Duality for Lie algebroids

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1 Introduction

The notion of Lie algebroids or Lie-Rinehart algebras constantly occurs in my research. It already occured in my Ph. D thesis (written under the direction of M. Duflo [1], [2], [3]) in the particular case of coinduced representations. During my stays at Harvard and at the university of Utrecht, I carried on my research in the direction of my Ph. D thesis ([4]). During these postdoctoral years, discussions with J. Kalkman, Y. Karshon, H. Duistermaat and S. Sternberg allowed me to learn the basis of symplectic geometry . During my stay in Utrecht, I had the idea of looking at induction of Lie algebras as a particular case of direct image. I was convinced that the duality property involving the direct image of $\mathcal{D}$-modules could be extended to the Lie algebroid setting without being able to achieve it. It was only when hired in Paris 6, with the help of P. Schapira, that I overcame the technical difficulties ([5], [7]). Then, benefiting from a visit of M. Kashiwara at Paris 6, I treated the case of the inverse image ([8]). Later, M. Duflo and T. Levasseur drew my attention on an article of A. Yekutieli having some link with my work. I got interested in the theory of rigid dualizing complexes developed by A. Yekutieli and M. van den Berg and I wrote [9].

Lie algebroids generalize at the same time finite dimensional Lie algebras and tangent bundles. Poisson manifolds and group actions provide examples of Lie algebroids. A Lie algebroid $\mathcal{L}_X$ gives rise to the sheaf of algebras of generalized differential operators generated by $\mathcal{O}_X$ and $\mathcal{L}_X$, $\mathcal{D}(\mathcal{L}_X)$. If $\mathcal{L}_X$ is the sheaf of vector fields over X, then $\mathcal{D}(\mathcal{L}_X)$ is the sheaf of differential operators over X. If X is a point, $\mathcal{L}_X$ is a Lie algebra and $\mathcal{D}(\mathcal{L}_X)$ is its enveloping algebra. Thus, $\mathcal{D}(\mathcal{L}_X)$ can be seen at the same time as a generalization of the enveloping algebra and a generalization of the sheaf of differential operators over X. I was interested in the second point of view and I extended a part of $\mathcal{D}$-modules theory (namely the operations) to $\mathcal{D}(\mathcal{L}_X)$-modules. This allowed me to prove duality properties for Lie algebroids and more particularly for Lie algebras.

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Four research teams hosted me and provided me a favourable and pleasant mathematical environment. In the chronological order : the research team of group theory, the department of mathematics of Harvard University, the department of mathematics of Utrecht University and the research team of algebraic analysis.

2 Lie algebroids

For sheaf theory we will follow the notaion of [K-S1].

2.1 Definitions

Let X be a $\mathcal{C}^\infty$, complex analytic or smooth complex algebraic manifold and let $\mathcal{O}_X$ be the sheaf of $\mathcal{C}^\infty$, holomorphic or regular functions over X. Let Θ_X be the $\mathcal{O}_X$-module of vector fields over X. Let $k = \mathbb{R}$ or $\mathbb{C}$ according to the setting.

Definition 2.1.1 *A Lie algebroid over X is a pair $(\mathcal{L}_X, \omega)$ where*

- *$\mathcal{L}_X$ is a locally free $\mathcal{O}_X$-module of finite constant rank*
- *$\mathcal{L}_X$ is a sheaf of k- Lie algebras*
- *$\omega :\ \mathcal{L}_X \to \Theta_X$ is an $\mathcal{O}_X$-linear morphism of sheaves of k- Lie algebras such that the following compatibility relation holds :*

$$\forall (\xi, \zeta) \in \mathcal{L}_X^2,\ \ \forall f \in \mathcal{O}_X,\ \ [\xi, f\zeta] = \omega(\xi)(f)\zeta + f[\xi, \zeta].$$

the morphism ω is called the anchor map. When there is no ambiguity, we will drop the anchor map in the notation of the Lie algebroid.

Instead of working in a sheaf setting, if we work in an algebraic setting, we get the notion of Lie-Rinehart algebra ([R]). Thus, if $(X, \mathcal{L}_X, \omega_X)$ is a Lie algebroid over X, then, for any open subset U of X, $(\mathcal{L}_X(U), \omega_X(U))$ is a $k - \mathcal{O}_X(U)$- Lie-Rinehart algebra.

A Lie algebroid $(\mathcal{L}_X, \omega)$ gives rise to the sheaf of generalized differential operators generated by $\mathcal{O}_X$ and $\mathcal{L}_X$ which is denoted by $\mathcal{D}(\mathcal{L}_X)$:

Definition 2.1.2 *$\mathcal{D}(\mathcal{L}_X)$ is the sheaf associated to the presheaf :*

$$U \mapsto T^+_{\mathbb{C}} \left(\mathcal{O}_X(U) \oplus \mathcal{L}_X(U)\right) / J_U$$

where J_U is the two-sided ideal generated by the relations

$$\forall (f, g) \in \mathcal{O}_X(U), \;\; \forall (\xi, \zeta) \in \mathcal{L}_X(U)^2$$

$$\begin{array}{l} 1) f \otimes g - fg \\ 2) f \otimes \xi - f\xi \\ 3) \xi \otimes \zeta - \zeta \otimes \xi - [\xi, \zeta] \\ 4) \xi \otimes f - f \otimes \xi - \omega(\xi)(f) \end{array}$$

The definition of $\mathcal{D}(\mathcal{L}_X)$ is similar to the one of the sheaf of differential operators over X, $\mathcal{D}_X$. This will allow us to extend $\mathcal{D}$-modules theory to the sheaf of generalized differential operators.

