,=\, \delta^0_0 + h\,\delta^0_1 + h^2\,\delta^0_2 + \cdots + h^r\,\delta^0_r + \cdots$ for suitable elements $\delta^0_1, \dots, \delta^0_r \in V^\ell(L)_h^{\widehat{\otimes}\, t'_0}$.

Now we can find $t'_1 \in \mathbb{N}$ such that $\overline{\big(id^{t'_0-1} \otimes \Delta^{t'_1}\big)(\delta^0_1)}$ contains at least r terms equal to 1 . We lift $\overline{\big(id^{t'_0-1} \otimes \Delta^{t'_1}\big)(\delta^0_0)}$ and $\overline{\big(id^{t'_0-1} \otimes \Delta^{t'_1}\big)(\delta^0_1)}$ to elements $\delta^1_0, \delta^1_1 \in V^\ell(L)_h^{\widehat{\otimes}\, t'_0+t'_1}$ which both contain at least r terms equal to 1 . Thus we find $\Delta^{t'_0+t'_1}(u) \,=\, \delta^1_0 + h\,\delta^1_1 + h^2\,\delta^1_2 + \cdots + h^r\,\delta^1_r + \cdots$ for suitable $\delta^1_2, \dots, \delta^1_r \in V^\ell(L)_h^{\widehat{\otimes}\, t'_0+t'_1}$. Iterating finitely many times, we complete the proof. $\square$

Notation 5.2. Before next lemma, we need some more notation. Given $V^\ell(L)_h \in (\text{LQUEAd})_{A_h}$, consider $K_h := V^\ell(L)_h^*$ and its subset $I_{K_h} := \big\{\, \chi \in K_h \,\big|\, \big\langle 1_{V^\ell(L)_h}, \chi \big\rangle \in h\,A_h \,\big\}$.

Remark 5.3. As $V^\ell(L)_h$ is a left bialgebroid, by §3.31 we know that its right dual $K_h := V^\ell(L)_h^*$ has a canonical structure of A^e–ring; then, with respect to this structure, one easily sees that I_{K_h} is a two-sided ideal of K_h . Moreover $\Delta(I_h) \subset I_{K_h} \otimes K_h + K_h \otimes I_{K_h}$. Indeed, given any $\phi \in I_{K_h}$, we write $\Delta(\phi) = \phi_{(1)} \otimes \phi_{(2)}$ — a formal series (in Σ–notation) — convergent in the I_{K_h}–adic topology of K_h . Writing $\phi_{(1)}$ and $\phi_{(2)}$ as

$$\phi_{(1)} \,=\, \phi^{+_s}_{(1)} + s^r\big(\partial_h\big(\phi_{(1)}\big)\big)\;, \quad \text{with} \quad \phi^{+_s}_{(1)} := \phi_{(1)} - s^r\big(\partial_h\big(\phi_{(1)}\big)\big) \,\in\, I_{K_h}$$
$$\phi_{(2)} \,=\, \phi^{+_t}_{(2)} + t^r\big(\partial_h\big(\phi_{(2)}\big)\big)\;, \quad \text{with} \quad \phi^{+_t}_{(2)} := \phi_{(2)} - t^r\big(\partial_h\big(\phi_{(2)}\big)\big) \,\in\, I_{K_h}$$

we can expand $\Delta(\phi) = \phi_{(1)} \otimes \phi_{(2)}$ as

$$\Delta(\phi) \,=\, \phi_{(1)} \otimes \phi_{(2)} \,=\, \phi^{+_s}_{(1)} \otimes \phi_{(2)} \,+\, s^r\big(\partial_h\big(\phi_{(1)}\big)\big) \otimes \phi^{+_t}_{(2)} \,+\, s^r\big(\partial_h\big(\phi_{(1)}\big)\big) \otimes t^r\big(\partial_h\big(\phi_{(2)}\big)\big) \,=$$
$$=\, \phi^{+_s}_{(1)} \otimes \phi_{(2)} + s^r\big(\partial_h\big(\phi_{(1)}\big)\big) \otimes \phi^{+_t}_{(2)} + s^r\big(\partial_h(x)\big) \otimes 1 \,\in\, \big(I_{K_h} \widetilde{\otimes}_{A_h} K_h + K_h \widetilde{\otimes}_{A_h} I_{K_h} + h\,s^r(A_h)\,\widetilde{\otimes}_{A_h} 1\big)$$

where we took into account the identity $s^r\big(\partial_h\big(x_{(1)}\big)\big) \otimes t^r\big(\partial_h\big(x_{(2)}\big)\big) \,=\, s^r\big(\partial_h(x)\big) \otimes 1$, due to Remarks 3.17*(e)*, and the fact that $\partial_h(x) \in h\,A_h$, since $\phi \in I_{K_h}$ by assumption.

Lemma 5.4. *Given* $V^\ell(L)_h \in (\mathrm{LQUEAd})_{A_h}$ *and* $K_h := V^\ell(L)_h^{\,*}$ *, consider the two-sided ideal* $I_{K_h} := \big\{\, \chi \in K_h \,\big|\, \langle 1_{V^\ell(L)_h}, \chi \rangle \in h\, A_h \,\big\}$ *of* K_h *, as well as its powers* $I_{K_h}^n$ $(n \in \mathbb{N})$ *. Then, for every* $u \in V^\ell(L)_h$ *and every* $r \in \mathbb{N}$ *, there exists* $t_r \in \mathbb{N}$ *such that* $\big\langle u \,, I_{K_h}^{t_r} \big\rangle \in h^r A_h$ *.*

The same property holds if one considers the left dual $K_h := \big(V^\ell(L)_h\big)_*$ *of* $V^\ell(L)_h$ *.*

Proof. We make use of the previous lemma. Indeed, the latter ensures that there exists $t_r \in \mathbb{N}$ such that

$$\Delta^{t_r}(u) \;=\; \delta_0 + h\,\delta_1 + h^2\,\delta_2 + \cdots + h^{r-1}\,\delta_{r-1} + h^r\,\delta_r$$

for some elements $\delta_0\,,\,\dots\,,\,\delta_r \in V^\ell(L)_h^{\widehat{\otimes}\,t_r}$ such that $\delta_0\,,\,\dots\,,\,\delta_{r-1}$ contain at least r terms equal to 1 . From this fact and the properties of the natural pairing $\langle\ ,\ \rangle$ between $V^\ell(L)_h$ and its right dual $K_h := V^\ell(L)_h^{\,*}$ it is easy to see that $\langle \phi\,,\, u \rangle \in h^r A_h$ for all $\phi \in I_{K_h}^{t_r}$, whence the claim. $\square$

We are now ready for our first important result about linear duality of "quantum groupoids". In a nutshell, it claims that the left and the right dual of a left, resp. right, quantum universal enveloping algebroid are both right, resp. left, quantum formal series algebroids.

Theorem 5.5.

(a) If $V^\ell(L)_h \in (\mathrm{LQUEAd})_{A_h}$ *, then* $V^\ell(L)_h^{\,*}\,, V^\ell(L)_{h\,*} \in (\mathrm{RQFSAd})_{A_h}$ *, with semiclassical limits (cf. §3.42)*

$$V^\ell(L)_h^{\,*} \Big/ h\,V^\ell(L)_h^{\,*} \;\cong\; V^\ell(L)^* \;=:\; J^r(L) \qquad \textit{and} \qquad V^\ell(L)_{h\,*} \Big/ h\,V^\ell(L)_{h\,*} \;\cong\; V^\ell(L)_* \cong J^r(L)$$

Therefore $V^\ell(L)_h^{\,*}$ *and* $V^\ell(L)_{h\,*}$ *are quantization of* $J^r(L)$*.*

Moreover, the structure of Lie-Rinehart algebra induced on L^* *by the quantization* $V^\ell(L)_h^{\,*}$ *of* $J^r(L)$ *— according to Theorem 4.12 — is the same as that induced by the quantization* $V^\ell(L)_h$ *of* $V^\ell(L)$ *— according to Theorem 4.4; therefore, the structure of Lie-Rinehart bialgebra induced on* L *is the same in either case.*

On the other hand, the structure of Lie-Rinehart algebra induced on L^* *by the quantization* $V^\ell(L)_{h\,*}$ *of* $V^\ell(L)_* \cong J^r(L)$ *is opposite to that induced by the quantization* $V^\ell(L)_h$ *of* $V^\ell(L)$ *. Thus the structures of Lie-Rinehart bialgebra induced on* L *in the two cases are coopposite to each other:* $V^\ell(L)_h$ *provides a quantization of the Lie-Rinehart bialgebra* L *, while* $V^\ell(L)_{h\,*}$ *provides a quantization of the coopposite Lie-Rinehart bialgebra* L_{coop} *— cf. Remarks 2.21(e).*

(b) If $V^r(L)_h \in (\mathrm{RQUEAd})_{A_h}$ *, then* ${}_*V^r(L)_h\,,\ {}^*V^r(L)_h \in (\mathrm{LQFSAd})_{A_h}$ *, with semiclassical limits (cf. §3.42)*

$${}_*V^r(L)_h \Big/ h\,{}_*V^r(L)_h \;\cong\; {}_*V^r(L) := J^\ell(L) \qquad \textit{and} \qquad {}^*V^r(L)_h \Big/ h\;{}^*V^r(L)_h \;\cong\; {}^*V^r(L) \cong J^\ell(L)$$

Therefore ${}_*V^r(L)_h$ *and* ${}^*V^r(L)_h$ *are quantizations of* $J^\ell(L) := {}_*V^r(L)$ *.*

Moreover, the structures of Lie-Rinehart algebra induced on L^* *by the quantization* ${}_*V^r(L)_h$ *of* $J^\ell(L)$ *— according to Theorem 4.13 — is the same as that induced by the quantization* $V^r(L)_h$ *of* $V^r(L)$ *— according to Theorem 4.7.*

On the other hand, the structure of Lie-Rinehart algebra induced on L^* *by the quantization* ${}^*V^r(L)_h$ *of* ${}^*V^r(L) \cong J^\ell(L)$ *is opposite to that induced by the quantization* $V^r(L)_h$ *of* $V^r(L)$ *. Thus the structures of Lie-Rinehart bialgebra induced on* L *in the two cases are coopposite to each other:* $V^\ell(L)_h$ *provides a quantization of the Lie-Rinehart bialgebra* L *, while* ${}^*V^r(L)_h$ *provides a quantization of the coopposite Lie-Rinehart bialgebra* L_{coop} *— cf. Remarks 2.21(e).*

Proof. (a) We shall start by proving that if $V^\ell(L)_h \in (\mathrm{LQUEAd})_{A_h}$, then $V^\ell(L)_h^{\,*} \in (\mathrm{RQFSAd})_{A_h}$.

As we saw in §5.2, the right dual $K_h := V^\ell(L)_h^{\,*}$ of $V^\ell(L)_h$ has a canonical structure of A^e–ring. Moreover, it is endowed with a map $\partial_h : V^\ell(L)_h^{\,*} \longrightarrow A_h$ $\big(\chi \mapsto \langle 1_{V^\ell(L)_h}, \chi \rangle\big)$, which has all the properties of a "counit" in a right bialgebroid and defines the two-sided ideal $I_{K_h} := \partial_h^{-1}\big(h\,A_h\big)$. What we still have to prove is that

- $K_h := V^\ell(L)_h^{\,*}$ is complete for the I_{K_h}–adic topology;

- there exists a suitable coproduct $\Delta_h : K_h := V^\ell(L)_h^* \to K_h \,\widetilde{\otimes}_{A_h} K_h = V^\ell(L)_h^* \,\widetilde{\otimes}_{A_h} V^\ell(L)_h^*$, which makes $K_h := V^\ell(L)_h^*$ into a topological right bialgebroid;
- $K_h \Big/ h\, K_h \,=\, V^\ell(L)_h^* \Big/ h\, V^\ell(L)_h^*$ is isomorphic to $V^\ell(L)^*$ as topological right bialgebroid.

We begin by looking for an isomorphism $V^\ell(L)_h^* \Big/ h\, V^\ell(L)_h^* \,\cong\, V^\ell(L)^*$. For this, we distinguish two cases, the *free* one and the *general* one.

— *Free case:* L *is a free A–module*, of finite type.

In this case, let us fix an A–basis $\{\,\overline{e}_1, \dots, \overline{e}_n\}$ of the A–module L ; then we lift each $\overline{e}_i$ to some $e_i \in V^\ell(L)_h$. Then any element of $V^\ell(L)_h$ can be written as the h–adic limit of elements of the form $\sum_{(a_1,\dots,a_n)\in\mathbb{N}^n} t_l(c_{a_1,\dots,a_n})\, e_1^{a_1} \cdots e_n^{a_n}$ in which almost all $c_{a_1,\dots,a_n}$'s are zero.

For a given $\lambda \in V^\ell(L)^*$, set $\overline{\alpha}_{a_1,\dots,a_n} := \lambda\big(\overline{e}_1^{\,a_1} \cdots \overline{e}_n^{\,a_n}\big) \in A$ for all $\underline{a} := (a_1, \dots, a_n) \in \mathbb{N}^n$. We lift each $\overline{\alpha}_{a_1,\dots,a_n}$ to some $\alpha_{a_1,\dots,a_n} \in A_h$, with the assumption that if $\overline{\alpha}_{a_1,\dots,a_n} = 0$ then we choose $\alpha_{a_1,\dots,a_n} = 0$. Now we set

$$\Lambda\left(\textstyle\sum_{(a_1,\dots,a_n)\in\mathbb{N}^n} t_l(c_{a_1,\dots,a_n})\, e_1^{a_1} \cdots e_n^{a_n}\right) \,:=\, \textstyle\sum_{(a_1,\dots,a_n)\in\mathbb{N}^n} c_{a_1,\dots,a_n}\, \alpha_{a_1,\dots,a_n}$$

This defines a map Λ from the right A_h–submodule of $V^\ell(L)_h$ spanned by all the monomials $e_1^{a_1} \cdots e_n^{a_n}$ to A_h : as the h–adic completion of this submodule is nothing but $V^\ell(L)_h$, this map uniquely extends (by continuity) to a map $\Lambda : V^\ell(L)_h \longrightarrow A_h$. By construction, we have $\Lambda \in V^\ell(L)_h^*$, and Λ is a lifting of λ , that is $\Lambda \mod h\, V^\ell(L)_h^* = \lambda$. This guarantees that the canonical map $V^\ell(L)_h^* \Big/ h\, V^\ell(L)_h^* \longrightarrow V^\ell(L)^*$, which is obviously injective, is also surjective.

— *General case:* L *is a projective A–module*, of finite type.

As in §2.12, we introduce a projective A–module Q such that $L \oplus Q = F$ is a finite free A–module. Fix an A–basis $\{b_1, \dots, b_n\}$ of F , and set $Y = k\, b_1 \oplus k\, b_2 \cdots \oplus k\, b_n$, so that $F = A \otimes_k Y$. The basis $\{b_1, \dots, b_n\}$ also defines a good basis $\{\overline{e}_i\}_{i \in T := \mathbb{N}\times\{1,\dots,n\}}$ of L_Q .

Now let $\lambda \in V(L)^*$. We extend λ to some $\lambda' \in V(L_Q)^*$ by setting $\lambda'\big|_{V(L)\otimes S(Z)^+} := 0$. Now we can adapt the arguments of the free case to construct a lifting $\Lambda' \in V(L)_{h,Y}{}^*$ of λ' . Then $\Lambda := \Lambda'\big|_{V(L)_h} \in V(L)_h^*$ is a lifting of λ as required.

Thus one sees again that the canonical map $V^\ell(L)_h^* \Big/ h\, V^\ell(L)_h^* \longrightarrow V^\ell(L)^*$ is a bijection.

On $V^\ell(L)_h^*$ we have already considered an algebraic structure of "A^e–ring with counit": the same structure then is inherited by its quotient $V^\ell(L)_h^* \Big/ h\, V^\ell(L)_h^*$. On the other hand, $V^\ell(L)^*$ is a right bialgebroid, hence in particular it is an "A^e–ring with counit" as well. The canonical bijection $V^\ell(L)_h^* \Big/ h\, V^\ell(L)_h^* \longrightarrow V^\ell(L)^*$ above is clearly compatible with this additional structure. In particular, this implies that $Ker(\partial_h) \mod h\, K_h \,\cong\, Ker(\partial_{V^\ell(L)}) =: \mathfrak{J}_{V^\ell(L)^*}$.

Now consider $I_{K_h} := \partial_h^{-1}\big(h\, A_h\big)$, which can be written as $I_{K_h} = Ker(\partial_h) \,+\, h\, K_h$. As we know that $V^\ell(L)^*$ is $\mathfrak{J}_{V^\ell(L)^*}$–adically complete (cf. §3.37), from $Ker(\partial_h) \mod h\, K_h \,\cong\, \mathfrak{J}_{V^\ell(L)^*}$ and $I_{K_h} = Ker(\partial_h) \,+\, h\, K_h$ we can easily argue that $K_h := V^\ell(L)_h^*$ is I_{K_h}–adically complete.

Now we look for a suitable coproduct. To this end, we shall show that the natural "coproduct" given by the recipe in §3.31 does the job. The problem is to prove the existence of an isomorphism from $V^\ell(L)_h^* {}_\blacktriangleleft\widetilde{\otimes}_\blacktriangleright V^\ell(L)_h^*$ — the completion of $V^\ell(L)_h^* \otimes V^\ell(L)_h^*$ with respect to the $\big(I_{K_h} \otimes K_h + K_h \otimes I_{K_h}\big)$–adic topology — to $Hom_{(-,A_h)}\big(\big(V^\ell(L)_h {}_\triangleleft\otimes_\blacktriangleright V^\ell(L)_h\big)_\triangleleft\, , \, A_h\big)$. More precisely, there exists (cf. §3.31) a natural map χ from $V^\ell(L)_h^* \otimes V^\ell(L)_h^*$ to $Hom_{(-,A_h)}\big(\big(V^\ell(L)_h {}_\triangleleft\otimes_\blacktriangleright V^\ell(L)_h\big)_\triangleleft\, , \, A_h\big)$; we now show that this χ actually extends to a (continuous) map — which we still denote by χ — from $V^\ell(L)_h^* {}_\blacktriangleleft\widetilde{\otimes}_\blacktriangleright V^\ell(L)_h^*$ to $Hom_{(-,A_h)}\big(\big(V^\ell(L)_h {}_\triangleleft\otimes_\blacktriangleright V^\ell(L)_h\big)_\triangleleft\, , \, A_h\big)$.

To begin with, fix $u \in V^\ell(L)_h$. For every $r \in \mathbb{N}$, there exists $t_r \in \mathbb{N}$ such that $\Delta^{t_r}(u)$ expands as $\Delta^{t_r}(u) = \delta_0 + h\, \delta_1 + h^2\, \delta_2 + \cdots + h^r\, \delta_r$ as in Lemma 5.1: in particular, every $\delta_i \in V^\ell(L)_h^{\otimes\, t_r}$ with $0 \leq i \leq r-1$ contains at least r terms equal to 1 . As the canonical evaluation pairing between $V^\ell(L)_h$ and $K_h := V^\ell(L)_h^*$ is a bialgebroid right pairing — in the sense of Definition

3.34 — the formulas for such pairings imply at once (by induction) that $\big\langle\, u\,, I_{K_h}^{\,t} \,\big\rangle \subseteq h^r A_h$ for all $t \geq t_r$. By the same arguments, given $v, w \in V^\ell(L)_h$ we see that, for every $r \in \mathbb{N}$, one has

$$\big\langle\, v \otimes w\,,\, I_{K_h}^{t'} \otimes I_{K_h}^{t''} \,\big\rangle \;\subseteq\; h^r A_h \qquad \text{for all} \quad t' + t'' \gg 0 \tag{4.1}$$

Now let $\Lambda \in V^\ell(L)_h^{\,*} \,{}_{\blacktriangleleft}\widetilde{\otimes}_{\blacktriangleright}\, V^\ell(L)_h^{\,*}$. Then Λ is the limit of a sequence $\big(\Lambda_n\big)_{n\in\mathbb{N}}$ — with $\Lambda_n \in V^\ell(L)_h^{\,*}\,{}_{\blacktriangleleft}\otimes_{\blacktriangleright} V^\ell(L)_h^{\,*}$ for all n — for the $\big(I_{K_h} \otimes K_h + K_h \otimes I_{K_h}\big)$–adic topology; in particular, for each $t \in \mathbb{N}$ one has

$$\big(\Lambda_{n'} - \Lambda_{n''}\big) \;\in\; \big(I_{K_h} \otimes K_h + K_h \otimes I_{K_h}\big)^t \;=\; \sum_{t'+t''=t} I_{K_h}^{t'} \otimes I_{K_h}^{t''} \qquad \text{for all} \quad n', n'' \gg 0 \tag{4.2}$$

By (4.1) and (4.2) together we get that for all $r \in \mathbb{N}$ one has

$$\big(\chi(\Lambda_{n'}) - \chi(\Lambda_{n''})\big)(v \otimes w) = \chi\big(\Lambda_{n'} - \Lambda_{n''}\big)(v \otimes w) := \big\langle\, v \otimes w\,,\, \Lambda_{n'} - \Lambda_{n''} \,\big\rangle \;\subseteq\; h^r A_h$$

for all $n' + n'' \gg 0$; in other words, $\big\{\chi\big(\Lambda_n\big)(v \otimes w)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence for the h–adic topology in A_h; as the latter is h–adically complete (and separated), there exists a unique, well-defined limit $\lim\limits_{n\to\infty} \chi\big(\Lambda_n\big)(v \otimes w) \in A_h$. In the end, we can set $\chi\big(\Lambda\big)(v \otimes w) := \lim\limits_{n\to\infty} \chi\big(\Lambda_n\big)(v \otimes w)$; this defines a (continuous) map extending the original one, namely

$$\chi \,:\, V^\ell(L)_h^{\,*} \,{}_{\blacktriangleleft}\widetilde{\otimes}_{\blacktriangleright}\, V^\ell(L)_h^{\,*} \;\longrightarrow\; \mathit{Hom}_{(-,A_h)}\Big(\big(V^\ell(L)_h \,{}_{\triangleleft}\otimes_{\blacktriangleright} V^\ell(L)_h\big)_{\triangleleft}\,,\, A_h\Big) \tag{4.3}$$

To complete our argument, we need a few more steps. To ease the notation, we shall write $X\big|_{h=0} := X \big/ h\,X$ for every $k[[h]]$–module X.

First, with the same arguments used to prove that $\mathit{Hom}_{(-,A_h)}\big(V^\ell(L)_h\,, A_h\big)\Big|_{h=0} = V^\ell(L)_h^{\,*}\Big|_{h=0}$ has a canonical bijection with $\mathit{Hom}_{(-,A)}\big(V^\ell(L)\,, A\,\big) =: V^\ell(L)^*$ we can also prove that

$$\mathit{Hom}_{(-,A_h)}\Big(\big(V^\ell(L)_h \,{}_{\triangleleft}\otimes_{\blacktriangleright} V^\ell(L)_h\big)_{\triangleleft}\,,\, A_h\Big)\Big|_{h=0} \;\cong\; \mathit{Hom}_{(-,A)}\big(V^\ell(L) \otimes V^\ell(L)\,,\, A\,\big) \tag{4.4}$$

Similarly, the same arguments once more can be adapted to prove that

$$V^\ell(L)_h^{\,*} \,{}_{\blacktriangleleft}\widetilde{\otimes}_{\blacktriangleright}\, V^\ell(L)_h^{\,*}\Big|_{h=0} \;\cong\; V^\ell(L)^* \,\widetilde{\otimes}\, V^\ell(L)^* \quad \big(\,=\, J^r(L) \,\widetilde{\otimes}\, J^r(L)\,, \;\; \text{cf. §3.37}\big) \tag{4.5}$$

Finally, by construction the reduction modulo h of the map χ in (4.3), call it $\overline{\chi}$, is nothing but the map

$$\widetilde{\vartheta} \,:\, J^r(L)_{\blacktriangleleft}\widetilde{\otimes}_{\blacktriangleright} J^r(L) \;=\; V^\ell(L)^*_{\blacktriangleleft}\widetilde{\otimes}_{\blacktriangleright} V^\ell(L)^* \;\longrightarrow\; \big(V^\ell(L)_{\triangleleft}\otimes_{\blacktriangleright} V^\ell(L)\big)^*$$

considered in §3.37. Therefore — since $\mathit{Hom}_{(-,A)}\big(V^\ell(L) \otimes V^\ell(L)\,,\, A\,\big) =: \big(V^\ell(L)_{\triangleleft}\otimes_{\blacktriangleright} V^\ell(L)\big)^*$, and taking into account the isomorphisms in (4.4–5) — as $\overline{\chi} = \widetilde{\vartheta}$ is a k–linear isomorphism we can deduce that χ is an isomorphism as well.

The outcome now is that $K_h := V^\ell(L)_h^{\,*}$ endowed with the previously constructed structure — including the coproduct map given by the recipe in §3.31 — is a topological right bialgebroid, complete with respect to the I_{K_h}–adic topology. In addition, the bijection $V^\ell(L)_h^{\,*} \big/ h\, V^\ell(L)_h^{\,*} \longrightarrow V^\ell(L)^*$ found above by construction happens to be a right bialgebroid isomorphism.

Our next task is the following. Denote by $\big(L^*, [\ ,\ ]', \omega'\big)$ and $\big(L^*, [\ ,\ ]'', \omega''\big)$ the structures of Lie-Rinehart bialgebras induced on L^* respectively by Theorem 4.12 — for $J^r(L) := V^\ell(L)_h^{\,*}$ — and by Theorem 4.4 — applied to $V^\ell(L)_h$. We must prove that $\omega' = \omega''$ and $[\ ,\ ]' = [\ ,\ ]''$. To this end, recall that, by Remarks 2.21*(b)*, ω'' and $[\ ,\ ]''$ are uniquely determined by the conditions

$$\omega''(\Phi)(a) \,=\, \big\langle \delta_L(a)\,, \Phi \big\rangle \;\;,\quad \big\langle\, \Theta\,, [\Phi\,, \Psi]'' \,\big\rangle \,=\, \omega''(\Phi)\big(\langle\, \Theta\,, \Psi\rangle\big) \,-\, \omega''(\Psi)\big(\langle \Theta\,, \Phi\rangle\big) \,-\, \big\langle \delta_L(\Theta)\,, \Phi \otimes \Psi \big\rangle$$

(for all $\Phi\,, \Psi \in L^*$, $\Theta \in L$, $a \in A$), where $\delta_L(a)$ and $\delta_L(a)$ are defined by the formula for δ in Theorem 4.4. Therefore, it is enough for us to prove that (for all $\Phi\,, \Psi \in L^*$, $\Theta \in L$, $a \in A$)

$$\omega'(\Phi)(a) \,=\, \big\langle \delta_L(a)\,, \Phi \big\rangle\,,\;\; \big\langle\, \Theta, [\Phi\,, \Psi]' \,\big\rangle \,=\, \omega''(\Phi)\big(\langle\, \Theta, \Psi\rangle\big) - \omega''(\Psi)\big(\langle \Theta, \Phi\rangle\big) - \big\langle \delta_L(\Theta)\,, \Phi \otimes \Psi \big\rangle \tag{4.6}$$

In order to prove (4.6), we choose liftings $\phi', \psi' \in J^r(L)_h := V^\ell(L)_h^{\,*}$, with the additional condition that $\phi', \psi' \in \mathfrak{J}_{J^r(L)_h} := Ker\big(\partial_{J^r(L)_h}\big)$ (such a choice is always possible), a lifting $\theta \in V^\ell(L)_h$ of Θ and a lifting $a' \in A_h$ of a. Now direct computation gives

$$\begin{aligned}
\omega'(\Phi)(a) \; &= \; \Big(\big(h^{-1}\big(\phi'\, t_r(a') - t_r(a')\,\phi'\big)\big) \mod h\, J^r(L)_h\Big) \mod \mathfrak{J}_{J(L)} \; = \\
&= \; \bigg(\frac{\phi'\, t_r(a') - t_r(a')\,\phi'}{h} \mod \mathfrak{J}_{J^r(L)_h}\bigg) \mod h\, A_h \; = \\
&= \; \partial_{J(L)_h}\!\bigg(\frac{\phi'\, t_r(a') - t_r(a')\,\phi'}{h}\bigg) \mod h\, A_h \; = \; \Big\langle 1\,, \frac{\phi'\, t_r(a') - t_r(a')\,\phi'}{h} \Big\rangle \mod h\, A_h \; = \\
&= \; \frac{a'\,\langle\, 1\,,\, \phi'\,\rangle \,-\, \langle\, 1\,,\, t_r(a')\,\phi'\,\rangle}{h} \mod h\, A_h \; = \; \frac{\langle\, 1\,,\, \phi'\,\rangle\, a' \,-\, \langle\, 1\,,\, t_r(a')\,\phi'\,\rangle}{h} \mod h\, A_h \; = \\
&= \; \frac{\langle\, t^\ell(a')\,, \phi'\,\rangle \,-\, \langle\, s^\ell(a')\,, \phi'\,\rangle}{h} \mod h\, A_h \; = \; \Big\langle \frac{t^\ell(a') - s^\ell(a')}{h}\,, \phi' \Big\rangle \mod h\, A_h \; = \; \big\langle\, \delta_L(a)\,, \Phi\,\big\rangle
\end{aligned}$$

where $\langle\ ,\ \rangle$ denotes the natural evaluation pairing between $V^\ell(L)_h$ and its right dual $V^\ell(L)_h^{\,*}$, we exploited the fact that this pairing is a *right* bialgebroid pairing (cf. Definitions 3.33 and 3.34) and the fact that $\big\langle 1_{V^\ell(L)_h}\,, \phi'\,\big\rangle =: \partial_{J^r(L)_h}(\phi') = 0$ because $\phi' \in \mathfrak{J}_{J^r(L)_h} := Ker\big(\partial_{J^r(L)_h}\big)$ by assumption. Thus the first identity in (4.6) is verified.

