DUALITY FEATURES OF LEFT HOPF ALGEBROIDS

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ABSTRACT. We explore special features of the pair (U^*, U_*) formed by the right and left dual over a (left) bialgebroid U in case the bialgebroid is, in particular, a left Hopf algebroid. It turns out that there exists a bialgebroid morphism S^* from one dual to another that extends the construction of the antipode on the dual of a Hopf algebra, and which is an isomorphism if U is both a left and right Hopf algebroid. This structure will be used to establish a categorical equivalence between left and right comodules over U without the need of a (Hopf algebroid) antipode, and also a categorical equivalence between left U-modules and left modules over certain biduals. In the applications, we illustrate the difference between this construction and those involving antipodes and also deal with dualising modules and their quantisation.

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1. INTRODUCTION

A characteristic feature in standard Hopf algebra theory is its self-duality, that is, the dual of a (finite-dimensional) Hopf algebra (over a field) is a Hopf algebra again. In particular, the antipode of this dual is nothing but the transpose of the original antipode; see, for example, [Sw]. In the broader setup of (*left* or *full*) Hopf algebroids over possibly

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noncommutative rings, this peculiar property is generally lost; see [B] or the main text for the precise definitions of these objects, we only mention here that, in contrast to full Hopf algebroids, there is no notion of antipode for left Hopf algebroids; nevertheless, left Hopf algebroids appear as the correct generalisation of Hopf algebras over non commutative rings, whereas full Hopf algebroids generalise Hopf algebras twisted by a character, see, for example, [Ko, §4.1.2].

Even worse, it is not known what kind of Hopf structure, if any, the dual of a left Hopf algebroid carries at all; for full Hopf algebroids, an answer has only been given in certain cases, more precisely, in the presence of integrals [BSz, §5]. The problem already starts on a rather fundamental level: instead of one dual, a left bialgebroid U rather possesses *two*, the *right dual* U^* and the *left dual* U_*, which, on top, live in a different category compared to U as they are both right bialgebroids [KadSz]. Hence, any question concerning "the dual of U" should be converted into a question about the pair (U^*, U_*). Dealing with full Hopf algebroids does notably worsen the situation as there are actually *four* duals to be taken into account, two of which are left and two of which are right bialgebroids.

1.1. **Aims and objectives.** As mentioned a moment ago, the object one should investigate to discover the limits of self-duality in (left) Hopf algebroid theory is a *pair* of duals. In short, our question reads as follows: if a left bialgebroid U is, in particular, a left (or right) Hopf algebroid, what extra structure, if any, can be found on the pair (U^*, U_*) of duals?

1.2. **Main results.** After highlighting in §3 a multitude of module structures that exist on Hom-spaces and tensor products in presence of a left Hopf algebroid structure and that will be used in the sequel, we give in §4.1 the following answer to the above problem (see the main text for all definitions and conventions used hereafter):

Theorem A. *If a left bialgebroid U is, in particular, a left Hopf algebroid, then there exists a morphism $S^* : U^* \to U_*$ of right bialgebroids between its duals. If the left bialgebroid is a right Hopf algebroid instead, then one obtains a morphism $S_* : U_* \to U^*$, and in case U is simultaneously both left and right Hopf, the two morphisms are inverse to each other and hence $U^* \simeq U_*$ as right bialgebroids.*

This is a straight analogue of the construction on the dual for a (finite-dimensional) Hopf algebra H (over a field) with antipode S in the following sense: here, one has $H^* = (H_*)^{\mathrm{op}}_{\mathrm{coop}}$ and S^* is exactly the transpose of S and therefore the antipode for the dual Hopf algebra.

Observe that the last case in Theorem A, *i.e.*, the presence of both a left and right Hopf structure is, for example, given when U is a full Hopf algebroid with bijective antipode but also in weaker cases such as for the universal enveloping of a Lie-Rinehart algebra. In this situation, U^* and U_* are additionally linked (in both directions) by the transposition ${}^tS$ of the antipode $S : U \to U^{\mathrm{op}}_{\mathrm{coop}}$. However, in Theorem 4.2.4 we show that the maps ${}^tS$ does in general not coincide with S^* or S_*, in contrast to the Hopf algebra case mentioned before. Moreover, if a left Hopf algebroid U is cocommutative, then $U^* = (U_*)_{\mathrm{coop}}$ is a full Hopf algebroid (with antipode given precisely by S^*), though U might be not.

Inspired by Theorem A, one might try to overcome the lack of self-duality of (full) Hopf algebroids by defining them more symmetrically: instead of having (*cf.* §4.2.1) *one* underlying algebra but *two* underlying coalgebra structures on the same k-module, one could start with both *two* underlying algebra and coalgebra structures and the antipode would intertwine these two; see Remark 4.2.5 for a couple of comments in this direction.

As an application of Theorem A, we establish in §5 the following statements:

Theorem B. *If a left bialgebroid (U, A) is, on top, a left Hopf algebroid that is finitely generated A-projective via the target map, then there exists a canonical functor $\mathbf{Comod}$-$U \to U$-$\mathbf{Comod}$. If (U, A) is right Hopf instead and finitely generated A-projective via the source map, then there exists a canonical functor U-$\mathbf{Comod} \to \mathbf{Comod}$-$U$. Hence, if U*

is biprojective over A and both a left and right Hopf algebroid, one obtains an equivalence of categories

$$U\text{-}\mathbf{Comod} \simeq \mathbf{Comod}\text{-}U.$$

Observe that this equivalence works without the help of an antipode as there are objects that are both left and right Hopf algebroids but not full Hopf algebroids (cocommutative left Hopf algebroids, for example). In a dual spirit, we prove in §5.2:

Theorem C. *For a left Hopf algebroid (U, A), there exists a canonical functor ${}^*(U^*)$-$\mathbf{Mod} \to U$-$\mathbf{Mod}$, and if U is finitely generated A-projective via the target map, then there also exists a canonical functor U-$\mathbf{Mod} \to {}_*(U_*)$-$\mathbf{Mod}$.*

Similarly, for a right Hopf algebroid one obtains functors ${}_*(U_*)$-$\mathbf{Mod} \to U$-$\mathbf{Mod}$, along with U-$\mathbf{Mod} \to {}^*(U^*)$-$\mathbf{Mod}$ if U is finitely generated A-projective via the source map. In particular, for a finitely generated biprojective left and right Hopf algebroid there exist equivalences of categories

$${}_*(U_*)\text{-}\mathbf{Mod} \simeq U\text{-}\mathbf{Mod} \quad \text{as well as} \quad U\text{-}\mathbf{Mod} \simeq {}^*(U^*)\text{-}\mathbf{Mod}.$$

In §6, we illustrate these results by considering some examples related to Lie-Rinehart algebras (or Lie algebroids) and their jet spaces, as well as their quantised versions. Finally, in §6.3 we consider further duality phenomena related to dualising modules, which appear in Poincaré duality, along with their quantisation.

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2. Preliminaries

We list here those preliminaries with respect to bialgebroids and their duals that are needed in this article; see, *e.g.*, [B] and references therein as an overview on this subject.

Fix an (associative, unital, commutative) ground ring k. Unadorned tensor products will always be meant over k. All other algebras, modules etc. will have an underlying structure of a k-module. Secondly, fix an associative and unital k-algebra A, *i.e.*, a ring with a ring homomorphism $\eta_A : k \to Z(A)$ to its centre. Denote by A^{op} the opposite and by $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ the enveloping algebra of A, and by A-$\mathbf{Mod}$ the category of left A-modules. Recall that an *A-ring* is a monoid in the monoidal category $(A^{\mathrm{e}}\text{-}\mathbf{Mod}, \otimes_A, A)$ of (A, A)-bimodules fulfilling the usual associativity and unitality axioms, whereas dually an *A-coring* is a comonoid in this category that is coassociative and counital.

2.1. **Bialgebroids.** Given an A^{e}-ring $\eta : A^{\mathrm{e}} \to U$, consider the restrictions $s := \eta(- \otimes 1_U)$ and $t := \eta(1_U \otimes -)$, called *source* and *target* map, respectively. Thus an A^{e}-ring U carries two A-module structures from the left and two from the right, namely

$$a \triangleright u \triangleleft b := s(a)t(b)u, \qquad a \blacktriangleright u \blacktriangleleft b := ut(a)s(b), \qquad \forall\, a, b \in A, u \in U.$$

If we let $U_\triangleleft \otimes_{A\,\triangleright} U$ be the corresponding tensor product of U (as an A^{e}-module) with itself, we define the *(left) Takeuchi-Sweedler product* as

$$U_\triangleleft \times_{A\,\triangleright} U \;:=\; \big\{ \textstyle\sum_i u_i \otimes u'_i \in U_\triangleleft \otimes_{A\,\triangleright} U \mid \sum_i (a \blacktriangleright u_i) \otimes u'_i = \sum_i u_i \otimes (u'_i \blacktriangleleft a),\ \forall a \in A \big\}.$$

By construction, $U_\triangleleft \times_{A\,\triangleright} U$ is an A^{e}-submodule of $U_\triangleleft \otimes_{A\,\triangleright} U$; it is also an A^e-ring via factorwise multiplication, with unit $1_U \otimes 1_U$ and $\eta_{U_\triangleleft \times_{A\,\triangleright} U}(a \otimes \tilde{a}) := s(a) \otimes t(\tilde{a})$.

Symmetrically, one can consider the tensor product $U_\blacktriangleleft \otimes_{B\,\blacktriangleright} U$ and define the *(right) Takeuchi-Sweedler product* as $U_\blacktriangleleft \times_{B\,\blacktriangleright} U$, which is an A^{e}-ring inside $U_\blacktriangleleft \otimes_{B\,\blacktriangleright} U$.

Definition 2.1.1. A *left bialgebroid* (U, A) is a k-module U with the structure of an A^{e}-ring (U, s^ℓ, t^ℓ) and an A-coring $(U, \Delta_\ell, \epsilon)$ subject to the following compatibility relations:

(*i*) the A^{e}-module structure on the A-coring U is that of ${}_\triangleright U_\triangleleft$;

(*ii*) the coproduct Δ_ℓ is a unital k-algebra morphism taking values in $U_{\triangleleft} \times_A {}_{\triangleright}U$;

(*iii*) for all $a, b \in A$, $u, u' \in U$, one has:

$$\epsilon(a \triangleright u \triangleleft b) = a\epsilon(u)b, \quad \epsilon(uu') = \epsilon\big(u \blacktriangleleft \epsilon(u')\big) = \epsilon\big(\epsilon(u') \blacktriangleright u\big).$$

A *morphism* between left bialgebroids (U, A) and (U', A') is a pair (F, f) of maps $F : U \to U'$, $f : A \to A'$ that commute with all structure maps in an obvious way.

As for any ring, we can define the categories U-$\mathbf{Mod}$ and $\mathbf{Mod}$-U of left and right modules over U. Note that U-$\mathbf{Mod}$ forms a monoidal category but $\mathbf{Mod}$-U usually does not. However, in both cases there is a forgetful functor U-$\mathbf{Mod} \to A^{\mathrm{e}}$-$\mathbf{Mod}$ resp. $\mathbf{Mod}$-$U \to A^{\mathrm{e}}$-$\mathbf{Mod}$: whereas we denote left and right action of a bialgebroid U on $M \in U$-$\mathbf{Mod}$ or $N \in \mathbf{Mod}$-U usually by juxtaposition, for the resulting A^{e}-module structures the notation

$$a \triangleright m \triangleleft b := s^\ell(a)t^\ell(b)m, \qquad a \blacktriangleright m \blacktriangleleft b := ns^\ell(b)t^\ell(a)$$

for $m \in M,\ n \in N,\ a, b \in A$ is used instead. For example, the base algebra A itself is a left U-module via the left action $u(a) := \epsilon(u \blacktriangleleft a) = \epsilon(a \blacktriangleright u)$ for $u \in U$ and $a \in A$, but in most cases there is no right U-action on A.