Denote by $\mathcal{D}(\mathcal{L}_X)^{op}$ the sheaf of opposite algebras so that a right $\mathcal{D}(\mathcal{L}_X)$-module is a (left) $\mathcal{D}(\mathcal{L}_X)^{op}$-module.

2.2 Lie algebroids morphisms

Definition 2.2.1 *Let $(\mathcal{L}_X, \omega_X)$ and $(\mathcal{L}_Y, \omega_Y)$ be Lie algebroids over complex manifolds X and Y respectively. A morphism Φ from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$ is a pair (f, F) such that*

- *$f \; : \; X \to Y$ is a morphism of $\mathcal{C}^\infty$, analytic or algebraic manifolds.*
- *$F \; : \; \mathcal{L}_X \to f^*\mathcal{L}_Y = \mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{L}_Y$ is a morphism of $\mathcal{O}_X$-modules such that the two following conditions are satisfied :*

1) The diagram

$$\begin{array}{ccc} \mathcal{L}_X & \xrightarrow{F} & f^*\mathcal{L}_Y \\ \omega_X \downarrow & & \downarrow f^*\omega_Y \\ \Theta_X & \xrightarrow[Tf]{} & f^*\Theta_Y \end{array}$$

commutes (Tf being the differential of f).

2) $\mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{D}(\mathcal{L}_Y)$ endowed with the two operations below is a left $\mathcal{D}(\mathcal{L}_X)$-module.

$$\forall (a,b) \in \mathcal{O}_X^2,\ \forall \xi \in \mathcal{L}_X,\ \forall v \in f^{-1}\mathcal{D}_Y$$

$$\begin{array}{l} a \cdot (b \otimes v) = ab \otimes v \\ \xi \cdot (b \otimes v) = \omega_X(\xi)(b) \otimes v + \sum_i ba_i \otimes \xi_i v \end{array}$$

(where $F(\xi) = \sum\limits_i a_i \otimes \xi_i$ with $a_i \in \mathcal{O}_X$ and $\xi_i \in f^{-1}\mathcal{L}_Y$).

Our definition ([5], [7]) coincides with Almeida et de Kumpera's one ([A-K]).

Notation :
The $\mathcal{D}(\mathcal{L}_X) \otimes_k f^{-1}\mathcal{D}(\mathcal{L}_Y)^{op}$-module $\mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{D}(\mathcal{L}_Y)$ (the right $f^{-1}\mathcal{D}(\mathcal{L}_Y)$-module structure is given by right multiplication) will be denoted $\mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_Y}$ (as for $\mathcal{D}$-modules [Bo], [7]).

The composition of two Lie algebroid morphisms is a Lie algebroid morphism.

2.3 Examples

1) The sheaf of generalized differential operators generalizes at the same time the sheaf of differential operators and the enveloping algebra of a Lie algebra. Lie algebroid morphisms generalize Lie algebra morphisms and morphisms of manifolds. The notions of relative manifolds and relative differential operators ([S1], [S-S]) fits the Lie algebroid setting.

2) Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Suppose that there exists a Lie algebra morphism $\sigma : \mathfrak{g} \to \Theta_X$. Then $\mathcal{O}_X \otimes \mathfrak{g}$ is endowed with a natural Lie algebroid structure with anchor map ω defined by:

$$\forall f \in \mathcal{O}_X,\ \forall \xi \in \mathfrak{g},\ \ \omega(f \otimes \xi) = f\sigma(\xi).$$

The Lie bracket on $\mathcal{O}_X \otimes \mathfrak{g}$ is given by

$$[f \otimes \xi, g \otimes \eta] = f\sigma(\xi)(g) \otimes \eta - g\sigma(\eta)(f) \otimes \xi + fg \otimes [\xi, \eta].$$

3) Let X be a Poisson manifold. Write $\{\ ,\ \}$ for the Poisson bracket over $\mathcal{O}_X$. The $\mathcal{O}_X$-module of differential forms of degree 1 , Ω^1_X, is endowed with a natural Lie algebroid structure ([Hu1]) with anchor map

$$\begin{array}{ccc} \Omega^1_X & \to & \Theta_X \\ f\mathrm{d}g & \mapsto & f\{g, \bullet\}. \end{array}$$

Let recall that the Lie bracket over Ω^1_X is given by

$$[f \otimes \mathrm{d}a, g \otimes \mathrm{d}b] = fg \otimes \mathrm{d}\{a, b\} + f\{a, g\} \otimes \mathrm{d}b - g\{b, f\} \otimes \mathrm{d}a.$$

The Lie algebroid (X, Ω^1_X) allows to express ([Hu1]) the canonical homology ([Br], [Ko], [Li]) and the canonical cohomology of Poisson manifolds as derived functors. Let Y be another Poisson manifold and let $f : X \to Y$ be a Poisson map. There is no reason why f should define a Lie algebroid morphism from (X, Ω^1_X) to (Y, Ω^1_Y). Nevertheless f defines a correspondence between (X, Ω^1_X) and (Y, Ω^1_Y) ([8]) .

Other examples of Lie algebroids can be found in [M], [7], [8].