As to the second identity, we write $\Delta(\theta) = \theta_{(1)} \otimes \theta_{(2)}$ as $\Delta(\theta) = \theta \otimes 1 + 1 \otimes \theta + h \sum_{[\theta]} \theta_{[1]} \otimes \theta_{[2]}$, so that $\big(\sum_{[\theta]} \theta_{[1]} \otimes \theta_{[2]}\big) \mod h\, V^\ell(L)_h \,{}_{\triangleleft}\!\underset{A}{\times}\!{}_{\triangleright} V^\ell(L)_h =: \Delta^{[1]}(\Theta)$ — notation of Definition 4.4. Then by direct computation we find

$$\begin{aligned}
\Big\langle \Theta\,, [\Phi, \Psi]' \Big\rangle \; &= \; \Big\langle \theta\,, \frac{\phi'\,\psi' \,-\, \psi'\,\phi'}{h} \Big\rangle \mod h\, A_h \; = \; h^{-1}\Big(\big\langle\, \theta\,,\, \phi'\,\psi'\,\big\rangle - \big\langle\, \theta\,,\, \psi'\phi'\,\big\rangle\Big) \mod h\, A_h \; = \\
&= \; h^{-1}\Big(\big\langle\, \theta_{(2)}\,,\, t_r\big(\langle\, \theta_{(1)}\,, \phi'\,\rangle\big)\, \psi'\,\big\rangle \,-\, \big\langle\, \theta_{(2)}\,,\, t_r\big(\langle\, \theta_{(1)}\,, \psi'\,\rangle\big)\, \phi'\,\big\rangle\Big) \mod h\, A_h \; = \\
&= \; h^{-1}\Big(\big\langle\, 1\,,\, t_r\big(\langle\, \theta\,, \phi'\,\rangle\big)\, \psi'\,\big\rangle \,+\, \big\langle\, \theta\,,\, t_r\big(\langle\, 1\,, \phi'\,\rangle\big)\, \psi'\,\big\rangle \,+\, h \textstyle\sum_{[\theta]} \big\langle\, \theta_{[2]}\,,\, t_r\big(\langle\, \theta_{[1]}\,, \phi'\,\rangle\big)\, \psi'\,\big\rangle \,- \\
&\qquad -\, \big\langle\, 1\,,\, t_r\big(\langle\, \theta\,, \psi'\,\rangle\big)\, \phi'\,\big\rangle \,-\, \big\langle\, \theta\,,\, t_r\big(\langle\, 1\,, \psi'\,\rangle\big)\, \phi'\,\big\rangle \,-\, h \textstyle\sum_{[\theta]} \big\langle\, \theta_{[2]}\,,\, t_r\big(\langle\, \theta_{[1]}\,, \psi'\,\rangle\big)\, \phi'\,\big\rangle\Big) \mod h\, A_h \; = \\
&= \; h^{-1}\Big(\big\langle\, 1\,,\, t_r\big(\langle\, \theta\,, \phi'\,\rangle\big)\, \psi'\,\big\rangle \,-\, \big\langle\, 1\,,\, t_r\big(\langle\, \theta\,, \psi'\,\rangle\big)\, \phi'\,\big\rangle\Big) \mod h\, A_h \,+ \\
&\qquad +\, \textstyle\sum_{[\theta]} \Big(\big\langle\, \theta_{[2]}\,,\, t_r\big(\langle\, \theta_{[1]}\,, \phi'\,\rangle\big)\, \psi'\,\big\rangle \,-\, \big\langle\, \theta_{[2]}\,,\, t_r\big(\langle\, \theta_{[1]}\,, \psi'\,\rangle\big)\, \phi'\,\big\rangle\Big) \mod h\, A_h \; = \\
&= \; h^{-1}\Big(\big\langle\, s^\ell\big(\langle\, \theta\,, \phi'\,\rangle\big)\,,\, \psi'\,\big\rangle \,-\, \big\langle\, s^\ell\big(\langle\, \theta\,, \psi'\,\rangle\big)\,,\, \phi'\,\big\rangle\Big) \mod h\, A_h \,+ \\
&\qquad +\, \textstyle\sum_{[\theta]} \Big(\big\langle\, s^\ell\big(\langle\, \theta_{[1]}\,, \phi'\,\rangle\big)\, \theta_{[2]}\,,\, \psi'\,\big\rangle \,-\, \big\langle\, s^\ell\big(\langle\, \theta_{[1]}\,, \psi'\,\rangle\big)\, \theta_{[2]}\,,\, \phi'\,\big\rangle\Big) \mod h\, A_h \; = \\
&= \; \Big(h^{-1}\Big\langle\, s^\ell\big(\langle\, \theta\,, \phi'\,\rangle\big) - t^\ell\big(\langle\, \theta\,, \phi'\,\rangle\big)\,,\, \psi'\,\Big\rangle \,-\, h^{-1}\Big\langle\, s^\ell\big(\langle\, \theta\,, \psi'\,\rangle\big) - t^\ell\big(\langle\, \theta\,, \psi'\,\rangle\big)\,,\, \phi'\,\Big\rangle\Big) \mod h\, A_h \,+ \\
&\qquad +\, \textstyle\sum_{[\theta]} \Big(\big\langle\, t^\ell\big(\langle\, \theta_{[1]}\,, \phi'\,\rangle\big)\, \theta_{[2]}\,,\, \psi'\,\big\rangle \,-\, \big\langle\, t^\ell\big(\langle\, \theta_{[1]}\,, \psi'\,\rangle\big)\, \theta_{[2]}\,,\, \phi'\,\big\rangle\Big) \mod h\, A_h \; = \\
&= \; \bigg(\Big\langle \frac{s^\ell\big(\langle\, \theta\,, \phi'\,\rangle\big) - t^\ell\big(\langle\, \theta\,, \phi'\,\rangle\big)}{h}\,, \psi' \Big\rangle \,-\, \Big\langle \frac{s^\ell\big(\langle\, \theta\,, \psi'\,\rangle\big) - t^\ell\big(\langle\, \theta\,, \psi'\,\rangle\big)}{h}\,, \phi' \Big\rangle\bigg) \mod h\, A_h \,+ \\
&\qquad +\, \textstyle\sum_{[\theta]} \Big(\big\langle\, \theta_{[2]}\,,\, \psi'\,\big\rangle \big\langle\, \theta_{[1]}\,, \phi'\,\big\rangle \,-\, \big\langle\, \theta_{[2]}\,,\, \phi'\,\big\rangle \big\langle\, \theta_{[1]}\,, \psi'\,\big\rangle\Big) \mod h\, A_h \; = \\
&= \; -\big\langle\, \delta_L\big(\langle\, \Theta\,, \Phi\,\rangle\big)\,,\, \Psi\,\big\rangle \,+\, \big\langle\, \delta_L\big(\langle\, \Theta\,, \Psi\,\rangle\big)\,,\, \Phi\,\big\rangle \,+\, \big\langle \Delta^{[1]}(\Theta) - \Delta^{[1]}(\Theta)_{2,1}\,,\, \Phi \otimes \Psi \big\rangle \; = \\
&= \; \omega''(\Phi)\big(\langle\, \Theta\,, \Psi\,\rangle\big) \,-\, \omega''(\Psi)\big(\langle\, \Theta\,, \Phi\,\rangle\big) \,-\, \big\langle \delta_L(\Theta)\,,\, \Phi \otimes \Psi \big\rangle
\end{aligned}$$

where we exploited the properties of a right bialgebroid pairing — in particular, the identity $\big\langle\, t^\ell(\alpha)\,,\, \chi'\,\big\rangle = \big\langle\, 1\,,\, \chi'\,\big\rangle\, \alpha$ — the fact that $\big\langle\, 1\,, \phi'\,\big\rangle = \partial_{J^r(L)_h}(\phi') = 0$, $\big\langle\, 1\,, \psi'\,\big\rangle = \partial_{J^r(L)_h}(\psi') = 0$, the fact that $s^\ell(\kappa) \equiv t^\ell(\kappa) \mod h\, V^\ell(L)_h$ and the fact (already proved) that $\omega'' = \omega'$. This proves the second identity in (4.6).

Finally, we have to deal with $V^\ell(L)_{h\,*}$. Acting much like for $V^\ell(L)_h^{\,*}$, one proves that $V^\ell(L)_{h\,*}$ is indeed a topological right bialgebroid, whose specialization modulo h is just $V^\ell(L)_* \cong J^\ell(L)$, hence we can claim that $V^\ell(L)_{h\,*} \in (\mathrm{RQFSAd})_{A_h}$ is a quantization of $J^\ell(L)$.

As to the last part of claim *(a)*, concerning the two Lie-Rinehart algebra structures induced on L^*, we can again proceed like for ${V^\ell(L)_h}^*$: the difference in the outcome — a minus sign — now is due to the fact that the natural pairing (given by evaluation) among the left bialgebroid $V^\ell(L)_h$ and the right bialgebroid ${V^\ell(L)_h}_*$ is a *left* bialgebroid pairing (cf. Definitions 3.33 and 3.34) — whereas in the case of $V^\ell(L)_h$ and ${V^\ell(L)_h}^*$ it is a *right* bialgebroid pairing.

In detail, direct computation (adopting similar notation as above) gives

$$\begin{aligned}
\omega'(\Phi)(a) \;&=\; \Big(\big(h^{-1}\big(\phi'\,s_r(a')-s_r(a')\,\phi'\big)\big) \mod h\,J^r(L)_h\Big) \mod \mathfrak{J}_{J(L)} \;= \\
&= \partial_{J(L)_h}\!\Big(\frac{\phi'\,s_r(a')-s_r(a')\,\phi'}{h}\Big) \mod h\,A_h \;=\; \Big\langle 1_{V^\ell(L)_h}\,,\,\frac{\phi'\,s_r(a')-s_r(a')\,\phi'}{h}\Big\rangle \mod h\,A_h \;= \\
&= h^{-1}\Big(\big\langle 1\,,\,\phi'\big\rangle\,a' - \big\langle t^\ell(a')\,,\,\phi'\big\rangle\Big) \mod h\,A_h \;=\; h^{-1}\Big(a'\,\big\langle 1\,,\,\phi'\big\rangle - \big\langle t^\ell(a')\,,\,\phi'\big\rangle\Big) \mod h\,A_h \;= \\
&= h^{-1}\Big(\big\langle s^\ell(a')\,,\,\phi'\big\rangle - \big\langle t^\ell(a')\,,\,\phi'\big\rangle\Big) \mod h\,A_h \;=\; \Big\langle \frac{s^\ell(a')-t^\ell(a')}{h}\,,\,\phi'\Big\rangle \mod h\,A_h \;= \\
&= -\Big\langle \frac{t^\ell(a')-s^\ell(a')}{h}\,,\,\phi'\Big\rangle \mod h\,A_h \;=\; -\big\langle \delta_L(a)\,,\Phi\big\rangle \;=\; -\,\omega''(\Phi)(a)
\end{aligned}$$

which proves that $\omega' = -\omega''$, and also

$$\begin{aligned}
\Big\langle \Theta\,,\,[\Phi,\Psi]'\Big\rangle \;&=\; \Big\langle \theta\,,\,\frac{\phi'\,\psi'\,-\,\psi'\,\phi'}{h}\Big\rangle \mod h\,A_h \;=\; h^{-1}\Big(\big\langle \theta\,,\,\phi'\,\psi'\big\rangle - \big\langle \theta\,,\,\psi'\phi'\big\rangle\Big) \mod h\,A_h \;= \\
&= h^{-1}\Big(\big\langle \theta_{(1)}\,,\,s_r\big(\langle \theta_{(2)}\,,\phi'\rangle\big)\,\psi'\big\rangle \,-\, \big\langle \theta_{(1)}\,,\,s_r\big(\langle \theta_{(2)}\,,\psi'\rangle\big)\,\phi'\big\rangle\Big) \mod h\,A_h \;= \\
&= h^{-1}\Big(\big\langle \theta\,,\,s_r\big(\langle 1\,,\phi'\rangle\big)\,\psi'\big\rangle + \big\langle 1\,,\,s_r\big(\langle \theta\,,\phi'\rangle\big)\,\psi'\big\rangle + h\textstyle\sum_{[\theta]}\big\langle \theta_{[1]}\,,\,s_r\big(\langle \theta_{[2]}\,,\phi'\rangle\big)\,\psi'\big\rangle \,- \\
&\quad -\big\langle \theta\,,\,s_r\big(\langle 1\,,\psi'\rangle\big)\,\phi'\big\rangle - \big\langle 1\,,\,s_r\big(\langle \theta\,,\psi'\rangle\big)\,\phi'\big\rangle - h\textstyle\sum_{[\theta]}\big\langle \theta_{[1]}\,,\,s_r\big(\langle \theta_{[2]}\,,\psi'\rangle\big)\,\phi'\big\rangle\Big) \mod h\,A_h \;= \\
&= h^{-1}\Big(\big\langle 1\,,\,s_r\big(\langle \theta\,,\phi'\rangle\big)\,\psi'\big\rangle - \big\langle 1\,,\,s_r\big(\langle \theta\,,\psi'\rangle\big)\,\phi'\big\rangle\Big) \mod h\,A_h \,+ \\
&\quad + \textstyle\sum_{[\theta]}\Big(\big\langle \theta_{[1]}\,,\,s_r\big(\langle \theta_{[2]}\,,\phi'\rangle\big)\,\psi'\big\rangle - \big\langle \theta_{[1]}\,,\,s_r\big(\langle \theta_{[2]}\,,\psi'\rangle\big)\,\phi'\big\rangle\Big) \mod h\,A_h \;= \\
&= h^{-1}\Big(\big\langle t^\ell\big(\langle \theta\,,\phi'\rangle\big)\,,\,\psi'\big\rangle - \big\langle t^\ell\big(\langle \theta\,,\psi'\rangle\big)\,,\,\phi'\big\rangle\Big) \mod h\,A_h \,+ \\
&\quad + \textstyle\sum_{[\theta]}\Big(\big\langle t^\ell\big(\langle \theta_{[2]}\,,\phi'\rangle\big)\,\theta_{[1]}\,,\,\psi'\big\rangle - \big\langle t^\ell\big(\langle \theta_{[2]}\,,\psi'\rangle\big)\,\theta_{[1]}\,,\,\phi'\big\rangle\Big) \mod h\,A_h \;= \\
&= \Big(h^{-1}\Big\langle t^\ell\big(\langle \theta\,,\phi'\rangle\big) - s^\ell\big(\langle \theta\,,\phi'\rangle\big)\,,\,\psi'\Big\rangle - h^{-1}\Big\langle t^\ell\big(\langle \theta\,,\psi'\rangle\big) - s^\ell\big(\langle \theta\,,\psi'\rangle\big)\,,\,\phi'\Big\rangle\Big) \mod h\,A_h \,+ \\
&\quad + \textstyle\sum_{[\theta]}\Big(\big\langle s^\ell\big(\langle \theta_{[2]}\,,\phi'\rangle\big)\,\theta_{[1]}\,,\,\psi'\big\rangle - \big\langle s^\ell\big(\langle \theta_{[2]}\,,\psi'\rangle\big)\,\theta_{[1]}\,,\,\phi'\big\rangle\Big) \mod h\,A_h \;= \\
&= \Big(\Big\langle \frac{t^\ell\big(\langle \theta\,,\phi'\rangle\big) - s^\ell\big(\langle \theta\,,\phi'\rangle\big)}{h}\,,\,\psi'\Big\rangle - \Big\langle \frac{t^\ell\big(\langle \theta\,,\psi'\rangle\big) - s^\ell\big(\langle \theta\,,\psi'\rangle\big)}{h}\,,\,\phi'\Big\rangle\Big) \mod h\,A_h \,+ \\
&\quad + \textstyle\sum_{[\theta]}\Big(\big\langle \theta_{[2]}\,,\,\phi'\big\rangle\big\langle \theta_{[1]}\,,\psi'\big\rangle - \big\langle \theta_{[2]}\,,\,\psi'\big\rangle\big\langle \theta_{[1]}\,,\phi'\big\rangle\Big) \mod h\,A_h \;= \\
&= \big\langle \delta_L\big(\langle \Theta\,,\Phi\rangle\big)\,,\,\Psi\big\rangle - \big\langle \delta_L\big(\langle \Theta\,,\Psi\rangle\big)\,,\,\Phi\big\rangle + \big\langle \Delta^{[1]}(\Theta)_{2,1} - \Delta^{[1]}(\Theta)\,,\,\Phi\otimes\Psi\big\rangle \;= \\
&= -\Big(\omega''(\Phi)\big(\langle \Theta\,,\Psi\rangle\big) - \omega''(\Psi)\big(\langle \Theta\,,\Phi\rangle\big) - \big\langle \delta_L(\Theta)\,,\,\Phi\otimes\Psi\big\rangle\Big)
\end{aligned}$$

which proves that $[\ ,\ ]' = -[\ ,\ ]''$, q.e.d.

(b) The proof given for claim *(a)* clearly adapts to claim *(b)* as well, by the same arguments. Otherwise, one can deduce claim *(b)* directly from claim *(a)* using general isomorphisms such as ${}_*\big(U^{op}_{coop}\big) \cong (U^*)^{op}_{coop}$ and ${}^*\big(U^{op}_{coop}\big) \cong (U_*)^{op}_{coop}$ (see Remark 3.36) as follows.

First of all, $\big(V^r(L)_h\big)^{op}_{coop}$ is obviously a quantization of $V^r(L)^{op}_{coop} = V^\ell(L^{op})$. Second, the structure of Lie bialgebra defined by $\big(V^r(L)_h\big)^{op}_{coop}$ and by $\big({}_*V^r(L)_h\big)^{op}_{coop} = \Big(\big(V^r(L)_h\big)^{op}_{coop}\Big)^*$ is L^{op}_{coop}. Hence the structure of Lie bialgebra defined by $V^r(L)_h$ and ${}_*\big(V^r(L)_h\big)$ is L.

Similarly, the structure of Lie bialgebra defined by $\big(V^r(L)_h\big)^{op}_{coop}$ and by $\big({}^*\big(V^r(L)_h\big)\big)^{op}_{coop} = \Big(\big(V^r(L)_h\big)^{op}_{coop}\Big)_*$ are respectively L^{op}_{coop} and $\big(L^{op}_{coop}\big)_{coop} = L^{op}$. Thus the structure of Lie bialgebra defined by $V^r(L)_h$ and by ${}^*\big(V^r(L)_h\big)$ are L and L_{coop} respectively. $\square$

5.2 Linear duality for quantum QFSAd's

Much like for their classical counterparts, the duals for QFSAD's have to be meant in topological sense. Indeed, we introduce now a suitable definition of "continuous" dual of a (R/L)QFSAd:

Definition 5.6. *Let* $K_h \in (\mathrm{RQFSAd})_{A_h}$. *Let* $I_h := \big\{ \lambda \in K_h \,\big|\, \partial_h(\lambda) \in hA_h \big\}$.

We denote by ${}_\star K_h$ *the* $k[[h]]$*–submodule of* ${}_* K_h$ *of all (left* A_h*–linear) maps from* K_h *to* A_h *which are continuous for the* I_h*–adic topology on* K_h *and the* h*–adic topology on* A_h .

We denote by ${}^\star K_h$ *the* $k[[h]]$*–submodule of* ${}^* K_h$ *of all (right* A_h*–linear) maps from* K_h *to* A_h *which are continuous for the* I_h*–adic topology on* K_h *and the* h*–adic topology on* A_h .

In a similar way, we define also "continuous dual" objects $K_{h\,\star}$ $\big(\subseteq K_{h\,*}\big)$ *and* $K_h^\star$ $\big(\subseteq K_h^*\big)$ *for every* $K_h \in (\mathrm{LQFSAd})_{A_h}$.

It is time for our second result about linear duality of "quantum groupoids". In short, it claims that the left and the right continuous dual of a left, resp. right, quantum formal series algebroid are both right, resp. left, quantum universal enveloping algebroids.

Theorem 5.7.

(a) If $J^r(L)_h \in (\mathrm{RQFSAd})_{A_h}$, *then* ${}_\star J^r(L)_h \, , {}^\star J^r(L)_h \in (\mathrm{LQUEAd})_{A_h}$, *with semiclassical limits (cf. Remark 3.45)*

$${}_\star J^r(L)_h \Big/ h\, {}_\star J^r(L)_h \;\cong\; {}_\star J^r(L) \;=\; V^\ell(L) \qquad \text{and} \qquad {}^\star J^r(L)_h \Big/ h\, {}^\star J^r(L)_h \;\cong\; {}^\star J^r(L) \;\cong\; V^\ell\big(L\big)$$

Therefore ${}_\star J^r(L)_h$ *and* ${}^\star J^r(L)_h$ *are quantizations of* $V^\ell(L) = {}_\star J^r(L)$.

Moreover, the structure of Lie-Rinehart bialgebra induced on L *by the quantization* ${}_\star J^r(L)_h$ *of* $V^\ell(L)$ *— according to Theorem 4.4 — is the same as that induced by the quantization* $J^r(L)_h$ *of* $J^r(L)$ *— according to Theorem 4.12.*

On the other hand, the structure of Lie-Rinehart algebra induced on L^* *by the quantization* ${}^\star J^r(L)_h$ *of* $V^\ell\big(L\big)$ *is opposite to that induced by the quantization* $J^r(L)_h$ *of* $J^r(L)$. *Therefore, the structures of Lie-Rinehart bialgebra induced on* L *in the two cases are coopposite to each other: in short,* $J^r(L)_h$ *provides a quantization of the Lie-Rinehart bialgebra* L , *while* ${}^\star J^r(L)_h$ *provides a quantization of the coopposite Lie-Rinehart bialgebra* L_{coop} *— cf. Remarks 2.21(e).*

(b) If $J^\ell(L)_h \in (\mathrm{LQFSAd})_{A_h}$, *then* $J^\ell(L)_h^\star \, , J^\ell(L)_{h\,\star} \in (\mathrm{RQUEAd})_{A_h}$, *with semiclassical limits (cf. Remark 3.45)*

$$J^\ell(L)_h^\star \Big/ h\, J^\ell(L)_h^\star \;\cong\; J^\ell(L)^\star \;=\; V^r(L) \qquad \text{and} \qquad J^\ell(L)_{h\,\star} \Big/ h\, J^\ell(L)_{h\,\star} \;\cong\; J^\ell(L)_\star \;\cong\; V^r\big(L\big)$$

Therefore $J^\ell(L)_h^\star$ *and* $J^\ell(L)_{h\,\star}$ *are quantization of* $V^r(L) = J^\ell(L)^\star$.

Moreover, the structure of Lie-Rinehart bialgebra induced on L *by the quantization* $J^\ell(L)_h^\star$ *of* $V^r(L)$ *— according to Theorem 4.7 — is the same as that induced by the quantization* $J^\ell(L)_h$ *of* $J^\ell(L)$ *— according to Theorem 4.13.*

On the other hand, the structure of Lie-Rinehart algebra induced on L^* *by the quantization* $J^\ell(L)_{h\,\star}$ *of* $V^r\big(L\big)$ *is opposite to that induced by the quantization* $J^\ell(L)_h$ *of* $J^\ell(L)$. *Therefore, the structures of Lie-Rinehart bialgebra induced on* L *in the two cases are coopposite to each other: in short,* $J^\ell(L)_h$ *provides a quantization of the Lie-Rinehart bialgebra* L , *while* $J^\ell(L)_{h\,\star}$ *provides a quantization of the coopposite Lie-Rinehart bialgebra* L_{coop} *— cf. Remarks 2.21(e).*

Proof. (a) To simplify notation, we write $K_h := J^r(L)_h$. We begin with the case of ${}_\star K_h$, moving through several steps.

The main point in the proof is the following. By definition, ${}_\star K_h$ is contained in ${}_* K_h$: by the recipe in §3.5, the latter is "almost" a left bialgebroid over A_h , as it has a natural structure of A_h^e–ring with "counit", and also a "candidate" as coproduct. Then the natural pairing among ${}_* K_h$ and K_h (given by evaluation), hereafter denoted $\langle\ ,\ \rangle$, is an A_h^e–right pairing (cf. Definition 3.33), and also a bialgebroid right pairing (cf. Definition 3.34) — as far as this makes sense. Basing on this, we shall presently show that this structure on ${}_* K_h$ — which makes it an A_h^e–ring and even "almost a left A_h^e–bialgebroid", actually does restrict to ${}_\star K_h$, making it into a left A_h^e–bialgebroid. Also, the evaluation will then provide a natural bialgebroid right pairing between ${}_\star K_h$ and K_h .

Along the way, we shall prove also that ${}_\star K_h$ has semiclassical limit $V^\ell(L)$, and finally that the Lie-Rinehart bialgebra structure on L induced by it is the same as that induced by $K_h := J^r(L)_h$.

(1) First we prove that the source and target maps of ${}_* K_h$ (as given in §3.5) actually map into ${}_\star K_h$, that is $s^\ell_*(A_h) \subseteq {}_\star K_h$ and $t^\ell_*(A_h) \subseteq {}_\star K_h$. We shall prove it by showing that, for any $a \in A_h$, one has $\langle s^\ell_*(a)\,,\,I_h^n\rangle \subseteq h^n A_h$, $\langle t^\ell_*(a)\,,\,I_h^n\rangle \subseteq h^n A_h$, for all $n \in \mathbb{N}$.

For t^ℓ_*, if $\phi \in I_h$, then $\langle\, t^\ell_*(a)\,,\,\phi\,\rangle = \langle\, 1\,,\phi\,\rangle\, a \,\in\, h\,A_h = h^1 A_h$; thus $\langle\, t^\ell_*(a)\,,\,I_h^1\,\rangle \subseteq h^1 A_h$.

Now assume by induction that $\langle\, t^\ell_*(a)\,,\,I_h^m\rangle \subseteq h^m A_h$. Let $\psi \in I_h^m$ and $\chi \in I_h$; then

$$\langle\, t^\ell_*(a)\,,\psi\,\chi\,\rangle \;=\; \langle\, 1\,,\psi\,\chi\,\rangle\, a \;=\; \langle\, s^\ell_*\big(\langle\, 1\,,\psi\,\rangle\big)\,1\,,\,\chi\,\rangle\, a$$

thus by the induction hypothesis and the case $m=1$ we see that $\langle t^\ell_*(a)\,,\psi\,\chi\rangle \in h^{m+1} A_h$.