Dually, one can introduce the categories U-$\mathbf{Comod}$ and $\mathbf{Comod}$-U of left resp. right U-comodules, both of which are monoidal; here again, one has forgetful functors U-$\mathbf{Comod} \to A^{\mathrm{e}}$-$\mathbf{Mod}$ and $\mathbf{Comod}$-$U \to A^{\mathrm{e}}$-$\mathbf{Mod}$.

The notion of a *right bialgebroid* is obtained if one starts with the A^{e}-module structure given by $\blacktriangleright$ and $\blacktriangleleft$ instead of $\triangleright$ and $\triangleleft$. We will refrain from giving the details here and refer to [KadSz] instead.

Remark 2.1.2. The *opposite* of a left bialgebroid $(U, A, s^\ell, t^\ell, \Delta_\ell, \epsilon)$ yields a *right* bialgebroid $(U^{\mathrm{op}}, A, t^\ell, s^\ell, \Delta_\ell, \epsilon)$. The *coopposite* of a left bialgebroid is the *left* bialgebroid given by $(U, A^{\mathrm{op}}, t^\ell, s^\ell, \Delta_\ell^{\mathrm{coop}}, \epsilon)$.

2.2. **Pairings of U-modules and dual bialgebroids.** Let (U, A) be a left bialgebroid, $M, M' \in U$-$\mathbf{Mod}$ be left U-modules, and $N, N' \in \mathbf{Mod}$-U be right U-modules. Define

$$\begin{aligned} \mathrm{Hom}_{A^{\mathrm{op}}}(M, M') &:= \mathrm{Hom}_{A^{\mathrm{op}}}(M_{\triangleleft}, M'_{\triangleleft}), & \mathrm{Hom}_A(M, M') &:= \mathrm{Hom}_A({}_{\triangleright}M, {}_{\triangleright}M'), \\ \mathrm{Hom}_{A^{\mathrm{op}}}(N, N') &:= \mathrm{Hom}_{A^{\mathrm{op}}}(N_{\blacktriangleleft}, N'_{\blacktriangleleft}), & \mathrm{Hom}_A(N, N') &:= \mathrm{Hom}_A({}_{\blacktriangleright}N, {}_{\blacktriangleright}N'). \end{aligned}$$

In particular, for $M' := A$ we set $M_* := \mathrm{Hom}_A(M, A)$ and $M^* := \mathrm{Hom}_{A^{\mathrm{op}}}(M, A)$, called, respectively, the *left* and *right* dual of M. Observe that analogous duals cannot be defined for right modules as A is in general not a right U-module.

The notion of *pairing* between A^{e}-bimodules is also useful (see, for instance, [ChGa]):

Definition 2.2.1. Let U and W be two A^{e}-bimodules.

(*i*) A *left A^{e}-pairing* is a k-bilinear map $\langle\ ,\ \rangle : U \times W \to A$ such that, for any $u \in U$, $w \in W$ and $a \in A$, one has

$$\begin{aligned} &\langle u, a \triangleright w\rangle = \langle u \triangleleft a, w\rangle, \quad \langle u, w \triangleleft a\rangle = \langle a \blacktriangleright u, w\rangle, \quad \langle u, a \blacktriangleright w\rangle = \langle u \blacktriangleleft a, w\rangle, \\ &\langle u, w \blacktriangleleft a\rangle = \langle u, w\rangle a, \qquad \langle a \triangleright u, w\rangle = a\langle u, w\rangle. \end{aligned}$$

(*ii*) A *right A^{e}-pairing* is a k-bilinear map $\langle\ ,\ \rangle : U \times W \to A$ such that, for any $u \in U$, $w \in W$ and $a \in A$, one has

$$\begin{aligned} &\langle u, w \triangleleft a\rangle = \langle a \triangleright u, w\rangle, \quad \langle u, a \triangleright w\rangle = \langle u \blacktriangleleft a, w\rangle, \quad \langle u, w \blacktriangleleft a\rangle = \langle a \blacktriangleright u, w\rangle, \\ &\langle u, a \blacktriangleright w\rangle = a\langle u, w\rangle, \qquad \langle u \triangleleft a, w\rangle = \langle u, w\rangle a. \end{aligned}$$

2.2.2. Duals of bialgebroids. Let U_* resp. U^* be the left and right dual of a left bialgebroid. If ${}_{\triangleright}U$ is finitely generated projective, then U_* is canonically endowed with a *right* bialgebroid structure [KadSz] such that the evaluation pairing between U and U_* is a (nondegenerate) *left* pairing; similarly, if $U_{\triangleleft}$ is finitely generated projective, then U^* has a canonical *right* bialgebroid structure for which the natural pairing between U and U^* is a *right* pairing. In §5.2 we will use the analogous fact that the two duals

${}_*W := \mathrm{Hom}_B({}_{\blacktriangleright}W, B)$ and ${}^*W := \mathrm{Hom}_{B^{\mathrm{op}}}(W_{\blacktriangleleft}, B)$ of a right bialgebroid (W, B) yield left bialgebroids in turn.

2.3. **Left Hopf algebroids.** For any left bialgebroid U, define the *Hopf-Galois maps*

$$\begin{array}{rclcrcl} \alpha_\ell : {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} & \to & U_{\triangleleft} \otimes_A {}_{\triangleright}U, & & u \otimes_{A^{\mathrm{op}}} v & \mapsto & u_{(1)} \otimes_A u_{(2)} v, \\ \alpha_r : U_{\blacktriangleleft} \otimes^A {}_{\triangleright}U & \to & U_{\triangleleft} \otimes_A {}_{\triangleright}U, & & u \otimes^A v & \mapsto & u_{(1)} v \otimes_A u_{(2)}. \end{array}$$

Similar maps can be defined for right bialgebroids but we refrain from writing them down explicitly. With the help of these maps, we can make the following definition due to Schauenburg [Sch]:

Definition 2.3.1. A left bialgebroid U is called a *left Hopf algebroid* if α_ℓ is a bijection. Likewise, it is called a *right Hopf algebroid* if α_r is so. In either case, we adopt for all $u \in U$ the following (Sweedler-like) notation

$$u_+ \otimes_{A^{\mathrm{op}}} u_- := \alpha_\ell^{-1}(u \otimes_A 1), \qquad u_{[+]} \otimes^A u_{[-]} := \alpha_r^{-1}(1 \otimes_A u), \tag{2.1}$$

and call both maps $u \mapsto u_+ \otimes_{A^{\mathrm{op}}} u_-$ and $u \mapsto u_{[+]} \otimes^A u_{[-]}$ *translation maps*.

Analogous notions exist with respect to an underlying *right* bialgebroid structure, but we will not give the details here.

Remark 2.3.2.

(*i*) In case $A = k$ is central in U, one can show that α_ℓ is invertible if and only if U is a Hopf algebra, and the translation map reads $u_+ \otimes u_- := u_{(1)} \otimes S(u_{(2)})$, where S is the antipode of U. If the antipode is invertible, one also obtains the bijectivity of α_r, where $u_{[+]} \otimes u_{[-]} := u_{(2)} \otimes S^{-1}(u_{(1)})$. In either case, this shows that for bialgebras the additional property of being a left or right Hopf algebroid is equivalent to being a Hopf algebra.

(*ii*) By definition, a left bialgebroid U is a left Hopf algebroid if and only if U_{coop} is a right Hopf algebroid. In particular, for a cocommutative U both notions of left and right Hopf algebroid coincide.

(*iii*) The underlying left bialgebroid in a *full* Hopf algebroid with bijective antipode is both a left and right Hopf algebroid (but not necessarily vice versa); see [BSz, Prop. 4.2] for the details of this construction.

The following proposition collects some properties of the translation maps [Sch]:

Proposition 2.3.3. *Let U be a left bialgebroid.*

(*i*) *If U is a left Hopf algebroid, the following hold:*

$$\begin{aligned} u_+ \otimes_{A^{\mathrm{op}}} u_- &\in U \times_{A^{\mathrm{op}}} U, && (2.2)\\ u_{+(1)} \otimes_A u_{+(2)} u_- &= u \otimes_A 1 \quad \in U_{\triangleleft} \otimes_A {}_{\triangleright}U, && (2.3)\\ u_{(1)+} \otimes_{A^{\mathrm{op}}} u_{(1)-} u_{(2)} &= u \otimes_{A^{\mathrm{op}}} 1 \quad \in {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft}, && (2.4)\\ u_{+(1)} \otimes_A u_{+(2)} \otimes_{A^{\mathrm{op}}} u_- &= u_{(1)} \otimes_A u_{(2)+} \otimes_{A^{\mathrm{op}}} u_{(2)-}, && (2.5)\\ u_+ \otimes_{A^{\mathrm{op}}} u_{-(1)} \otimes_A u_{-(2)} &= u_{++} \otimes_{A^{\mathrm{op}}} u_- \otimes_A u_{+-}, && (2.6)\\ (uv)_+ \otimes_{A^{\mathrm{op}}} (uv)_- &= u_+ v_+ \otimes_{A^{\mathrm{op}}} v_- u_-, && (2.7)\\ u_+ u_- &= s^\ell(\varepsilon(u)), && (2.8)\\ \varepsilon(u_-) \blacktriangleright u_+ &= u, && (2.9)\\ (s^\ell(a) t^\ell(b))_+ \otimes_{A^{\mathrm{op}}} (s^\ell(a) t^\ell(b))_- &= s^\ell(a) \otimes_{A^{\mathrm{op}}} s^\ell(b), && (2.10) \end{aligned}$$

where in (2.2) we mean the Sweedler-Takeuchi product

$$U \times_{A^{\mathrm{op}}} U := \big\{ \textstyle\sum_i u_i \otimes v_i \in {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} \mid \sum_i u_i \triangleleft a \otimes v_i = \sum_i u_i \otimes a \blacktriangleright v_i, \ \forall a \in A \big\}.$$

(*ii*) *Analogously, if U is a right Hopf algebroid, one has:*

$$u_{[+]} \otimes^A u_{[-]} \quad \in \quad U \times^A U, \tag{2.11}$$
$$u_{[+](1)} u_{[-]} \otimes_A u_{[+](2)} \quad = \quad 1 \otimes_A u \quad \in U_{\triangleleft} \otimes_{A} {}_{\triangleright}U, \tag{2.12}$$
$$u_{(2)[-]} u_{(1)} \otimes^A u_{(2)[+]} \quad = \quad 1 \otimes^A u \quad \in U_{\blacktriangleleft} \otimes^A {}_{\triangleright}U\,, \tag{2.13}$$
$$u_{[+](1)} \otimes^A u_{[-]} \otimes_A u_{[+](2)} \quad = \quad u_{(1)[+]} \otimes^A u_{(1)[-]} \otimes_A u_{(2)}, \tag{2.14}$$
$$u_{[+][+]} \otimes^A u_{[+][-]} \otimes_A u_{[-]} \quad = \quad u_{[+]} \otimes^A u_{[-](1)} \otimes_A u_{[-](2)}, \tag{2.15}$$
$$(uv)_{[+]} \otimes^A (uv)_{[-]} \quad = \quad u_{[+]} v_{[+]} \otimes^A v_{[-]} u_{[-]}, \tag{2.16}$$
$$u_{[+]} u_{[-]} \quad = \quad t^\ell(\varepsilon(u)), \tag{2.17}$$
$$u_{[+]} \blacktriangleleft \varepsilon(u_{[-]}) \quad = \quad u, \tag{2.18}$$
$$(s^\ell(a) t^\ell(b))_{[+]} \otimes^A (s^\ell(a) t^\ell(b))_{[-]} \quad = \quad t^\ell(b) \otimes^A t^\ell(a), \tag{2.19}$$

where in (2.11) we mean the Sweedler-Takeuchi product

$$U \times^A U := \big\{ \textstyle\sum_i u_i \otimes v_i \in U_{\blacktriangleleft} \otimes^A {}_{\triangleright}U \mid \sum_i a \triangleright u_i \otimes v_i = \sum_i u_i \otimes v_i \blacktriangleleft a, \, \forall a \in A \big\}.$$