2.4 Properties of $\mathcal{D}(\mathcal{L}_X)$

$\mathcal{D}(\mathcal{L}_X)$ is endowed with the filtration $(\mathcal{F}_n\mathcal{D}(\mathcal{L}_X))_{n\in\mathbb{N}}$ defined as follows :

$$\begin{array}{l} \mathcal{F}_0\mathcal{D}(\mathcal{L}_X) = \mathcal{O}_X \\ \mathcal{F}_n\mathcal{D}(\mathcal{L}_X) = \mathcal{F}_{n-1}\mathcal{D}(\mathcal{L}_X) \cdot \mathcal{L}_X + \mathcal{F}_{n-1}\mathcal{D}(\mathcal{L}_X) \end{array}$$

As we assume that $\mathcal{L}_X$ is a locally free $\mathcal{O}_X$-module, the sheaf of generalized differential operators satisfies the Poincaré - Birkhoff - Witt theorem ([R]) :

Theorem 2.4.1 *The $\mathcal{O}_X$-algebras $S_{\mathcal{O}_X}(\mathcal{L}_X)$ and $Gr\mathcal{F}\mathcal{D}(\mathcal{L}_X)$ are isomorphic.*

Consequently, $\mathcal{D}(\mathcal{L}_X)$ has the same homological properties as $\mathcal{D}_X$. The Poincaré-Birkhoff-Witt also allows to define the notion of characteristic variety and to reduce some proofs to the commutative case.

3 Operations for $\mathcal{D}(\mathcal{L}_X)$-modules

In this section, we generalize basic notions of $\mathcal{D}$-modules theory (due to Bernstein and Kashiwara) to $\mathcal{D}(\mathcal{L}_X)$-modules. We refer the reader to [Bj], [Bo], [Ho], [S2] and [Ka2] for an exposition. The result of this section can be found in [4], [5], [7], [8].

3.1 Left and right modules

The following proposition, well known for $\mathcal{D}$-modules, can easily be generalized to Lie algebroids ([5], [7]).

Proposition 3.1.1 *a) Let $\mathcal{N}$ and $\mathcal{N}'$ be two $\mathcal{D}(\mathcal{L}_X)$-left modules. Then $\mathcal{N} \underset{\mathcal{O}_X}{\otimes} \mathcal{N}'$ is a left $\mathcal{D}(\mathcal{L}_X)$-module with the two following operations :*

$$\begin{aligned}
&\forall a \in \mathcal{O}_X,\ \forall n \in \mathcal{N}, \forall n' \in \mathcal{N}', \forall D \in \mathcal{L}_X \\
&a \cdot (n \otimes n') = a \cdot n \otimes n' = n \otimes a \cdot n' \\
&D \cdot (n \otimes n') = D \cdot n \otimes n' + n \otimes D \cdot n'.
\end{aligned}$$

b) Let $\mathcal{M}$ (respectively $\mathcal{N}$) be a right (respectively left) $\mathcal{D}(\mathcal{L}_X)$-module. Then $\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{N}$ is a right $\mathcal{D}(\mathcal{L}_X)$-module with the following operations

$$\begin{aligned}
&\forall a \in \mathcal{O}_X,\ \forall m \in \mathcal{M}, \forall n \in \mathcal{N}, \forall D \in \mathcal{L}_X \\
&(m \otimes n) \cdot a = m \otimes a \cdot n = m \cdot a \otimes n \\
&(m \otimes n) \cdot D = m \cdot D \otimes n - m \otimes D \cdot n.
\end{aligned}$$

c) Let $\mathcal{M}$ and $\mathcal{M}'$ be two right $\mathcal{D}(\mathcal{L}_X)$-modules. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}')$ is a left $\mathcal{D}(\mathcal{L}_X)$-module with the following operations

$$\begin{aligned}
&\forall \phi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}'),\ \forall m \in \mathcal{M}, \forall a \in \mathcal{O}_X, \forall D \in \mathcal{L}_X \\
&(a \cdot \phi)(m) = \phi(m) \cdot a \\
&(D \cdot \phi)(m) = -\phi(m) \cdot D + \phi(m \cdot D).
\end{aligned}$$

d) If $\mathcal{N}$ and $\mathcal{N}'$ are two left $\mathcal{D}(\mathcal{L}_X)$-modules. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{N}')$ is a left $\mathcal{D}(\mathcal{L}_X)$-module with the following operations

$$\begin{aligned}
&\forall \phi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{N}'),\ \forall m \in \mathcal{M}, \forall a \in \mathcal{O}_X, \forall D \in \mathcal{L}_X \\
&(a \cdot \phi)(m) = a \cdot \phi(m) \\
&(D \cdot \phi)(m) = D \cdot \phi(m) - \phi(D \cdot m).
\end{aligned}$$

The following theorem is a consequence of the previous proposition.