As to s^ℓ_*, if $\phi \in I_h$, then we have

$$\langle\, s^\ell_*(a)\,,\,\phi\,\rangle \;=\; \langle\, 1\,,t_r(a)\,\phi\,\rangle \;\in\; \langle\, 1\,,\phi\,t_r(a)\,\rangle \,+\, h\,A_h \;=\; a\,\langle\, 1\,,\phi\,\rangle \,+\, h\,A_h \;=\; h\,A_h$$

because $t_r(a)\,\phi \,-\, \phi\,t_r(a) \,\in\, h\,K_h$ and $\langle\, 1\,,\phi\,\rangle \in h\,A_h$.

Now assume by induction that $\langle\, s^\ell_*(a)\,,\,I_h^m\rangle \subseteq h^m A_h$. Let $\psi \in I_h^m$ and $\chi \in I_h$; then

$$\langle\, s^\ell_*(a)\,,\psi\,\chi\,\rangle \;=\; \langle\, 1\,,t_r(a)\,\psi\,\chi\,\rangle \;=\; \langle\, s^\ell_*\big(\langle\, 1\,,t_r(a)\,\psi\,\rangle\big)\,1\,,\,\chi\,\rangle \;=\; \langle\, s^\ell_*\big(\langle\, s^\ell_*(a)\,,\psi\,\rangle\big)\,1\,,\,\chi\,\rangle$$

thus by the induction hypothesis and the case $m=1$ we argue that $\langle s^\ell_*(a)\,,\psi\,\chi\rangle \in h^{m+1} A_h$.

(2) Let us show that if $\omega, \omega' \in {}_\star K_h$, then $\omega\,\omega' \in {}_\star K_h$. Given $n \in \mathbb{N}$, let $p, q \in \mathbb{N}$ be such that $\langle \omega\,,I_h^p\,\rangle \in h^n A_h$ and $\langle \omega',I_h^q\,\rangle \in h^n A_h$. Now take $\eta \in I_h^{p+q}$. Then the identity

$$\langle\, \omega\,\omega',\,\eta\,\rangle \;=\; \langle\, \omega\, t^\ell_*\big(\langle \omega',\,\eta_{(2)}\rangle\big)\,,\,\eta_{(1)}\rangle \;=\; \langle \omega\,,\,\eta_{(1)}\,s_r\big(\langle \omega',\,\eta_{(2)}\rangle\big)\rangle$$

taking into account that $\Delta\big(I_h^{p+q}\big) \subseteq \sum_{r+s=p+q} I_h^r \otimes_{A_h} I_h^s$ because $\Delta(I_h) \subseteq K_h \otimes_{A_h} I_h + I_h \otimes_{A_h} K_h$, proves that $\langle\, \omega\,\omega',\,I_h^{p+q}\,\rangle \in h^n A_h$. Thus ${}_\star K_h$ is a subring of ${}_* K_h$ — even an A_h^e–subring, by *(1)*.

(3) Let us show that ${}_\star K_h$ is topologically free. First we prove that it is complete for the h–adic topology. Indeed, as K_h is topologically free (for its own h–adic topology), so is $\mathit{Hom}_{k[[h]]}(K_h\,,A_h)$ as well. Now let $(\lambda_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of elements in ${}_\star K_h$; then this sequence converges to a unique limit $\lambda \in \mathit{Hom}_{k[[h]]}(K_h\,,A_h)$. Then it is easy to see that $\lambda \in \mathit{Hom}_{A_h}(K_h\,,A_h)$.

Now we show that $\lambda \in {}_\star K_h$. Given $n \in \mathbb{N}$, there exists $n_1 \in \mathbb{N}$ such that $\lambda_{n_1} - \lambda$ takes values in $h^n A_h$. As $\lambda_{n_1} \in {}_\star K_h$, there exists $n_2 \in \mathbb{N}$ such that $\langle \lambda_{n_1}\,,I_h^{n_2}\rangle \in h^n A_h$. But then we have $\langle \lambda\,,I_h^{n_2}\rangle \in h^n A_h$ and so we conclude that $\lambda \in {}_\star K_h$.

Finally, as ${}_\star K_h$ is complete for the h–adic topology and without torsion, it is topologically free.

(4) Now we show that ${}_\star K_h \Big/ h\,{}_\star K_h \;=\; {}_\star\big(K_h\big/\,h\,K_h\big) \;=\; {}_\star J^r(L) \;=\; V^\ell(L)$.

Let $\lambda \in {}_\star K_h$, so that λ as a map from K_h (with the I_h–adic topology) to A_h (with the h–adic topology) is continuous. Then λ induces (modulo h) a map $\overline{\lambda} : J^r(L) \longrightarrow A$ which is 0 on $\mathfrak{J}^n$ for $n \gg 0$, where $\mathfrak{J} := \mathit{Ker}\big(\partial_{J^r(L)}\big)$. We claim that the kernel of the map $\chi : \lambda \mapsto \overline{\lambda}$ is $h\,{}_\star K_h$: indeed, it is obvious that $h\,{}_\star K_h \subseteq \mathit{Ker}(\chi)$, and the inverse inclusion follows from the fact that A_h is topologically free. Therefore we have an injective map $\overline{\chi} : {}_\star K_h\big/h\,{}_\star K_h \longrightarrow {}_\star\big(K_h\big/\,h\,K_h\big) = {}_\star J^r(L) \,=\, V^\ell(L)$ induced by χ (modulo h), and we are left to show that $\overline{\chi}$ is surjective too.

We distinguish two cases:

Finite free case: Assume that L as an A–module is *free of finite type*. Let $\{\overline{e}_1\,,\dots,\overline{e}_n\}$ be an A–basis of L. Then $\big\{\, \overline{e}^{\,\underline{\alpha}} := \overline{e}_1^{\,\alpha_1} \cdots \overline{e}_n^{\,\alpha_n} \,\big|\, \underline{\alpha} := (\alpha_1\,,\dots,\,\alpha_n) \in \mathbb{N}^n \,\big\}$ is a basis of $V^\ell(L)$, by the Poincaré-Birkhoff-Witt theorem. Define $\overline{\xi}_i \in K := J^r(L)$ by $\big\langle\, \overline{\xi}_i\,,\overline{e}_1^{\,\alpha_1} \cdots \overline{e}_n^{\,\alpha_n}\big\rangle \,=\, \prod_{j=1}^n \delta_{\alpha_j\,,\,\delta_{i,j}}$.

Let $\xi_i \in K_h$ be a lifting of $\overline{\xi}_i$ such that $\partial_h(\xi_i) = 0$. We denote (ordered) monomials in the $\overline{\xi}_i$'s or in the ξ_i's by $\underline{\overline{\xi}}^{\,\underline{\alpha}} := \overline{\xi}_1^{\,\alpha_1} \cdots \overline{\xi}_n^{\,\alpha_n}$ and $\underline{\xi}^{\,\underline{\alpha}} := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ respectively. Note that $\underline{\xi}^{\,\underline{\alpha}} \in I_h^{|\underline{\alpha}|}$, where $|\underline{\alpha}| := \sum_{i=1}^n \alpha_i$. Let $\lambda \in {}_\star\big(K_h\big/\,h\,K_h\big) = {}_\star J^r(L) = V^\ell(L)$ be given: we write $\bar{a}_{\underline{\alpha}} := \big\langle\, \lambda\,,\,\underline{\overline{\xi}}^{\,\underline{\alpha}}\,\big\rangle \in A$, and note that all but finitely many of the $\bar{a}_{\underline{\alpha}}$'s are zero. Let $a_{\underline{\alpha}} \in A_h$ be any lifting of $\bar{a}_{\underline{\alpha}}$ (for all $\underline{\alpha} \in \mathbb{N}^n$), with the condition that whenever $\bar{a}_{\underline{\alpha}} = 0$ we take also $a_{\underline{\alpha}} = 0$. Now we define $\Lambda \in {}_* K_h$

by setting $\big\langle \Lambda\,,\underline{\xi}^{\underline{\alpha}} \big\rangle := a_{\underline{\alpha}}$. As $I_h^m = \sum\limits_{|\underline{\alpha}|+s\geq m} h^s\,\underline{\xi}^{\underline{\alpha}}\,t_r(A_h)$, it is easy to check that if $n\in\mathbb{N}$ then $\big\langle \Lambda, I_h^m \big\rangle \subseteq h^n A_h$ for $m\gg 0$. Hence $\Lambda \in {}_\star K_h$, and by construction $\overline{\chi}\,\big(\Lambda \bmod h\,{}_\star K_h\big)=\lambda$, so that the map $\overline{\chi}$ is onto, q.e.d.

General case: By our overall assumption, L as an A–module is *projective of finite type*. Then we resume the setup and notation of in §2.12: there exists a finitely generated projective A–module Q such that $L\oplus Q=F$ is a finite free A–module, and we consider the free A–module $L_Q := L\oplus\big(A\otimes_k Z\big)$ with $Z:=Y\oplus Y\oplus Y\oplus\cdots$. From an A–basis $\{b_1\,,\dots,\,b_n\}$ of Y we get a "good basis" of elements $\bar{e}_t$ indexed by $T:=\mathbb{N}\times\{1,\dots,n\}$, i.e. $L_Q=\bigoplus\limits_{t\in T} k\,\bar{e}_t$. Fixing on T any total order, the PBW theorem yields $\big\{\,\underline{\bar{e}}^{\,\underline{\alpha}} := \prod_{t\in T}\bar{e}_t^{\,\alpha_t}\,\big|\,\underline{\alpha}=(\alpha_t)_{t\in T}\in T^{(\mathbb{N})}\big\}$ is an A–basis of $V^\ell(L_Q)$. Let $\overline{\xi}_j$ be the element of $J_f^r(L_Q)$ defined by $\big\langle \overline{\xi}_j\,,\underline{\bar{e}}^{\,\underline{\alpha}}\big\rangle = 1$ if $\underline{\alpha}=(\alpha_t=\delta_{t,j})_{t\in T}$, $\big\langle \overline{\xi}_j\,,\underline{\bar{e}}^{\,\underline{\alpha}}\big\rangle = 0$ otherwise. If $A\big[\big[\{X_t\}_{t\in T}\big]\big]_f := \bigcup\limits_{i_1<\cdots<i_n} A[[X_{i_1}\,,\dots,X_{i_n}]]$, then one has $J_f^r(L_Q)=A\big[\big[\{\overline{\xi}_t\}_{t\in T}\big]\big]_f$.

Now consider the quantization $K_{h,Y}$ of $J_f^r(L_Q)$ — cf. §4.17. Recall (cf. §4.17) that

$$K_{h,Y} \;:=\; h\text{–adic completion of } \;\textstyle\sum_{n\in\mathbb{N}} K_h\widetilde{\otimes}_k S(Y^*)^{\widetilde{\otimes}\,n}\otimes 1\otimes 1\otimes 1\cdots$$

where $K_h\,\widetilde{\otimes}_k\,S(Y^*)^{\widetilde{\otimes}\,n}\otimes 1\otimes 1\cdots$ is the $\big(\big(S(Y^*)^{\otimes n}\big)^+\otimes 1\otimes 1\cdots\big)$–adic completion of $K_h\otimes_k S(Y^*)^{\otimes n}\otimes 1\otimes 1\cdots$. By construction, every $\overline{\xi}_i$ belongs to some $K\widetilde{\otimes}_k S(Y^*)^{\widetilde{\otimes}\,n_i}\otimes_k 1\otimes_k 1\otimes_k\cdots$ ($n_i\in\mathbb{N}$). Let ξ_i be any lifting of $\overline{\xi}_i$ in $K_h\widetilde{\otimes}_k S(Y^*)^{\widetilde{\otimes}\,n_i}\otimes_k 1\otimes_k 1\otimes_k\cdots$ such that $\big(\partial_h\otimes\epsilon_{S(Z^*)}\big)(\xi_i)=0$. Given $a\in A_h$, we denote again by a the element $t^r(a)\in t^r(A_h)\subseteq K_h$. Let also $\sigma: A\lhook\joinrel\longrightarrow A_h$ be a section of the natural projection map from A_h to A , let $\mathfrak{I}_{h,Y}:=Ker(\partial_h)=\partial_h^{-1}\big(\{0\}\big)$ and $I_{h,Y}:=\partial_h^{-1}\big(h\,A_h\big)$: taking into account that $t^r(A)=A^{op}$ and $t^r(A_h)=A_h^{op}$, one has

$$K_{h,Y} \;=\; \Big\{\textstyle\sum_{n\in\mathbb{N}} h^n P_n^\sigma\big(\underline{\xi}\big)\,\Big|\,P_n\in\big[\big[\{X_t\}_{t\in T}\big]\big]_f\,A^{op}\Big\} \;=\; \Big\{\textstyle\sum_{n\in\mathbb{N}} h^n P_n\big(\underline{\xi}\big)\,\Big|\,P_n\in\big[\big[\{X_t\}_{t\in T}\big]\big]_f\,A_h^{op}\Big\}$$

$$I_{h,Y} \;=\; \big(h\,,\{\xi_t\}_{t\in T}\big) \qquad,\qquad \mathfrak{I}_{h,Y} \;=\; \textstyle\sum_{t\in T}\xi_t\,K_{h,Y}$$

where round braces stand for "two-sided ideal generated by", and $\big[\big[\{X_t\}_{t\in T}\big]\big]_f\,A$, respectively $\big[\big[\{X_t\}_{t\in T}\big]\big]_f\,A_h$, denotes the ring of formal power series with coefficients *on the right* chosen in A , respectively in A_h , each one involving only finitely many indeterminates X_t .

Now, L_Q as an A–module is free but not finite; however, $J_f^r(L_Q)$ and its quantization $K_{h,Y}$ have enough "finiteness" behavior as to let the arguments for the finite free case apply again. In other words, the analysis we carried on for the finite free case can be applied again in the present, general context working with $K_{h,Y}$. Indeed, let us remark that

$$I_{h,Y} \;:=\; h\text{–adic completion of } \bigg(\sum_{n\in\mathbb{N}} I_h\underset{k}{\widetilde{\otimes}} S(Y^*)^{\widetilde{\otimes}_k n}\underset{k}{\otimes}1\underset{k}{\otimes}\cdots + \sum_{n\in\mathbb{N}_+} K_h\underset{k}{\widetilde{\otimes}}\Big(S(Y^*)^{\widetilde{\otimes}_k n}\Big)^+\underset{k}{\otimes}1\underset{k}{\otimes}\cdots\bigg)$$

while on the other hand $V^\ell(L_Q)=V^\ell(L)\oplus\big(V^\ell(L)\otimes_k S(Z)^+\big)$. Now let $K:=J^r(L)$ and $\lambda\in{}_\star K$. As $J_f^r(L_Q)=K\oplus\sum_{n\in\mathbb{N}_+}K\widetilde{\otimes}_k\big(S(Y^*)^{\widetilde{\otimes}\,n}\big)^+\otimes_k 1\otimes_k\cdots$, we can extend λ to an element $\mu\in{}_\star J_f^r(L_Q)$ by setting $\mu\big|_{\sum_{n\in\mathbb{N}_+}K\,\widetilde{\otimes}_k\,(S(Y^*)^{\otimes n})^+\otimes_k 1\otimes_k\cdots} := 0$ and $\mu\big|_K:=\lambda$. By the arguments used in the finite free case, μ can be lifted to an element $M\in{}_{\star_f}K_{h,Y}$; then $\Lambda=M\big|_{K_h}\in{}_\star K_h$ is a lift of λ . So that the (injective) map $\overline{\chi}:{}_\star K_h\big/h\,{}_\star K_h\longrightarrow{}_\star\big(K_h\big/h\,K_h\big)={}_\star J^r(L)=V^\ell(L)$ is surjective, q.e.d.

(5) Let us now show that $\Delta\big({}_\star K_h\big)\subseteq{}_\star K_h\,\widehat{\otimes}_{A_h}{}_\star K_h$ for the "coproduct map" Δ given by the transpose map of the multiplication in K_h .

Let $\Lambda\in{}_\star K_h$. We know that modulo h one has $\overline{\Delta(\Lambda)}\in{}_\star K\otimes_A{}_\star K$. Now write $\overline{\Delta(\Lambda)}=\sum\lambda^{(1)}\otimes\lambda^{(2)}$ (a finite sum) with $\lambda^{(1)},\lambda^{(2)}\in{}_\star K$, and let $\Lambda_h^{(1)}$ and $\Lambda_h^{(2)}$ in ${}_\star K_h$ be liftings of $\lambda^{(1)}$ and $\lambda^{(2)}$, i.e. $\overline{\Lambda_h^{(1)}}=\lambda^{(1)}$ and $\overline{\Lambda_h^{(2)}}=\lambda^{(2)}$: then $\Delta(\Lambda_h)-\sum\Lambda_h^{(1)}\otimes\Lambda_h^{(2)}\in h\,\big({}_*K_h\,\widehat{\otimes}_{A_h}{}_*K_h\big)$, so that $h^{-1}\big(\Delta(\Lambda_h)-\sum\Lambda_h^{(1)}\otimes\Lambda_h^{(2)}\big)\in{}_*K_h\,\widehat{\otimes}_{A_h}{}_*K_h$. In addition, whenever $p+q\gg 0$ one has also $\big\langle\Delta(\Lambda_h)-\sum\Lambda_h^{(1)}\otimes\Lambda_h^{(2)}\,,I_h^p\otimes I_h^q\big\rangle\in h^2A_h$; therefore we find that

$$\overline{h^{-1}\big(\Delta(\Lambda_h)-\sum\Lambda_h^{(1)}\otimes\Lambda_h^{(2)}\big)}\in{}_\star K\otimes_A{}_\star K$$

We can carry on this argument and eventually show that $\Delta(\Lambda_h)\in{}_\star K_h\,\widehat{\otimes}_{A_h}{}_\star K_h$, q.e.d.

(6) Altogether, the steps *(1)–(5)* above prove that ${}_\star K_h$ is a LQUEAd (over A_h), whose semiclassical limit ${}_\star K_h \big/ h\, {}_\star K_h$ is exactly isomorphic (as a left bialgebroid over A) to $V^\ell(L)$. Now we show that the structure of Lie-Rinehart bialgebra induced on L by the quantization ${}_\star K_h$ of $V^\ell(L)$ is the same as that induced by the quantization K_h of $J^r(L)$. To this end, let $[\ ,\ ]'$, ω', be the Lie bracket and the anchor map on L^* induced by ${}_\star K_h$, and $[\ ,\ ]''$, ω'', those induced by K_h.

We proceed like in the proof of Theorem 5.5. Our goal is to prove that $\omega' = \omega''$ and $[\ ,\ ]' = [\ ,\ ]''$; thus recall that (cf. Remarks 2.21*(b)*) ω' and $[\ ,\ ]'$ are uniquely determined by

$$\omega'(\Phi)(a) = \big\langle \delta_L(a)\,, \Phi \big\rangle \quad , \quad \big\langle\, \Theta\,, [\Phi\,, \Psi]' \,\big\rangle = \omega'(\Phi)\big(\langle\, \Theta\,, \Psi \rangle\big) \,-\, \omega'(\Psi)\big(\langle \Theta\,, \Phi \rangle\big) \,-\, \big\langle \delta_L(\Theta)\,, \Phi \otimes \Psi \big\rangle$$

(for all $\Phi\,, \Psi \in L^*\,,\ \Theta \in L\,,\ a \in A$), where $\delta_L(a)$ and $\delta_L(\Theta)$ are defined by the formula for δ in Theorem 4.4. Thus it is enough to prove that (for all $\Phi\,, \Psi \in L^*\,,\ \Theta \in L\,,\ a \in A$)

$$\omega''(\Phi)(a) = \big\langle \delta_L(a)\,, \Phi \big\rangle\,,\ \ \big\langle\, \Theta, [\Phi\,, \Psi]'' \,\big\rangle = \omega'(\Phi)\big(\langle\, \Theta, \Psi\rangle\big) - \omega'(\Psi)\big(\langle \Theta, \Phi\rangle\big) - \big\langle \delta_L(\Theta)\,, \Phi \otimes \Psi \big\rangle \qquad (4.7)$$

To prove (4.7), choose liftings $\phi', \psi' \in J^r(L)_h =: K_h$, with the additional condition that $\phi', \psi' \in \mathfrak{J}_{J^r(L)_h} := \mathit{Ker}\big(\partial_{J^r(L)_h}\big)$ (this is always possible), a lifting $\theta \in V^\ell(L)_h := {}_\star J^r(L)_h$ of Θ and a lifting $a' \in A_h$ of a. Now direct computation gives

$$\begin{aligned} \omega'(\Phi)(a) \;&=\; \big\langle\, \delta_L(a)\,, \Phi \big\rangle \;=\; \bigg\langle\, \frac{t^\ell_*\big(a'\big) - s^\ell_*\big(a'\big)}{h}\,, \phi' \,\bigg\rangle \ \bmod h\, A_h \;= \\ &=\; h^{-1} \big\langle\, t^\ell_*\big(a'\big) - s^\ell_*\big(a'\big)\,, \phi' \,\big\rangle \ \bmod h\, A_h \;=\; h^{-1} \big\langle\, 1\,, \phi'\, s_r\big(a'\big) - s_r\big(a'\big)\, \phi' \,\big\rangle \ \bmod h\, A_h \;= \\ &=\; \bigg\langle 1\,, \frac{\phi'\, s_r\big(a'\big) - s_r\big(a'\big)\, \phi'}{h} \bigg\rangle \ \bmod h\, A_h \;= \\ &=\; \bigg(\frac{\phi'\, s_r\big(a'\big) - s_r\big(a'\big)\, \phi'}{h} \ \bmod h\, K_h \bigg) \ \bmod \mathfrak{J}_{J^r(L)} \;=\; \omega''(\Phi)(a) \end{aligned}$$

where we exploited the fact that the involved pairing a *right* bialgebroid pairing (cf. Definitions 3.33 and 3.34). Thus the first identity in (4.7) is verified.

As to the rest, we write $\Delta(\theta) = \theta_{(1)} \otimes \theta_{(2)}$ as $\Delta(\theta) = \theta \otimes 1 + 1 \otimes \theta + h\, \theta_{[1]} \otimes \theta_{[2]}$, so that $\big(\theta_{[1]} \otimes \theta_{[2]}\big) \ \bmod h\, V^\ell(L)_h \,{}_\triangleleft\!\underset{A}{\times}{}_\triangleright V^\ell(L)_h \;=:\; \Delta^{[1]}(\Theta)$, as in Definition 4.4. Moreover, let us set $\phi := \phi' \bmod h\, J^r(L)_h\,,\ \psi := \psi' \bmod h\, J^r(L)_h$, which are lifts of Φ and Ψ in $J^r(L)$, and actually belong to $\mathfrak{J}_{J^r(L)}$. Then direct computation gives

$$\Big\langle \Theta\,, [\Phi, \Psi]'' \Big\rangle \;=\; \big\langle\, \Theta\,, \{\phi, \psi\} \,\big\rangle \;=\; \bigg\langle \theta\,, \frac{\phi'\, \psi' \,-\, \psi'\, \phi'}{h} \bigg\rangle \ \bmod h\, A_h$$

Now, in the proof of Theorem 5.5 — namely, to prove the second part of (4.6) — we saw that

$$\bigg\langle \theta\,, \frac{\phi'\, \psi' \,-\, \psi'\, \phi'}{h} \bigg\rangle \ \bmod h\, A_h \;=\; \omega'(\Phi)\big(\langle\, \Theta\,, \Psi \,\rangle\big) \,-\, \omega'(\Psi)\big(\langle\, \Theta\,, \Phi \,\rangle\big) \,-\, \big\langle \delta_L(\Theta)\,, \Phi \otimes \Psi \big\rangle$$

so that the second identity in (4.7) is proved.

At last, let now cope with the case of ${}^\star K_h$. Clearly, we can proceed much like for ${}_\star K_h$: one proves that ${}^\star K_h = {}^\star J^r(L)_h$ is a topological left bialgebroid, whose specialization modulo h is ${}^\star J^r(L) \cong V^\ell(L)$, hence we can claim that ${}^\star J^r(L)_h \in (\mathrm{LQUEAd})_{A_h}$ is a quantization of $V^\ell(L)$.

A difference arises about the last part of claim *(a)*, concerning the two Lie-Rinehart algebra structures induced on L^*: indeed, the difference in the outcome — a minus sign — is due to the fact that the natural pairing (given by evaluation) among the left bialgebroid ${}^\star J^r(L)_h$ and the right bialgebroid $J^r(L)_h$ is now a *left* bialgebroid pairing (cf. Definitions 3.33 and 3.34) — while for ${}_\star J^r(L)_h$ and $J^r(L)_h$ it is a *right* one. Explicit computations are (again) much like those in the proof of Theorem 5.5 (for the very last part of claim *(a)*), just as it occurs for ${}_\star K_h = {}_\star J^r(L)_h$.

(b) The arguments used to prove claim *(a)* clearly adapt to claim *(b)* as well. Otherwise, one can deduce *(b)* directly from claim *(a)* using general isomorphisms such as ${}_\star\big(U^{op}_{coop}\big) \cong \big(U^\star\big)^{op}_{coop}$ and ${}^\star\big(U^{op}_{coop}\big) \cong \big(U_\star\big)^{op}_{coop}$ — see Remark 3.36. $\square$

5.3 Functoriality of linear duality for quantum groupoids

The results in Sections 5.1 and 5.2 about the duality constructions for quantum bialgebroids can be improved. Indeed, they can be cast in the following, functorial version (cf. Definition 4.2 and 4.9 for notation), which is the main outcome of this section:

Theorem 5.8. *Left and right duals yield pairs of well-defined contravariant functors*

$$\begin{array}{llll}
(\mathrm{LQUEAd})_{A_h} \longrightarrow (\mathrm{RQFSAd})_{A_h}\,, & H_h \mapsto H_h^*\,, & (\mathrm{RQFSAd})_{A_h} \longrightarrow (\mathrm{LQUEAd})_{A_h}\,, & K_h \mapsto {}_\star K_h \\
(\mathrm{LQUEAd})_{A_h} \longrightarrow (\mathrm{RQFSAd})_{A_h}\,, & H_h \mapsto H_{h\,*}\,, & (\mathrm{RQFSAd})_{A_h} \longrightarrow (\mathrm{LQUEAd})_{A_h}\,, & K_h \mapsto {}^\star K_h \\
(\mathrm{RQUEAd})_{A_h} \longrightarrow (\mathrm{LQFSAd})_{A_h}\,, & H_h \mapsto {}^*H_h\,, & (\mathrm{LQFSAd})_{A_h} \longrightarrow (\mathrm{RQUEAd})_{A_h}\,, & K_h \mapsto K_{h\,\star} \\
(\mathrm{RQUEAd})_{A_h} \longrightarrow (\mathrm{LQFSAd})_{A_h}\,, & H_h \mapsto {}_*H_h\,, & (\mathrm{LQFSAd})_{A_h} \longrightarrow (\mathrm{RQUEAd})_{A_h}\,, & K_h \mapsto K_h^\star
\end{array}$$

which are (pairwise) inverse to each other, hence yield pairs of antiequivalences of categories.

Proof. It is clearly enough to present the proof for just one pair of functors, say those in first line.

Let $H_h = V^\ell(L)_h \in (\mathrm{LQUEAd})_{A_h}$. For any $\lambda \in H_h^*$ and any $\eta \in H_h$, let $ev_\eta(\lambda) := \lambda(\eta)$, and consider the map $H_h \longrightarrow {}_\star\big(H_h^*\big)$ given by $\eta \mapsto ev_\eta$; note that *a priori* this map takes values in ${}_*\big(H_h^*\big)$, but Lemma 5.4 actually proves that every ev_η belongs to ${}_\star(H^*)$.