These two structures are not entirely independent:

Lemma 2.3.4. *The following mixed relations hold among left and right translation maps:*

$$u_{+[+]} \otimes_{A^{\mathrm{op}}} u_- \otimes^A u_{+[-]} \quad = \quad u_{[+]+} \otimes_{A^{\mathrm{op}}} u_{[+]-} \otimes^A u_{[-]}, \tag{2.20}$$
$$u_+ \otimes_{A^{\mathrm{op}}} u_{-[+]} \otimes^A u_{-[-]} \quad = \quad u_{(1)+} \otimes_{A^{\mathrm{op}}} u_{(1)-} \otimes^A u_{(2)}, \tag{2.21}$$
$$u_{[+]} \otimes^A u_{[-]+} \otimes_{A^{\mathrm{op}}} u_{[-]-} \quad = \quad u_{(2)[+]} \otimes^A u_{(2)[-]} \otimes_{A^{\mathrm{op}}} u_{(1)}, \tag{2.22}$$

where, for example, in the first equation (2.20) the second tensor product relates the first component with the third, and mutatis mutandum *for the other identities.*

Proof. In order to prove (2.20), we apply $\alpha_\ell \otimes \mathrm{id}$ to both sides (note that this operation is well-defined on the considered tensor products); for the right hand side we obtain, by definition,

$$(\alpha_\ell \otimes \mathrm{id})(u_{[+]+} \otimes_{A^{\mathrm{op}}} u_{[+]-} \otimes^A u_{[-]}) = u_{[+]} \otimes_A (1 \otimes^A u_{[-]}),$$

and for the left hand side we have

$$\begin{aligned} &(\alpha_\ell \otimes \mathrm{id})(u_{+[+]} \otimes_{A^{\mathrm{op}}} u_- \otimes^A u_{+[-]}) = u_{+[+](1)} \otimes_A (u_{+[+](2)} u_- \otimes^A u_{+[-]}) \\ &= u_{+(1)[+]} \otimes_A (u_{+(2)} u_- \otimes^A u_{+(1)[-]}) = u_{[+]} \otimes_A (1 \otimes^A u_{[-]}), \end{aligned}$$

using (2.14) along with (2.3). Applying α_ℓ^{-1} to the first and the third component of this identity gives the equality we are looking for. Since α_ℓ is assumed to be an isomorphism, this proves (2.20).

Let us also prove (2.21); the remaining identity will be left to the reader. To this end, apply $\mathrm{id} \otimes \alpha_r$ to both sides in (2.21): for the left hand side, we obtain

$$\begin{aligned} (\mathrm{id} \otimes \alpha_r)(u_+ \otimes_{A^{\mathrm{op}}} u_{-[+]} \otimes^A u_{-[-]}) &= u_+ \otimes_{A^{\mathrm{op}}} (u_{-[+](1)} u_{-[-]} \otimes_A u_{-[+](2)}) \\ &= u_+ \otimes_{A^{\mathrm{op}}} (1 \otimes_A u_-) \end{aligned}$$

by (2.12), and where in the second equation the first tensor product relates the first component with the third. As for the right hand side, we compute:

$$\begin{aligned} &(\mathrm{id} \otimes \alpha_r)(u_{(1)+} \otimes_{A^{\mathrm{op}}} u_{(1)-} \otimes^A u_{(2)}) = u_{(1)+} \otimes_{A^{\mathrm{op}}} (u_{(1)-(1)} u_{(2)} \otimes_A u_{(1)-(2)}) \\ &= u_{(1)++} \otimes_{A^{\mathrm{op}}} (u_{(1)-} u_{(2)} \otimes_A u_{(1)+-}) = u_+ \otimes_{A^{\mathrm{op}}} (1 \otimes_A u_-), \end{aligned}$$

using (2.6) and (2.4) in the last step as follows: Eq. (2.4) yields $u_{(1)+} \otimes_{A^{\mathrm{op}}} u_{(1)-} u_{(2)} \otimes_A 1 = u \otimes_{A^{\mathrm{op}}} 1 \otimes_A 1$ and applying α_ℓ^{-1} to the first and the third component gives the required equality. ∎

3. Modules over left or right Hopf algebroids

In this section we collect some general results about modules over left and right Hopf algebroids. Some of them are known, while others seem to have passed unnoticed so far.

3.1. **Module structures on** Hom**-spaces and tensor products.** Similarly as for bialgebras, the tensor product $M_{\triangleleft} \otimes_A {}_{\triangleright}M'$ of two left U-modules with left U-module structure given by

$$u(m \otimes_A m') := u_{(1)}m \otimes_A u_{(2)}m' \tag{3.1}$$

equips the category U-$\mathbf{Mod}$ for a left bialgebroid U with a monoidal structure. On the other hand, for $M \in U$-$\mathbf{Mod}$ and $N \in \mathbf{Mod}$-U, the A^{e}-module $\mathrm{Hom}_{A^{\mathrm{op}}}(M_{\triangleleft}, N_{\blacktriangleleft})$ is a right U-module via

$$(fu)(m) := f(u_{(1)}m)u_{(2)}.$$

The existence of a translation map if U is, on top, a left or right Hopf algebroid makes it possible to endow Hom-spaces and tensor products of U-modules with further natural U-module structures. The proof of the following proposition is straightforward.

Proposition 3.1.1. *Let (U, A) be a left bialgebroid, $M, M' \in U$-$\mathbf{Mod}$ and $N, N' \in \mathbf{Mod}$-U be left resp. right U-modules, denoting the respective actions by juxtaposition.*

(*i*) *Let (U, A) be additionally a left Hopf algebroid.*

(a) *The A^{e}-module $\mathrm{Hom}_{A^{\mathrm{op}}}(M, M')$ carries a left U-module structure given by*

$$(uf)(m) := u_+\big(f(u_-m)\big). \tag{3.2}$$

In particular, M^* is endowed with a left U-module structure.

(b) *The A^{e}-module $\mathrm{Hom}_A(N, N')$ carries a canonical left U-module structure via*

$$(u \rhd f)(n) := \big(f(nu_+)\big)u_-. \tag{3.3}$$

(c) *The A^{e}-module ${}_{\blacktriangleright}N \otimes_{A^{\mathrm{op}}} M_{\triangleleft}$ carries a canonical right U-module structure via*

$$(n \otimes_{A^{\mathrm{op}}} m) \lhd u := nu_+ \otimes_{A^{\mathrm{op}}} u_-m \tag{3.4}$$

(*ii*) *Let (U, A) be a right Hopf algebroid instead.*

(a) *The A^{e}-module $\mathrm{Hom}_A(M, M')$ carries a left U-module structure given by*

$$(uf)(m) := u_{[+]}\big(f(u_{[-]}m)\big). \tag{3.5}$$

In particular, M_* is naturally endowed with a left U-module structure.

(b) *The A^{e}-module $\mathrm{Hom}_{A^{\mathrm{op}}}(N, N')$ carries a left U-module structure given by*

$$(u \rhd f)(n) := \big(f(nu_{[+]})\big)u_{[-]}. \tag{3.6}$$

(c) *The A^{e}-module $N_{\blacktriangleleft} \otimes_A {}_{\triangleright}M$ carries a right U-module structure given by*

$$(n \otimes^A m) \lhd u := nu_{[+]} \otimes^A u_{[-]}m. \tag{3.7}$$

Note 3.1.2. These structures are well-known for D-modules (that is, when $U = D_X$, see [Bo, Ka]) and were later extended to $V^\ell(L)$-modules in [Ch1], [Ch3]. The results about tensor products can be found in [KoKr].

Of course, for any bialgebroid U, the A^{e}-modules $\mathrm{Hom}_{A^{\mathrm{op}}}(M, M')$ and $N \otimes_A M$ as in (3.2) resp. (3.3) carry left resp. right U-module structures in a standard way if M happens to be a right U-module. In case U is a left Hopf algebroid, these structures commute with those of Proposition 3.1.1:

Proposition 3.1.3. *Let $M' \in U$-$\mathbf{Mod}$ and $N \in \mathbf{Mod}$-U over a left bialgebroid (U, A).*

(*i*) *In case M is a (U,U)-bimodule and U a left Hopf algebroid,*

$$(u \vdash f)(m) := f(mu) \tag{3.8}$$

defines a left U-action on $\mathrm{Hom}_{A^{\mathrm{op}}}(M, M')$ that commutes with the action (3.2) such that $\mathrm{Hom}_{A^{\mathrm{op}}}(M, M')$ becomes a left $(U \otimes U)$-module. Likewise, if U is a right Hopf algebroid, the left action

$$(u \vdash f')(m) := f'(mu) \tag{3.9}$$

on $\mathrm{Hom}_A(M, M')$ commutes with the action (3.5) such that $\mathrm{Hom}_A(M, M')$ becomes a left $(U \otimes U)$-module.

(*ii*) *Similarly, for a (U,U)-bimodule M over a left Hopf algebroid U,*

$$(n \otimes_{A^{\mathrm{op}}} m) \dashv u := n \otimes_{A^{\mathrm{op}}} mu$$

defines a right action on ${}_\blacktriangleright N \otimes_{A^{\mathrm{op}}} M_\triangleleft$ that commutes with the action (3.4) (and likewise for a right Hopf algebroid and (3.7) on $N_\blacktriangleleft \otimes^A {}_\triangleright M$) such that ${}_\blacktriangleright N \otimes_{A^{\mathrm{op}}} M_\triangleleft$ (resp. $N_\blacktriangleleft \otimes^A {}_\triangleright M$) becomes a right $(U \otimes U)$-module.

(*iii*) *If U is simultaneously a left and a right Hopf algebroid, the map*

$$\Phi : N_\blacktriangleleft \otimes^A {}_\triangleright U \to {}_\blacktriangleright N \otimes_{A^{\mathrm{op}}} U_\triangleleft, \quad n \otimes_A u \mapsto nu_+ \otimes_{A^{\mathrm{op}}} u_-$$

is an isomorphism of right $(U \otimes U)$-modules from $(N_\blacktriangleleft \otimes^A {}_\triangleright U, \lhd, \dashv)$ to $({}_\blacktriangleright N \otimes_{A^{\mathrm{op}}} U_\triangleleft, \dashv, \lhd)$.

Proof. Only part (iii) is not entirely straightforward; there, also the claim $\Phi\big((n \otimes^A v) \dashv u\big) = \Phi(n \otimes^A v) \lhd u$ for $u, v \in U$ and $n \in N$ is immediate from (2.7). On the other hand, a direct calculation gives

$$\begin{aligned}\Phi((n \otimes^A v) \lhd u) &= \Phi(nu_{[+]} \otimes^A u_{[-]}v) = nu_{[+]}u_{[-]+}v_+ \otimes_{A^{\mathrm{op}}} v_-u_{[-]-} \\ &\overset{(2.22)}{=} nu_{(2)[+]}u_{(2)[-]}v_+ \otimes_{A^{\mathrm{op}}} v_-u_{(1)} \overset{(2.17)}{=} nt^\ell(\epsilon(u_{(2)}))v_+ \otimes_{A^{\mathrm{op}}} v_-u_{(1)} \\ &\overset{(2.2)}{=} nv_+ \otimes_{A^{\mathrm{op}}} v_-u = \Phi(n \otimes^A v) \dashv u.\end{aligned}$$

Finally, a straightforward computation shows that Φ is invertible with inverse

$${}_\blacktriangleright N \otimes_{A^{\mathrm{op}}} U_\triangleleft \to N_\blacktriangleleft \otimes^A {}_\triangleright U, \quad n \otimes_{A^{\mathrm{op}}} u \mapsto nu_{[+]} \otimes^A u_{[-]},$$

which concludes the proof. □

3.2. **Switching left and right modules: dualising modules.** We investigate now conditions which imply an equivalence between the categories of left and of right U-modules for a left bialgebroid U which is simultaneously a left and right Hopf algebroid. As in other frameworks, this is guaranteed by the existence of a suitable *dualising module*. This is the content of the next result, which generalises the well-known equivalence of categories between left and right $\mathcal{D}$-modules (due to Borel [Bo] and Kashiwara [Ka]). It also generalises the equivalence between left and right modules over a Lie-Rinehart algebra, *cf.* [Ch1].