Theorem 3.1.2 *Let $\mathcal{E}$ be a right $\mathcal{D}(\mathcal{L}_X)$-module which is a rank one locally free $\mathcal{O}_X$-module. The functor $\mathcal{N} \mapsto \mathcal{E} \underset{\mathcal{O}_X}{\otimes} \mathcal{N}$ establishes an equivalence of categories between left $\mathcal{D}(\mathcal{L}_X)$-modules and right $\mathcal{D}(\mathcal{L}_X)$-modules. Its inverse functor is given by $\mathcal{M} \mapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M})$.*

It is well known that $\Omega_X^{\dim X}$ (The sheaf of maximal degree differential forms) is endowed with a right $\mathcal{D}_X$-module structure (see [S2] p.9, [Bo] p. 226). Using the morphism $\mathcal{D}(\mathcal{L}_X) \to \mathcal{D}_X$, one endows $\Omega_X^{\dim X}$ with a right $\mathcal{D}(\mathcal{L}_X)$-module structure. Theorem 3.1.2 applies in particular if $\mathcal{E} = \Omega_X^{dimX}$. This construction can be extended to Lie algebroids. Set

$$\mathcal{L}_X^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_X, \mathcal{O}_X)$$

and denote by $d_{\mathcal{L}_X}$ the rank of $\mathcal{L}_X$. Using the Lie derivative in the Lie algebroid setting, one endows $\Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*)$ with a right $\mathcal{D}(\mathcal{L}_X)$-module ([4]).

Consider

$$\begin{aligned} \mathcal{H}_{\mathcal{L}_X} &= \mathcal{H}om_{\mathcal{O}_X}\left(\Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*), \mathcal{D}(\mathcal{L}_X)\right) = \mathcal{D}(\mathcal{L}_X) \underset{\mathcal{O}_X}{\otimes} \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X) \\ \mathcal{K}_{\mathcal{L}_X} &= \Omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}(\mathcal{L}_X). \end{aligned}$$

$\mathcal{H}_{\mathcal{L}_X}$ is endowed with a natural left $\mathcal{D}(\mathcal{L}_X) \otimes_k \mathcal{D}(\mathcal{L}_X)$-module structure (the first $\mathcal{D}(\mathcal{L}_X)$-module structure is given by left multiplication, the second one is obtained from right multiplication by proposition 3.1.1 c) . Similarly, $\mathcal{K}_{\mathcal{L}_X}$ is endowed with a natural $\mathcal{D}(\mathcal{L}_X)^{op} \otimes_k \mathcal{D}(\mathcal{L}_X)^{op}$ left module structure (the first right $\mathcal{D}(\mathcal{L}_X)$-module structure is given by right multiplication, the second one is obtained from left multiplication by proposition 3.1.1b).

3.2 Duality functor

Let Mod $(\mathcal{D}(\mathcal{L}_X))$ be the abelian category of (left) $\mathcal{D}(\mathcal{L}_X)$-modules and $D^b(\mathcal{D}(\mathcal{L}_X))$ be its bounded derived category. Denote by $D^b_{coh}(\mathcal{D}(\mathcal{L}_X))$ the full subcategory of $D^b(\mathcal{D}(\mathcal{L}_X))$ consisting of objects with coherent cohomology. If $\mathcal{N}^\bullet$ and $\mathcal{M}^\bullet$ are objects of $D^b_{coh}(\mathcal{D}(\mathcal{L}_X))$ and $D^b_{coh}(\mathcal{D}(\mathcal{L}_X)^{op})$ respectively, one puts

$$\begin{aligned} \underline{D}_{\mathcal{L}_X}(\mathcal{N}^\bullet) &= R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{N}^\bullet, \mathcal{H}_{\mathcal{L}_X})[d_{\mathcal{L}_X}] \\ \underline{\Delta}_{\mathcal{L}_X}(\mathcal{M}^\bullet) &= R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{M}^\bullet, \mathcal{K}_{\mathcal{L}_X})[dimX]. \end{aligned}$$

As the natural arrow $\mathcal{N}^\bullet \mapsto \underline{D}_{\mathcal{L}_X}\left(\underline{D}_{\mathcal{L}_X}\left(\mathcal{N}^\bullet\right)\right)$ is an isomorphism (see [7]), one says that $\underline{D}_{\mathcal{L}_X}$ is a duality functor. Similarly, $\underline{\Delta}_{\mathcal{L}_X}$ is a duality functor in $D^b\left(\mathcal{D}(\mathcal{L}_X)^{op}\right)$.

Proposition 3.2.1 *If $\mathcal{N}$ is a left $\mathcal{D}(\mathcal{L}_X)$-module which is free as an $\mathcal{O}_X$-module, then $\underline{D}_{\mathcal{L}_X}(\mathcal{N})$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{O}_X)$ are isomorphic in $D^b\left(\mathcal{D}(\mathcal{L}_X)\right)$.*

One can give a meaning to this proposition in the case where $\mathcal{N}$ is only $\mathcal{O}_X$-coherent ([8]). This proposition is well known for $\mathcal{D}$-modules (voir [Ho] p. 93).

3.3 Direct images

In this paragraph, we recall the results of [5] and [7].

Let $\Phi = (f, F)$ be a Lie algebroid morphism from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$. Let $\mathcal{M}^\bullet$ be an object of $D^b\left(\mathcal{D}(\mathcal{L}_X)^{op}\right)$. In [7], the direct image functor is defined by

$$\underline{\Phi}_!(\mathcal{M}^\bullet) = Rf_!\left(\mathcal{M}^\bullet \overset{L}{\underset{\mathcal{D}(\mathcal{L}_X)}{\otimes}} \mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_Y}\right).$$

Then $\underline{\Phi}_!(\mathcal{M}^\bullet)$ is in $D^b\left(\mathcal{D}(\mathcal{L}_Y)^{op}\right)$. If $\Phi = (f, Tf)$, we recover the $\mathcal{D}$-modules construction (see [S2] for example). In this case, $\mathcal{D}_{\Theta_X \to \Theta_Y}$ is denoted by $\mathcal{D}_{X \to Y}$ and $\underline{\Phi}_!$ is denoted by $\underline{f}_!$.