We claim that this map is an isomorphism in $(\mathrm{LQUEAd})_{A_h}$. Indeed, by construction this is a morphisms of left bialgebroids; moreover, it is even an isomorphism, because both H_h and ${}_\star\big(H_h^*\big)$ are topologically free $k[[h]]$–modules, and the reduction modulo h of $H_h \longrightarrow {}_\star\big(H_h^*\big)$ is an isomorphism. Indeed, to prove the latter claim, note that the specialization of H_h modulo h is just $H_h \big/ h\,H_h = V^\ell(L)_h \big/ h\,V^\ell(L)_h = V^\ell(L)$, by assumption, and that of ${}_\star\big(H_h^*\big)$ is nothing but ${}_\star\big(H_h^*\big) \big/ h\,{}_\star\big(H_h^*\big) \cong {}_\star\Big(H_h^* \big/ h\,H_h^*\Big) \cong {}_\star\Big(V^\ell(L)_h^* \big/ h\,V^\ell(L)_h^*\Big) \cong {}_\star\big(V^\ell(L)^*\big)$, by Theorem 5.7 and Theorem 5.5. In addition, the specialization modulo h of the given map is nothing but

$$H_h \big/ h\,H_h = V^\ell(L) \longrightarrow {}_*\big(V^\ell(L)^*\big) \cong {}_\star\big(H_h^*\big) \big/ h\,{}_\star\big(H_h^*\big) \quad , \qquad \overline{\eta} \mapsto ev_{\overline{\eta}}$$

directly by construction, and this is known to be an isomorphism — see Theorem 3.44*(a)*.

Let $K_h = J^r(L)_h \in (\mathrm{RQFSAd})_{A_h}$. We prove now that $K_h = \big({}_\star K_h\big)^*$. As ${}_\star K_h$ is a LQUEAd, it follows that $\big({}_\star K_h\big)^*$ is a RQFSAd. Like above, there is a natural morphism $K_h \longrightarrow \big({}_\star K_h\big)^*$ — in $(\mathrm{RQFSAd})_{A_h}$ — given by $\sigma \mapsto \big(ev_\sigma : \lambda \mapsto \lambda(\sigma)\big)$. Like before, since K_h and $\big({}_\star K_h\big)^*$ are two topologically free $k[[h]]$–modules, in order to show that this map is an isomorphism it is enough to show that it is an isomorphism modulo h.

By assumption we have $K_h / h\,K_h \cong J^r(L)$; then Theorem 5.7 implies also that ${}_\star K_h$ is a quantization of $V^\ell(L)$, i.e. its semiclassical limit is $V^\ell(L)$ itself. Now we have $K_h \big/ h\,K_h = J^r(L)$ and $\big({}_\star K_h\big)^* \big/ h\,\big({}_\star K_h\big)^* \cong \Big({}_\star K_h \big/ h\,{}_\star K_h\Big)^* \cong V^\ell(L)^* = J^r(L)$ because we know — by the assumptions and Theorem 5.7 — that ${}_\star K_h$ is a quantization of $V^\ell(L)$. We conclude like before.

Finally, both recipes $H_h \mapsto H_h^*$ and $K_h \mapsto {}_\star K_h$ are clearly (contravariantly) functorial. □

6 Drinfeld functors and quantum duality

In this section we present the main new contribution in this paper, namely the definition of Drinfeld functors and the equivalences — instead of antiequivalences! — of categories established via them among (left or right) QUEAd's and QFSAd's.

6.1 The Drinfeld functor $(\ )^\vee$

Let $K_h \in (\mathrm{RQFSAd})_{A_h}$. We set $K_F := k((h)) \underset{k[[h]]}{\otimes} K_h$, and $I_h := \partial_h^{-1}(h\,A_h)$, $\mathfrak{I}_h := Ker(\partial_h)$, where ∂_h is the counit map of K_h. If $\lambda \in K_h$, one has $\lambda = \lambda - s^r\big(\partial_h(\lambda)\big) + s^r\big(\partial_h(\lambda)\big)$ with

$\lambda - s^r\big(\partial_h(\lambda)\big) \in \mathfrak{J}_h$; thus one has $I_h = \mathfrak{J}_h + h\,K_h$. Define $K_h^\times := s^r(A_h) + \sum_{n\in\mathbb{N}_+} h^{-n} I_h^n$, which is a $k[[h]]$–submodule of K_F ; then note that we have also $K_h^\times = s^r(A_h) + \sum_{n\in\mathbb{N}_+} h^{-n} s^r(A_h)\,\mathfrak{J}_h^n$.

Definition 6.1. *Given* $K_F := k((h)) \underset{k[[h]]}{\otimes} K_h$ *, let* $I_h := \partial_h^{-1}(h\,A_h)$ *and* $\mathfrak{J}_h := Ker(\partial_h)$ *, with* $I_h = \mathfrak{J}_h + h\,K_h$ *. Then we define*

$$K_h^\times \;:=\; s^r(A_h) + \sum_{n\in\mathbb{N}_+} h^{-n}\, I_h^n \;=\; s^r(A_h) + \sum_{n\in\mathbb{N}_+} h^{-n} s^r(A_h)\,\mathfrak{J}_h^n$$

$$K_h^\vee \;:=\; h\text{–adic completion of the } k[[h]]\text{–module } K_h^\times$$

Moreover, in an entirely similar way we define $K_h^\vee$ *for any* $K_h \in (\mathrm{LQFSAd})_{A_h}$.

Remarks 6.2.

(a) Note that $\mathfrak{J}_h$ is not an $(A_h\,,A_h)$–subbimodule of K_h , in general. Indeed, if $a \in A_h$ and $\psi \in \mathfrak{J}_h$, it is clear (from the properties of the counit of a right bialgebroid) that $\psi\, s_r(a)$, $\psi\, t_r(a)$ in $\mathfrak{J}_h$; but we cannot prove in general that $s_r(a)\,\psi$ and $t_r(a)\,\psi$ belong to $\mathfrak{J}_h$. On the other hand, one has that I_h instead *is* definitely an $(A_h\,,A_h)$–subbimodule. For this reason, it is better to (define and) describe $K_h^\times$ and $K_h^\vee$ using I_h than using $\mathfrak{J}_h$.

(b) Let K be a LQFSAd, respectively a RQFSAd. Then $(K_h)_{coop}^{op}$ is a RQFSAd, respectively a LQFSAd. It easily follows from definitions that $\big((K_h)_{coop}^{op}\big)^\vee = \big(K_h^\vee\big)_{coop}^{op}$.

6.3. Description of $K_h^\vee$. Directly from its very definition, we can find out a description of $K_h^\vee$. This is very neat in the case when the Lie-Rinehart algebra L — such that K_h is a quantization of $J^r(L)$ or $J^\ell(L)$ — is of finite free type (as an A–module), and can be reduced somehow to that case when L instead is just of finite projective type. Thus we distinguish these two cases.

(a) <u>*Finite free case:*</u> Let us assume that L (as an A–module, of finite type) is free. Then we can explicitly describe $K_h^\vee$, as follows. Fix an A–basis $\{\overline{e}_1\,,\dots,\,\overline{e}_n\}$ of L , and let $\bar{\xi}_i$ be the element of $Hom\big(V^\ell(L),A\big) = V^\ell(L)^* = J^r(L)$ defined (using standard multiindex notation) by

$$\big\langle\, \bar{\xi}_i\,,\, \overline{e}^{\,\underline{\alpha}} \,\big\rangle \;=\; \big\langle\, \bar{\xi}_i\,,\, \overline{e}_n^{\,\alpha_n} \cdots \overline{e}_n^{\,\alpha_n} \,\big\rangle \;:=\; \delta_{\alpha_1,0} \cdots \delta_{\alpha_i,1} \cdots \delta_{\alpha_n,0} \qquad \forall\; \underline{\alpha} = (\alpha_1\,,\dots,\,\alpha_n) \in \mathbb{N}^n$$

Let ξ_i be an element of K_h lifting $\bar{\xi}_i$ and such that $\partial_h(\xi_i) = 0$. If $a \in A_h$, we shall write again a to denote the element $t^r(a) \in K_h$. We have the following descriptions

$$K_h \;=\; \big\{ \textstyle\sum_{\underline{d}\in\mathbb{N}^n} \xi_1^{d_1} \cdots \xi_n^{d_n}\, a_{\underline{d}} \;\big|\; a_{\underline{d}} \in A_h^{op} \,,\; \forall\; \underline{d} \in \mathbb{N}^n \big\} \;\cong\; A[[\,X_1,\dots,X_n]][[h\,]]$$

$$I_h \;=\; \big(h\,,\xi_1\,,\dots,\xi_n\big) \qquad , \qquad \mathfrak{J}_h \;=\; \textstyle\sum_{i=1}^n \xi_i\, K_h$$

where the first line item is a (right) A_h^{op}–module of formal power series (convergent in the I_h–adic topology) and the last isomorphism is one of topological k–modules, while round braces in second line stand once again for “two-sided ideal generated by”. By this and the very definition it follows that, writing $\check{\xi}_i := h^{-1}\xi_i$, one has (the last isomorphism being one of topological k–modules)

$$K_h^\vee \;=\; \Big\{ \textstyle\sum_{\underline{b}\in\mathbb{N}^{n+1}} h^{b_0}\, \check{\xi}_1^{\,b_1} \cdots \check{\xi}_n^{\,b_n}\, a_{\underline{b}} \;\Big|\; a_{\underline{b}} \in A_h^{op} \,,\; \forall\; \underline{b} \;\Big\} \;\cong\; A\big[\check{X}_1,\dots,\check{X}_n\big][[h]]$$

where the sum denotes formal series which are convergent in the h–adic topology, and then also

$$\mathfrak{J}_h^\vee \;:=\; h^{-1}\mathfrak{J}_h \;=\; \textstyle\sum_{i=1}^n \check{\xi}_i\, K_h^\vee \;=\; \text{right ideal of } K_h^\vee \text{ generated by the } \check{\xi}_i\text{'s}$$

(b) <u>*Finite projective case:*</u> Assume now that L (as an A–module) is just finite projective (as usual in this work). Like in Subsection 4.2, we fix a finite projective A–module Q such that $L \oplus Q = F$ is a finite free A–module, we write $F = A \otimes_k Y$ where Y is the k–span of an A–basis of F , and we construct the (infinite dimensional) Lie-Rinehart algebra $L_Q = L \oplus \big(A \otimes_k Z\big)$ with $Z = Y \oplus Y \oplus Y \oplus \cdots$. Then, for $J^r(L)_h := K_h$, we can introduce the right bialgebroid $K_{h,Y} := J^r(L)_{h,Y}$ as in §4.17: namely (with notation as in §4.17), we recall that

$$K_{h,Y} \;\;:=\;\; h\text{–adic completion of } \;\textstyle\sum_{n\in\mathbb{N}} K_h \widetilde{\otimes}_k S(Y^*)^{\widetilde{\otimes}\, n} \otimes 1 \otimes 1 \otimes 1 \cdots$$

$\big(S(Y^*)^{\otimes n}\big)^+$ being the kernel of the natural counit map of $S(Y^*)^{\otimes n}$ and $K_h \widetilde{\otimes}_k S(Y^*)^{\widetilde{\otimes}\, n} \otimes 1 \cdots$ the $\big(\big(S(Y^*)^{\otimes n}\big)^+ \otimes 1 \otimes 1 \cdots\big)$–adic completion of $K_h \otimes_k S(Y^*)^{\otimes n} \otimes 1 \otimes 1 \cdots$.

Furthermore, let ∂_h be the counit of $K_{h,Y}$, and $I_{h,Y} := \partial_h^{-1}\big(h\, A_h\big)$. Then we have also

$$I_{h,Y} \;:=\; h\text{–adic completion of } \Big(\sum_{r \in \mathbb{N}} I_h \mathop{\widetilde{\otimes}}_k S(Y^*)^{\widetilde{\otimes}\, r} \otimes 1 \otimes \cdots + \sum_{s \in \mathbb{N}} K_h \mathop{\widetilde{\otimes}}_k \big(S(Y^*)^{\widetilde{\otimes}\, s}\big)^+ \otimes 1 \otimes \cdots\Big)$$

Basing upon these remarks, we can define $K_{h,Y}^\vee$ and describe it as above: namely, one has

$$K_{h,Y}^\vee \;=\; h\text{–adic completion of } \sum_{n,m} \sum_{r+s=n} h^{-n}\, \mathfrak{J}_h^r \widetilde{\otimes}_k \Big(\big(S(Y^*)^{\widetilde{\otimes} m}\big)^+\Big)^s \otimes 1 \otimes 1 \otimes \cdots \;=\; K_h^\vee \,\widehat{\otimes}_k\, S(Z^{*_f})$$

where $Z^{*_f} = Y^* \oplus Y^* \oplus Y^* \oplus .....$

Let now $\{e_t\}_{t \in T := \mathbb{N} \times \{1,\dots,n\}}$ be a good basis of the A–module L_Q. From the proof of Theorem 5.7 (step *(4)* for the general case) we can select elements $\xi_t \in K_{h,Y}$ $(t \in T)$ such that

$$K_{h,Y} \;=\; \Big\{ \textstyle\sum_{n \in \mathbb{N}} h^n P_n\big(\{\xi_t\}_{t \in T}\big) \,\Big|\, P_n \in \big[\big[\{X_t\}_{t \in T}\big]\big]_f\, A_h^{op} \Big\} \;\cong\; A\big[\big[\{X_t\}_{t \in T}\big]\big]_f[[h]]$$

$$I_{h,Y} \;=\; \big(h\,,\{\xi_t\}_{t \in T}\big) \qquad , \qquad \mathfrak{J}_{h,Y} \;=\; \textstyle\sum_{t \in T} \xi_t\, K_{h,Y}$$

where $\big[\big[\{X_t\}_{t \in T}\big]\big]_f\, A_h$ denotes the ring of formal power series with coefficients *on the right* chosen in A_h involving only finitely many indeterminates X_t. One can then easily find, letting $\check{\xi}_t := h^{-1}\xi_t$ for all $t \in T$, that

$$K_{h,Y}^\vee \;=\; \Big\{ \textstyle\sum_{n \in \mathbb{N}} h^n\, P_n\big(\{\check{\xi}_t\}_{t \in T}\big) \,\Big|\, P_n \in \big[\{\,\underline{\check{\xi}}\,\}_{t \in T}\big] A_h^{op},\ \forall\, n \in \mathbb{N} \Big\} \;\cong\; A\big[\{\check{X}_t\}_{t \in T}\big][[h]]$$

where the sum denotes formal series convergent in the h–adic topology, and $\big[\{X_t\}_{t \in T}\big]\, A_h^{op}$ denotes the ring of polynomials with coefficients *on the right* chosen in A_h^{op}. We find also

$$\mathfrak{J}_{h,Y}^\vee \;:=\; h^{-1}\mathfrak{J}_{h,Y} \;=\; \textstyle\sum_{t \in T} \check{\xi}_t\, K_{h,Y}^\vee \;=\; \text{right ideal of } K_{h,Y}^\vee \text{ generated by the } \check{\xi}_t\text{'s} \qquad \diamondsuit$$

It is time for the main result of this subsection. In short, it claims that the construction $K_h \mapsto K_h^\vee$, starting from a quantization of L — of type $J^{r/\ell}(L)$ — provides a quantization of the *dual* Lie-Rinehart bialgebra L^* — of type $V^{r/\ell}(L^*)$; moreover, this construction is functorial.

Theorem 6.4.

(a) Let $J^r(L)_h \in (\mathrm{RQFSAd})_{A_h}$, where L is a Lie-Rinehart algebra which, as an A–module, is projective of finite type. Then:

— (a.1) $J^r(L)_h^\vee \in (\mathrm{RQUEAd})_{A_h}$, with semiclassical limit $J^r(L)_h^\vee \big/ h\, J^r(L)_h^\vee \;\cong\; V^r(L^*)$. Moreover, the structure of Lie-Rinehart bialgebra induced on L^* by the quantization $J^r(L)_h^\vee$ of $V^r(L^*)$ — as in Theorem 4.7 — is dual to that induced on L by the quantization $J^r(L)_h$ of $J^r(L)$ — as in Theorem 4.12;

— (a.2) the definition of $J^r(L)_h \mapsto J^r(L)_h^\vee$ extends to morphisms in (RQFSAd), so that we have a well defined (covariant) functor $(\ )^\vee : (\mathrm{RQFSAd}) \longrightarrow (\mathrm{RQUEAd})$.

(b) Let $J^\ell(L)_h \in (\mathrm{LQFSAd})_{A_h}$, where L is a Lie-Rinehart algebra which, as an A–module, is projective of finite type. Then:

— (b.1) $J^\ell(L)_h^\vee \in (\mathrm{LQUEAd})_{A_h}$, with semiclassical limit $J^\ell(L)_h^\vee \big/ h\, J^\ell(L)_h^\vee \;\cong\; V^\ell(L^*)$. Moreover, the structure of Lie-Rinehart bialgebra induced on L^* by the quantization $J^\ell(L)_h^\vee$ of $V^\ell(L^*)$ — as in Theorem 4.4 — is dual to that induced on L by the quantization $J^\ell(L)_h$ of $J^\ell(L)$ — as in Theorem 4.13;

— (b.2) the definition of $J^\ell(L)_h \mapsto J^\ell(L)_h^\vee$ extends to morphisms in (LQFSAd), so that we have a well defined (covariant) functor $(\ )^\vee : (\mathrm{LQFSAd}) \longrightarrow (\mathrm{LQUEAd})$.

Proof. (a) To ease notation, let us write $K_h := J^r(L)_h$.

By definition, $K_h^\times$ is the unital $k[[h]]$–subalgebra of $\big(K_h\big)_F := k((h)) \otimes_{k[[h]]} K_h$ generated by $h^{-1}I_h$ and $s^r(A_h)$: thus it is automatically a unital $k[[h]]$–algebra. It follows that $K_h^\vee$ is a unital $k[[h]]$–algebra too, complete in the h–adic topology. Moreover, I_h is an (A_h, A_h)–subbimodule of K_h : this implies at once that $K_h^\times$ and $K_h^\vee$ are (A_h, A_h)–bimodules too. As $\big(K_h\big)_F$ is torsionless, so are $K_h^\times$ and its completion $K_h^\vee$; also, $K_h^\vee$ is separated and complete, so it is topologically free.

Let us now see that the coproduct in K_h induces a coproduct — in a suitable, h–adical sense — for $K_h^\vee$ as well. Given any $\phi \in I_h$, we write $\Delta(\phi) = \phi_{(1)} \otimes \phi_{(2)}$ — a formal series (in Σ–notation) — convergent in the I_h–adic topology of K_h . Writing $\phi_{(1)}$ and $\phi_{(2)}$ as

$$\begin{aligned} \phi_{(1)} \;&=\; \phi_{(1)}^{+_s} + s^r\big(\partial_h\big(\phi_{(1)}\big)\big) \; , \quad \text{with} \quad \phi_{(1)}^{+_s} \;:=\; \phi_{(1)} - s^r\big(\partial_h\big(\phi_{(1)}\big)\big) \;\in\; \mathfrak{J}_h \;\subseteq\; I_h \\ \phi_{(2)} \;&=\; \phi_{(2)}^{+_t} + t^r\big(\partial_h\big(\phi_{(2)}\big)\big) \; , \quad \text{with} \quad \phi_{(2)}^{+_t} \;:=\; \phi_{(2)} - t^r\big(\partial_h\big(\phi_{(2)}\big)\big) \;\in\; \mathfrak{J}_h \;\subseteq\; I_h \end{aligned}$$

we have seen that

$$\Delta(\phi) \;=\; \phi_{(1)}^{+_s} \otimes \phi_{(2)} + s^r\big(\partial_h\big(\phi_{(1)}\big)\big) \otimes \phi_{(2)}^{+_t} + s^r\big(\partial_h(x)\big) \otimes 1 \;\in\; \big(I_h \,\widetilde{\otimes}_{A_h} K_h + K_h \,\widetilde{\otimes}_{A_h} I_h + h\, s^r(A_h) \,\widetilde{\otimes}_{A_h} 1\big)$$

All this implies

$$\Delta\big(h^{-1}\phi\big) \;\in\; \big(h^{-1}I_h \,\widetilde{\otimes}_{A_h} K_h + K_h \,\widetilde{\otimes}_{A_h} h^{-1} I_h + \, s^r(A_h) \,\widetilde{\otimes}_{A_h} 1\big) \;\subseteq\; K_h^\vee \,\widetilde{\otimes}_{A_h} K_h^\vee$$

In addition, we must observe the following. Every $\phi \in I_h$ expands as an I_h–adically convergent series $\phi = \sum_{n \in \mathbb{N}_+} \phi_n$ with $\phi_n \in I_h^n$ for all $n \in \mathbb{N}_+$; but then $\phi_n \in I_h^n = h^n \big(h^{-1} I_h\big)^n$ for every n and so $h^{-1}\phi$ expands as a series $h^{-1}\phi = \sum_{n \in \mathbb{N}_+} h^{n-1} \big(h^{-1} I_h\big)^n$ which is convergent in the h–adic topology of $K_h^\vee$. As a byproduct of this analysis, we can apply the same argument to $\Delta\big(h^{-1}\phi\big)$ and thus realize that it is actually a well defined element of $K_h^\vee \,\widehat{\otimes}_{A_h} K_h^\vee$, the h–adic completion of $K_h^\vee \otimes_{A_h} K_h^\vee$. Finally, it is clear that in fact $\Delta\big(h^{-1}\phi\big)$ even belongs to the Takeuchi product inside $K_h^\vee \otimes_{A_h} K_h^\vee$, as the parallel property is true for $\Delta(\phi)$ inside $K_h \,\widetilde{\otimes}_{A_h} K_h$.

As $K^\times$ is generated — as an algebra — by $h^{-1}I_h$ and $s^r(A_h)$, and $K_h^\vee$ is its h–adic completion, we finally conclude that the coproduct of K_h does provide a well defined coproduct for $K_h^\vee$, making it into a (topological) right bialgebroid over A_h .

Moreover, by construction $K_h^\vee$ is isomorphic (as a $k[[h]]$–module) to $\big(K_h^\vee / h\, K_h^\vee\big)[[h]]$.

What we are left to prove — for claim *(a.1)* — is that $\overline{K_h^\vee} := K_h^\vee \Big/ h\, K_h^\vee$ be isomorphic to $V^r(L')$ for some Lie-Rinehart bialgebra, and that such L' — with its structure of (Lie-Rinehart) *bialgebra* induced by this very quantization — is isomorphic to L^* with its structure of Lie-Rinehart bialgebra dual to that induced on L by the quantization $K_h := J^r(L)_h$ we started from.

We follow the strategy in [13] and [10]. By the analysis we did so far we know that $K_h^\vee$ is a deformation of the right bialgebroid $K_h^\vee \Big/ h\, K_h^\vee$: then we shall apply Proposition 3.21 (and the remarks following it) to show that the latter is indeed of the form $V^r(L')$, with $L' \cong L^*$. For computations hereafter we fix some notation: $\mathfrak{J}_h := Ker(\partial_h)$, $K := K_h \Big/ h\, K_h$ and $\mathfrak{J} := Ker(\partial)$ for $\partial := \partial_K$. Also, from Theorem 5.7*(a)* we consider $V^\ell(L)_h := {}_\star K_h = {}_\star J^r(L)_h \in (\text{LQUEAd})_{A_h}$ so that $J^r(L)_h = V^l(L)_h^*$.

We proceed in several steps.

— • For all $a \in A_h$, we have $s_r(a) \equiv t_r(a) \mod h\, K_h^\vee$.

Indeed, one has $\big(s^r(a) - t^r(a)\big) \in \mathfrak{J}_h \;\subseteq\; I_h \;=\; h\, h^{-1} I_h \;\subseteq\; h\, K_h^\vee$, whence the claim.

— • For every $\bar{a} \in A$, the map $\overline{t^r(a)} : A \longrightarrow K_h^\vee \Big/ h\, K_h^\vee$ induced modulo h by t^r is injective.

In fact, let $\bar{a} \in Ker\big(\overline{t^r}\big)$, and let $a \in A_h$ be a lifting of $\bar{a}$ in A_h . Then for all $u \in V^\ell(L)_h$ we have $\big\langle\, \epsilon \,,\, s^l(a)\, u \,\big\rangle \mod h = \big\langle\, t^r(a) \,, u \,\big\rangle \mod h = \big\langle\, \overline{t^r(a)} \,, \bar{u} \,\big\rangle = \bar{a} < \epsilon, \bar{u} >= 0$. This implies $a \in h\, A_h$, so $\bar{a} = 0$.

— • The set $P^r\big(\overline{K_h^\vee}\big)$ of (right) primitive elements of $\overline{K_h^\vee} := K_h^\vee \Big/ h\, K_h^\vee$ — cf. Proposition 3.21 — has a natural structure of right Lie-Rinehart algebra, induced by specialization from $K_h^\vee$.

Indeed, this is entirely standard. Both the Lie bracket $[\ ,\ ]$ and the anchor map ω are recovered as semiclassical limits of commutators from the multiplicative structure and source/target structure of the "quantum right bialgebroid" $K_h^\vee$. Namely, for any $x, y \in P^r\big(\overline{K_h^\vee}\big)$ and $a \in A$, choose any lifts $x', y' \in K_h^\vee$ and $a' \in A_h$ of them: then defining

$$a.x := x'\, s^r(a') \mod h\, K_h^\vee \quad , \qquad [x,y] := x'\, y' - y'\, x' \mod h\, K_h^\vee$$
$$\omega(x)(a) := \partial_h\big(x'\, s^r(a') - s^r(a')\, x'\big) \mod h\, A_h$$

it is a routine matter to check that $P^r\big(\overline{K_h^\vee}\big)$ is made into a Lie-Rinehart algebra over A.

— • Set $\mathfrak{J}_h^\vee := h^{-1}\mathfrak{J}_h \big(\subseteq K_h^\vee\big)$ and $\overline{\mathfrak{J}_h^\vee} := \mathfrak{J}_h^\vee \mod h\, K_h^\vee$; then $\overline{\mathfrak{J}_h^\vee}$ is a Lie-Rinehart subalgebra of $P^r\big(\overline{K_h^\vee}\big)$.

Indeed, let $\phi \in \mathfrak{J}_h$, and set $\phi^\vee := h^{-1}\phi \in \mathfrak{J}_h^\vee$. Then acting as in the first part of the proof (with notation introduced therein) we get

$$\begin{aligned}\Delta(\phi) \,=\, \phi_{(1)} \otimes \phi_{(2)} \,=\, \big(\phi_{(1)} - s^r\big(\partial_h\big(\phi_{(1)}\big)\big) + s^r\big(\partial_h\big(\phi_{(1)}\big)\big)\big) \otimes \big(\phi_{(2)} - t^r\big(\partial_h\big(\phi_{(2)}\big)\big) + t^r\big(\partial_h\big(\phi_{(2)}\big)\big)\big) \,= \\ =\, s^r\big(\partial_h\big(\phi_{(1)}\big)\big) \otimes t^r\big(\partial_h\big(\phi_{(2)}\big)\big) + \phi_{(1)}^{+s} \otimes t^r\big(\partial_h\big(\phi_{(2)}\big)\big) + s^r\big(\partial_h\big(\phi_{(1)}\big)\big) \otimes \phi_{(2)}^{+t} + \phi_{(1)}^{+s} \otimes \phi_{(2)}^{+t} \,= \\ =\, \partial_h(\phi) \otimes 1 + \phi_{(1)}^{+s} \otimes t^r\big(\partial_h\big(\phi_{(2)}\big)\big) + s^r\big(\partial_h\big(\phi_{(1)}\big)\big) \otimes \phi_{(2)}^{+t} + \phi_{(1)}^{+s} \otimes \phi_{(2)}^{+t} \,= \\ =\, \phi \otimes 1 + 1 \otimes \phi + \phi_{(1)}^{+s} \otimes \phi_{(2)}^{+t} \,\in\, \phi \otimes 1 \,+\, 1 \otimes \phi \,+\, \mathfrak{J}_h \,\widetilde{\otimes}_{A_h} \mathfrak{J}_h\end{aligned}$$

thanks to the assumption $\phi \in \mathfrak{J}_h$ (and to several identities holding true in any right bialgebroid). As $\mathfrak{J}_h = h\, h^{-1}\mathfrak{J}_h = h\, \mathfrak{J}_h^\vee \subseteq h\, K_h^\vee$, we end up with $\Delta\big(\phi^\vee\big) = \phi^\vee \otimes 1 + 1 \otimes \phi^\vee + h\,\big(K_h^\vee \,\widehat{\otimes}_{A_h} K_h^\vee\big)$, so that $\overline{\phi^\vee} := \phi^\vee \mod h\, K_h^\vee$ is primitive in $\overline{K_h^\vee}$. This proves that $\overline{\mathfrak{J}_h^\vee} \subseteq P^r\big(\overline{K_h^\vee}\big)$.