Proposition 3.2.1. *Let (U, A) be simultaneously a left and right Hopf algebroid. Assume that there exists a right U-module P, where $P_\blacktriangleleft$ is finitely generated projective over A^{op}, such that*

(*i*) *the left U-module morphism*

$$A \to \mathrm{Hom}_{A^{\mathrm{op}}}(P, P), \quad a \mapsto \{p \mapsto a \blacktriangleright p\}$$

is an isomorphism of k-modules;

(*ii*) *the evaluation map*

$${}_\blacktriangleright P \otimes_{A^{\mathrm{op}}} \mathrm{Hom}_{A^{\mathrm{op}}}(P, N)_\triangleleft \to N, \quad p \otimes_{A^{\mathrm{op}}} \phi \mapsto \phi(p) \tag{3.10}$$

is an isomorphism for any $N \in \mathbf{Mod}\text{-}U$.

Then

$$U\text{-}\mathbf{Mod} \to \mathbf{Mod}\text{-}U, \quad M \mapsto {}_{\blacktriangleright}P \otimes_{A^{\mathrm{op}}} M_{\triangleleft}$$

is an equivalence of categories with quasi inverse given by $N' \mapsto \mathrm{Hom}_{A^{\mathrm{op}}}(P, N')$.

Proof. For $M \in U\text{-}\mathbf{Mod}$ and $N, N' \in \mathbf{Mod}\text{-}U$, one checks with (2.22) that the map

$$M_{\triangleleft} \otimes_{A} {}_{\triangleright}\mathrm{Hom}_{A^{\mathrm{op}}}(N, N') \to \mathrm{Hom}_{A^{\mathrm{op}}}(N, {}_{\blacktriangleright}N' \otimes_{A^{\mathrm{op}}} M_{\triangleleft}), \quad m \otimes_A \chi \mapsto \{n \mapsto \chi(n) \otimes_{A^{\mathrm{op}}} m\}$$

is a morphism of left U-modules, where the left U-module structure on the left hand side is given by (3.1) combined with (3.6), and on the right hand side by (3.6) combined with (3.4). It is even an isomorphism if $N_{\blacktriangleleft}$ is finitely generated projective over A. On the other hand, using (2.21) and (2.8), one easily sees that the evaluation (3.10) is a morphism of right U-modules; it is then an isomorphism by hypothesis, which finishes the proof. □

Remark 3.2.2. Of course, a similar statement holds when interchanging the rôles of source and target, *i.e.*, if ${}_{\blacktriangleright}P$ is projective of finite type and if the U-module morphism $A \to \mathrm{Hom}_A(P, P)$, $a \mapsto \{p \mapsto p \blacktriangleleft a\}$ is an isomorphism, but we will not give the details here.

Remark 3.2.3. A right U-module P with the properties as in the above proposition appeared in various contexts in the literature and is usually called a *dualising module*. We refer to [Ch1, KoKr, Hue, VdB] for applications and further details, and in particular to the situation in §6.3.

4. Linking structure for the duals of left Hopf algebroids

As mentioned in the Introduction, it is not known what kind of Hopf structure (if any) carry the (left or right) duals of a (left or right) Hopf algebroid. In this section—the core of the present work—, we find that the translation map of a left Hopf algebroid induces a special map linking its right to its left dual, which is apparently as close as one can get to an antipode kind-of structure on the dual. Note, however, that even in the case of a full Hopf algebroid this map is not simply the transpose of the antipode, as discussed in §4.2. In some sense, this amounts to sort of a generalisation of (the antipode in) a full Hopf algebroid as explained in Remark 4.2.5.

4.1. **Morphisms between left and right duals.** Let (U, A) be a left bialgebroid. If it is additionally a left Hopf algebroid, its right dual U^* (see §2.2) carries a canonical left U-module structure as in (3.2); define

$$S^*(\phi)(u) := (u\phi)(1_U) = \epsilon_U\big(u_+ t^\ell(\phi(u_-))\big), \qquad \forall \phi \in U^*,\ u \in U. \tag{4.1}$$

Likewise, if the left bialgebroid (U, A) is a right Hopf algebroid instead, its left dual U_* (see §2.2 again) carries a canonical left U-module structure as in (3.5), with the help of which one defines

$$S_*(\psi)(u) := (u\psi)(1_U) = \epsilon_U\big(u_{[+]} s^\ell(\psi(u_{[-]}))\big), \qquad \forall \psi \in U_*,\ u \in U. \tag{4.2}$$

The following result presents the key properties of the maps S^* and S_*:

Theorem 4.1.1. *Let (U, A) be a left bialgebroid.*

(*i*) *If (U, A) is moreover a left Hopf algebroid, Eq. (4.1) defines a map $S^* : U^* \to U_*$. In particular, $(S^*, \mathrm{id}_A) : (U^*, A) \to (U_*, A)$ is a morphism of right bialgebroids.*

(*ii*) *If (U, A) is a right Hopf algebroid instead, Eq. (4.2) defines a map $S_* : U_* \to U^*$. In particular, $(S_*, \mathrm{id}_A) : (U_*, A) \to (U^*, A)$ is a morphism of right bialgebroids.*

Proof. We only prove part *(i)* as *(ii)* follows *mutatis mutandum*. For the explicit computations, we will use the notation and description of the structure maps of the two right bialgebroids $(U_*, A, s^r_*, t^r_*, \Delta^r_*, \partial_*)$ and $(U^*, A, s^*_r, t^*_r, \Delta^*_r, \partial^*)$ as given in detail in [Ko, §3.1], together with the respective properties of left and right pairings $\langle ., . \rangle$ as in Definition 2.2.1. Direct verification shows that S^* takes values in U_*. Besides, for S^* to be a bialgebroid morphism, we need to show the following properties:

(i) $S^* s^*_r = s^r_*, \quad S^* t^*_r = t^r_*$,

(ii) $S^*\big(\phi\phi'\big) = S^*(\phi)S^*(\phi')$,

(iii) $\Delta^r_* S^* = (S^* \otimes S^*)\Delta^*_r, \quad \partial_* S^* = \partial^*$.

As for *(i)*, we find for $u \in U$, $a \in A$ by direct computation using (2.9) and (2.10):

$$S^*\big(s^*_r(a)\big)(u) = \epsilon\big(u_+ t^\ell\big(s^*_r(a)(u_-)\big)\big) = \epsilon\big(u_+ t^\ell\big(\epsilon(u_- s^\ell(a))\big)\big) = \epsilon(u)a = s^r_*(a)(u).$$

Likewise, the second identity follows from

$$S^*\big(t^*_r(a)\big)(u) = \epsilon\big(u_+ t^\ell\big(t^*_r(a)(u_-)\big)\big) = \epsilon\big(u_+ t^\ell(a\epsilon(u_-))\big) = \epsilon(ut^\ell(a)) = t^r_*(a)(u)$$

As for *(ii)*, let us first more generally compute an element $S^*(\phi)\psi$ for $\phi \in U^*$ and $\psi \in U_*$: by [Ko, Eq. (3.1.1)], Eq. (2.5), and the properties of a bialgebroid counit, we have

$$\begin{aligned}
\langle S^*(\phi)\psi, u\rangle &= \langle \psi, t^\ell(\langle u_{(2)}, S^*(\phi)\rangle)u_{(1)}\rangle = \langle \psi, t^\ell(\langle \epsilon, u_{(2)+} t^\ell(\langle \phi, u_{(2)-}\rangle)\rangle)u_{(1)}\rangle \\
&= \langle \psi, t^\ell(\langle \epsilon, u_{+(2)} t^\ell(\langle \phi, u_-\rangle)\rangle)u_{+(1)}\rangle \\
&= \langle \psi, t^\ell(\langle \epsilon, u_{+(2)} s^\ell(\langle \phi, u_-\rangle)\rangle)u_{+(1)}\rangle \\
&= \langle \psi, t^\ell(\langle \epsilon, u_{+(2)}\rangle)u_{+(1)} t^\ell(\langle \phi, u_-\rangle)\rangle = \langle \psi, u_+ t^\ell(\langle \phi, u_-\rangle)\rangle.
\end{aligned}$$

With the help of this property, by [Ko, Eq. (3.1.2)] along with (2.6), (2.10), and the fact that the counit in U gives the unit in U_*, one sees that for all $\phi, \phi' \in U^*$

$$\begin{aligned}
\langle S^*(\phi\phi'), u\rangle &= \langle \epsilon, u_+ t^\ell(\langle \phi\phi', u_-\rangle)\rangle = \langle \epsilon, u_+ t^\ell(\langle \phi', s^\ell\phi(u_{-(1)})u_{-(2)}\rangle)\rangle \\
&= \langle \epsilon, u_{++} t^\ell(\langle \phi', s^\ell\phi(u_-)u_{+-}\rangle)\rangle \\
&= \langle \epsilon, (u_+ t^\ell\phi(u_-))_+ t^\ell(\langle \phi', (u_+ t^\ell\phi(u_-))_-\rangle)\rangle \\
&= \langle S^*(\phi')\epsilon, u_+ t^\ell\phi(u_-)\rangle = \langle S^*(\phi)S^*(\phi'), u\rangle.
\end{aligned}$$

For proving the first identity in *(iii)*, recall that the nondegenerate pairing between U_* and U also defines a similar pairing between $U_{*\blacktriangleleft} \otimes_A {}_\blacktriangleright U_*$ and $U_\blacktriangleleft \otimes^A {}_\triangleright U$ by

$$\langle \psi \otimes_A \psi', u \otimes^A v\rangle := \langle \psi', us^\ell(\langle \psi, v\rangle)\rangle,$$

which is used to define the coproduct in U_* via the product in U (see, for example, [Ko, §3.1]). Denoting the right coproduct on U_* resp. U^* by Sweedler superscripts, we then have

$$\begin{aligned}
\langle S^*(\phi)^{(1)} \otimes_A S^*(\phi)^{(2)}, u \otimes^A v\rangle &= \langle S^*(\phi)^{(2)}, us^\ell\big(\langle S^*(\phi)^{(1)}, v\rangle\big)\rangle \\
&= \langle S^*(\phi), uv\rangle \\
&\overset{(2.7)}{=} \big\langle \epsilon, u_+ v_+ t^\ell(\langle \phi, v_- u_-\rangle)\big\rangle \\
&= \big\langle \epsilon, u_+ s^\ell\big(\epsilon\big(v_+ s^\ell\big(\langle \phi^{(1)}, v_- t^\ell(\langle \phi^{(2)}, u_-\rangle)\rangle\big)\big)\big)\big\rangle \\
&\overset{(2.2)}{=} \big\langle \epsilon, u_+ s^\ell\big(\epsilon\big(t^\ell(\langle \phi^{(2)}, u_-\rangle)v_+ s^\ell(\langle \phi^{(1)}, v_-\rangle)\big)\big)\big\rangle \\
&= \big\langle \epsilon, u_+ s^\ell\big(\langle S^*(\phi^{(1)}), v\rangle\big)s^\ell(\langle \phi^{(2)}, u_-\rangle)\big\rangle \\
&= \big\langle S^*(\phi^{(2)}), us^\ell\big(\langle S^*(\phi^{(1)}), v\rangle\big)\big\rangle \\
&= \langle S^*(\phi^{(1)}) \otimes_A S^*(\phi^{(2)}), u \otimes^A v\rangle.
\end{aligned}$$

Finally, for $\phi \in U^*$, the line

$$\partial_* S^*(\phi) = S^*(\phi)(1_U) = \phi(1_U) = \partial^*\phi$$

proves the second identity in (iii), and this finishes the proof. ☐

Remark 4.1.2. When U is just a Hopf algebra over $A := k$ with antipode S, we have $U^* = (U_*)^{\mathrm{op}}_{\mathrm{coop}}$, and S^* is nothing but the transpose of S. If U^* itself is in turn a Hopf algebra — namely, if the transpose of the multiplication m_U in U takes values in the tensor square of U^* —, then S^* is just the antipode of this dual Hopf algebra U^*. In this perspective, Theorem 4.1.1 just expresses the fact that the antipode in a Hopf algebra is an antimorphism of algebras and of coalgebras.