To define the direct image of an object of $D^b\left(\mathcal{D}(\mathcal{L}_Y)\right)$, as in the $\mathcal{D}$-modules case, one uses the $\left(f^{-1}\mathcal{D}(\mathcal{L}_Y) \otimes \mathcal{D}(\mathcal{L}_X)^{op}\right)$-bimodule $\mathcal{D}_{\mathcal{L}_Y \leftarrow \mathcal{L}_X}$ defined by

$$\mathcal{D}_{\mathcal{L}_Y \leftarrow \mathcal{L}_X} = \Lambda^{d_{\mathcal{L}_X}}\left(\mathcal{L}_X^*\right) \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_Y} \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\Lambda^{d_{\mathcal{L}_Y}}\left(\mathcal{L}_Y\right).$$

Proposition 3.3.1 *Let Φ and Ψ be two Lie algebroid morphisms from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$ and from $(\mathcal{L}_Y, \omega_Y)$ to $(\mathcal{L}_Z, \omega_Z)$ respectively, then*

$$\underline{\Psi}_! \circ \underline{\Phi}_! = \underline{(\Psi \circ \Phi)}_!\,.$$

The proof of proposition 3.3.1 is similar to the $\mathcal{D}$-modules case (see [Bo] p. 251).

Remark : The Kashiwara theorem

In the $\mathcal{D}$-modules case ([Bo]), the Kashiwara theorem states that, for a closed immersion $X \hookrightarrow Y$, one has an equivalence of categories between the category of $\mathcal{D}_Y$-modules with support in X and the category of $\mathcal{D}_X$-modules. Let $\Phi = (f, F)$ be a Lie algebroid morphism from $(X, \mathcal{L}_X)$ to $(Y, \mathcal{L}_Y)$. One says that Φ is a closed immersion if f is a closed immersion. The Kashiwara theorem is not true for any closed immersion of Lie algebroids (since it is wrong in the $\mathcal{O}_X$-modules setting). In [5], we give a sufficient condition so that a closed immersion between Lie algebroids satisfies the Kashiwara theorem. This condition was given by Levasseur ([Le]) in the particular case where X is a point.

In general, the direct image of a coherent module is not coherent. Introducing the notion of "good" $\mathcal{D}(\mathcal{L}_X)$-modules (due to Kashiwara [S-S]), we will provide a sufficient condition so that the direct image functor preserves coherence.

If X is algebraic, a coherent $\mathcal{D}(\mathcal{L}_X)$-module admits a global good filtration. In the analytic case, it is wrong. It is even wrong in the neighborhood of any compact subset. For a $\mathcal{D}(\mathcal{L}_X)$-module, the property of being "good" is a refinement of the property of having a good filtration in the neighborhood of any compact subset. As we already noticed it, if X is an algebraic manifold, all the coherent $\mathcal{D}(\mathcal{L}_X)$-modules are "good". Let us denote by $D^b_{good}\left(\mathcal{D}(\mathcal{L}_Y)^{op}\right)$ the full subcategory of $D^b\left(\mathcal{D}(\mathcal{L}_X)^{op}\right)$ consisting of objects with "good" cohomology.

Theorem 3.3.2 *Assume that $\mathcal{M}^\bullet$ is in $D^b_{good}\left(\mathcal{D}(\mathcal{L}_X)^{op}\right)$ and that f is proper on Supp$(\mathcal{M})$, then $\underline{\Phi}_!(\mathcal{M})$ is in $D^b_{good}\left(\mathcal{D}(\mathcal{L}_Y)^{op}\right)$.*

The proof of Schneiders ([S2] p. 38) in the $\mathcal{D}$-modules case extends without any change to our situation. The particular case where f is projective and $\mathcal{M}$ has a global good filtration was treated in [Ka1].

3.4 Inverse image ([8])

Let $\Phi = (f, F)$ be a Lie algebroid morphism from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$. Let $\mathcal{R}^\bullet$ be an object of $D^b\left(\mathcal{D}(\mathcal{L}_Y)\right)$. Set

$$\underline{\Phi}^{-1}(\mathcal{R}^\bullet) = \mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_Y} \overset{L}{\underset{f^{-1}\mathcal{D}(\mathcal{L}_Y)}{\otimes}} f^{-1}\mathcal{R}^\bullet.$$

Then $\underline{\Phi}^{-1}(\mathcal{R}^\bullet)$ is in $D^b\left(\mathcal{D}(\mathcal{L}_X)\right)$. We will call it the inverse image of $\mathcal{R}^\bullet$ by Φ. If $\Phi = (f, Tf)$, we recover the $\mathcal{D}$-modules construction (see [S2] for example). The inverse image functor behaves well with respect to composition of morphisms.

The following question arises naturally : Let $\mathcal{R}^\bullet$ be an object of $D^b_{coh}(\mathcal{D}(\mathcal{L}_Y))$. Give a sufficient condition so that $\underline{\Phi}^{-1}(\mathcal{R}^\bullet)$ is in $D^b_{coh}(\mathcal{D}(\mathcal{L}_X))$. For that purpose, we introduce, as in the $\mathcal{D}$-modules case, the notion of non caracteristicity.