Finally, $\overline{\mathfrak{J}_h^\vee}$ is a Lie-Rinehart subalgebra of $P^r\big(\overline{K_h^\vee}\big)$ if and only if it is a (right) A–submodule, closed for the Lie bracket. Now, by definition $\mathfrak{J}_h$ is a right ideal in K_h, and this implies — by construction — that $\overline{\mathfrak{J}_h^\vee}$ is a (right) A–submodule. As to the Lie bracket, if $x, y \in \overline{\mathfrak{J}_h^\vee}$ we have by definition $[x,y] := x'\, y' - y'\, x' \mod h\, K_h^\vee$ for any choice of liftings $x', y' \in K_h^\vee$ of x and y. On the other hand, we can clearly choose $x', y' \in \mathfrak{J}_h^\vee$, so that $x' = h^{-1}\chi$, $y' = h^{-1}\eta$, for some $\chi, \eta \in \mathfrak{J}_h$; then we have

$$x'\, y' - y'\, x' \,=\, h^{-2}(\chi\,\eta - \eta\,\chi) \,\in\, h^{-2}\big(\mathfrak{J}_h \textstyle\bigcap h\, K_h\big) \,=\, h^{-2} h\, \mathfrak{J}_h \,=\, h^{-1}\mathfrak{J}_h \,=:\, \mathfrak{J}_h^\vee$$

since $\mathfrak{J}_h$ is a right ideal and $K_h \big/ h\, K_h \cong J^r(L)$ is commutative. It follows that $[x,y] \in \overline{\mathfrak{J}_h^\vee}$, q.e.d.

— • We will now show that $\mathfrak{J}_h^\vee \bigcap h\, K_h^\vee \,=\, \mathfrak{J}_h + \mathfrak{J}_h^\vee\, \mathfrak{J}_h \,=\, h\, \mathfrak{J}_h^\vee + h\,\big(\mathfrak{J}_h^\vee\big)^2$.

Indeed, the second identity in the claim is a trivial consequence of $\mathfrak{J}_h^\vee := h^{-1}\mathfrak{J}_h$. As to the first one, as $K_h = J^r(L)_h$, we distinguish two cases: either L is free (as an A–module), or not.

If L is free, then the identity $\mathfrak{J}_h^\vee \bigcap h\, K_h^\vee \,=\, \mathfrak{J}_h + \mathfrak{J}_h^\vee\, \mathfrak{J}_h$ is an easy, direct consequence of the description of $\mathfrak{J}_h^\vee$ given in §6.3 here above in the free case — i.e. part *(a)*.

If instead L is not free, then we proceed as follows. First consider $K_{h,Y}$ and $\mathfrak{J}_{h,Y}$, and construct from them $K_{h,Y}^\vee$ and $\mathfrak{J}_{h,Y}^\vee$. In this case, the description of $\mathfrak{J}_{h,Y}^\vee$ given in §6.3, part *(b)*, implies again easily the identity $\mathfrak{J}_{h,Y}^\vee \bigcap h\, K_{h,Y}^\vee \,=\, \mathfrak{J}_{h,Y} + \mathfrak{J}_{h,Y}^\vee\, \mathfrak{J}_{h,Y}$. Now consider the map $\pi^Y : K_{h,Y} \longrightarrow K_h$, introduced in §4.17*(b)*, for $J^r(L)_h := K_h$ and $J^r(L)_{h,Y} := K_{h,Y}$: this is a an epimorphism of right bialgebroids, thus in particular $\pi^Y\big(\mathfrak{J}_{h,Y}\big) = \mathfrak{J}_h$. Then it follows at once that π^Y canonically induces another epimorphism of right bialgebroids $\check{\pi}^Y : K_{h,Y}^\vee \longrightarrow K_h^\vee$ such that $\check{\pi}^Y\big(\mathfrak{J}_{h,Y}^\vee\big) = \mathfrak{J}_h^\vee$. But then, using π^Y and $\check{\pi}^Y$ and the identity $\mathfrak{J}_{h,Y}^\vee \bigcap h\, K_{h,Y}^\vee \,=\, \mathfrak{J}_{h,Y} + \mathfrak{J}_{h,Y}^\vee\, \mathfrak{J}_{h,Y}$ we easily deduce the identity $\mathfrak{J}_h^\vee \bigcap h\, K_h^\vee \,=\, \mathfrak{J}_h + \mathfrak{J}_h^\vee\, \mathfrak{J}_h$ we were looking for.

— • There exists an A–linear isomorphism $\psi : \mathfrak{J}_h^\vee \Big/ \Big(h\, \mathfrak{J}_h^\vee + h\,\big(\mathfrak{J}_h^\vee\big)^2\Big) \,\cong\, \overline{\mathfrak{J}_h^\vee}$ — hence hereafter we shall identify $\overline{\mathfrak{J}_h^\vee}$ and $\mathfrak{J}_h^\vee \Big/ \Big(h\, \mathfrak{J}_h^\vee + h\,\big(\mathfrak{J}_h^\vee\big)^2\Big)$ via ψ and ψ^{-1}.

Indeed, the natural projection map $K_h^\vee \twoheadrightarrow \overline{K_h^\vee} := K_h^\vee \big/ h\, K_h^\vee$, whose kernel is $h\, K_h^\vee$, yields by restriction a similar map $\mathfrak{J}_h^\vee \twoheadrightarrow \overline{\mathfrak{J}_h^\vee} := \mathfrak{J}_h^\vee \Big/ \big(\mathfrak{J}_h^\vee \bigcap h\, K_h^\vee\big)$ whose kernel is $\big(\mathfrak{J}_h^\vee \bigcap h\, K_h^\vee\big)$. By the previous step, we have $\mathfrak{J}_h^\vee \bigcap h\, K_h^\vee \,=\, h\, \mathfrak{J}_h^\vee + h\,\big(\mathfrak{J}_h^\vee\big)^2$, whence we get an A–linear isomorphism.

— • There exists an A–linear isomorphism $\sigma : \overline{\mathfrak{J}_h^\vee} \cong \mathfrak{J}_h^\vee \Big/ \Big(h\, \mathfrak{J}_h^\vee + h\, \big(\mathfrak{J}_h^\vee\big)^2 \Big) \cong \mathfrak{J} \Big/ \mathfrak{J}^{\,2} =: L^*$, where $\mathfrak{J} \equiv \mathfrak{J}_{J^r(L)} := Ker\big(\partial_{J^r(L)}\big)$, given by $\overline{h^{-1}y} \mapsto \sigma\big(\overline{h^{-1}y}\,\big) := \overline{y} \mod \mathfrak{J}^{\,2}$.

Indeed, there exists a natural projection map $\sigma'' : \mathfrak{J}_h \relbar\joinrel\twoheadrightarrow \mathfrak{J}_h \big/ h\,\mathfrak{J}_h = \mathfrak{J} \relbar\joinrel\twoheadrightarrow \mathfrak{J}\big/\mathfrak{J}^{\,2} =: L^*$, whose kernel is $\big(h\,\mathfrak{J}_h + \mathfrak{J}_h^2\big)$. Then $\sigma' : \mathfrak{J}_h^\vee := h^{-1}\mathfrak{J}_h \relbar\joinrel\twoheadrightarrow \mathfrak{J}\big/\mathfrak{J}^{\,2} =: L^* \ \big(h^{-1}y \mapsto \sigma'\big(h^{-1}y\big) := \sigma''(y)\big)$ is a well defined k–linear map, whose kernel is $\big(\mathfrak{J}_h + h^{-1}\mathfrak{J}_h^2\big) = \Big(h\,\mathfrak{J}_h^\vee + h\,\big(\mathfrak{J}_h^\vee\big)^2\Big)$. Therefore σ' canonically induces a k–linear isomorphism $\sigma : \overline{\mathfrak{J}_h^\vee} \cong \mathfrak{J}_h^\vee \Big/ \Big(h\,\mathfrak{J}_h^\vee + h\,\big(\mathfrak{J}_h^\vee\big)^2\Big) \xhookrightarrow{\ \cong\ } \mathfrak{J}\big/\mathfrak{J}^{\,2} =: L^*$ given by $\overline{h^{-1}y} \mapsto \sigma\big(\,\overline{h^{-1}y}\,\big) := \overline{\sigma''(y)}$; also, it is straightforward to check that this is A–linear too.

— • We have $\overline{K_h^\vee} \in (\mathrm{RQUEAd})_{A_h}$, namely $\overline{K_h^\vee} \cong V^r\big(L'\big)$ for the Lie-Rinehart A–algebra $L' := \overline{\mathfrak{J}_h^\vee}$ (with the Lie-Rinehart structure mentioned above).

Indeed, what we proved so far show that $L' := \overline{\mathfrak{J}_h^\vee}$ is a Lie-Rinehart subalgebra of $P^r\big(\,\overline{K_h^\vee}\,\big)$, which together with A generates $\overline{K_h^\vee}$ (as an algebra) and is finite projective as an A–module (since it is isomorphic, as an A–module, to L^* , see above). Therefore, all conditions in Remark 3.22 are fulfilled, so it applies and gives $\overline{K_h^\vee} \cong V^r\big(L'\big)$ for $L' := \overline{\mathfrak{J}_h^\vee} = P^r\big(\,\overline{K_h^\vee}\,\big)$.

— • There exists on the Lie-Rinehart algebra L' a unique structure of Lie-Rinehart bialgebra, canonically induced from the quantization $K_h^\vee$ of $V^r\big(L'\big)$.

In fact, this is just a direct consequence of Theorem 4.7.

— • The A–linear isomorphism $\sigma : \overline{\mathfrak{J}_h^\vee} \cong \mathfrak{J}_h^\vee \Big/ \Big(h\,\mathfrak{J}_h^\vee + h\,\big(\mathfrak{J}_h^\vee\big)^2\Big) \cong \mathfrak{J}\Big/\mathfrak{J}^{\,2} =: L^*$ is actually an isomorphism of Lie-Rinehart bialgebras over A .

In order to prove this, we must show that σ preserves the Lie bracket, the anchor map and the differential δ (cf. Definition 2.20) on either side.

For the Lie bracket, let $x, y \in \overline{\mathfrak{J}_h^\vee}$: given $\chi, \eta \in \mathfrak{J}_h$ such that $x = h^{-1}\chi$, $y = h^{-1}\eta$, we have

$$[x,y] \;=\; h^{-2}(\chi\,\eta - \eta\,\chi) \mod h\,K_h^\vee \;=\; h^{-2}h\,\zeta \mod h\,K_h^\vee \;=\; h^{-1}\zeta \mod h\,K_h^\vee$$

for some $\zeta \in \mathfrak{J}_h$. But then also $\overline{\zeta} := \zeta \mod h\,K_h =: \big\{\overline{\chi},\overline{\eta}\big\}$ — where $\overline{\alpha} := \alpha \mod h\,K_h$ for all $\alpha \in K_h$ — by Theorem 4.12. Now the Poisson bracket of $K_h\big/h\,K_h$ restricted to $\mathfrak{J}_h$ pushes down to the Lie bracket of $\mathfrak{J}_h\big/\mathfrak{J}_h^2 =: L^*$; thus setting $X := \overline{\chi} \mod \mathfrak{J}^{\,2}$, $Y := \overline{\eta} \mod \mathfrak{J}^{\,2}$ $\big(\in \mathfrak{J}\big/\mathfrak{J}^{\,2} =: L^*\big)$, we have $[X,Y] = \big\{\overline{\chi},\overline{\eta}\big\} \mod \mathfrak{J}^{\,2} = Z$. Now, by construction we have $X = \sigma(x)$, $Y = \sigma(y)$, and the previous analysis eventually gives also $\sigma\big([x,y]\big) = Z = [X,Y] = \big[\sigma(x),\sigma(y)\big]$, q.e.d.

For the anchor map, let $x \in \overline{\mathfrak{J}_h^\vee}$, $\chi \in \mathfrak{J}_h$, $X \in \mathfrak{J}\big/\mathfrak{J}^{\,2} = L^*$ as above, and take $a \in A$ and $a' \in A_h$ such that $a' \mod h\,A_h = a$. Then direct computations give

$$\begin{aligned} \omega(x)(a) \;&=\; \partial_h\big(h^{-1}\chi\,s^r(a') - s^r(a')\,h^{-1}\chi\big) \mod h\,A_h \;= \\ &=\; \partial\Big(h^{-1}\big(\chi\,s^r(a') - s^r(a')\,\chi\big) \mod h\,K_h\Big) \;=\; \omega(X)(a) \end{aligned}$$

which means $\omega(x) = \omega(X) = \omega\big(\sigma(x)\big)$, that is σ preserves the anchor, q.e.d.

Finally, in order to compare the two differentials on $\overline{\mathfrak{J}_h^\vee}$ and L^* , respectively denoted δ' and δ'' , recall that in any Lie-Rinehart bialgebra $\big(\mathcal{L}\,,\mathcal{A}\big)$ — in the present case (L^*, A) — the differential $\delta_{\mathcal{L}}$ is related with the Lie bracket and the anchor map by the identities

$$\big\langle\, \mathrm{f}, \delta_{\mathcal{L}}(a)\,\big\rangle = \omega_{\mathcal{L}^*}(\lambda)(a)\;,\quad \big\langle\, \mathrm{f}\otimes\mu, \delta_{\mathcal{L}}(x)\,\big\rangle = \omega_{\mathcal{L}^*}(\mathrm{f})\big(\langle\,\mathrm{m},x\rangle\big) - \omega_{\mathcal{L}^*}(\mathrm{m})\big(\langle\,\mathrm{f},x\rangle\big) - \big\langle [\,\mathrm{f},\mathrm{m}]_{\mathcal{L}^*},x\big\rangle \quad (5.1)$$

for all $x \in \mathcal{L}$, $\mathrm{f}, \mathrm{m} \in \mathcal{L}^*$, $a \in A$ — see Remarks 2.21*(b)*. We apply this to $\big(\mathcal{L}\,,\mathcal{A}\big) = (L^*, A)$.

For the differential on A , we must prove that $\sigma\big(\delta'(a)\big) = \delta''(a)$ for all $a \in A$, which amounts to show that $\big\langle\,\mathrm{f}\,,\,\sigma\big(\delta'(a)\big)\big\rangle = \big\langle\,\mathrm{f}\,,\,\delta''(a)\big\rangle$ for all $a \in A$ and all $\mathrm{f} \in L$. For this comparison, recall that $V^\ell(L)_h := {}_\star J^r(L)_h \in (\mathrm{LQUEAd})_{A_h}$ is a quantization of $V^\ell(L)$, by Theorem 5.7*(a)*; moreover, the natural pairing between $V^\ell(L)_h$ and $J^r(L)_h$ (given by evaluation) is a right bialgebroid pairing. Now choose a lifting $a' \in A_h$ of $a \in A$ and a lifting $f' \in V^\ell(L)_h$ of $\mathrm{f} \in L$: more precisely, we choose $f' \in Ker\big(\epsilon_{V^\ell(L)_h}\big)$. Then direct computation gives

$$\begin{aligned} \big\langle\,\mathrm{f}\,,\,\sigma\big(\delta'(a)\big)\big\rangle \;&=\; h\cdot\big\langle\,f'\,,\,\delta'(a)\big\rangle \ mod\, h\,A_h = \; h\cdot\Big\langle\,f'\,,\,\frac{s^r\big(a'\big) - t^r\big(a'\big)}{h}\Big\rangle \mod h\,A_h \;= \\ &=\; \big\langle\,f'\,,\,s^r\big(a'\big) - t^r\big(a'\big)\big\rangle \mod h\,A_h \;=\; \big\langle\,f'\,s^\ell\big(a'\big) - s^\ell\big(a'\big)\,f'\,,\,1\,\big\rangle \mod h\,A_h \;= \end{aligned}$$

$$= \big\langle f' s^\ell(a') - t^\ell(a') f' , 1 \big\rangle \mod h A_h = \Big(\big\langle f' s^\ell(a') , 1 \big\rangle - \big\langle t^\ell(a') f' , 1 \big\rangle \Big) \mod h A_h =$$

$$= \Big(\big\langle f' s^\ell(a') , 1 \big\rangle - \big\langle f' , 1 \big\rangle a' \Big) \mod h A_h = \big\langle f' s^\ell(a') , 1 \big\rangle \mod h A_h =$$

$$= \epsilon_{V^\ell(L)_h}\big(f' s^\ell(a')\big) \mod h A_h = \epsilon_{V^\ell(L)}(\mathrm{f} a) = \omega_L(\mathrm{f})(a) = \big\langle \mathrm{f} , \delta''(a) \big\rangle$$

(cf. §3.12 for the last but one identity). This proves that $\sigma\big(\delta'(a)\big) = \delta''(a)$ for all $a \in A$.

For the differential on L^*, consider $x := \overline{\chi^\vee} = \overline{h^{-1}\chi} \in \overline{\mathfrak{J}}_h^\vee$, with $\chi \in \mathfrak{J}_h$; then we have $\sigma(x) := \overline{\chi} \mod \mathfrak{J}^{\,2} =: X \in \mathfrak{J}/\mathfrak{J}^{\,2} = L^*$. Our goal is to prove that $(\sigma \otimes \sigma)\big(\delta'(x)\big) = \delta''\big(\sigma(x)\big)$.

Write $\Delta(\chi) = \chi_{(1)} \otimes \chi_{(2)}$ as $\Delta(\chi) = \chi \otimes 1 + 1 \otimes \chi + \sum_{[\theta]} \chi_{[1]} \otimes \chi_{[2]}$; then we have $\Delta\big(\chi^\vee\big) = \chi^\vee \otimes 1 + 1 \otimes \chi^\vee + h \sum_{[\theta]} \chi_{[1]}^\vee \otimes \chi_{[2]}^\vee$ — where $\chi_{[i]}^\vee := h^{-1}\chi_{[i]}$, for $i \in \{1,2\}$ — so that $\delta'(x) := -\sum_{[\theta]} x_{[1]} \otimes x_{[2]} + \sum_{[\theta]} x_{[2]} \otimes x_{[1]}$ with $x_{[i]} := \overline{\chi_{[i]}^\vee}$ for $i \in \{1,2\}$. In all this, $\chi^\vee := h^{-1}\chi$ is a lifting of $x \in L'$ in $V^r(L')_h := J^r(L)_h^\vee$, and χ is a lifting of $X := \sigma(x)$ in $J^r(L)_h$; in addition, we can assume that $\partial_h(\chi) = 0$. We adopt similar remarks, and notation, for $x_{[i]}$, $\chi_{[i]}$ and $X_{[i]} := \sigma(x_{[i]})$ with $i \in \{1,2\}$. Now for $\mathrm{f}, \mathrm{m} \in L$ and liftings $f', m' \in V^\ell(L)_h$ of them, direct calculation yields

$$\big\langle \mathrm{f} \otimes \mathrm{m} , \delta''\big(\sigma(x)\big) \big\rangle = \big\langle \mathrm{f} \otimes \mathrm{m} , \delta''(X) \big\rangle = \omega''_L(\mathrm{f})\big(\langle \mathrm{m} , X \rangle\big) - \omega''_L(\mathrm{m})\big(\langle \mathrm{f} , X \rangle\big) - \big\langle [\mathrm{f},\mathrm{m}]''_L , X \big\rangle =$$

$$= \epsilon_{V^\ell(L)}\big(\mathrm{f}\langle \mathrm{m} , X \rangle\big) - \epsilon_{V^\ell(L)}\big(\mathrm{m}\langle \mathrm{f} , X \rangle\big) - \big\langle \mathrm{f}\mathrm{m} - \mathrm{m}\mathrm{f} , X \big\rangle =$$

$$= \Big(\big\langle f' \cdot t^\ell\big(\langle m' , \chi \rangle\big) , 1 \big\rangle - \big\langle m' \cdot t^\ell\big(\langle f' , \chi \rangle\big) , 1 \big\rangle - \big\langle f' m' - m' f' , \chi \big\rangle \Big) \mod h A_h =$$

$$= \Big(\big\langle f' \cdot t^\ell\big(\langle m' , \chi \rangle\big) , 1 \big\rangle - \big\langle m' \cdot t^\ell\big(\langle f' , \chi \rangle\big) , 1 \big\rangle -$$

$$- \big\langle f' \cdot t^\ell\big(\langle f' , \chi_{(2)} \rangle\big) , \chi_{(1)} \big\rangle + \big\langle m' \cdot t^\ell\big(\langle f' , \chi_{(2)} \rangle\big) , \chi_{(1)} \big\rangle \Big) \mod h A_h =$$

$$= \Big(\big\langle f' \cdot t^\ell\big(\langle m' , \chi \rangle\big) , 1 \big\rangle - \big\langle m' \cdot t^\ell\big(\langle f' , \chi \rangle\big) , 1 \big\rangle - \big\langle f' \cdot t^\ell\big(\langle m' , 1 \rangle\big) , \chi \big\rangle +$$

$$+ \big\langle m' \cdot t^\ell\big(\langle f' , 1 \rangle\big) , \chi \big\rangle - \big\langle f' \cdot t^\ell\big(\langle m' , \chi \rangle\big) , 1 \big\rangle + \big\langle m' \cdot t^\ell\big(\langle f' , \chi \rangle\big) , 1 \big\rangle -$$

$$- \big\langle f' \cdot t^\ell\big(\langle m' , \chi_{[2]} \rangle\big) , \chi_{[1]} \big\rangle + \big\langle m' \cdot t^\ell\big(\langle f' , \chi_{[2]} \rangle\big) , \chi_{[1]} \big\rangle \Big) \mod h A_h =$$

$$= \Big(\big\langle m' \cdot t^\ell\big(\langle f' , \chi_{[2]} \rangle\big) , \chi_{[1]} \big\rangle - \big\langle f' \cdot t^\ell\big(\langle m' , \chi_{[2]} \rangle\big) , \chi_{[1]} \big\rangle \Big) \mod h A_h =$$

$$= \Big(\big\langle m' , \chi_{[1]} s^r\big(\langle f' , \chi_{[2]} \rangle\big) \big\rangle - \big\langle f' , \chi_{[1]} s^r\big(\langle m' , \chi_{[2]} \rangle\big) \big\rangle \Big) \mod h A_h =$$

$$= \Big(\big\langle m' , \chi_{[1]} t^r\big(\langle f' , \chi_{[2]} \rangle\big) \big\rangle - \big\langle f' , \chi_{[1]} t^r\big(\langle m' , \chi_{[2]} \rangle\big) \big\rangle \Big) \mod h A_h =$$

$$= \Big(\big\langle m' , \chi_{[1]} \big\rangle \big\langle f' , \chi_{[2]} \big\rangle - \big\langle f' , \chi_{[1]} \big\rangle \big\langle m' , \chi_{[2]} \big\rangle \Big) \mod h A_h =$$

$$= \big\langle \mathrm{m} , \sigma\big(x_{[1]}\big) \big\rangle \big\langle \mathrm{f} , \sigma\big(x_{[2]}\big) \big\rangle - \big\langle \mathrm{f} , \sigma\big(x_{[1]}\big) \big\rangle \big\langle \mathrm{m} , \sigma\big(x_{[2]}\big) \big\rangle =$$

$$= \big\langle \mathrm{f} \otimes \mathrm{m} , (\sigma \otimes \sigma)\big(\Delta^{[1]}(x)_{2,1} - \Delta^{[1]}(x)\big) \big\rangle = \big\langle \mathrm{f} \otimes \mathrm{m} , (\sigma \otimes \sigma)\big(\delta'(x)\big) \big\rangle$$

Here above we used the fact that $s^r\big(\langle f' , \chi_{[2]} \rangle\big) - t^r\big(\langle m' , \chi_{[2]} \rangle\big)$ belongs to $\mathfrak{J}_h$, so that we have $\chi_{[1]}\big(s^r\big(\langle f' , \chi_{[2]} \rangle\big) - t^r\big(\langle f' , \chi_{[2]} \rangle\big)\big) \in \mathfrak{J}_h^2$ and $\big\langle m' , \chi_{[1]}\big(s^r\big(\langle f' , \chi_{[2]} \rangle\big) - t^r\big(\langle f' , \chi_{[2]} \rangle\big)\big) \big\rangle = 0 \mod h A_h$. Thus $\big\langle \mathrm{f} \otimes \mathrm{m} , \delta''\big(\sigma(x)\big) \big\rangle = \big\langle \mathrm{f} \otimes \mathrm{m} , (\sigma \otimes \sigma)\big(\delta'(x)\big) \big\rangle$ for $\mathrm{f}, \mathrm{m} \in L$, so $\delta''\big(\sigma(x)\big) = (\sigma \otimes \sigma)\big(\delta'(x)\big)$.

In the end, all the above eventually completes the proof of claim *(a.1)*.

As to claim *(a.2)*, let $\big(K_h , A_h , s^r_{K_h} , t^r_{K_h} , \Delta , \partial_{K_h}\big)$ and $\big(\varGamma_h , B_h , s^r_{\varGamma_h} , t^r_{\varGamma_h} , \Delta , \partial_{\varGamma_h}\big)$ be two RQFSAd's, and let $(f, \phi) : K_h \longrightarrow \varGamma_h$ be a morphism between them in (RQFSAd). The very definition of morphism in (RQFSAd) imply at once that $\phi\big(s^r_{K_h}(A_h)\big) \subseteq s^r_{\varGamma_h}(B_h)$ — because $\phi \circ s^r_{K_h} = s^r_{\varGamma_h} \circ f$ — and $\phi\big(I_{K_h}\big) \subseteq I_{\varGamma_h}$ — because $\partial_{\varGamma_h} \circ \phi = \partial_{K_h}$ — hence also $\phi\big(h^{-1} I_{K_h}\big) \subseteq h^{-1} I_{\varGamma_h}$ for the natural, $k((h))$–linear extension of $\phi : K_h \longrightarrow \varGamma_h$ to $\phi^\times : (K_h)_F \longrightarrow (\varGamma_h)_F$. By construction, this implies that $\phi^\times$ defines by restriction a morphism $\phi^\times : K_h^\times \longrightarrow \varGamma_h^\times$, and this in turn extends by h–adic continuity to a well defined morphism $\phi^\vee : K_h^\vee \longrightarrow \varGamma_h^\vee$ in the category (RQUEAd).

(b) A direct proof of claim *(b)* can be given mimicking stepwise the proof of claim *(a)*. Otherwuse, it can be deduced from claim *(a)* (and, clearly, the rôles of the two results in this deduction can be reversed) as follows.

If $\varGamma_h := J^\ell(L)_h \in (\mathrm{LQFSAd})_{A_h}$, then $(\varGamma_h)^{op}_{coop} \in (\mathrm{RQFSAd})_{A_h}$; thus by claim *(a)* we have that $\big((\varGamma_h)^{op}_{coop}\big)^\vee \in (\mathrm{RQUEAd})_{A_h}$. Now, by construction $\big((\varGamma_h)^{op}_{coop}\big)^\vee = \big(\varGamma_h^\vee\big)^{op}_{coop}$, hence we deduce that $\varGamma_h^\vee \in (\mathrm{LQUEAd})_{A_h}$. All other aspects of the claim also follow from this argument. $\square$

6.2 The Drinfeld functor(s) $(\ )' = {}'(\ )$

We introduce now a second type of Drinfeld functor, denoted $H \mapsto H'$. Just like for the functor $H \mapsto H^\vee$, this also is inspired by the similar notion introduced for "quantum" Hopf algebras (see [13]); nevertheless, in this case we must be more careful, as we shall presently explain.

Let H_h be a left (or a right) bialgebroid. If $s_h^\ell = t_h^\ell =: \iota_h^\ell$, then we can define H' as in the "classical" framework of quantum Hopf algebra deformations. Let us recall it.