The next statement concerns the left U-actions (3.3), (3.6) along with (3.8), (3.9) on U^* resp. U_* considered in Proposition 3.1.3, claiming that either S^* or S_* intertwines them. The proof is straightforward, hence left to the reader.

Proposition 4.1.3. *Let U be a left bialgebroid.*

- (i) *If U is additionally a left Hopf algebroid, then S^* is a left U-module morphism from $(U^*, \rhd)$ to $(U_*, \vdash)$. If U is a right Hopf algebroid instead, then S_* is a left U-module morphism from $(U_*, \rhd)$ to $(U^*, \vdash)$.*
- (ii) *Assume that U is both a left and right Hopf algebroid. Then S^* and S_* are $(U \otimes U)$-bimodule morphisms respectively from $(U^*, \rhd, \vdash)$ to $\big(U_*, \vdash, \rhd\big)$, and vice versa.*

In particular, in case U is both a left and right Hopf algebroid we have:

Theorem 4.1.4. *Let (U, A) be a left and right Hopf algebroid. Then S^* and S_* are inverse to each other. In other words, the pair (S^*, id_A) is an isomorphism of right bialgebroids between (U^*, A) and (U_*, A), whose inverse is (S_*, id_A).*

Proof. As for the first statement, we directly compute by means of the bialgebroid axioms along with (2.22) and (2.17), for any $\phi \in U^*$:

$$\begin{aligned}(S_* S^* \phi)(u) &= \epsilon\big(u_{[+]} s^\ell (S^* \phi(u_{[-]}))\big) = \epsilon\big(u_{[+]} s^\ell\big(\epsilon_U(u_{[-]+} t^\ell \phi(u_{[-]-}))\big)\big) \\ &= \epsilon\big(u_{[+]} u_{[-]+} t^\ell \phi(u_{[-]-})\big) = \epsilon\big(u_{(2)[+]} u_{(2)[-]} t^\ell \phi(u_{(1)})\big) \\ &= \phi(u_{(1)}) \epsilon(u_{(2)}) = \phi(u),\end{aligned}$$

which proves that $S_* \circ S^* = \mathrm{id}_{U^*}$. Likewise, one shows that $S^* \circ S_* = \mathrm{id}_{U_*}$. □

4.2. **The case of a full Hopf algebroid.** If H is a full Hopf algebroid with bijective antipode S in the sense of [BSz], then it is, in particular, both a left and right left bialgebroid (see the short summary in §4.2.1 below): therefore, there is a right bialgebroid analogue to the previous constructions concerning the maps S^* and S_*. On the other hand, the antipode S induces by transposition new maps S^t, ${}^tS$, etc., for the dual spaces. Hereafter we discuss the links between these various maps, in particular showing that, while for the Hopf algebra case one has identities like $S^* = {}^tS$ (*cf.* Remark 4.1.2), this is no longer the case for the general setup of full Hopf algebroids as illustrated in §6.1 below.

4.2.1. Reminder on full Hopf algebroids. Recall that a full Hopf algebroid structure (see, for example, [B]) on a k-module H consists of the following data:

- (i) a left bialgebroid structure $H^\ell := (H, A, s^\ell, t^\ell, \Delta_\ell, \epsilon)$ over a k-algebra A;
- (ii) a right bialgebroid structure $H^r := (H, B, s^r, t^r, \Delta_r, \partial)$ over a k-algebra B;
- (iii) the assumption that the k-algebra structures for H in *(i)* and in *(ii)* be the same;
- (iv) a k-module map $S : H \to H$;
- (v) some compatibility relations between the previously listed data for which we refer to *op. cit.*

We shall denote by lower Sweedler indices the left coproduct Δ_ℓ and by upper indices the right coproduct Δ_r, that is, $\Delta_\ell(h) =: h_{(1)} \otimes_A h_{(2)}$ and $\Delta_r(h) = h^{(1)} \otimes_B h^{(2)}$ for any $h \in H$. As said before, a full Hopf algebroid (with bijective antipode) is both a left and right Hopf algebroid but not necessarily vice versa (as illustrated in §6.1). In this case, the translation maps in (2.1) are given by

$$h_+ \otimes_{A^{\mathrm{op}}} h_- = h^{(1)} \otimes_{A^{\mathrm{op}}} S(h^{(2)}) \quad \text{and} \quad h_{[+]} \otimes_{B^{\mathrm{op}}} h_{[-]} = h^{(2)} \otimes_{B^{\mathrm{op}}} S^{-1}(h^{(1)}), \tag{4.3}$$

formally similar as for Hopf algebras.

The following lemma [B, BSz] will be needed to prove the main result in this subsection.

Lemma 4.2.2. *Let H be any Hopf algebroid. Then* (*i*) *the maps* $\nu := \partial s^\ell : A \to B^{\mathrm{op}}$ *and* $\mu := \epsilon s^r : B \to A^{\mathrm{op}}$ *are isomorphisms of* k*-algebras;*

(*ii*) *the pair of maps* $(S, \partial s^\ell) : H^\ell \to (H^r)^{\mathrm{op}}_{\mathrm{coop}}$ *gives an isomorphism of left bialgebroids;*

(*iii*) *the pair of maps* $(S, \epsilon s^r) : H^r \to (H^\ell)^{\mathrm{op}}_{\mathrm{coop}}$ *gives an isomorphism of right bialgebroids.*

The next observation might let us consider S^* and S_* as sort of an analogue of the antipode on the dual:

Proposition 4.2.3. *Let* (U, A) *be a cocommutative left bialgebroid (in particular,* A *is commutative and* $s^\ell = t^\ell$*). Then* (U, A) *is a left Hopf algebroid if and only if it is a right Hopf algebroid; in this case,* $(U^*, A) = ((U_*)_{\mathrm{coop}}, A)$ *is a full Hopf algebroid with involutive antipode* $\mathscr{S} := S^* = S_*$*.*

Proof. The first claim directly holds true by the very definitions. The rest of the proof follows *verbatim* in the footsteps of the one of Theorem 3.17 in [KoP], which considers the special case for $U := V^\ell(L)$. □

As mentioned before, one can also link the duals of a Hopf algebroid (H, S) by transposed maps ${}^tS$, which usually do not coincide with S^* or S_* (see also §6.1). The next result explains the precise connection between them.

Theorem 4.2.4. *Let* H *be a Hopf algebroid. Then the diagram*

$$\begin{array}{ccc}
\big((H^r)^{\mathrm{op}}_{\mathrm{coop}}\big)^* & \xrightarrow{\ {}^tS\ } & (H^\ell)^* \\
{\scriptstyle S_r^*}\downarrow & & \downarrow{\scriptstyle S_\ell^*} \\
\big((H^r)^{\mathrm{op}}_{\mathrm{coop}}\big)_* & \xrightarrow[{}^tS]{} & (H^\ell)_*
\end{array}$$

of right bialgebroid morphisms commutes.

Proof. Let us identify B^{op} and A by means of the k-algebra isomorphism $\nu : A \to B^{\mathrm{op}}$ mentioned above; then the left algebroid $(H^r)^{\mathrm{op}}_{\mathrm{coop}}$ is described by the sextuple

$$\big((H^r)^{\mathrm{op}}, \hat{s^\ell} := s^r\nu, \hat{t^\ell} := t^r\nu, \Delta_r^{\mathrm{coop}}, \hat{\epsilon} := \nu^{-1}\partial\big).$$

Moreover, the Hopf algebroid $\big((H^r)^{\mathrm{op}}_{\mathrm{coop}}, (H^\ell)^{\mathrm{op}}_{\mathrm{coop}}, (\mu, S) : (H^r)^{\mathrm{op}}_{\mathrm{coop}} \to H^\ell\big)$ is the one we have to consider to compute S_r^*. For $\phi \in \big((H_r)^{\mathrm{op}}_{\mathrm{coop}}\big)_*$ and $h \in H$ we have

$$\begin{aligned}
\langle({}^tS \circ S_r^*)(\phi), h\rangle &= \hat{\epsilon}\big(S(h)_{(2)}\hat{t^\ell}\big(\langle\phi, S(S(h)_{(1)})\rangle\big) = \big(\nu^{-1}\partial S\big)\big(h^{(1)}t^\ell\big(\langle\phi, S^2(h^{(2)})\rangle\big) \\
&= \epsilon\big(h^{(1)}t^\ell\big(\langle\phi, S^2(h^{(2)})\rangle\big)\big) \\
&= \epsilon\big(h^{(1)}t^\ell\big(\langle{}^tS(\phi), S(h^{(2)})\rangle\big)\big) = \langle(S_\ell^* \circ {}^tS)(\phi), h\rangle,
\end{aligned}$$

where we used the explicit form (4.3) of the translation map and the fact that S is an anti-coring morphism between left and right coproduct, which proves ${}^tS \circ S_r^* = S_\ell^* \circ {}^tS$ as claimed. □

Remark 4.2.5. In general, both maps S^* or S_* can be thought of as an extension of the notion of antipode for a Hopf algebroid, in the following sense. As mentioned in Lemma 4.2.2, the antipode in a Hopf algebroid yields a bialgebroid morphism $S : H^\ell \to (H^r)^{\mathrm{op}}_{\mathrm{coop}}$. On the other hand, if U is a left Hopf algebroid, then we have a similar situation replacing $(H^\ell, (H^r)^{\mathrm{op}}_{\mathrm{coop}}, S)$ with the triple $((U^*)^{\mathrm{op}}, U_*, S^*)$, and one might be tempted to define a Hopf algebroid as a triple (U, V, S) of a left resp. right bialgebroid U resp. V where the underlying ring structure is *not* the same. More precisely, the apparent asymmetry of a Hopf algebroid consisting of two coring structures but only one ring structure (that makes it difficult to obtain self-duality) would be somewhat attenuated. In this spirit, Theorem 4.1.4 can be seen as the statement that either the triple $((U^*)^{\rm op}, U_*, S^*)$ or $((U_*)^{\rm op}, U^*, S_*)$ for a left and right Hopf algebroid U yields sort of a generalised Hopf algebroid structure on the dual with invertible antipode.

5. Categorical fallouts

For a given left Hopf algebroid U, the existence of the right bialgebroid morphism $S^* : U^* \to U_*$ implies nontrivial consequences on the representation-theoretical setup, *i.e.*, for U-modules and U-comodules.

5.1. **A categorical equivalence for comodules.** Classically [Ca], if U happens to be a finite dimensional algebra over a field k and $DU := \mathrm{Hom}_k(U, k)$ is its dual (carrying the structure of a coassociative coalgebra [Sw, 1.1.2]), right DU-modules naturally correspond to left U-comodules, *i.e.*, one has a categorical equivalence

$$\mathbf{Mod}\text{-}DU \simeq U\text{-}\mathbf{Comod}.$$

The situation in the bialgebroid context is richer, as summarised by the following theorem that was proven in detail in [Ko, Theorem 3.1.11]:

Theorem 5.1.1. *Let (U, A) be a left bialgebroid.*

(*i*) *There exists a canonical functor $\mathbf{Comod}\text{-}U \to \mathbf{Mod}\text{-}U_*$. If ${}_\triangleright U$ is finitely generated A-projective, this functor has a quasi-inverse $\mathbf{Mod}\text{-}U_* \to \mathbf{Comod}\text{-}U$ such that there is an equivalence*

$$\mathbf{Comod}\text{-}U \simeq \mathbf{Mod}\text{-}U_*$$

of categories.