Let L_X (respectively L_Y) be the fiber bundle associated to $\mathcal{L}_X$ (respectively $\mathcal{L}_Y$). We have the following diagram :

$$L_X^* \xleftarrow{{}^tF} X \times_Y L_Y^* \xrightarrow{F_\pi} L_Y^*$$

where, for $x \in X$ and $\lambda \in L^*_{f(x)}$, one has

$$\begin{aligned} {}^tF\,(x, f(x), \lambda) &= (x, {}^tF(\lambda)) \\ F_\pi\,(x, f(x), \lambda) &= (f(x), \lambda)\,. \end{aligned}$$

Denote by $NS(X \times_Y L_Y^*)$ the zero section of $X \times_Y L_Y^*$. Let $\mathcal{R}^\bullet$ be an object of $D^b_{coh}(\mathcal{D}(\mathcal{L}_Y))$. Denote by $Char(\mathcal{R})$ its characteristic variety. One has $Char(\mathcal{R}) \subset L_Y^*$. We will say that $\mathcal{R}^\bullet$ is non caracteristic with respect to Φ if the following inclusion holds

$$F_\pi^{-1}\,(char(\mathcal{R}^\bullet)) \bigcap \{(x, f(x), \lambda) \in X \times_Y L_Y^* \mid \lambda \circ F_x = 0\} \subset NS(X \times_Y L_Y^*).$$

Kashiwara has shown, in the $\mathcal{D}$-modules case, that the non characteristicity notion ensures that $\underline{\Phi}^{-1}(\mathcal{R}^\bullet)$ is in $D^b_{coh}(\mathcal{D}(\mathcal{L}_X))$ ([S2]). We have generalized this result to Lie algebroids. Our proof consists in working with filtered $\mathcal{D}(\mathcal{L}_X)$-modules and reduce to the commutative case.

4 The duality theorems

4.1 Duality theorem for direct image ([7])

Theorem 4.1.1 *Let X and Y be two complex manifolds. Let $(\mathcal{L}_X, \omega_X)$ and $(\mathcal{L}_Y, \omega_Y)$ be Lie algebroids over X and Y respectively. Let $\Phi = (f, F)$ be a Lie algebroid morphism from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$. Let $\mathcal{M}^\bullet$ be an object of $D^b_{good}(\mathcal{D}(\mathcal{L}_X)^{op})$ such that f is proper on the support of $\mathcal{M}^\bullet$. Then there exists a functorial isomorphism from $\underline{\Phi_!}\Delta_{\mathcal{L}_X}(\mathcal{M}^\bullet)$ to $\Delta_{\mathcal{L}_Y}\underline{\Phi_!}(\mathcal{M}^\bullet)$ in $D^b_{good}(\mathcal{D}(\mathcal{L}_Y)^{op})$.*

Theorem 4.1.1 generalizes a result of Schneiders' thesis [S1] (see also [S-S]) where the case of relative differential operators is treated. The smooth algebraic case had been previously treated by Bernstein (in the $\mathcal{D}$-modules case [Be], [Bo], [Ho]) for a proper map. Moreover Mebkhout had treated the absolute case (i.e Y consists in one point [Me1], [Me2]). If $\mathcal{L}_X = \mathcal{L}_Y = \{0\}$, we recover the Ramis-Ruget-Verdier duality in the case of analytic manifolds.

Corollary 4.1.2 *Let X be an analytic complex compact manifold of complex dimension x and let $(\mathcal{L}_X, \omega_X)$ be a Lie algebroid of rank $d_{\mathcal{L}_X}$ over X. Let $\mathcal{N}$ be a left $\mathcal{D}(\mathcal{L}_X)$-module which is a finite rank locally free $\mathcal{O}_X$-module. Then, for any i in $\mathbb{Z}$,* $\mathrm{Ext}^i_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X, \mathcal{N})$ *is of finite dimension and*

$$\mathrm{Ext}^{d_{\mathcal{L}_X}+x-i}_{\mathcal{D}(\mathcal{L}_X)}\left(\mathcal{O}_X, \mathcal{N}^* \underset{\mathcal{O}_X}{\otimes} \mathcal{H}om_{\mathcal{O}_X}\left(\Lambda^{d_X}\mathcal{L}^*_X, \Omega_X\right)\right) \simeq \mathrm{Ext}^i_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X, \mathcal{N})^* .$$

Remarks :

1) If X is a point, we recover the Poincaré duality for finite dimensional Lie algebras.
2) If $\mathcal{L}_X = 0$, we recover Serre duality.
3) The $\mathcal{C}^\infty$ case has been studied in [E-L-W].
4) Corollary 4.1.2 was conjectured independently by Huebschmann in [Hu2].

4.2 The duality theorem for inverse image ([8])

Theorem 4.2.1 *Let Φ be a Lie algebroid morphism from $(X, \mathcal{L}_X)$ to $(Y, \mathcal{L}_Y)$. Let $\mathcal{R}^\bullet$ be an object of $D^b_{coh}(\mathcal{D}(\mathcal{L}_Y))$ which is supposed to be non characteristic with respect to Φ. There exists a functorial isomorphism from $\underline{D}_{\mathcal{L}_X}\underline{\Phi}^{-1}(\mathcal{R}^\bullet)$ to $\underline{\Phi}^{-1}\underline{D}_{\mathcal{L}_Y}(\mathcal{R}^\bullet)$.*

This theorem generalizes a duality theorem due to Kashiwara, Kawai and Sato in the $\mathcal{D}$-modules case ([SKK], [Ka2]). Nevertheless, the proof of Kashiwara-Kawai-Sato does not extend to Lie algebroids and, even in the $\mathcal{D}$-modules case, my proof is different from Kashiwara-Kawai-Sato's one. My proof consists in working with filtered modules in order to reduce to the commutative case.