First recall some notation. For any non empty ordered subset $E = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \cdots < i_k$, define the morphism $j_E : H^{\otimes k} \longrightarrow H^{\otimes n}$ by

$$j_E(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n \text{ with } b_i := 1 \text{ if } i \notin E \text{ and } b_{i_m} := a_m \text{ for } 1 \leq m \leq k.$$

For each $n \in \mathbb{N}$, define the "iterated coproducts" $\Delta^n : H \longrightarrow H^{\otimes n}$ by $\Delta^0 := \iota_h^\ell \circ \epsilon$, $\Delta^1 := id_H$, $\Delta^n := \big(\Delta \otimes id_H^{\otimes (n-2)}\big) \circ \Delta^{n-1}$ for all $n \geq 2$. Then set

$$\Delta_E := j_E \circ \Delta^{|E|} \quad , \quad \Delta_\emptyset := \Delta^0 \quad , \qquad \text{and} \qquad \delta_E := \textstyle\sum_{E' \subseteq E} (-1)^{n-|E'|} \Delta_{E'} \quad , \quad \delta_\emptyset := \iota_h^\ell \circ \epsilon$$

By the inclusion-exclusion principle, the inverse formula $\Delta_E = \sum_{E' \subseteq E} \delta_\Psi$ holds. As usual, we introduce the notation $\delta_0 := \delta_\emptyset$, $\delta_n := \delta_{\{1,\dots,n\}}$. Then, we define

$$H' := \big\{\, a \in H \,\big|\, \delta_n(a) \in h^n H^n \;\; \forall\, n \in \mathbb{N} \,\big\} \;\subseteq\; H$$

We have also the formula $\delta_n = (id_H - s^\ell \circ \epsilon)^{\otimes n} \circ \Delta^n$, where $(id_H - s^\ell \circ \epsilon)^{\otimes n}$ is the projection of $H^{\otimes n}$ onto $\mathfrak{J}^{\otimes n}$ defined by the decomposition $H = \mathfrak{J} \oplus s_\ell(A)$, with $\mathfrak{J} := Ker(\epsilon)$ as usual.

If s^ℓ and t^ℓ *do not* coincide, then j_E is not well-defined, nor is $j_E \circ \Delta^n$. The projection of $H^{\otimes n}$ onto $\mathfrak{J}^{\otimes n}$ is not defined either, because the $\big(A_h \otimes A_h^{op}\big)$–module $\mathfrak{J}_h$ does not have a complement in H_h . Therefore, as s^ℓ and t^ℓ do not necessarily coincide, we adopt the following definition:

Definition 6.5. *As above, we use notation* $(H_h)_F := k((h)) \otimes_{k[[h]]} H_h$.

(a) If $H_h \in (\mathrm{LQUEAd})_{A_h}$ *, we define*

$$H_h' := \big\{\, \eta \in (H_h)_F \,\big|\, \big\langle\, \eta \,, (H_h^*)^\times \,\big\rangle \in A_h \,\big\} \quad , \qquad {}'H_h := \big\{\, \eta \in (H_h)_F \,\big|\, \big\langle\, \eta \,, ((H_h)_*)^\times \,\big\rangle \in A_h \,\big\}$$

(b) If $H_h \in (\mathrm{RQUEAd})_{A_h}$ *, we define*

$$H_h' := \big\{\, \eta \in (H_h)_F \,\big|\, \big\langle\, \eta \,, ({}^*H_h)^\times \,\big\rangle \in A_h \,\big\} \quad , \qquad {}'H_h := \big\{\, \eta \in (H_h)_F \,\big|\, \big\langle\, \eta \,, ({}_*(H_h))^\times \,\big\rangle \in A_h \,\big\}$$

Proposition 6.6. *Let* $H_h \in (\mathrm{LQUEAd})_{A_h}$ *. Then*

(a) $\quad H_h' \subseteq H_h \,, \qquad {}'H_h \subseteq H_h$

$$H_h' = {}_*\big((H_h^*)^\times\big) = {}_*\big((H_h^*)^\vee\big) \,, \qquad {}'H_h = {}^*\big((H_h)_*^{\,\times}\big) = {}^*\big((H_h)_*^{\,\vee}\big)$$

$$H_h' = \big\{\, \lambda : H_h^* \to A_h \,\big|\, \lambda(u+u') = \lambda(u) + \lambda(u') \,,\; \lambda(u\, t^r(a)) = a\, \lambda(u) \,,\; \lambda\big(I_{H_h^*}^n\big) \subseteq h^n A_h \;\, \forall\, n \,\big\}$$

$${}'H_h = \big\{\, \lambda : H_{h*} \to A_h \,\big|\, \lambda(u+u') = \lambda(u) + \lambda(u') \,,\; \lambda(u\, s^r(a)) = \lambda(u)\, a \,,\; \lambda\big(I_{H_{h*}}^n\big) \subseteq h^n A_h \;\, \forall\, n \,\big\}$$

(b) The analogous results hold if $H_h \in (\mathrm{RQUEAd})_{A_h}$.

Proof. We reproduce the proof of [13]. Consider H_h' in case $H_h \in (\mathrm{LQUEAd})_{A_h}$ and set $K_h := H_h^*$: we prove that $H_h' = {}_*\big(K_h^\times\big) = {}_*\big((H_h^*)^\times\big)$, the rest being entirely similar (left to the reader).

It is clear that $H_h' \subseteq {}_*\big(K_h^\times\big) = {}_*\big((H_h^*)^\times\big)$, by definition. For the converse inclusion, consider $f \in {}_*\big(K_h^\times\big)$: it is determined by $f\big|_{K_h}$. As $f\big(h^{-n} I_h^n\big) \subseteq A_h$, one has $f\big(I_h^n\big) \subseteq h^n A_h$: therefore $f \in {}_\star K_h = {}_\star\big(H_h^*\big) = H_h$. In addition, $f\big(K_h^\times\big) \subseteq A_h$ yields $f \in H_h'$, q.e.d.

Finally, it is clear that ${}_*\big(K_h^\times\big) = {}_*\big(K_h^\vee\big)$ because every linear functional on $K_h^\vee$ restricts to a similar functional on $K_h^\times$ and, conversely every linear functional on $K_h^\times$ uniquely extends (by h–adic continuity) to a similar functional on $K_h^\vee$.

All the above proves all aspects of the statement concerning H_h'; by similar arguments, one also proves those about ${}'H_h$ as well. ∎

Remark 6.7. If $H_h \in (\mathrm{LQUEAd})_{A_h}$, then $\big((H_h)^{op}_{coop}\big)' = \big({}'H_h\big)^{op}_{coop}$. This follows from the following three remarks:

— if U is any left bialgebroid, then $(U_*)^{op}_{coop} \cong {}^*(U^{op}_{coop})$ as left bialgebroids;

— if W is any right bialgebroid, then $({}^*W)^{op}_{coop} \cong \big(W^{op}_{coop}\big)_*$ as right bialgebroids;

— the functor $(\ )^\vee$ commutes with the functor $(\ )^{op}_{coop}$.

Similarly, one has ${}'\big((H_h)^{op}_{coop}\big) = (H_h')^{op}_{coop}$. Finally, in the same way one finds also the parallel identities $\big((H_h)^{op}_{coop}\big)' = \big({}'H_h\big)^{op}_{coop}$ and ${}'\big((H_h)^{op}_{coop}\big) = (H_h')^{op}_{coop}$ for every $H_h \in (\mathrm{RQUEAd})_{A_h}$.

6.8. Explicit description of ${}'H_h$. For a given $H_h \in (\mathrm{LQUEAd})_{A_h}$, we shall now provide an explicit description of ${}'H_h$.

Write $H_h = \mathfrak{I}_h \oplus s_\ell(A_h)$, and let π_s be the projection of H_h onto $\mathfrak{I}_h$: note that this is *not* a morphism of $\big(A_h \otimes A_h^{op}\big)$–modules. We need the following lemma, whose proof is left to the reader:

Lemma 6.9. *For any $u \in H_h$ and $a \in A_h$, one has $\pi_s\big(s^\ell(a)\, u\big) = s^\ell(a)\, \pi_s(u)$.*

If in addition $t^\ell(a) - s^\ell(a) = h\, j$ for some $j \in \mathfrak{I}_h$, then $\pi_s\big(t^\ell(a)\, u\big) = s^\ell(a)\, \pi_s(u) + h\, \pi_s(j\, u)$.

The operator $\pi_s^{\otimes n}$ is not defined on $H_h \otimes_{A_h} H_h \otimes_{A_h} \cdots \otimes_{A_h} H_h$. If $u_1 \otimes \cdots \otimes u_n \in H_h \otimes_{A_h} H_h \otimes_{A_h} \cdots \otimes_{A_h} H_h$, then $\pi_s(u_1) \otimes \cdots \otimes \pi_s(u_n)$ depends on the way of writing of $u_1 \otimes \cdots \otimes u_n$. We will say that the component of $\sum u_1 \otimes \cdots \otimes u_n$ in $\mathfrak{I}_h^{\otimes n}$ is defined up to $h^n \mathfrak{I}_h^{\otimes n}$ if $\sum u_1 \otimes \cdots \otimes u_n = \sum v_1 \otimes \cdots \otimes v_n$ implies $\sum \pi_s(u_1) \otimes \cdots \otimes \pi_s(u_n) - \sum \pi_s(v_1) \otimes \cdots \otimes \pi_s(v_n) \in h^n \mathfrak{I}_h^{\otimes n}$.

Lemma 6.10. *Let $u \in H_h$ and $n \in \mathbb{N}_+$. If the component of $\Delta^n(u)$ in $\mathfrak{I}_h^{\otimes n}$ is defined up to $h^n \mathfrak{I}_h^{\otimes n}$ and belongs to $h^n \mathfrak{I}_h^{\otimes n}$, then the component of $\Delta^{n+1}(u)$ is defined up to $h^{n+1} \mathfrak{I}_h^{\otimes (n+1)}$ — hence it makes sense to say that it belongs to $h^{n+1} \mathfrak{I}_h^{\otimes (n+1)}$.*

Proof. If the component of $\Delta^n(u)$ in $\mathfrak{I}_h^{\otimes n}$ is in $h^n \mathfrak{I}_h^{\otimes n}$, then $\Delta^n(u)$ can be written as

$$\Delta^n(u) \;=\; \sum h^n\, \phi_1 \otimes \cdots \otimes \phi_n \;+\; \textit{other terms}$$

where all the ϕ_i's are in $\mathfrak{I}_h$ and "other terms" stands for a sum of homogeneous tensors containing (as tensor factors) elements of $s_\ell(A_h)$ which do not occur in the computation of the component of $\Delta^{n+1}(u)$ in $\mathfrak{I}_h^{n+1}$. Assume that $\Delta^{n+1}(u)$ can be written, for some $a \in A$, as

$$\Delta^{n+1}(u) \;=\; \sum h^n\, \chi_1 \otimes \cdots \otimes t^\ell(a)\chi_i \otimes \chi_{i+1} \otimes \cdots \otimes \chi_{n+1} \;+\; \textit{other terms}$$

or

$$\Delta^{n+1}(u) \;=\; \sum h^n\, \chi_1 \otimes \cdots \otimes \chi_i \otimes s^\ell(a)\, \chi_{i+1} \otimes \cdots \otimes \chi_{n+1} \;+\; \textit{other terms}$$

and let us compute $\pi_s^{\otimes(n+1)}\big(\Delta^{n+1}(u)\big)$ in both cases.

In the second case, $\pi_s^{\otimes(n+1)}\big(\Delta^{n+1}(u)\big)$ can be written as

$$\pi_s^{\otimes(n+1)}\big(\Delta^{n+1}(u)\big) \;=\; \sum h^n\, \pi_s(\chi_1) \otimes \cdots \otimes \pi_s(\chi_i) \otimes s_\ell(a)\, \pi_s(\chi_{i+1}) \otimes \cdots \otimes \pi_s(\chi_{n+1})$$

In the first case, if we write $t^\ell(a) - s^\ell(a) = h\,j$ (with $j \in \mathfrak{J}_h$) and use the previous lemma, we get

$$\begin{aligned}\pi_s^{\otimes(n+1)}\big(\Delta^{n+1}(u)\big) \; &= \; \sum h^n\,\pi_s(\chi_1) \otimes \cdots \otimes t^\ell(a)\,\pi_s(\chi_i) \otimes \pi_s(\chi_{i+1}) \otimes \cdots \otimes \pi_s(\chi_{n+1}) \,+ \\ &+ \sum h^n\,\pi_s(\chi_1) \otimes \cdots \otimes \pi_s(\chi_{i-1}) \otimes h\,\big(-j\,\pi_s(\chi_i) + \pi_s(j\,\chi_i)\big) \otimes \pi_s(\chi_{i+1}) \cdots \otimes \pi_s(\chi_{n+1})\end{aligned}$$

Taking the difference between the two computations we find

$$h^n\,\pi_s(\chi_1) \otimes \cdots \otimes \pi_s(\chi_{i-1}) \otimes h\,\big(-j\,\pi_s(\chi_i) + \pi_s(j\,\chi_i)\big) \otimes \pi_s(\chi_{i+1}) \cdots \otimes \pi_s(\chi_{n+1})$$

which does belong to $h^{n+1}\mathfrak{J}^{\otimes(n+1)}$, q.e.d. $\square$

Notation: If the component of $\Delta^n(u)$ in $\mathfrak{J}_h^{\otimes n}$ is defined up to $h^n\mathfrak{J}^{\otimes n}$, we shall write it as $\delta_s^n(u)$. Then the condition $\delta_s^n(u) \in h^n\mathfrak{J}^{\otimes n}$ perfectly makes sense. Hereafter we shall write $\delta_s^n(u) \in h^n\mathfrak{J}^{\otimes n}$ to mean that $\delta_s^n(u)$ is well defined — i.e., the component of $\Delta^n(u)$ in $\mathfrak{J}^{\otimes n}$ is well defined — up to $h^n\mathfrak{J}^{\otimes n}$ *and* it belongs to $h^n\mathfrak{J}^{\otimes n}$.

For the rest of the discussion, we introduce also the following notation:

$$\delta_s(H_h) \; := \; \big\{\, u \in H_h \,\big|\, \delta_s^n(u) \in h^n\mathfrak{J}_h^{\otimes n} \;\; \forall\, n \in \mathbb{N}_+ \big\}$$

We need again a couple of technical results:

Proposition 6.11. *Let* $u \in \delta_s(H_h)$ *. Then* $\Delta(u)$ *can be written as*

$$\Delta(u) \; = \; u \otimes 1 + \sum u'_{(1)} \otimes u'_{(2)} \qquad \textit{with} \quad u'_{(1)} \in \delta_s(H_h) \quad \textit{and} \quad u'_{(2)} \in h\,\mathfrak{J}_h$$

Proof. *First case:* L is a finite free as an A–module.

Let $\big\{\overline{e}_1\,, \overline{e}_2\,, \dots\,, \overline{e}_n\big\}$ be a basis of the A–module L: we lift each $\overline{e}_i$ to an element $e_i \in H_h$ such that $\epsilon(e_i) = 0$. Let $u \in \delta^s(H_h)$. We write $\Delta(u)$ as

$$\Delta(u) \;\; = \;\; u' \otimes 1 \, + \, \textstyle\sum_{\underline{\alpha}\in\mathbb{N}^n\setminus\{\underline{0}\}} u_{\underline{\alpha}} \otimes e^{\underline{\alpha}} \;\text{ with } \displaystyle\lim_{|\underline{\alpha}|\to+\infty} \| \, u_{\underline{\alpha}} \, \| = 0.$$

for suitable $u', u_{\underline{\alpha}} \in H_h$. The relation $m_{H_h}\big((s^\ell \circ \epsilon) \otimes id\big)\big(\Delta(u)\big) = u$ gives $u' = u$. Thus we have

$$\Delta(u) \;\; = \;\; u \otimes 1 \, + \, \textstyle\sum_{\underline{\alpha}\in\mathbb{N}^n\setminus\{\underline{0}\}} u_{\underline{\alpha}} \otimes e^{\underline{\alpha}}$$

The relation $m_{H_h}\big((s^\ell \circ \epsilon) \otimes id\big)\big(\Delta(u)\big) = u$ yields the identity

$$\textstyle\sum_{\underline{\alpha}\in\mathbb{N}^n\setminus\{\underline{0}\}} s^\ell\big(\epsilon(u_{\underline{\alpha}})\big)\, e^{\underline{\alpha}} \;\; = \;\; u - s^\ell\big(\epsilon(u)\big)$$

As $u \in \delta_s(H_h)$, one has $u - s^\ell\big(\epsilon(u)\big) \in h\,\mathfrak{J}_h$, which implies that $s^\ell\big(\epsilon(u_{\underline{\alpha}})\big) \in h\,H_h$; hence $\overline{s^\ell\big(\epsilon(u_{\underline{\alpha}})\big)} = \overline{s_\ell}\big(\overline{\epsilon(u_{\underline{\alpha}})}\big) = 0 \in H_h \big/ h\,H_h$. As $\overline{s_\ell}$ is injective, we get $\overline{\epsilon(u_{\underline{\alpha}})} = 0$, that is to say $\epsilon(u_{\underline{\alpha}}) \in h\,A_h$.

If $n > 1$, one has

$$\delta_s^n(u) \;\; = \;\; \textstyle\sum_{\underline{\alpha}\in\mathbb{N}^n} \delta_s^{n-1}(u_{\underline{\alpha}}) \otimes e^{\underline{\alpha}} \;\; \in \;\; h^n\,\mathfrak{J}^{\otimes n}$$

which implies $\delta_s^{n-1}(u_{\underline{\alpha}}) \in h^n\,\mathfrak{J}^{\otimes(n-1)}$ and $u_{\underline{\alpha}} \in \delta_s(H_h)$. Let $\widetilde{u}_{\underline{\alpha}} = \pi_s(u_{\underline{\alpha}}) = u_{\underline{\alpha}} - s^\ell\big(\epsilon(u_{\underline{\alpha}})\big)$. For all $n \geq 1$, one has $\delta_s^n(\widetilde{u}_{\underline{\alpha}}) = \delta_s^n(u_{\underline{\alpha}}) \in h^{n+1}\,\mathfrak{J}^n$. In particular for $n = 1$ we get $\widetilde{u}_{\underline{\alpha}} = h\,w_\alpha$ for some $w_\alpha \in \delta_s(H_h)$. The element $u_{\underline{\alpha}}$ can be written as $u_{\underline{\alpha}} = h\,\big(w_{\underline{\alpha}} + s^\ell\big(h^{-1}\epsilon(u_{\underline{\alpha}})\big)\big) \in h\,\delta_s(H_h)$.

Second case: L is finite projective as an A–module.

Like in Subsection 4.2, we fix a finite projective A–module Q such that $L \oplus Q = F$ is a finite free A–module. We fix an A–basis $B := \{e_1\,, \dots\,, e_n\}$ of F: then we call Y the k–span of B , so that we can write $F = A \otimes_k Y$. Moreover, we construct the (infinite dimensional) Lie-Rinehart algebra $L_Q = L \oplus \big(A \otimes_k Z\big)$, with $Z = Y \oplus Y \oplus Y \oplus \cdots$, which has a good basis $\{e_i\}_{i\in T:=\mathbb{N}\times\{1,\dots,n\}}$ defined by B . Like in §4.15, we can define $H_{h,Y}$ and $\delta_s(H_{h,Y})$. Now given $u \in \delta_s(H_{h,Y})$, we can write $\Delta(u)$ as follows:

$$\Delta(u) \;\; = \;\; u \otimes 1 \, + \, \textstyle\sum_{\underline{\alpha}\in T^{(\mathbb{N})}\setminus\{\underline{0}\}} u_{\underline{\alpha}} \otimes e^{\underline{\alpha}} \qquad \text{with} \qquad \displaystyle\lim_{|\underline{\alpha}|+\varpi(\underline{\alpha})\to+\infty} \big\| u_{\underline{\alpha}} \big\| = 0$$

Then the same reasoning as above shows that the proposition is true for $H_{h,Y}$ in the rôle of H_h.

Recall (cf. §4.15) that $H_{h,Y} = H_h \oplus \big(H_h \,\widehat{\otimes}_k\, S(Z)^+\big)$ where $H_h \,\widehat{\otimes}_k\, S(Z)^+$ is the h–adic completion of $H_h \otimes_k S(Z)^+$, with $Z = Y \oplus Y \oplus Y \oplus \cdots$; the natural projection $\pi_Y : H_{h,Y} \twoheadrightarrow H_h$ is then a morpism of left bialgebroids. Moreover, if $\mathfrak{J}_{h,Y}$ is the kernel of the counit of $H_{h,Y}$, we have $\mathfrak{J}_{h,Y} = \mathfrak{J}_h \oplus \big(H_h \,\widehat{\otimes}_k\, S(Z)^+\big)$. Now it is easy to see that, if $v \in \delta_s(H_{h,Y})$, then $\pi_Y(v) \in \delta_s(H_h)$.

Now let $u \in \delta_s(H_h)$. By the result for $H_{h,Y}$, we know that $\Delta(u)$ can be written as

$$\Delta(u) \;=\; u \otimes 1 + \sum u'_{(1)} \otimes u'_{(2)} \qquad \text{with} \quad u'_{(1)} \in \delta_s(H_{h,Y}) \quad \text{and} \quad u'_{(2)} \in h\,\mathfrak{J}_{h,Y}$$

As $\pi_Y(u) = u$, applying $\pi_Y \otimes \pi_Y$ to the previous identity we get

$$\Delta(u) \;=\; u \otimes 1 + \sum \pi_Y(u'_{(1)}) \otimes \pi_Y(u'_{(2)})$$

with $\pi_Y(u'_{(1)}) \in \pi_Y\big(\delta_s(H_{h,Y})\big) = \delta_s(H_h)$ and $\pi_Y(u_{(2)}) \in h\,\pi_Y\big(\mathfrak{J}_{h,Y}\big)\big) = h\,\mathfrak{J}_h$, q.e.d. $\square$

Lemma 6.12. *$s^\ell(A_h) \cdot \delta_s(H_h) \subseteq \delta_s(H_h)$ and $t^\ell(A_h) \cdot \delta_s(H_h) \subseteq \delta_s(H_h)$.*

Proof. Let $u \in \delta_s(H_h)$ and $a \in A_h$. The properties $s^\ell(a)\,u \in \delta_s(H_h)$ follows from the following properties: $\pi_s\big(s^\ell(a)\,u\big) = s^\ell(a)\,\pi_s(u)$ and $\Delta^n\big(s^\ell(a)\big) = s^\ell(a) \otimes 1 \otimes \cdots \otimes 1$. Let us now show that $\delta_s^n\big(t^\ell(a)\,u\big) \in h^n\,\mathfrak{J}^{\otimes n}$ for all $n \in \mathbb{N}$. Write $t^\ell(a) - s^\ell(a) = h\,j$ with $j \in \mathfrak{J}_h$.

For $n = 1$, by Lemma 6.9 we have $\pi_s\big(t^\ell(a)\,u\big) = s^\ell(a)\,\pi_s(u) + h\,\pi_s\big(j\,u\big) \in h\,\mathfrak{J}$.

For $n > 1$, let us show that $\delta_s^n\big(t^\ell(a)\big) \in h^n\,\mathfrak{J}^{\otimes n}$. Set $\Delta(u) \;=\; u \otimes 1 + u'_{(1)} \otimes u'_{(2)}$ with $u'_{(1)} \in \delta_s(H_h)$, $u'_{(2)} \in h\,\mathfrak{J}_h$ (cf. Proposition 6.11). Then $\Delta\big(t^\ell(a)\,u\big) \;=\; u \otimes t^l(a) + u'_{(1)} \otimes t^\ell(a)\,u'_{(2)}$, hence $\delta_s^n\big(t^\ell(a)\,u\big) \;=\; \delta_s^{n-1}(u) \otimes \pi_s\big(t^\ell(a)\big) + \delta_s^{n-1}(u'_{(1)}) \otimes \pi_s\big(t^\ell(a)\,u'_{(2)}\big)$, thus $\delta_s^n\big(t^\ell(a)\,u\big) \in h^n\,\mathfrak{J}^{\otimes n}$. $\square$

We are now ready for the first key result of this subsection:

Theorem 6.13. *With assumptions and notation as above, we have*

$${}'H_h \;=\; \big\{\, u \in H_h \,\big|\, \delta_s^n(u) \in h^n\,\mathfrak{J}_h^{\otimes n} \;\forall\, n \in \mathbb{N}_+ \,\big\} \;=:\; \delta_s(H_h)$$

Proof. To begin with, we show that $\delta_s(H_h) \subseteq {}'H_h$. To this end, we prove that for any $u \in \delta_s(H)$ we have $\big\langle\, u\,, I^n_{(H_h)_*} \big\rangle \subseteq h^n A_h$ for all $n \in \mathbb{N}_+$, using induction on n.

Take $n = 1$. As $u \in \delta_s(H)$, note that $\delta^1(u) \in h\,\mathfrak{J}_h$ implies $u = h\,j + s^\ell\big(\epsilon(u)\big)$ with $j \in \mathfrak{J}_h$. Then one has

$$\big\langle\, u\,, I_{H_{h*}} \big\rangle \;=\; h \big\langle\, j\,, I_{H_{h*}} \big\rangle + \epsilon(u) \big\langle\, 1\,, I_{H_{h*}} \big\rangle \;\in\; h\,A_h$$

Now assume $n > 1$. For our $u \in \delta_s(H)$, set $\Delta(u) = u \otimes 1 + u'_{(1)} \otimes u'_{(2)}$ with $u'_{(1)} \in \delta_s(H_h)$ and $u'_{(2)} \in h\,\mathfrak{J}_h$ as in Proposition 6.11. Let $\alpha \in I^n_{H_{h*}}$ be of the form $\alpha = \alpha_1\,\alpha_2$ with $\alpha_1 \in I_{H_{h*}}$ and $\alpha_2 \in I^{n-1}_{H_{h*}}$: then, as the pairing $\langle\ ,\ \rangle$ between H_h and H_{h*} is a left bialgebroid pairing, we have

$$\big\langle\, u\,,\, \alpha_1\,\alpha_2 \,\big\rangle \;=\; \big\langle\, t^\ell\big(\langle\, u'_{(2)}\,,\, \alpha_1 \,\rangle\big)\, u'_{(1)}\,,\, \alpha_2 \,\big\rangle + \big\langle\, t^\ell\big(\langle\, 1\,,\, \alpha_1 \,\rangle\big)\, u\,,\, \alpha_2 \,\big\rangle \;\in\; h^n\,A_h$$

by the induction hypothesis and the case $n = 1$ (also using the two previous lemmas).

Conversely, let us now show that ${}'H_h \subseteq \delta_s(H_h)$. To this end, we prove (by induction on n) that for any $u \in {}'H_h$ one has $\delta_s^n(u) \in h^n \mathfrak{J}_h^{\otimes n}$ for all $n \in \mathbb{N}$.

For $n = 1$. As $u \in {}'H_h$ we have $\big\langle\, u\,, I_{H_{h*}} \big\rangle \subseteq h\,A_h$; on the other hand, $\delta_s^1(u) = u - s^\ell\big(\epsilon(u)\big)$ by definition. Then we have $\big\langle\, \delta_s^1(u)\,, \lambda \,\big\rangle \in h\,A_h$, if $\lambda \in I_{H_{h*}}$ because

$$\big\langle u - s^\ell\big(\epsilon(u)\big)\,, \lambda \big\rangle \;=\; \big\langle u\,, \lambda \big\rangle - \big\langle s^\ell\big(\epsilon(u)\big)\,, \lambda \big\rangle \;=\; \big\langle u\,, \lambda \big\rangle - \epsilon(u) \big\langle 1\,, \lambda \big\rangle \;=\; \big\langle u\,, \lambda \big\rangle - \epsilon(u)\,\partial(\lambda) \;\in\; h\,A_h$$

On the other hand, clearly $\delta_s^1(u) = u - s^\ell\big(\epsilon(u)\big) \in \mathfrak{J}_h$, hence $\delta_s^1(u) \in \mathfrak{J}_h \cap h\,H_h \;=\; h\,\mathfrak{J}_h$.

Let now $n > 1$, and assume by induction that $\delta_s^{n-1}(u') \in h^{n-1} \mathfrak{J}^{\otimes (n-1)}$ for all $u' \in {}'H_h$. For our $u \in {}'H_h$, write $\Delta(u) = u_{(1)} \otimes u_{(2)}$ with $u_{(1)}$ and $u_{(2)} \in {}'H$. As $\Delta^n(u) = \Delta^{n-1}\big(u_{(1)}\big) \otimes u_{(2)}$, we deduce that $\delta_s^n(u) \;=\; \delta_s^{n-1}\big(u_{(1)}\big) \otimes \delta_s^1\big(u_{(2)}\big) \;\in\; h^n \mathfrak{J}^{\otimes n}$ by the induction hypothesis and the case $n = 1$. $\square$

6.14. Explicit description of H_h'. We shall now give an explicit description of H_h': this will be entirely similar to that for ${}'H_h$, thus we shall only outline the main steps, without dwelling into details — which can be easily filled in by the reader.