(*ii*) *Likewise, there exists a canonical functor $U\text{-}\mathbf{Comod} \to \mathbf{Mod}\text{-}U^*$. If $U_\triangleleft$ is finitely generated A-projective, this functor has a quasi-inverse $\mathbf{Mod}\text{-}U^* \to U\text{-}\mathbf{Comod}$ such that there is an equivalence*

$$U\text{-}\mathbf{Comod} \simeq \mathbf{Mod}\text{-}U^*$$

of categories.

An explicit description of all involved functors is given in [Ko, §3.1]. A direct consequence follows at once when looking at left Hopf algebroids:

Theorem 5.1.2. *Let (U, A) be a left bialgebroid.*

(*i*) *Let (U, A) be additionally a left Hopf algebroid, where $U_\triangleleft$ is finitely generated projective over A. Then there exists a canonical functor $\mathbf{Comod}\text{-}U \to U\text{-}\mathbf{Comod}$.*

(*ii*) *Let (U, A) be a right Hopf algebroid instead, where ${}_\triangleright U$ is finitely generated projective over A. Then there exists a canonical functor $U\text{-}\mathbf{Comod} \to \mathbf{Comod}\text{-}U$.*

As a consequence, if U is both a left and right Hopf algebroid and both $U_\triangleleft$ and ${}_\triangleright U$ are finitely generated projective over A, then the functors mentioned in (i) and (ii) are quasi-inverse to each other and we have an equivalence

$$U\text{-}\mathbf{Comod} \simeq \mathbf{Comod}\text{-}U$$

of categories.

Proof. Part *(i)* is proven by combining Theorem 5.1.1 *(i)* with the fact that the right bialgebroid morphism given by $S^* : U^* \to U_*$ induces in an obvious way a canonical functor $\mathbf{Mod}\text{-}U_* \to \mathbf{Mod}\text{-}U^*$ and then finally applying Theorem 5.1.1 *(ii)*. The proof of *(ii)* is clearly similar, and the last claim follows by the preceding two. □

Remark 5.1.3. Note that this does *not* boil down to the usual equivalence of left and right comodules via the antipode (as there is no antipode for left or right Hopf algebroids, not even if the bialgebroid is simultaneously both). Even if we dealt with a full Hopf algebroid, this is still a different kind of equivalence, as follows from the considerations in §4.2 and §6.1 below.

5.2. **A categorical equivalence for modules.** In this section, we dually record functors (possibly equivalences) concerning modules over left or right Hopf algebroids. We first recall a standard fact that (under suitable projective finiteness assumptions) the canonical morphism between any left bialgebroid and its (suitably chosen) bidual (which is a left bialgebroid again as mentioned in §2.2.2) is in fact an isomorphism:

Lemma 5.2.1. *Let (U, A) be a left bialgebroid and let*

$$\begin{aligned} \Phi : U \to {}_*(U^*), &\quad u \mapsto \{\phi \mapsto \mathrm{ev}_u(\phi) := \phi(u)\}, \\ \Psi : U \to {}^*(U_*), &\quad u \mapsto \{\psi \mapsto \mathrm{ev}_u(\psi) := \psi(u)\} \end{aligned}$$

be the canonical left bialgebroid morphisms from U to its biduals ${}_*(U^*)$ resp. ${}^*(U_*)$. If $U_\triangleleft$ resp. ${}_\triangleright U$ is finitely generated projective over A, then Φ resp. Ψ is an isomorphism.

This allows us to state the main result of this subsection:

Theorem 5.2.2.

(*i*) *Let (U, A) be a left Hopf algebroid. Then there exists a canonical functor*

$${}^*(U^*)\text{-}\mathbf{Mod} \to U\text{-}\mathbf{Mod}.$$

If $U_\triangleleft$ is finitely generated A-projective, then there also exists a canonical functor

$$U\text{-}\mathbf{Mod} \to {}_*(U_*)\text{-}\mathbf{Mod}.$$

(*ii*) *Let (U, A) be a right Hopf algebroid instead. Then there exists a canonical functor*

$${}_*(U_*)\text{-}\mathbf{Mod} \to U\text{-}\mathbf{Mod}.$$

If ${}_\triangleright U$ is finitely generated A-projective, then there also exists a canonical functor

$$U\text{-}\mathbf{Mod} \to {}^*(U^*)\text{-}\mathbf{Mod}.$$

(*iii*) *Let (U, A) be a left and right Hopf algebroid such that $U_\triangleleft$ and ${}_\triangleright U$ are finitely generated projective over A. Then there exist equivalences of categories*

$${}_*(U_*)\text{-}\mathbf{Mod} \simeq U\text{-}\mathbf{Mod} \quad \textit{as well as} \quad U\text{-}\mathbf{Mod} \simeq {}^*(U^*)\text{-}\mathbf{Mod}.$$

Proof. As for *(i)*, observe that the transpose of the right bialgebroid morphism $S^* : U^* \to U_*$ is a left bialgebroid morphism ${}^t(S^*) : {}^*(U_*) \to {}^*(U^*)$, which by restriction of coefficients defines a functor ${}^*(U^*)\text{-}\mathbf{Mod} \to {}^*(U_*)\text{-}\mathbf{Mod}$. On the other hand, the canonical evaluation morphism $\Psi : U \to {}^*(U_*)$ provides another functor ${}^*(U_*)\text{-}\mathbf{Mod} \to U\text{-}\mathbf{Mod}$, which leads to the desired result. As for the second claim in *(i)*, observe that although S^* via ${}^t(S^*) : {}_*(U_*) \to {}_*(U^*)$ also yields a functor ${}_*(U^*)\text{-}\mathbf{Mod} \to {}_*(U_*)\text{-}\mathbf{Mod}$, for reversing the evaluation morphism Φ, the projective finiteness assumption is needed. Part *(ii)* is proven in the same spirit and part *(iii)* then follows from Theorem 4.1.4. □

6. Examples and applications

In this section we present some applications to specific examples.

6.1. **Lie-Rinehart algebras and their jet spaces.** Let (A, L) be a Lie-Rinehart algebra (*cf.* [Ri], geometrically a Lie algebroid). Then its (left) universal enveloping algebra $V^\ell(L)$ carries not only the structure of a left bialgebroid over the commutative algebra A [Xu] but also of a left Hopf algebroid [KoKr]; as it is cocommutative, it is also a right Hopf algebroid.

Full Hopf algebroid structures on $V^\ell(L)$ are in bijection with right $V^\ell(L)$-module structures on A which play the rôle of possible right counits, expressed by suitable maps $\partial : V^\ell(L) \to A$ (*cf.* [Ko, §4.2] or [KoP] for more information). The corresponding antipode $S : V^\ell(L) \to V^\ell(L)^{\mathrm{op}}_{\mathrm{coop}}$ is then uniquely determined by the prescriptions

$$S(a) = a, \quad S(X) = -X + \partial(X), \quad \forall a \in A, \forall X \in L, \tag{6.1}$$

on generators. For a general Lie-Rinehart algebra (which does not arise from a Lie algebroid), such a map ∂ and hence the antipode might or might not exist.

As A is commutative, one can think of L as being a *right* Lie-Rinehart algebra in the sense specified in [ChGa, §2.1.8]. The difference this makes here only appears on the level of the universal enveloping object: one obtains a different algebra $V^r(L)$, which in turn is a right bialgebroid, see *op. cit.*: one has $V^r(L) = V^\ell(L^{\mathrm{op}})^{\mathrm{op}}$, where L^{op} is the Lie Rinehart algebra opposite to L obtained from L by taking the opposite bracket and the opposite anchor.

If L is finitely generated projective as an A-module, then the *jet space* $J^r(L) := V^\ell(L)^*$ is a right bialgebroid (in a topological sense as the coproduct takes values in a *topological* tensor product). In particular, $J^r(L)$ is commutative, so it is also a left bialgebroid. There is a canonical identification of $V^\ell(L)^* = J^r(L)$ and $V^\ell(L)_*$: as $V^\ell(L)$ is cocommutative, the maps S^* and S_* are equal and yield an antipode for $J^r(L)$, which in this way becomes a full Hopf algebroid (*cf.* [KoP] for details).

6.1.1. *Difference between S^* and ${}^tS$.* In this specific example, one can explicitly observe the difference between S^* and the transpose of the antipode S on $V^\ell(L)$ in (6.1). Apart from the fact mentioned above that S^* always exists while ${}^tS$ does not, this is already clear on an abstract level since these are maps of different nature as pointed out in Theorem 4.2.4. Nevertheless, one directly sees here that with respect to the A-module structures coming from left and right multiplication in $V^\ell(L)$, the map $S^*(\phi)$ is left A-linear whereas ${}^tS(\phi)$ is A-linear from the right, for $\phi \in V^\ell(L)^*$. Evaluating both maps on an element in $L \subset V^\ell(L)$, one obtains

$${}^tS(\phi)(X) = -\phi(X) + \partial(X)\phi(1) \qquad \forall \phi \in V^\ell(L)^*, X \in L,$$

on one hand, and on the other hand:

$$S^*(\phi)(X) = -\phi(X) + X(\phi(1)) \qquad \forall \phi \in V^\ell(L)^*, X \in L,$$

where $L \to \mathrm{Der}(A, A)$, $X \mapsto \{a \mapsto X(a)\}$ denotes the anchor of the Lie-Rinehart algebra (A, L). Using the property $Xa - aX = X(a)$ with respect to the product in $V^\ell(L)$ as well as the right A-linearity of ∂, one obtains $\partial(aX) = \partial(X)a - X(a)$ and therefore ${}^tS(\phi)(X) - S^*(\phi)(X) = \partial(\phi(1)X)$, which in general does not vanish.

6.1.2. *(Co)module theoretical fallouts.* Actually, the left bialgebroid $V^\ell(L)$ is *not* finitely generated projective over A; nevertheless, a suitable version of Lemma 5.2.1, namely

$$V^\ell(L) \cong {}_\star(V^\ell(L)^*) \tag{6.2}$$

still holds true, where the left subscript "$\star$" denotes a suitable *topological* dual (see, for instance, [ChGa] for details). Hence, we still have an equivalence

$$V^\ell(L)\text{-}\mathbf{Mod} \to {}_\star(V^\ell(L)_*)\text{-}\mathbf{Mod}$$

of categories as in Theorem 5.2.2*(ii)*. Moreover, one also has canonical isomorphisms

$${}_\star(V^\ell(L)_*) \cong (V^r(L))^{\mathrm{op}} \cong V^\ell(L^{\mathrm{op}}). \tag{6.3}$$

Overall, this yields category equivalences of the form

$$V^\ell(L)\text{-}\mathbf{Mod} \to (V^r(L))^{\mathrm{op}}\text{-}\mathbf{Mod}, \qquad V^\ell(L)\text{-}\mathbf{Mod} \to V^\ell(L^{\mathrm{op}})\text{-}\mathbf{Mod}.$$

In particular, the right hand side can be considered as induced by the isomorphism

$$\sigma : V^\ell(L) \xrightarrow{\cong} V^\ell(L^{\mathrm{op}})$$

of algebras, which on generators $X \in L$ and $a \in A$ is explicitly given as

$$X \mapsto -X, \quad a \mapsto a.$$

Indeed, via the identifications (6.2) and (6.3), this is nothing but the transpose of $S_* : V^\ell(L)_* \to V^\ell(L)^*$. In particular, when L is a Lie algebra, *i.e.*, with zero anchor, the map $\sigma : V^\ell(L) \xrightarrow{\cong} V^\ell(L^{\mathrm{op}})$ is simply the well-known antipode of the Hopf algebra $V^\ell(L) := U(L)$, the usual universal enveloping algebra of a Lie algebra.

6.1.3. *Universal enveloping algebras and deformations.* Following [ChGa], one can consider a quantum deformation $V^\ell(L)_h$ of $V^\ell(L)$: by definition, this is a left bialgebroid (in a topological sense), but as its "limit at $h = 0$" (that is, $V^\ell(L)$) is both a left and right Hopf algebroid, it follows automatically (by a standard argument in deformation theory) that $V^\ell(L)_h$ is also both a left and right Hopf algebroid.