Combining theorems 4.1.1 and 4.2.1, we obtain adjunction formulas ([8]) which generalize those existing in the $\mathcal{D}$-modules case ([K-S 2] , chapitre 6).

5 A duality property in coinduced representations of Lie superalgebras

In this section, we will denote by k a commutative field of characteristic zero.

A superspace does not admit a maximal wedge but the notion of Berezinian superspace is a generalization to the supercase of the maximal wedge ([Ma] page 172). If M is a finite dimensional k-superspace, we will write $Ber(M)$ for its Berezinian superspace.

Let $\mathfrak{g}$ be a k-Lie superalgebra, $\mathfrak{h}$ a Lie sub-superalgebra of $\mathfrak{g}$ and (π, V) a representation of $\mathfrak{h}$ in a superspace V. One defines the superspace coinduced from π, $Coind_{\mathfrak{h}}^{\mathfrak{g}}(\pi)$, by

$$Coind_{\mathfrak{h}}^{\mathfrak{g}}(\pi) = Hom_{U(\mathfrak{h})}\left(U(\mathfrak{g}), V\right).$$

$\mathfrak{g}$ acts on $Coind_{\mathfrak{h}}^{\mathfrak{g}}(\pi)$ by the transpose of right multiplication.

Denote by $\check{}$ the antiautomorphism of $U(\mathfrak{g})$ defined as follows. If X is in $\mathfrak{g}$, one has $\check{X} = -X$ and if u and v are two homogeneous elements of $U(\mathfrak{g})$, one has $(uv)\check{} = (-1)^{|v||u|}\check{v}\check{u}$. Let I_π ($I_\pi \subset U(\mathfrak{g})$) be the kernel of the representation coinduced from (π, V). The contragredient representation of π will be denoted by π^*. From now on, we will assume that $\mathfrak{h}$ is of finite codimension. The Lie superalgebra $\mathfrak{h}$ acts naturally on the superspace $Ber\left((\mathfrak{g}/\mathfrak{h})^*\right)$ by the character $-strad_{\mathfrak{g}/\mathfrak{h}}$. In [1], [2] et [3], we give two proofs of the following theorem.

Theorem 5.0.1:
If $\mathfrak{h}$ is of finite codimension, one has the relation

$$\check{I}_\pi = I_{\pi^* \otimes Ber((\mathfrak{g}/\mathfrak{h})^*)}.$$

Theorem 5.0.1 was proved by M. Duflo [D1] in the case of a finite dimensional Lie algebra. The proof of [D1] does not extend to the case where only $\mathfrak{g}/\mathfrak{h}$ is finite dimensional.

Let us describe briefly our two proofs. Both of them are linked to $\mathcal{D}$-modules theory and its extension to Lie-Rinehart algebras.

First proof :

Write $\mathcal{D}_\pi$ for the superalgebra of differential operators over $Coind_{\mathfrak{h}}^{\mathfrak{g}}(\pi)$. Let X be an element of $\mathfrak{g}$. Its action on $Coind_{\mathfrak{h}}^{\mathfrak{g}}(\pi)$ defines a differential operator of $Coind_{\mathfrak{h}}^{\mathfrak{g}}(\pi)$ of degree inferior or equal to 1, which will be denoted by $D_\pi(X)$. We endow $Coind_{\mathfrak{h}}^{\mathfrak{g}}(\pi^* \otimes Ber((\mathfrak{g}/\mathfrak{h})^*))$ with a right $\mathcal{D}_\pi$-module structure such that : for any X in $\mathfrak{g}$ and any λ in $Coind_{\mathfrak{h}}^{\mathfrak{g}}(\pi^* \otimes Ber((\mathfrak{g}/\mathfrak{h})^*))$, the following relation holds

$$\lambda \cdot D_\pi(X) = -(-1)^{|\lambda||X|} X \cdot \lambda.$$

From this, we deduce the inclusion $\check{I}_\pi \subset I_{\pi^* \otimes Ber((\mathfrak{g}/\mathfrak{h})^*)}$. Applying the same relation to the representation $\pi^* \otimes Ber((\mathfrak{g}/\mathfrak{h})^*)$, we get the equality.

Second proof :

The second proof is a consequence of the realization of the induced representation of a Lie superalgebra in terms of Grothendieck local cohomology ([2]). This realization is established for $\pi = 0$ and $\mathfrak{g}$ a finite dimensional Lie algebra in [B-B (théorème 3.5)] . It is also partly reproved in [Le (théorème 6.2)] as a corollary of a criterium of induction.

6 Poincaré duality

6.1 Statement

In this section, k will be a field of characteristic zero.