Write $H_h = \mathfrak{J}_h \oplus t_\ell(A_h)$, and let π_t be the projection of H_h onto $\mathfrak{J}_h$: once again, this is *not* a morphism of $\big(A_h \otimes A_h^{op}\big)$–modules. The operator $\pi_t^{\otimes n}$ is not defined on $H_h \otimes_{A_h} H_h \otimes_{A_h} \cdots \otimes_{A_h} H_h$: indeed, if $u_1 \otimes \cdots \otimes u_n \in H_h \otimes_{A_h} H_h \otimes_{A_h} \cdots \otimes_{A_h} H_h$, then $\pi_t(u_1) \otimes \cdots \otimes \pi_t(u_n)$ depends on the way of writing $u_1 \otimes \cdots \otimes u_n$. We say that the component of $\sum u_1 \otimes \cdots \otimes u_n$ in $\mathfrak{J}_h^{\otimes n}$ is defined up to $h^n \mathfrak{J}_h^{\otimes n}$ if $\sum u_1 \otimes \cdots \otimes u_n = \sum v_1 \otimes \cdots \otimes v_n$ yields $\sum \pi_t(u_1) \otimes \cdots \otimes \pi_t(u_n) - \sum \pi_t(v_1) \otimes \cdots \otimes \pi_t(v_n) \in h^n \mathfrak{J}_h^{\otimes n}$.

The following lemma is the parallel of Lemma 6.10, with similar proof. *Note that the statement is formally the same*, but actually the "componentes" to which one refers in the two claims are defined with respect to different projectors — namely $\pi_s^{\otimes n}$ or $\pi_t^{\otimes n}$ — in the two cases.

Lemma 6.15. *Let $u \in H_h$. If the component of $\Delta^n(u)$ in $\mathfrak{J}_h^{\otimes n}$ is defined up to $h^n \mathfrak{J}_h^{\otimes n}$ and belongs to $h^n \mathfrak{J}_h^{\otimes n}$, then the component of $\Delta^{n+1}(u)$ is defined up to $h^{n+1} \mathfrak{J}_h^{\otimes (n+1)}$ — hence it makes sense to say that it belongs to $h^{n+1} \mathfrak{J}_h^{\otimes (n+1)}$.*

Notation: If the component of $\Delta^n(u)$ in $\mathfrak{J}_h^{\otimes n}$ is defined up to $h^n \mathfrak{J}^{\otimes n}$ (in the above sense), we shall write it as $\delta_t^n(u)$. Then the condition $\delta_t^n(u) \in h^n \mathfrak{J}^{\otimes n}$ perfectly makes sense. Thus we shall write $\delta_t^n(u) \in h^n \mathfrak{J}^{\otimes n}$ to mean that $\delta_t^n(u)$ is well defined (i.e., the component of $\Delta^n(u)$ in $\mathfrak{J}^{\otimes n}$, in the above sense, is well defined) up to $h^n \mathfrak{J}^{\otimes n}$ and it belongs to $h^n \mathfrak{J}^{\otimes n}$. Also, we set

$$\delta_t(H_h) \;:=\; \big\{\, u \in H_h \,\big|\, \delta_t^n(u) \in h^n \mathfrak{J}_h^{\otimes n} \;\; \forall\, n \in \mathbb{N}_+ \big\}$$

Arguing like for ${}'H_h$, we can then prove the following, analogous characterization of H_h':

Theorem 6.16. *With assumptions and notation as above, we have*

$$H_h' \;=\; \big\{\, u \in H_h \,\big|\, \delta_t^n(u) \in h^n \mathfrak{J}_h^{\otimes n} \; \forall\, n \in \mathbb{N}_+ \big\} \;=:\; \delta_t(H_h)$$

Remark 6.17. The study of ${}'H_h$ and H_h' we have done for LQUEAd holds for RQUEAd as well. One can check it directly (via the same arguments) or, besides, deducing the results for RQUEAd's from those for LQUEAd's in force of the general identities $\big(H_h'\big)_{coop}^{op} = {}'\big((H_h)_{coop}^{op}\big)$.

Thanks to the characterizations in Theorem 6.13 and Theorem 6.16 we can eventually prove the following remarkable result:

Theorem 6.18. *Let H_h be a LQUEAd or a RQUEAd. Then $H_h' = {}'H_h$.*

Proof. We begin with H_h being an LQUEAd. We show, that for any $u \in \delta_s(H_h)$ we have $\delta_t^n(u) \in h^n \mathfrak{J}^{\otimes n}$ for all $n \in \mathbb{N}$, by induction on n.

For $n = 1$, one has

$$\delta_t^1(u) \;=\; u - t_l\big(\epsilon(u)\big) \;=\; u - s_\ell\big(\epsilon(u)\big) + s_\ell\big(\epsilon(u)\big) - t_\ell\big(\epsilon(u)\big)$$

As $s^\ell - t^\ell = 0 \mod h$, one has $s^\ell\big(\epsilon(u)\big) - t^\ell\big(\epsilon(u)\big) \in h\,A_h$. Moreover, we have also $\epsilon\big(s^\ell\big(\epsilon(u)\big) - t^\ell\big(\epsilon(u)\big)\big) = 0$, so that $s^\ell\big(\epsilon(u)\big) - t^\ell\big(\epsilon(u)\big)$ belongs to $\mathfrak{J}_h \cap h\,A_h = h\,\mathfrak{J}_h$. Thus $\delta_t^1(u) \in h\,\mathfrak{J}_h$, q.e.d.

For $n > 1$, let us write $\Delta(u) = u \otimes 1 + u'_{(1)} \otimes u'_{(2)}$ with $u'_{(1)} \in \delta_s(H_h)$ and $u'_{(2)} \in h\,\mathfrak{J}_h$ as in Proposition 6.11. Then one has $\delta_t^n(u) = \delta_t^{n-1}\big(u'_{(1)}\big) \otimes \delta_t^1\big(u'_{(2)}\big)$, which is an element of $h^n \mathfrak{J}^{\otimes n}$ thanks to the induction hypothesis.

By the above we have proved the inclusion $\delta_s(H_h) \subseteq \delta_t(H_h)$; the reverse inclusion can be shown in the same way, so to give $\delta_s(H_h) = \delta_t(H_h)$. By Theorem 6.13 — giving ${}'H_h = \delta_t(H_h)$ — and Theorem 6.16 — giving $H_h' = \delta_s(H_h)$ — this eventually implies $H_h' = {}'H_h$.

For H_h a RQUEAd, we can provide a direct proof by the same arguments used for a LQUEAd; otherwise, we can deduce the result for RQUEAd's from that for LQUEAd's, as follows.

If H_h is a RQUEAd, then $(H_h)_{coop}^{op}$ is a LQUEAd; then we have the chain of identities $(H_h')_{coop}^{op} = {}'\big((H_h)_{coop}^{op}\big) = \big((H_h)_{coop}^{op}\big)' = \big({}'H_h\big)_{coop}^{op}$, whence ${}'H_h = H_h'$ follows too. $\square$

We are now ready for the main result of this subsection. In short, it claims that the construction $H_h \mapsto {}'H_h = H_h'$, starting from a quantization of L — of type $V^{\ell/r}(L)$ — provides a quantization of the *dual* Lie-Rinehart bialgebra L^* — of type $J^{\ell/r}(L^*)$; moreover, this construction is functorial.

Theorem 6.19. *(a) Let $V^\ell(L)_h \in (\mathrm{LQUEAd})_{A_h}$, where L is a Lie-Rinehart algebra which, as an A–module, is projective of finite type. Then:*

— (a.1) ${}'V^\ell(L)_h = V^\ell(L)_h' \in (\mathrm{LQFSAd})_{A_h}$, with semiclassical limit $V^\ell(L)_h' \big/ h\, V^\ell(L)_h' \cong J^\ell(L^*)$. Moreover, the structure of Lie-Rinehart bialgebra induced on L^* by the quantization $V^\ell(L)_h'$ of $J^\ell(L^*)$ is dual to that on L by the quantization $V^\ell(L)_h$ of $V^\ell(L)$;

— (a.2) the definition of $V^\ell(L)_h \mapsto {}'V^\ell(L)_h = V^\ell(L)_h'$ extends to morphisms in (LQUEAd), so that we have a well defined (covariant) functor ${}'(\ ) = (\ )' : (\mathrm{LQUEAd}) \longrightarrow (\mathrm{LQFSAd})$.

(b) Let $V^r(L)_h \in (\mathrm{RQUEAd})_{A_h}$, where L is a Lie-Rinehart algebra which, as an A–module, is projective of finite type. Then:

— (b.1) ${}'V^r(L)_h = V^r(L)_h' \in (\mathrm{RQFSAd})_{A_h}$, with semiclassical limit $V^r(L)_h' \big/ h\, V^r(L)_h' \cong J^r(L^*)$. Moreover, the structure of Lie-Rinehart bialgebra induced on L^* by the quantization $V^r(L)_h'$ of $J^r(L^*)$ is dual to that on L by the quantization $V^r(L)_h$ of $V^r(L)$;

— (b.2) the definition of $V^r(L)_h \mapsto {}'V^r(L)_h = V^r(L)_h'$ extends to morphisms in (RQUEAd), so that we have a well defined (covariant) functor ${}'(\ ) = (\ )' : (\mathrm{RQUEAd}) \longrightarrow (\mathrm{RQFSAd})$.

Proof. (a) Given $V^\ell(L)_h \in (\mathrm{LQUEAd})_{A_h}$, we know that $J^r(L)_h := V^\ell(L)_h^* \in (\mathrm{RQFSAd})_{A_h}$, by Theorem 5.5*(a)*; then $V^r(L^*)_h := J^r(L)_h^\vee \in (\mathrm{RQUEAd})_{A_h}$ is a quantization of $V^r(L^*)$, by Theorem 6.4. Now, there is a unique way to endow $\big(V^\ell(L)_h\big)'$ with a left bialgebroid structure such that the standard (non degenerate) pairing between $\big(V^\ell(L)_h\big)'$ and $\big((V^\ell(L)_h)^*\big)^\vee = J^r(L)_h^\vee$ is a bialgebroid left pairing: indeed, by Proposition 6.6 this pairing identifies $\big(V^\ell(L)_h\big)'$ with the left dual of $\big((V^\ell(L)_h)^*\big)^\vee = J^r(L)_h^\vee$, that is $\big(V^\ell(L)_h\big)' = {}_*\big(J^r(L)_h^\vee\big)$ as right bialgebroids. Then

$$\big(V^\ell(L)_h\big)' \;=\; {}_*\big(J^r(L)_h^\vee\big) \;=\; {}_*\big(V^r(L^*)_h\big) \;=\; J^\ell(L^*)_h$$

where $J^\ell(L^*)_h = {}_*\big(V^r(L^*)_h\big) \in (\mathrm{LQFSAd})_{A_h}$ is a quantization of $J^\ell(L^*)$, by Theorem 5.5. In all this, L^* stands for the A–module dual to L endowed with the Lie-Rinehart bialgebra structure dual to that defined on L by scratch and by the quantization $V^\ell(L)_h$ — according to Theorem 4.4.

This completes the proof of *(a.1)*.

(a.2) Let $H_h = V^\ell\big(L_A\big)_h$ be a LQUEAd over A_h and $\varGamma_h = V^\ell\big(L_B\big)_h$ a LQUEAd over B_h, and let $\phi := (f, F)$ be a morphism of left bialgebroids among them. Set $\mathfrak{J}_{H_h} := Ker(\epsilon_{H_h})$ and $\mathfrak{J}_{\varGamma_h} := Ker(\epsilon_{\varGamma_h})$. Then $F\big(\mathfrak{J}_{H_h}\big) \subseteq \mathfrak{J}_{\varGamma_h}$ by the property $\epsilon_{\varGamma_h} \circ F = f \circ \epsilon_{H_h}$ of a morphism of bialgebroids. Similarly, one has $F^{\otimes n} \circ \Delta^n_{H_h} = \Delta^{\otimes n}_{\varGamma_h} \circ F$ and $F \circ s^\ell_{H_h} = s^\ell_{\varGamma_h}$; from this, one easily sees that $\delta^n_s\big(F(u)\big) = F^{\otimes n}\big(\delta^n_s(u)\big)$. From all this we get $F(H_h') \subseteq \varGamma_h'$, so the restriction of the morphism (f, F) between H_h and $\varGamma_h$ provides a morphism in (LQFSAd) between H_h' and $\varGamma_h'$.

(b) A direct proof for claim *(b)* can be given by the same arguments used for *(a)*. Otherwise, we can deduce *(b)* from *(a)* as follows.

If $H_h \in (\mathrm{RQUEAd})$, then $(H_h)^{op}_{coop} \in (\mathrm{LQUEAd})$ and $\big((H_h)^{op}_{coop}\big)' = \big({}'H_h\big)^{op}_{coop}$, so that $H_h' = {}'H_h = \big(\big((H_h)^{op}_{coop}\big)'\big)^{op}_{coop}$. From this we can easily deduce claim *(b)* from claim *(a)*. □

6.20. Description of $V^\ell(L)_h'$ when L is a (finite type) free A–module. Let L be a Lie-Rinehart algebra which, as an A–module, is free of finite type. Let $V^\ell(L)_h \in (\mathrm{LQUEAd})_{A_h}$ be a quantization of $V^\ell(L)$. By the freeness of L, we can provide an explicit description of $V^\ell(L)_h'$, much like that given in [13] for the similar case of quantum universal enveloping algebras.

First of all, consider $K_h := V^\ell(L)_h^* \equiv J^r(L)_h \in (\mathrm{RQFSAd})_{A_h}$, which (cf. Theorem 5.5) is a quantization of $J^r(L)$. From Proposition 6.6 we have $V^\ell(L)_h' \cong {}_*\big(K_h^\vee\big)$.

Let $\{\overline{e}_i\}_{i\in\{1,\dots,n\}}$ be a basis of the free A–module L. Then (by the Poincaré-Birkhoff-Witt theorem) the set of ordered monomials $\big\{\overline{e}^{\,\underline{\alpha}}\big\}_{\underline{\alpha}\in\mathbb{N}^n}$ is an A–basis of $V^\ell(L)$, where $\overline{e}^{\,\underline{\alpha}} := \overline{e}_1^{\,\alpha_1}\cdots\overline{e}_n^{\,\alpha_n}$.

Let $\overline{\xi}_i \in Hom\big(V^\ell(L),A\big) \cong \widehat{S_A(L^*)}$ be defined by $\big\langle \overline{\xi}_i\,, \overline{e}_1^{\,\underline{\alpha}} \big\rangle = \delta_{\alpha_1,0}\cdots\delta_{\alpha_i,1}\cdots\delta_{\alpha_n,0}$. Then the ordered monomials $\frac{1}{\underline{\alpha}!}\,\overline{\xi}^{\,\underline{\alpha}}$ (with $\underline{\alpha}! := \alpha_1!\cdots\alpha_n!$) is a pseudobasis — i.e., a basis in topological sense — of the A–module $J^r(L)$ dual to the PBW basis $\big\{\overline{e}^{\,\underline{\alpha}}\big\}_{\underline{\alpha}\in\mathbb{N}^n}$.

Now lift $\{\overline{e}_i\}_{i\in\{1,\dots,n\}}$ to a subset $\{e_i\}_{i\in\{1,\dots,n\}}$ in $V^\ell(L)_h$, such that $\overline{e}_i = e_i \mod h\,V^\ell(L)_h$ and $\epsilon(e_i)=0$; similarly, lift $\{\overline{\xi}_i\}_{i\in\{1,\dots,n\}}$ to a subset $\{\xi_i\}_{i\in\{1,\dots,n\}}$ in $J^r(L)_h = K_h$ such that $\partial_h(\xi_i)=0$. Then $\big\{e^{\underline{\alpha}}\big\}_{\underline{\alpha}\in\mathbb{N}^n}$ is a topological basis of $V^\ell(L)_h$ and $\big\{\frac{1}{\underline{\alpha}!}\,\xi^{\underline{\alpha}}\big\}_{\underline{\alpha}\in\mathbb{N}^n}$ is a topological pseudo-basis of $J^r(L)_h = K_h$: indeed, these two (pseudo-)bases are dual to each other module h, i.e. $\big\langle e^{\underline{\alpha}}, \xi^{\underline{\beta}} \big\rangle = \delta_{\underline{\alpha},\underline{\beta}} + h\,A_h$.

Let $\big\{\theta^{\underline{\alpha}}\big\}_{\underline{\alpha}\in\mathbb{N}^n}$ be the topological basis of $V^\ell(L)_h$ dual to $\big\{\frac{1}{\underline{\alpha}!}\,\xi^{\,\underline{\alpha}}\big\}_{\underline{\alpha}\in\mathbb{N}^n}$. By the duality modulo h mentioned above, we see that the elements in this basis are of the form

$$\theta_{\underline{\alpha}} \;=\; e^{\underline{\alpha}} \,+\, \textstyle\sum_{n=1}^{+\infty} h^n \sum_{\underline{\alpha}'} s_\ell\big(c^n_{\underline{\alpha},\underline{\alpha}'}\big)\, e^{\underline{\alpha}'}$$

so $\big\{\theta^{\underline{\alpha}}\big\}_{\underline{\alpha}\in\mathbb{N}^n}$ is a lift of the PBW basis $\big\{\overline{e}^{\,\underline{\alpha}}\big\}_{\underline{\alpha}\in\mathbb{N}^n}$ of $V^\ell(L)$. By construction, it is clear that $\big\{h^{-|\underline{\alpha}|}\,\frac{1}{\underline{\alpha}!}\,\xi^{\,\underline{\alpha}}\big\}_{\underline{\alpha}\in\mathbb{N}^n}$, where $|\underline{\alpha}| := \sum_{i=1}^n \alpha_i$, is a topological basis of the (topologically free) left A_h–module $K_h^\vee = J^r(L)_h^\vee$. But then, it is also clear that the dual pseudobasis of ${}_*\big(K_h^\vee\big) = V^\ell(L)_h'$ to this basis is $h^{|\underline{\alpha}|}\big\{\theta_{\underline{\alpha}}\big\}_{\underline{\alpha}\in\mathbb{N}^n}$. Therefore

$$V^\ell(L)_h' \;=\; \Big\{ \textstyle\sum_{\underline{\alpha}} t^\ell(a_{\underline{\alpha}})\, h^{|\underline{\alpha}|}\,\theta_{\underline{\alpha}} \;\Big|\; a_{\underline{\alpha}} \in A_h \Big\}$$

where the summation symbol denotes h–adically convergent series. A similar reasoning shows that

$${}'V^\ell(L)_h \;=\; \Big\{ \textstyle\sum_{\underline{\alpha}} s^\ell(a_{\underline{\alpha}})\, h^{|\underline{\alpha}|}\,\theta_{\underline{\alpha}} \;\Big|\; a_{\underline{\alpha}} \in A_h \Big\}$$

6.3 Quantum duality for quantum groupoids

We consider now the composition of two Drinfeld functors. We shall prove that the functors $(\ )^\vee$ and $(\ )' = {}'(\ )$ are actually inverse to each other, so that they establish equivalences of categories (RQFSAd) $\cong$ (RQUEAd) and (LQFSAd) $\cong$ (LQUEAd).

We begin with an auxiliary result (cf. [13], Lemma 3.3, for the well known case of QUEA's):

Proposition 6.21. *Let $V^r(L)_h$ be a quantization of $V^r(L)$ and let $x' \in \big(V^r(L)_h\big)'$. Let $x \in V^r(L)_h \setminus h\,V^r(L)_h$, $n\in\mathbb{N}$, be such that $x' = h^n x$, and set $\overline{x} := x \mod h\,V^r(L)_h \;\big(\in V^r(L)\big)$. Then $\overline{x} \in V^r_n(L)$, the n–th piece of the standard filtration (cf. Remark 2.18) of $V^r(L)$.*

A similar, parallel result holds with $V^\ell(L)_h$ and $V^\ell(L)$ replacing $V^r(L)_h$ and $V^r(L)$ respectively.

Proof. Set $K_h := {}_*\big(V^r(L)_h\big)$, and $I_h := \partial_{K_h}^{-1}\big(h\,A_h\big)$. One has $\big\langle h^n x\,, I_h^{n+1} \big\rangle \subseteq h^{n+1}A_h$, hence $\big\langle x\,, I_h^{n+1} \big\rangle \subseteq h\,A_h$ which implies $\big\langle \overline{x}\,, I_K^{n+1} \big\rangle = 0$. Therefore $\overline{x} \in V^r_n(L)$, q.e.d. $\square$

We are ready for the main result:

Theorem 6.22.

(a) If $K_h \in$ (RQFSAd), then $\big(K_h^\vee\big)' = K_h = {}'\big(K_h^\vee\big)$.

(b) If $K_h \in$ (LQFSAd), then $\big(K_h^\vee\big)' = K_h = {}'\big(K_h^\vee\big)$.

(c) If $H_h \in$ (LQUEAd), then $\big(H_h'\big)^\vee = H_h = \big({}'H_h\big)^\vee$.

(d) If $H_h \in$ (RQUEAd), then $\big(H_h'\big)^\vee = H_h = \big({}'H_h\big)^\vee$.

(e) The functors $(\ )^\vee$: (RQFSAd) $\to$ (RQUEAd) and $(\ )' = {}'(\ )$: (RQUEAd) $\to$ (RQFSAd) are inverse to each other, hence they are equivalences of categories. Similarly for the functors $(\ )^\vee$: (LQFSAd) $\longrightarrow$ (LQUEAd) and $(\ )' = {}'(\ )$: (LQUEAd) $\longrightarrow$ (LQFSAd).

Proof. Clearly, claim *(e)* is just a consequence of the previous items in the statement. We begin by focusing on claim *(a)*: we assume that $K_h \in (\mathrm{RQFSAd})_{A_h}$ and we shall prove that $\big(K_h^\vee\big)' = K_h$.

Let us show that $K_h \subseteq \big(K_h^\vee\big)'$.

Given λ is in K_h, consider its n–th iterated coproduct $\Delta^n(\lambda) = \lambda_{(1)} \otimes \cdots \otimes \lambda_{(n)}$; if we write every $\lambda_{(i)}$ as $\lambda_{(i)} = \lambda'_{(i)} + \lambda''_{(i)}$ with $\lambda'_{(i)} := \lambda_{(i)} - s_h^r\big(\partial(\lambda_{(i)})\big) \in \mathfrak{J}_h := \mathit{Ker}\big(\partial_{K_h}\big)$ and $\lambda''_{(i)} := s_h^r\big(\partial(\lambda_{(i)})\big) \in s_h^r(A_h)$, then expanding again $\Delta^n(\lambda)$ we can write it as a sum $\Delta^n(\lambda) = \sum \lambda^\circ_{(1)} \otimes \cdots \otimes \lambda^\circ_{(n)}$ in which $\lambda^\circ_i \in \mathfrak{J}_h$ or $\lambda^\circ_i \in s_h^r(A_h)$ for every $i = 1, \dots, n$.

Now let $\alpha_1, \dots, \alpha_n \in I_{{}_*(K_h^\vee)} := \epsilon^{-1}_{{}_*(K_h^\vee)}\big(h\, A_h\big)$. As every α_j belongs to ${}_*\big(K_h^\vee\big)$, it defines a map from $\mathfrak{J}_h$ to $h\, A_h$. Hence $\big\langle \alpha_i \, , \lambda_j \big\rangle \in h\, A_h$ and one has $\big\langle \alpha_1 \cdots \alpha_n \, , \, \lambda \big\rangle \in h^n A_h$. Thus, for any $n \in \mathbb{N}$, we have that λ defines a map $\Lambda_n : h^{-n}\, I^{\,n}_{{}_*(K_h^\vee)} \longrightarrow A_h$. Clearly all these Λ_n's match together to define an element $\Lambda \in \Big(\big({}_*\big(K_h^\vee\big)\big)^\vee\Big)^* = \big(K_h^\vee\big)'$; thus we end up with a natural map $K_h \longrightarrow \big(K_h^\vee\big)'$ $\big(\lambda \mapsto \Lambda\big)$, which is clearly injective. This yields the inclusion $K_h \subseteq \big(K_h^\vee\big)'$.

To prove the converse inclusion $K_h \supseteq \big(K_h^\vee\big)'$, we proceed like in [13].

Let $x' \in \big(K_h^\vee\big)' \setminus \{0\}$ be given. Then there exists $x \in K_h^\vee \setminus h\, K_h^\vee$ such that $x' = h^n x$. By Theorem 6.4 we know that $K_h^\vee \big/ h\, K_h^\vee \cong V^r\big(L^*\big)$. Setting $\overline{x} := x \mod h\, K_h^\vee \in K_h^\vee \big/ h\, K_h^\vee \cong V^r\big(L^*\big)$, by construction and by Proposition 6.21 we know that $\overline{x} \in V_n^r\big(L^*\big)$. Using the PBW Theorem for $V^r\big(L^*\big)$, and noting that every element of L^* can be lifted to an element of $\mathfrak{J}_h^\vee$, we see that there exists $x_0 \in K_h^\vee$ such that

- $x_0 = \sum_{s=0}^d h^{-s} j_s$ with $j_s \in \mathfrak{J}_h^{\,s}$, and $d \leq n$,
- $x = x_0 + h\, x_{\langle 1 \rangle}$ or $x' = h^n\, x_0 + h^{n+1} x_{\langle 1 \rangle}$ for some $x_{\langle 1 \rangle} \in K_h^\vee$.

As $K_h \subseteq \big(K_h^\vee\big)'$, as we already showed, we get $h^{n+1} x_{\langle 1 \rangle} = h^n (x - x_0) = x' - h^n\, x_0 \in \big(K_h^\vee\big)'$.

If $x_{(1)} := h^{n+1} x_{\langle 1 \rangle}$ is zero, then we are done. If not, we can repeat the argument with $x_{(1)}$ in the role of $x_{(0)} := x'$. We write $x_{(1)} = h^{n+n_1} x_{\langle\langle 1 \rangle\rangle}$ with $n_1 > 0$ and $x_{\langle\langle 1 \rangle\rangle} \notin h\, K_h^\vee$. This will provide us with an $x_1 \in K_h^\times$ and an $x_{\langle 2 \rangle} \in K_h^\vee$ such that $x_{(1)} = h^{n+n_1} x_1 + h^{n+n_1+1} x_{\langle 2 \rangle}$ with $h^{n+n_1} x_1 \in K_h$ and $h^{n+n_1+1} x_{\langle 2 \rangle} \in \big(K_h^\vee\big)'$. Iterating this procedure, we will eventually find a sequence $\{x_s\}_{s \in \mathbb{N}} \in K_h^\times$ such that

- $h^{n+n_1+n_2+\cdots+n_s} x_s \in K_h$ for all $s \in \mathbb{N}$,
- $x' = \sum_{s=0}^{+\infty} h^{n+n_1+n_2+\cdots+n_s} x_s$ where the right-hand side converges to x' in $K_h^\vee$.

As $h^{n+n_1+n_2+\cdots+n_s} x_s \in I_h^{n+n_1+n_2+\cdots+n_s}$ and K_h is complete with respect to the I_h–adic topology, the series $\sum_{s=0}^{+\infty} h^{n+n_1+n_2+\cdots+n_s} x_s$ does converge to x' in K_h . This completes the proof of *(a)*.

The proof of *(b)* is analogous to the proof of *(a)*, so we leave it to the reader.