On the other hand, the dual (right) bialgebroids $J^r(L)_h := (V^\ell(L)_h)^*$ and ${}^rJ(L)_h := (V^\ell(L)_h)_*$ are deformations of $J^r(L) := V^\ell(L)^* = (V^\ell(L)_*)_{\mathrm{coop}}$. This common "limit" is a full Hopf algebroid (with bijective antipode) by the above, hence in particular it is a left and right Hopf algebroid with respect to the underlying right bialgebroid structure. It follows then that the same is true for the right bialgebroids $J^r(L)_h$ and ${}^rJ(L)_h$, but usually they are not full Hopf algebroids. Nonetheless, we can apply our constructions of §4.1 to $U := V^\ell(L)_h$ to find the maps S^* and S_*. Theorem 4.1.4 then assures that the two deformations $J^r(L)_h$ and ${}^rJ(L)_h$ of $V^\ell(L)^* = V^\ell(L)_*$ are isomorphic (as right bialgebroids) via S^* and S_*.

6.2. **Mixed distributive law between duals.** A particular kind of mixed distributive law (or entwining) in the sense of Beck [Be] can be constructed via the following recipe. Combining a morphism $(\phi_1, \phi_0) : (V, B) \to (V', B')$ of right (say) bialgebroids with a Hopf-Galois map yields

$$\chi : V'_{\blacktriangleleft} \otimes^B {}_{\triangleright}V \to V_{\blacktriangleleft} \otimes_B {}_{\blacktriangleright}V', \quad v' \otimes^B v \mapsto v^{(1)} \otimes_B v'\phi(v^{(2)}),$$

which can be easily seen to define a mixed distributive law between V' (thought of as a coalgebra) and V (thought of as an algebra, although its coproduct appears in χ). Applying this to the two duals of a left bialgebroid U along with S^*, one obtains

$$\chi : U_{*\blacktriangleleft} \otimes^A {}_{\triangleright}U^* \to U^*{}_{\blacktriangleleft} \otimes_A {}_{\blacktriangleright}U_*, \qquad \psi \otimes^A \phi \mapsto \phi^{(1)} \otimes_A \psi S^*(\phi^{(2)})$$

as a mixed distributive law between U^* and U_*, to which any standard construction based on mixed distributive laws could be applied.

6.3. **Cases where a dualising module exists.** In this section, we will come back to the situation of dualising modules as in §3.2 by investigating their (deformation) quantisation. To this end, we first need to introduce some notation and definitions with respect to decreasing filtrations; see, for example, [Ch2, Schn] for further basic results and details.

6.3.1. *Notation and terminology.* Let A be an algebra endowed with a decreasing filtration $(F_nA)_{n\in\mathbb{N}}$ and consider a filtered FA-module denoted by FM, whereas its underlying A-module will be denoted by M. If FM and FN are two filtered FA-modules, then a filtered morphism $Fu : FM \to FN$ is a morphism $u : M \to N$ of the underlying A-modules such that $u(F_sM) \subset F_sN$. A filtered morphism $Fu : FM \to FN$ is *strict* if it satisfies $u(F_sM) = u(M) \cap F_sN$. An exact sequence of FA-modules is a sequence

$$FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP \tag{6.4}$$

such that $\operatorname{Ker} F_sv = \operatorname{Im} F_su$, where $F_sv = v|_{F_sN}$; hence Fu is strict. If moreover Fv is also strict, (6.4) is a called a *strict exact sequence*.

The filtration of a filtered module gives rise to a topology and even a metric if the filtered module is separated, that is, if $\bigcap_{n\in\mathbb{N}} F_nM = \{0\}$. For any $r \in \mathbb{Z}$ and for any FA-module FM, we define the *shifted module* $FM(r)$ as the module M endowed with the filtration

$(F_{s+r}M)_{s\in\mathbb{Z}}$. An FA-module is called *finite free* if it is isomorphic to an FA-module of the type $\bigoplus_{i=1}^{p} FA(-d_i)$, where $d_1, \ldots, d_p \in \mathbb{Z}$. An FA-module FM is called *of finite type* if one can find $m_1 \in F_{d_1}M, \ldots, m_p \in F_{d_p}M$ such that any $m \in F_dM$ may be written as

$$m = \sum_{i=1}^{p} a_{d-d_i} m_i,$$

where $a_{d-d_i} \in F_{d-d_i}A$.

For example, considering a bialgebroid (U, A) and its quantisation (U_h, A_h), the algebra U_h is endowed with the h-adic filtration and A_h, endowed with the h-adic filtration, becomes a filtered FU_h-module. Observe that the natural left U_h-module structure on A_h quantises that of U on A.

Theorem 6.3.1. *Let (U, A) be a left bialgebroid, where U is assumed to be a k-Noetherian algebra. Assume that there exists an integer d satisfying*

$$\mathrm{Ext}^i_U(A, U) = \begin{cases} 0 & \text{if } i \neq d, \\ \Lambda & \text{if } i = d. \end{cases}$$

Then there exists an A_h-module Λ_h that is a quantisation of Λ such that

$$\mathrm{Ext}^i_{U_h}(A_h, U_h) = \begin{cases} 0 & \text{if } i \neq d, \\ \Lambda_h & \text{if } i = d, \end{cases}$$

where the right action of U_h on $\mathrm{Ext}^d_{U_h}(A_h, U_h)$ is a quantisation of the right action of U on $\mathrm{Ext}^d_U(A, U)$ given by right multiplication.

We remind the reader here that Λ_h is $\Lambda[[h]]$ as a $k[[h]]$-module. This theorem is proven in [Ch2] in the case where $A_h = k[[h]]$. For the proof of the general case, we will need the following auxiliary statement:

Lemma 6.3.2. *There exists a resolution of the U_h-module A_h by finite rank free (filtered) FU_h-modules*

$$\ldots \xrightarrow{\partial_{i+1}} FL^i \xrightarrow{\partial_i} \ldots \xrightarrow{\partial_2} FL^1 \xrightarrow{\partial_1} FL^0 \longrightarrow A_h \longrightarrow \{0\},$$

where FL^i is $(U_h)^{d_i}$ endowed with the h-adic filtration such that the associated graded complex

$$\ldots GL^i \xrightarrow{G\partial_i} \ldots \to GL^1 \xrightarrow{G\partial_1} GL^0 \longrightarrow A[h] \longrightarrow \{0\}$$

is a resolution of the $U[h]$-module $A[h]$.

Proof. We will construct the p-th module FL^p by induction on p: for $p = 0$, one may take $FL^0 := U_h$ and $\partial_0 := \epsilon$, endowed with the h-adic topology. Assume then that $FL^0, FL^1, \ldots, FL^p$ are already constructed along with $\partial_0, \partial_1, \ldots, \partial_p$. As FL^p is topologically free, the induced filtration and the h-adic filtration coincide on $\mathrm{Ker}\,\partial_p$. As $\mathrm{Ker}\,\partial_p$ is closed for the topology defined by the induced filtration, it is complete. This $k[[h]]$-module is topologically free as it is complete for the h-adic topology and also torsion free; set $\mathrm{Ker}\,\partial_p := V_p[[h]]$. Since $GU_h = U[h]$ is Noetherian, the (filtered) algebra U_h is (filtered) Noetherian [Ch2, Prop. 3.0.7] and the U_h-module $\mathrm{Ker}\,\partial_p$ is finitely generated so that the U-module V_p is finitely generated as well. Let $(\overline{v_1}, \ldots, \overline{v_{d_{p+1}}})$ be a generating system of the U-module V_p and let $(v_1, \ldots, v_{d_{p+1}}) \in (\mathrm{Ker}\,\partial_p)^{d_{p+1}}$ be a lift of $(\overline{v_1}, \ldots, \overline{v_{d_{p+1}}})$. Moreover, introduce the U_h-module morphism

$$\partial_{p+1} : (U_h)^{d_{p+1}} \to \mathrm{Ker}\,\partial_p, \quad (u_1, \ldots, u_{p+1}) \mapsto \sum u_i v_i,$$

which is a strict morphism of filtered modules. The filtered exact sequence

$$(U_h)^{p+1} \xrightarrow{\partial_{p+1}} (U_h)^p \xrightarrow{\partial_p} (U_h)^{p-1}$$

is strict exact so that the sequence

$$(GU_h)^{p+1} \xrightarrow{G\partial_{p+1}} (GU_h)^p \xrightarrow{G\partial_p} (GU_h)^{p-1}$$

is exact (*cf.* [Ch2, Prop. 3.0.2]). ☐

Proof of Theorem 6.3.1. The $\mathrm{Ext}^\bullet_{U_h}(A_h, U_h)$-groups can be computed via the complex $M^\bullet := \big(\mathrm{Hom}_{U_h}(L^\bullet, U_h), \partial_\bullet\big)$. Its components are endowed with the natural filtration

$$F_s \mathrm{Hom}_{U_h}(L^i, U_h) := \{\lambda \in \mathrm{Hom}_{U_h}(L^i, U_h) \mid \lambda(F_p L^i) \subset F_{s+p} U_h\},$$

and the right FA-modules $F\,\mathrm{Hom}_{U_h}(L^i, U_h)$ are isomorphic to $(U_h)^{d_i}$ endowed with the h-adic filtration. On the other hand, the filtration of the $M^i := \mathrm{Hom}_{U_h}(L^i, U_h)$ induces a filtration on $\mathrm{Ext}^i_{U_h}(A_h, U_h)$ as follows:

$$F_s\mathrm{Ext}^i_{U_h}(A_h, U_h) := \frac{\mathrm{Ker}\,{}^t\partial_i \cap F_s M^i + \mathrm{Im}\,{}^t\partial_{i-1}}{\mathrm{Im}\,{}^t\partial_{i-1}} \simeq \frac{\mathrm{Ker}\,{}^t\partial_i \cap F_s M^i}{\mathrm{Im}\,{}^t\partial_{i-1} \cap F_s M^{i-1}}.$$

The filtration on the $\mathrm{Ext}^i_{U_h}(A_h, U_h)$-groups is nothing but the h-adic filtration. Reproducing the proof of [Ch2], one can see that:

- if $i \neq d$, then $\mathrm{Ext}^i_{U_h}(A_h, U_h) = \{0\}$;
- the maps ${}^t\partial_i$ are strict filtered morphisms;
- $\mathrm{Ext}^d_{U_h}(A_h, U_h)$ is complete for the h-adic filtration (as it is a finitely generated U_h^{op}-module, see [Ch2]). Moreover, $\mathrm{Ext}^d_{U_h}(A_h, U_h)/h\mathrm{Ext}^d_{U_h}(A_h, U_h) \simeq \mathrm{Ext}^d_U(A, U)$ as U^{op}-modules.