To prove claim *(c)*, consider $H_h \in (\mathrm{LQUEAd})$. We have $K_h := H_h^* \in (\mathrm{RQFSAd})$, and $H_h' = {}_*\Big(\big(H_h^*\big)^\vee\Big) = {}_*\big(K_h^\vee\big)$ by Proposition 6.6*(a)*. Now $\varGamma_h := K_h^\vee \in (\mathrm{RQUEAd})$ by Theorem 6.4, and then ${}'\varGamma_h = \big(\big({}_*\varGamma_h\big)^\vee\big)^*$ by Proposition 6.6*(b)*, which implies ${}_\star\big({}'\varGamma_h\big) = {}_\star\Big(\big(\big({}_*\varGamma_h\big)^\vee\big)^*\Big) = \big({}_*\varGamma_h\big)^\vee$. Altogether — also exploiting claim *(a)* — this gives

$$\big(H_h'\big)^\vee = \big({}_*\big(K_h^\vee\big)\big)^\vee = \big({}_*\varGamma_h\big)^\vee = {}_\star\big({}'\varGamma_h\big) = {}_\star\big(\varGamma_h'\big) = {}_\star\Big(\big(K_h^\vee\big)'\Big) = {}_\star K_h = {}_\star\big(H_h^*\big) = H_h$$

This proves *(c)*, and the proof of *(d)* is entirely similar. □

7 An example

In this last section we apply the main construction of the paper — duality functors and Drinfeld's functors to a toy model, namely a simple (yet non trivial!) quantum groupoid.

We consider the two dimensional Lie k–algebra $\mathfrak{g} = k\,e_1 \oplus k\,e_2$ with non zero bracket $[e_1, e_2] = e_1$. It is well known that $\mathfrak{g}^*$ is a Poisson manifold. We shall consider e_1 and e_2 as coordinates on $\mathfrak{g}^*$, denoting them by x_1 and x_2 respectively. The Poisson structure on $\mathfrak{g}^*$ is determined by $\{x_1, x_2\} = [e_1, e_2] = e_1$.

Let us introduce the Lie $k[[h]]$–algebra $\mathfrak{g}_h := k[[h]]\,e_1 \oplus k[[h]]\,e_2$ with non zero bracket $[e_1, e_2]_h := h\,e_1$. The enveloping algebra $A_h := U(\mathfrak{g}_h)$ is a quantization of the Poisson algebra of polynomial functions on $\mathfrak{g}^*$, namely $A = S(\mathfrak{g})$.

For simplicity, we shall write $\mathcal{D}$ for the ring of polynomial differential operators on $\mathfrak{g}^*$. It is the enveloping algebra of the Lie-Rinehart algebra $\big(S(\mathfrak{g}), Der\big(S(\mathfrak{g})\big), id\big)$. We endow it with the standard left algebroid stucture and denote by $\mathcal{D}[[h]]$ the trivial deformation of this structure.

Proposition 7.1. *Set* $\theta_1 = x_1 \dfrac{\partial}{\partial x_1}$. *Then* $\mathcal{F} := \sum_{n=0}^{\infty} \dfrac{1}{n!} \dfrac{h^n}{2^n} \Big(\theta_1 \otimes \dfrac{\partial}{\partial x_2} - \dfrac{\partial}{\partial x_2} \otimes \theta_1 \Big)^n$ *is a twist for* $\mathcal{D}[[h]]$.

Proof. It is a straightforward computation. □

We will now denote by $\mathcal{D}_h$ the twist of $\mathcal{D}[[h]]$ by $\mathcal{F}$. As an algebra, $\mathcal{D}_h$ is isomorphic to $\big(S(\mathfrak{g}) \otimes S(\mathfrak{g}^*)\big)[[h]]$. The deformation of $A = S(\mathfrak{g})$ defined by $\mathcal{F}$ is $A_h = U(\mathfrak{g}_h)$. The source map $s^l_{\mathcal{F}}$ (an algebra morphism) is determined by

$$s^l_{\mathcal{F}}(x_1) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \frac{h^n}{2^n}\, x_1 \frac{\partial^n}{\partial x_2^n} \quad , \qquad s^l_{\mathcal{F}}(x_2) \;=\; x_2 \,-\, h\, x_1 \frac{\partial}{\partial x_1}$$

The target $t^l_{\mathcal{F}}$ (an algebra antimorphism) is determined by

$$t^l_{\mathcal{F}}(x_1) \;=\; \textstyle\sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!} \dfrac{h^n}{2^n}\, x_1 \dfrac{\partial^n}{\partial x_2^n} \quad , \qquad t^l_{\mathcal{F}}(x_2) \;=\; x_2 \,+\, h\, x_1 \dfrac{\partial}{\partial x_1}$$

$$\Delta_{\mathcal{F}} \;=\; \mathcal{F}^{\#-1} \Delta \mathcal{F}$$

$$\epsilon\bigg(x_1^{\alpha_1} x_2^{\alpha_2} \frac{\partial^{\beta_1+\beta_2}}{(\partial x_1)^{\beta_1} (\partial x_2)^{\beta_2}} \bigg) \;=\; x_1^{\alpha_1} x_2^{\alpha_2} \frac{\partial^{\beta_1+\beta_2}}{(\partial x_1)^{\beta_1} (\partial x_2)^{\beta_2}} (1)$$

Let $\widetilde{\mathcal{F}}$ be the element of $\mathcal{D} \otimes_k \mathcal{D}[[h]]$ defined by

$$\widetilde{\mathcal{F}} \;=\; \exp\bigg(\frac{h}{2} \Big(\theta_1 \otimes \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2} \otimes \theta_1 \Big) \bigg) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \frac{h^n}{2^n} \Big(\theta_1 \otimes \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2} \otimes \theta_1 \Big)^n \;\in\; \mathcal{D} \otimes_k \mathcal{D}[[h]]$$

The element $\widetilde{\mathcal{F}}$ is invertible in $\big(\mathcal{D} \otimes_k \mathcal{D}\big)[[h]]$ and one has

$$\widetilde{\mathcal{F}}^{-1} \;=\; \exp\bigg(-\frac{h}{2} \Big(\theta_1 \otimes \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2} \otimes \theta_1 \Big) \bigg) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n h^n}{2^n} \Big(\theta_1 \otimes \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2} \otimes \theta_1 \Big)^n \;\in\; \big(\mathcal{D} \otimes_k \mathcal{D}\big)[[h]]$$

The element $\widetilde{\mathcal{F}}^{-1}$ defines an element

$$\mathcal{G} \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{h^n}{2^n\, n!} \Big(\theta_1 \otimes \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2} \otimes \theta_1 \Big)^n \;\in\; \mathcal{D}[[h]] \,\widehat{\otimes}_{A_{\mathcal{F}}}\, \mathcal{D}[[h]]$$

The map

$$\mathcal{F}^{\#} \;:\; \mathcal{D}[[h]] \,\widehat{\otimes}_{A_{\mathcal{F}}}\, \mathcal{D}[[h]] \longrightarrow \mathcal{D}[[h]] \,\widehat{\otimes}_{A}\, \mathcal{D}[[h]] \;, \qquad h_1 \otimes h_2 \;\mapsto\; \mathcal{F} \cdot (h_1 \otimes h_2)$$

is invertible and its inverse is

$$\mathcal{F}^{\#-1} \;:\; \mathcal{D}[[h]] \,\widehat{\otimes}_{A}\, \mathcal{D}[[h]] \longrightarrow \mathcal{D}[[h]] \,\widehat{\otimes}_{A_{\mathcal{F}}}\, \mathcal{D}[[h]] \;, \qquad h_1 \otimes h_2 \;\mapsto\; \mathcal{G} \cdot (h_1 \otimes h_2)$$

We will compute $\big(\mathcal{D}_h\big)_*$ and $\big(\mathcal{D}_h\big)^*$.

Computation of $\big(\mathcal{D}_h\big)_*$: We shall use the isomorphism

$$(\mathcal{D}_h)_* \longrightarrow Hom(\mathcal{D}, A)[[h]] \ , \qquad \lambda \mapsto \left(\frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \mapsto \left\langle \lambda \, , \, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle \right)$$

Let de_1 and de_2 be the elements of $\big(\mathcal{D}_h\big)_*$ such that

$$\left\langle de_1 \, , \, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle = \delta_{1,a}\,\delta_{0,b} \quad , \qquad \left\langle de_2 \, , \, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle = \delta_{0,a}\,\delta_{1,b}$$

Let e_1 and e_2 be the elements of $\big(\mathcal{D}_h\big)_*$ such that

$$\left\langle e_1 \, , \, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle = x_1\,\delta_{0,a}\,\delta_{0,b} \quad , \qquad \left\langle e_2 \, , \, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle = x_2\,\delta_{0,a}\,\delta_{0,b}$$

A computation shows that

$$\left\langle de_1 \cdot_h de_2 \, , \, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle = \begin{cases} 0 \ \text{ if } \ a \geq 2 \ \text{ or } \ b \geq 2 \\ 1 \ \text{ if } \ a = 1 \ \text{ and } \ b = 1 \\ -\dfrac{h}{2} \ \text{ if } \ a = 1 \ \text{ and } \ b = 0 \\ 0 \ \text{ if } \ a = 0 \ \text{ and } \ b = 1 \end{cases}$$

Similarly

$$\left\langle de_2 \cdot_h de_1 \, , \, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle = \begin{cases} 0 \ \text{ if } \ a \geq 2 \ \text{ or } \ b \geq 2 \\ 1 \ \text{ if } \ a = 1 \ \text{ and } \ b = 1 \\ \dfrac{h}{2} \ \text{ if } \ a = 1 \ \text{ and } \ b = 0 \\ 0 \ \text{ if } \ a = 0 \ \text{ and } \ b = 1 \end{cases}$$

Hence $de_1 \cdot_h de_2 - de_2 \cdot_h de_1 = -h\, de_1$. Set $\check{de}_i := h^{-1}\, de_i$. This equality can be written as

$$\check{de}_1 \cdot_h \check{de}_2 - \check{de}_2 \cdot_h \check{de}_1 = -\check{de}_1$$

A straightforward computation shows that

$$\left\langle de_1 \cdot_h e_2 \, , \, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle = \begin{cases} 0 \ \text{ if } \ (a,b) \neq (1,0),(0,0) \\ x_2 \ \text{ if } \ a = 1 \ \text{ and } \ b = 0 \\ 0 \ \text{ if } \ a = 0 \ \text{ and } \ b = 0 \end{cases}$$

A direct computation proves that

$$\left\langle e_2 \cdot_h e_1 \, , \, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle = \begin{cases} 0 \ \text{ if } \ (a,b) \neq (1,0),(0,0) \\ x_2 \ \text{ if } \ a = 1 \ \text{ and } \ b = 0 \\ h\,x_1 \ \text{ if } \ a = 0 \ \text{ and } \ b = 0 \end{cases}$$

Let us compute $(e_2 \cdot_h de_1)(1)$ as an example: we find

$$e_2 \cdot_h de_1(1) = (e_2 \otimes de_1)(1 \otimes 1) = de_1\big(t^\ell_{\mathcal{F}}(e_2)\big) = de_1\big(t^\ell_{\mathcal{F}}(e_2) - s^\ell_{\mathcal{F}}(e_2)\big) = de_1\left(h\,x_1 \frac{\partial}{\partial x_1}\right) = h\,x_1$$

Hence $de_1 \cdot_h e_2 - e_2 \cdot de_1 = -h\, e_1$. This equality can be written as

$$\check{de}_1 \cdot_h e_2 - e_2 \cdot \check{de}_1 = -e_1$$

Similarly, the following equalities can be established:

$$\begin{gathered} e_1 \cdot e_2 - e_2 \cdot_h e_1 = h\, e_1 \quad , \qquad \check{de}_1 \cdot_h e_1 = e_1 \cdot_h \check{de}_1 \\ \check{de}_2 \cdot_h e_2 = e_2 \cdot_h \check{de}_2 \quad , \qquad \check{de}_2 \cdot_h e_1 - e_1 \cdot_h \check{de}_2 = e_1 \\ s^r_*(e_i) = e_i \quad , \qquad t^r_*(e_i) = e_i + h\,\check{de}_i \end{gathered}$$

From the properties of the coproduct, one deduces

$$\Delta(e_i) \,=\, 1 \otimes e_i \quad , \qquad \Delta(e_i + h\, \check{de_i}) \,=\, (e_i + h\, \check{de_i}) \otimes 1$$

hence

$$\begin{array}{rcl} \Delta(h\, \check{de_i}) & = & e_i \otimes 1 + h\, \check{de_i} \otimes 1 - 1 \otimes e_i \\ & = & s^r(e_i) \otimes 1 + h\, \check{de_i} \otimes 1 - 1 \otimes e_i \\ & = & 1 \otimes t_r(e_i) + h\, \check{de_i} \otimes 1 - 1 \otimes e_i \\ & = & h\, \check{de_i} \otimes 1 + 1 \otimes h\, \check{de_i} \end{array}$$

which gives $\Delta(\check{de_i}) \,=\, \check{de_i} \otimes 1 + 1 \otimes \check{de_i}$. The coproduct on $\big((\mathcal{D}_h)_*\big)^\vee$ is now determined. Let us precise its counit: it is given by

$$\partial(\check{de_i}) \,=\, 0 \quad , \qquad \partial(e_i) \,=\, e_i$$

Remark 7.2. Let us introduce the Lie algebra $\mathfrak{g}_1$ such that $\mathfrak{g}_1 \cong \mathfrak{g} \,=\, k\, \check{de_1} \oplus k\, \check{de_2}$ (as a k–vector space) and $[\ ,\ ]_1 := -[\ ,\ ]_{\mathfrak{g}}$. Then $\mathfrak{g}_1$ acts on $\mathfrak{g}_h = k[[h]]e_1 \oplus k[[h]]e_2$ by derivations, via

$$\mathfrak{g} \longrightarrow Der\,(\mathfrak{g}_h)\ , \qquad \check{de_1} \mapsto \left\{ \begin{array}{l} e_1 \mapsto 0 \\ e_2 \mapsto -e_1 \end{array} \right. \quad , \qquad \check{de_2} \mapsto \left\{ \begin{array}{l} e_2 \mapsto 0 \\ e_1 \mapsto e_1 \end{array} \right.$$

We may perform the semi direct product $\mathfrak{g}_1 \ltimes \mathfrak{g}_h$ and $\big((\mathcal{D}_h)_*\big)^\vee$ is isomorphic to $U(\mathfrak{g}_1 \ltimes \mathfrak{g}_h)$ as an algebra but not as a bialgebroid.

Let us now compute ${}'\mathcal{D}_h$. We proceed in several steps.

- *Let us show that* $h\, \dfrac{\partial}{\partial x_2} \in {}'\mathcal{D}_h$.

We shall show that $\left\langle h\, \dfrac{\partial}{\partial x_2}\, ,\, de_1^{a_1}\, de_2^{a_2} \right\rangle = 0$ if $(a_1, a_2) \neq (0,1)$. We have three cases to consider:

First case: $a_2 = 0$. In this case it is obvious that $\left\langle \dfrac{\partial}{\partial x_2}\, ,\, de_1^{a_1} \right\rangle = 0$.

Second case: $a_2 = 1$. In this case we have

$$\left\langle \frac{\partial}{\partial x_2}\, ,\, de_1^{a_1}\, de_2 \right\rangle \quad = \quad \left\{ \begin{array}{ll} 0 & \text{if} \quad a_1 \neq 0 \\ 1 & \text{if} \quad a_1 = 0 \end{array} \right.$$

Third case: $a_2 > 1$. In this case the summands in $\Delta_{\mathcal{F}}\left(\dfrac{\partial}{\partial x_2}\right)$ that might bring a non zero contribution in the computation of $\left\langle \dfrac{\partial}{\partial x_2}\, ,\, de_1^{a_1}\, de_2^{a_2} \right\rangle$ are those of the form

$$\frac{\partial^{a_2}}{(\partial x_2)^{a_2}} \otimes \theta_1^{a_2' + a_2''}\, \frac{(-1)^{a_2''}}{a_2'!\, a_2''!}\, \frac{h^{a_2 - 1}}{2^{a_2 - 1}} \qquad \text{with} \qquad a_2' + a_2'' = a_2 - 1$$

but

$$\sum\nolimits_{a_2' + a_2'' = a_2 - 1} \frac{\partial^{a_2}}{(\partial x_2)^{a_2}} \otimes \theta_1^{a_2' + a_2''}\, \frac{(-1)^{a_2''}}{a_2'!\, a_2''!}\, \frac{h^{a_2 - 1}}{2^{a_2 - 1}} \quad = \quad 0$$

hence in the end we find again $\left\langle h\, \dfrac{\partial}{\partial x_2}\, ,\, de_1^{a_1}\, de_2^{a_2} \right\rangle \,=\, 0$, q.e.d.

- *Let us show that* $h\, \dfrac{\partial}{\partial x_1} \in {}'\mathcal{D}_h$. We will show that $\left\langle h\, \dfrac{\partial}{\partial x_1}\, ,\, de_1^{a_1}\, de_2^{a_2} \right\rangle \in h^{a_1 + a_2}\, A_h$.

We start by computing $\left\langle \theta_1\, ,\, de_1^{a_1}\, de_2^{a_2} \right\rangle$.

First case: $a_2 = 0$. It is easy to check that

$$\left\langle \theta_1\, ,\, (de_1)^{a_1} \right\rangle \quad = \quad \left\{ \begin{array}{ll} x_1 & \text{if} \quad a_1 = 1 \\ 0 & \text{otherwise} \end{array} \right.$$

Second case case: $a_2 \geq 1$. The summands in $\Delta_{\mathcal{F}}\left(\dfrac{\partial}{\partial x_2}\right)$ that might bring a non zero contribution in the computation of $\left\langle \theta_1 \, , \, de_1^{a_1}\, de_2^{a_2} \right\rangle$ are those of the form

$$\frac{\partial^{a_2}}{(\partial x_2)^{a_2}} \otimes \theta_1^{a_2'+a_2''+1} \, \frac{(-1)^{a_2''}}{a_2'!\, a_2''!} \, \frac{h^{a_2}}{2^{a_2}} \qquad \text{with} \qquad a_2' + a_2'' = a_2$$

but

$$\sum\nolimits_{a_2'+a_2''=a_2} \frac{\partial^{a_2}}{(\partial x_2)^{a_2}} \otimes \theta_1^{a_2'+a_2''+1} \, \frac{(-1)^{a_2''}}{a_2'!\, a_2''!} \, \frac{h^{a_2}}{2^{a_2}} \;\; = \;\; 0$$

hence in the end we get $\left\langle \theta_1 \, , \, (de_1)^{a_1}\, (de_2)^{a_2} \right\rangle \; = \; 0$.

In conclusion, we find

$$\left\langle \theta_1 \, , \, (de_1)^{a_1}\, (de_2)^{a_2} \right\rangle \quad = \quad \begin{cases} x_1 & \text{if} \quad (a_1,a_2) = (1,0) \\ 0 & \text{otherwise} \end{cases}$$

Let us now compute $\left\langle \dfrac{\partial}{\partial x_1} \, , \, de_1^{a_1}\, de_2^{a_2} \right\rangle$. Again we have several cases to consider.

First case: $a_2 = 0$. It is easy to check that

$$\left\langle \frac{\partial}{\partial x_1} \, , \, (de_1)^{a_1} \right\rangle \quad = \quad \begin{cases} 1 & \text{if} \quad a_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Second case: $a_2 \geq 1$. In this case one has

$$\begin{aligned} 0 \; = \; \left\langle \theta_1 \, , \, (de_1)^{a_1}\, (de_2)^{a_2} \right\rangle \; &= \; \left\langle s_{\mathcal{F}}^{l}(x_1)\, \frac{\partial}{\partial x_1} \; - \; \sum_{n=1}^{+\infty} \theta_1 \, \frac{\partial^n}{(\partial x_2)^n} \, \frac{h^n}{2^n\, n!} \, , \, (de_1)^{a_1}\, (de_2)^{a_2} \right\rangle \\ &= \; x_1 \left\langle \frac{\partial}{\partial x_1} \, , \, (de_1)^{a_1}\, (de_2)^{a_2} \right\rangle \; - \; \sum_{n=1}^{a_2-1} \left\langle \theta_1 \, \frac{\partial^n}{(\partial x_2)^n} \, \frac{h^n}{2^n\, n!} \, , \, (de_1)^{a_1}\, (de_2)^{a_2} \right\rangle \\ &\qquad + \left\langle \theta_1 \, \frac{\partial^{a_2}}{(\partial x_2)^{a_2}} \, \frac{h^{a_2}}{2^{a_2}\, a_2!} \, , \, (de_1)^{a_1}\, (de_2)^{a_2} \right\rangle \end{aligned}$$

If we choose $n \in [1, a_2 - 1]$, the unique summands in $\Delta_{\mathcal{F}}\left(\theta_1 \, \dfrac{\partial^n}{(\partial x_2)^n}\right)$ that brings a non zero contribution in the computation of $\left\langle \theta_1 \, \dfrac{\partial^n}{(\partial x_2)^n} \, , \, (de_1)^{a_1}\, (de_2)^{a_2} \right\rangle$ are those of the form

$$\frac{\partial^{a_2}}{(\partial x_2)^{a_2}} \otimes \theta_1^{a_2-n+1} \, \frac{h^{a_2-n}}{2^{a_2-n}} \, \frac{(-1)^{c_2}}{c_1!\, c_2!} \qquad \text{with} \qquad c_1 + c_2 = a_2 - n$$

But since

$$\sum\nolimits_{c_1+c_2=a_2-n} \frac{\partial^{a_2}}{(\partial x_2)^{a_2}} \otimes \theta_1^{a_2-n+1} \, \frac{h^{a_2-n}}{2^{a_2-n}} \, \frac{(-1)^{c_2}}{c_1!\, c_2!} \;\; = \;\; 0$$

Hence $\sum\nolimits_{n=1}^{a_2-1} \left\langle \theta_1 \, \dfrac{\partial^n}{(\partial x_2)^n} \, \dfrac{h^n}{2^n\, n!} \, , \, (de_1)^{a_1}\, (de_2)^{a_2} \right\rangle \; = \; 0$.

Finally, remark that $\left\langle \theta_1 \, \dfrac{\partial^{a_2}}{(\partial x_2)^{a_2}} \, , \, (de_1)^{a_1}\, (de_2)^{a_2} \right\rangle$ is zero if $a_1 \neq 1$. Hence, in any case, we have $\left\langle \theta_1 \, \dfrac{\partial^{a_2}}{(\partial x_2)^{a_2}} \, \dfrac{h^{a_2}}{2^{a_2}\, a_2!} \, , \, (de_1)^{a_1}\, (de_2)^{a_2} \right\rangle \; \in \; h^{a_1+a_2-1} A_h$.

In conclusion, we find that in all cases one has $\left\langle \dfrac{\partial}{\partial x_1} \, , \, (de_1)^{a_1}\, (de_2)^{a_2} \right\rangle \; \in \; h^{a_1+a_2-1}\, A_h$.

Now denote by $\big\{\eta_{a,b}\big\}_{(a,b)\in\mathbb{N}^2}$ the topological basis of $\mathcal{D}_h$ dual to the basis $\left\{ \dfrac{de_1^a}{a!} \, \dfrac{de_2^b}{b!} \right\}_{(a,b)\in\mathbb{N}^2}$. We know that

$${}'\mathcal{D}_h \;\; = \;\; \left\{ \sum\nolimits_{(a,b)\in\mathbb{N}^2} s_{\mathfrak{F}}^{l}(\alpha_{a,b})\, h^{a+b} \eta_{a,b} \;\middle|\; \alpha_{a,b} \in A_h \right\}$$

As $h^{a+b}\eta_{a,b} = h^{a+b}\,\dfrac{\partial^a}{(\partial x_1)^a}\,\dfrac{\partial^b}{(\partial x_2)^b}\mod h'\,\mathcal{D}_h$, we have

$$'\mathcal{D}_h \;=\; \left\{ \textstyle\sum_{(a,b)\in\mathbb{N}^2} s^l_{\mathfrak{F}}(\alpha_{a,b})\, h^{a+b}\,\dfrac{\partial^a}{(\partial x_1)^a}\,\dfrac{\partial^b}{(\partial x_2)^b} \;\middle|\; \alpha_{a,b}\in A_h \right\}$$

Computation of $(\mathcal{D}_h)^*$: We shall compute $(\mathcal{D}_h)^*$, using the isomorphism

$$(\mathcal{D}_h)^* \longrightarrow \mathit{Hom}(\mathcal{D},A)[[h]] \;, \qquad \lambda \mapsto \left(\frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \mapsto \left\langle \lambda\,,\, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle \right)$$

Let de_1 and de_2 be the elements of $(\mathcal{D}_h)^*$ such that

$$\left\langle de_1\,,\, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle \;=\; \delta_{1,a}\,\delta_{0,b} \quad , \qquad \left\langle de_2\,,\, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle \;=\; \delta_{0,a}\,\delta_{1,b}$$

Similarly, let e_1 and e_2 be the elements of $(\mathcal{D}_h)^*$ such that

$$\left\langle e_1\,,\, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle \;=\; x_1\,\delta_{0,a}\,\delta_{0,b} \quad , \qquad \left\langle e_2\,,\, \frac{\partial^a}{\partial x_1^a}\,\frac{\partial^b}{\partial x_2^b} \right\rangle \;=\; x_2\,\delta_{0,a}\delta_{0,b}$$

Now set $\check{de}_i := h^{-1}de_i$ for $i=1,2$. Then the following equalities can be established:

$$\begin{gathered} e_1\cdot e_2 - e_2\cdot_h e_1 \;=\; -h\,e_1 \quad , \qquad \check{de}_1\cdot_h \check{de}_2 - \check{de}_2\cdot_h \check{de}_1 \;=\; \check{de}_1 \\ \check{de}_1\cdot_h e_2 - e_2\cdot \check{de}_1 \;=\; e_1 \quad , \qquad \check{de}_1\cdot_h e_1 \;=\; e_1\cdot_h \check{de}_1 \\ \check{de}_2\cdot_h e_2 \;=\; e_2\cdot_h \check{de}_2 \quad , \qquad \check{de}_2\cdot_h e_1 - e_1\cdot_h \check{de}_2 \;=\; -e_1 \end{gathered}$$

Moreover, source and target are $s^*_r(x_i) \;=\; e_i + h\,\check{de}_i$, $t^*_r(x_i) \;=\; e_i$. From the properties of the coproduct, one has also

$$\Delta(e_i) \;=\; e_i\otimes 1 \quad , \qquad \Delta(\check{de}_i) \;=\; \check{de}_i\otimes 1 + 1\otimes \check{de}_i$$

Finally, the counit of $\big((\mathcal{D}_h)^*\big)^\vee$ is given by

$$\partial\big(\check{de}_i\big) \;=\; 0 \quad , \qquad \partial\big(e_i\big) \;=\; e_i$$

Remark 7.3. Let us introduce the Lie algebra $\mathfrak{g}_{h,1}$ such that $\mathfrak{g}_{h,1}\cong\mathfrak{g}_h = k[[h]]\,\check{de}_1\oplus k[[h]]\,\check{de}_2$ (as a k–vector space) and $[\ ,\ ]_1 := -[\ ,\ ]_{\mathfrak{g}_h}$. Then $\mathfrak{g}$ acts on $\mathfrak{g}_{h,1}$ by derivations, via

$$\mathfrak{g}\longrightarrow \mathit{Der}\,(\mathfrak{g}_{h,1})\;, \qquad \check{de}_1 \mapsto \begin{cases} e_1\mapsto 0 \\ e_2\mapsto e_1 \end{cases} \;, \qquad \check{de}_2 \mapsto \begin{cases} e_2\mapsto 0 \\ e_1\mapsto -e_1 \end{cases}$$

Now we may perform the semi direct product $\mathfrak{g}\ltimes\mathfrak{g}_{h,1}$; then $\big((\mathcal{D}_h)_*\big)^\vee$ is isomorphic to $U(\mathfrak{g}\ltimes\mathfrak{g}_{h,1})$ as an algebra but not as a bialgebroid.

A right bialgebroid isomorphism $\big((\mathcal{D}_h)_*\big)^\vee\cong\big((\mathcal{D}_h)^*\big)^\vee$. From the above analysis, one sees that there exists a unique isomorphism of right bialgebroids $\phi : \big((\mathcal{D}_h)_*\big)^\vee \longrightarrow \big((\mathcal{D}_h)^*\big)^\vee$ determined by

$$\phi(e_i) \;=\; e_i + h\check{de}_i \quad , \qquad \phi(\check{de}_i) \;=\; -\check{de}_i$$

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