Let us show that $\mathrm{Ext}^d_{U_h}(A_h, U_h)$ is h-torsion free. Let $[\sigma_d] \in \mathrm{Ext}^d_{U_h}(A_h, U_h)$, where $\sigma_d \in \mathrm{Ker}\,{}^t\partial_d$, be an h-torsion element in $\mathrm{Ext}^d_{U_h}(A_h, U_h)$. There exists a minimal $n \in \mathbb{N}^*$ such that $h^n[\sigma_d] = 0$. Let $\sigma_{d-1} \in \mathrm{Hom}_{U_h}(L^{d-1}, U_h)$ be such that $h^n\sigma_d = {}^t\partial_{d-1}(\sigma_{d-1})$. Then, by reduction modulo h, one obtains $\overline{{}^t\partial_{d-1}}(\overline{\sigma_{d-1}}) = 0$ and there exists $\overline{\sigma_{d-2}}$ such that $\overline{\sigma_{d-1}} = \overline{\partial_{d-2}}\,(\overline{\sigma_{d-2}})$. Let σ_{d-2} be a lift of $\overline{\sigma_{d-2}}$. Then there exists τ_{d-1} such that

$$\sigma_{d-1} = {}^t\partial_{d-2}(\sigma_{d-2}) + h\tau_{d-1}.$$

Hence $h^n\sigma_d = h\,{}^t\partial_{d-1}(\tau_{d-1})$, which gives (using the fact that $\mathrm{Hom}_{U_h}(L^d, U_h)$ is topologically free) $h^{n-1}\sigma_d = {}^t\partial_{d-1}(\tau_{d-1})$. This contradicts the minimality of n so that $\mathrm{Ext}^d_{U_h}(A_h, U_h)$ is h-torsion free. As $\mathrm{Ext}^d_{U_h}(A_h, U_h)$ is complete for the h-adic topology and h-torsion free, it is topologically free. ☐

Combining this result with the more general structure theory as in Proposition 3.2.1, one obtains:

Proposition 6.3.3. *Let U satisfy the conditions of Theorem 6.3.1. Assume moreover that*

(*i*) *A is noetherian;*
(*ii*) $\mathrm{Ext}_U(A, U)$ *is a dualising module for (U, A),* i.e., *satisfies the hypothesis of Proposition 3.2.1;*
(*iii*) $_\blacktriangleright\mathrm{Ext}_U(A, U)$ *is a finitely generated projective A-module.*

Then $\mathcal{P}_h = \mathrm{Ext}^d_{U_h}(A_h, U_h)$ is a dualising module for (U_h, A_h) and produces an equivalence between the categories of left resp. right U_h-modules.

Remark 6.3.4. Let $M_h := M[[h]]$ and $N_h := N[[h]]$ be two A_h^{op}-modules which are topologically free with respect to the h-adic topology. Assume moreover that M_h is finitely generated projective over A_h^{op}; then $\mathrm{Hom}_{A_h^{\mathrm{op}}}(M_h, N_h)$ is topologically free and and is isomorphic to $\mathrm{Hom}_{A^{\mathrm{op}}}(M, N)[[h]]$ as a $k[[h]]$-module: observe that $\mathrm{Hom}_{A_h^{\mathrm{op}}}(M_h, N_h)$ is complete for the induced topology as it is a closed subset of the topologically free $k[[h]]$-module $\mathrm{Hom}_{k[[h]]}(M_h, N_h)$. On the other hand, on $\mathrm{Hom}_{A_h^{\mathrm{op}}}(M_h, N_h)$, the induced topology coincides with the h-adic topology. Hence $\mathrm{Hom}_{A_h^{\mathrm{op}}}(M_h, N_h)$ is complete for the h-adic topology and since it is also torsion free, it is topologically free. Let us now show that $\mathrm{Hom}_{A_h^{\mathrm{op}}}(M_h, N_h)/h\,\mathrm{Hom}_{A_h^{\mathrm{op}}}(M_h, N_h)$ is isomorphic to $\mathrm{Hom}_{A^{\mathrm{op}}}(M, N)$: in fact, there exists an A_h^{op}-module M_h' and a finitely generated free A_h^{op}-module F_h such that $M_h \oplus M_h' = F_h$. Any element ϕ of $\mathrm{Hom}_{A^{\mathrm{op}}}(M, N)$ can be extended to an element of $\mathrm{Hom}_{A^{\mathrm{op}}}(F_h/hF_h, N)$, which, in turn, can be lifted to an element of $\mathrm{Hom}_{A_h^{\mathrm{op}}}(F_h, N_h)$ and produces (by restriction) a lift of ϕ.

Proof of Proposition 6.3.3. The module $(\mathcal{P}_h)_\blacktriangleleft$ is a finitely generated A_h^{op}-module as $\mathcal{P}_\blacktriangleleft := \mathrm{Ext}_U(A, U)_\blacktriangleleft$ is a finitely generated A^{op}-module (see Proposition 3.0.5 of the preprint version of [Ch2]).

Let N be a finitely generated A_h^{op}-module. It can be considered as a filtered FA_h^{op}-module as follows: one has an epimorphism $A_h^n \xrightarrow{p} N \to 0$, and we endow N with the filtration $p(FA_h^n)$. As $\mathcal{P}_\blacktriangleleft$ is a projective A^{op}-module, $\mathcal{P}[h]_\blacktriangleleft$ is a projective $A[h]^{\mathrm{op}}$-module, and Proposition 3.0.11 of the preprint version of [Ch2] shows that $\mathrm{Ext}^i_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N) = \{0\}$ if $i > 0$.

Let now N be any A_h-module. We have $N = \varinjlim N'$, where N' runs over all finitely generated submodules of N. Let $F^\bullet$ be a resolution of $\mathcal{P}$ by finitely generated free A_h-modules. We have

$$\begin{aligned}\mathrm{Ext}^j_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N) &= \mathrm{Ext}^j_{A_h^{\mathrm{op}}}(\mathcal{P}_h, \varinjlim N') = H^j\big(\mathrm{Hom}_{A_h^{\mathrm{op}}}(F^\bullet, \varinjlim N')\big)\\ &= H^j\big(\varinjlim \mathrm{Hom}_{A_h^{\mathrm{op}}}(F^\bullet, N')\big) = \varinjlim H^j\big(\mathrm{Hom}_{A_h^{\mathrm{op}}}(F^\bullet, N')\big)\\ &= \varinjlim \mathrm{Ext}^j_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N') = \{0\},\end{aligned}$$

where we used the fact that the functor $\varinjlim$ is exact because the set of finitely generated submodules of M is a directed set, *cf.* [Ro, Prop. 5.33]. Thus we have proved that if N is any A_h-module, then

$$\mathrm{Ext}^j_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N) = \{0\} \quad \text{if } j > 0.$$

Consequently, $(\mathcal{P}_h)_\blacktriangleleft$ is a projective A_h-module; similarly, ${}_\blacktriangleright\mathrm{Ext}_{U_h}(A_h, U_h)$ is a projective A_h-module.

The assertion with respect to the evaluation map yet is true if N is a topologically free U_h-module as it is true modulo h, see Remark 6.3.4. Moreover, the functor $N \mapsto \mathrm{Hom}_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N) \otimes_{A_h} \mathcal{P}_h$ is exact as $\mathcal{P}_{h\blacktriangleleft}$ and ${}_\blacktriangleright\mathcal{P}_h$ are projective A_h^{op}- resp. A_h-modules.

Let now N be a finitely generated U_h-module. Using a finite free resolution of N, one can show (by a diagram chasing argument) that the evaluation map is an isomorphism (as it is an isomorphism for any component of the resolution). If N is any U_h-module instead, one can write $N = \varinjlim N'$, where N' runs over all finitely generated submodules of N. Since $\mathcal{P}_h$ is a finitely generated A_h^{op}-module, any element $\phi \in \mathrm{Hom}_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N)$ can be considered as an element of $\mathrm{Hom}_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N')$ for a well-chosen finitely generated A_h^{op}-module N'. Using the finitely generated case, one can see that the evaluation map is an isomorphism for any U_h-module N.

As $\mathcal{P}_h$ is a finitely generated projective A_h^{op}-module, the natural left U_h-module map

$$A_h \to \mathrm{Hom}_{A_h^{\mathrm{op}}}(\mathcal{P}_h, \mathcal{P}_h), \quad a \mapsto (p \mapsto a \blacktriangleright p)$$

of Proposition 3.2.1 is an isomorphism as it is an isomorphism modulo h. This concludes the proof. $\square$

Example 6.3.5. For example, $U := V^\ell(L)$ satisfies the conditions of Theorem 6.3.1 if A is the algebra of regular functions on a smooth affine variety X and L the vector fields over it. More generally, for any Lie-Rinehart algebra (A, L), where L is finitely generated projective of constant rank d over a Noetherian algebra A, the pair $\big(A, V^\ell(L)\big)$ fulfils the conditions of Theorem 6.3.1 and one obtains $\mathrm{Ext}^d_{V^\ell(L)}(A, V^\ell(L)) = \bigwedge_A^d \mathrm{Hom}_A(L, A)$ for the dualising module (see [Ch1, Hue] for more details in this direction). Under the same conditions, also the pair $\big(A, J^r(L)\big)$ fits into Theorem 6.3.1 with $\mathrm{Ext}^d_{J^r(L)}(A, J^r(L)) = \bigwedge_A^d L$. Then, for any quantisation $V^\ell(L)_h$ of $V^\ell(L)$ resp. $J^r(L)_h$ of $J^r(L)$, Proposition 6.3.3 leads to an equivalence of categories between left and right $V^\ell(L)_h$-modules and likewise for left and right $J^r(L)_h$-modules.

REFERENCES

[Be] J. Beck, *Distributive laws*, Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), Springer, Berlin, 1969, pp. 119–140.

[B] G. Böhm, *Hopf algebroids*, Handbook of algebra, Vol. 6, North-Holland, Amsterdam, 2009, pp. 173–236.

[BSz] G. Böhm and K. Szlachányi, *Hopf algebroids with bijective antipodes: axioms, integrals, and duals*, J. Algebra **274** (2004), no. 2, 708–750.

[Bo] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers, *Algebraic D-modules*, Perspectives in Mathematics, vol. 2, Academic Press, Inc., Boston, MA, 1987.

[Ca] P. Cartier, *Cohomologie des coalgèbres, exposés 4, 5*, Séminaire Sophus Lie, tome 2 (1955–1956), Facultè des Sciences de Paris, 1957.

[Ch1] S. Chemla, *Poincaré duality for k-A Lie superalgebras*, Bull. Soc. Math. France **122** (1994), no. 3, 371–397.

[Ch2] ———, *Duality properties for quantum groups*, Pacific J. Math. **252** (2011), no. 2, 313–341 (a more detailed version is available at `arXiv:0911.2860`).

[Ch3] ———, *Rigid dualizing complex for quantum enveloping algebras and algebras of generalized differential operators*, Journal of algebra **276** (2004), 80-102.

[ChGa] S. Chemla and F. Gavarini, *Duality functors for quantum groupoids*, preprint `arXiv:1211.3773` (2012), to appear in J. Noncomm. Geom.

[Hue] J. Huebschmann, *Duality for Lie-Rinehart algebras and the modular class*, J. Reine Angew. Math. **510** (1999), 103–159.

[KadSz] L. Kadison and K. Szlachányi, *Bialgebroid actions on depth two extensions and duality*, Adv. Math. **179** (2003), no. 1, 75–121.

[Ka] M. Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003.

[Ko] N. Kowalzig, *Hopf algebroids and their cyclic theory*, Ph. D. thesis, Universiteit Utrecht and Universiteit van Amsterdam, 2009.

[KoKr] N. Kowalzig and U. Krähmer, *Duality and products in algebraic (co)homology theories*, J. Algebra **323** (2010), no. 7, 2063–2081.

[KoP] N. Kowalzig and H. Posthuma, *The cyclic theory of Hopf algebroids*, J. Noncomm. Geom. **5** (2011), no. 3, 423–476.

[Ri] G. Rinehart, *Differential forms on general commutative algebras*, Trans. Amer. Math. Soc. **108** (1963), 195–222.

[Ro] J. Rotman, *An introduction to homological algebra*, second ed., Universitext, Springer, New York, 2009.

[Sch] P. Schauenburg, *Duals and doubles of quantum groupoids ($\times_R$-Hopf algebras)*, New trends in Hopf algebra theory (La Falda, 1999), Contemp. Math., vol. 267, Amer. Math. Soc., Providence, RI, 2000, pp. 273–299.

[Schn] J.-P. Schneiders, *An introduction to $\mathscr{D}$-modules*, Bull. Soc. Roy. Sci. Liège **63** (1994), no. 3-4, 223–295, Algebraic Analysis Meeting (Liège, 1993).

[Sw] M. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969.

[VdB] M. Van den Bergh, *A relation between Hochschild homology and cohomology for Gorenstein rings*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1345–1348, Erratum: Proc. Amer. Math. Soc. **130**, no. 9, 2809–2810 (electronic) (2002).

[Xu] P. Xu, *Quantum groupoids*, Comm. Math. Phys. **216** (2001), no. 3, 539–581.

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