Let $\mathfrak{g}$ be a finite dimensional k-Lie superalgebra. We will consider $Ber(\mathfrak{g}^*)$ as a right $U(\mathfrak{g})$-module as we have done for $\Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*)$ (in paragraph 3.1). If M is a left $U(\mathfrak{g})$-module, we will consider $Ber(\mathfrak{g}^*) \otimes M$ as a right $U(\mathfrak{g})$-module in the following way :

$$\forall m \in M,\ \forall X \in \mathfrak{g},\ \forall \omega \in Ber(\mathfrak{g}^*),$$
$$(\omega \otimes m) \cdot X = -(-1)^{|X||m|} \omega \otimes X \cdot m + (-1)^{|X||m|} strad(X) \omega \otimes m.$$

Theorem 6.1.1 *Denote by d_0 the dimension of the even part of $\mathfrak{g}$. Let M be a left $U(\mathfrak{g})$-module and let i be in $\mathbb{Z}$. There exists a superspace morphism, $\Psi_i(M)$, from $Tor_{d_0-i}^{U(\mathfrak{g})}(Ber(\mathfrak{g}^*) \otimes M, k)$ to $Ext^i_{U(\mathfrak{g})}(k, M)$ which is an isomorphism if M is of finite projective dimension.*

Remarks

1) The last assertion can not be extended to any complex of $D^-(U(\mathfrak{g}))$. One can construct easily a counterxample by taking $\mathfrak{g}$ to be the completely odd Lie superalgebra Πk and M to be trivial module k.

2) In [4], we prove this theorem in the more general setting of Lie-Rinehart superalgebras.

6.2 Duality properties for induced representations of Lie superalgebras

From Poincaré duality, we deduce ([4]) duality theorems for induced representations of Lie superalgebras.

Theorem 6.2.1 *Let $\mathfrak{g}$ be a k-Lie superalgebra. Let $\mathfrak{h}$ and $\mathfrak{t}$ be two finite dimensional Lie sub-superalgebras of $\mathfrak{g}$. Put $h_0 = dim\mathfrak{h}_{\bar{0}}$ and $s_0 = dim\mathfrak{t}_{\bar{0}}$. Let V (respectively W) be a finite dimensional $\mathfrak{h}$-module (respectively $\mathfrak{t}$-module). Then, for any n in $\mathbb{Z}$, we have*

$$Ext^{n-s_0}_{U(\mathfrak{g})}\left(U(\mathfrak{g})\underset{U(\mathfrak{h})}{\otimes}V\ ,\ U(\mathfrak{g})\underset{U(\mathfrak{t})}{\otimes}W\right)\simeq$$
$$Ext^{n-h_0}_{U(\mathfrak{g})}\left((ber(\mathfrak{t}^*)\otimes W^*)\underset{U(\mathfrak{t})}{\otimes}U(\mathfrak{g})\ ,\ (ber(\mathfrak{h}^*)\otimes V^*)\underset{U(\mathfrak{h})}{\otimes}U(\mathfrak{g})\right).$$

Remarks :

1) As corollary of theorem 4.1.1, we obtain a version of theorem 6.2.1 in the Lie algebroid setting ([7]). Nevertheless, in the case of a Lie algebra $\mathfrak{g}$, this second proof allows us to get rid of the hypothesis on the finiteness of the dimension of $\mathfrak{g}$.

2) Generalizing a result of G. Zuckerman ([B-C]), A. Gyoja ([G]) proved this theorem in the following particular case : $\mathfrak{g}$ is split semi-simple, $\mathfrak{h} = \mathfrak{t}$ is a parabolic subalgebra of $\mathfrak{g}$, $n = h_0 = n_0$. D. H. Collingwood and B. Shelton proved also a duality of this type in a slightly different context but still in the semi-simple setting ([C-S]). M. Duflo ([D2]) had obtained this duality property in the following case : $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h} = \mathfrak{t}$, $V^* = W = k_\lambda$ is a one dimensional representation with character λ. My proof is inspired by M. Duflo's one.

In the case where $\mathfrak{t} = \{0\}$ et $W = \{0\}$, $n = h_0$, we can improve the result :

Theorem 6.2.2 *Let $\mathfrak{g}$ be a k-Lie superalgebra and $\mathfrak{h}$ be a finite dimensional Lie subsuperalgebra of $\mathfrak{g}$. Put $h_0 = dim\mathfrak{h}_{\bar{0}}$. Let V be a finite dimen-*

sional $U(\mathfrak{h})$*-module. The right* $U(\mathfrak{g})$*-modules* $Ext^{h_0}\left(U(\mathfrak{g}) \underset{U(\mathfrak{h})}{\otimes} V\ ,\ U(\mathfrak{g})\right)$ *and* $(ber(\mathfrak{h}^*) \otimes V^*) \underset{U(\mathfrak{h})}{\otimes} U(\mathfrak{g})$ *are isomorphic.*

Remarks :

1) This theorem follows also easily from proposition 3.2.1.

2) This result was proved by Brown and Levasseur [B-L p. 410] and by Kempf [Ke] in the case where $\mathfrak{g}$ is a finite dimensional semi-simple Lie algebra and $U(\mathfrak{g}) \underset{U(\mathfrak{h})}{\otimes} V$ is a Verma module.

7 Computations of some rigid dualizing complexes

7.1 Dualizing complexes

Grothendieck duality involves dualizing complexes. The extension of the definition of dualizing complexes to the non commutative setting is due to Yekutieli ([Y1]). Let k be a commutative field. If A is a k-algebra, we put $A^e = A \otimes_k A^{op}$. Let $D(A^e)$ (respectively $D^b(A^e)$) be the derived category (respectively the bounded derived category) of the category of (left) A^e-modules.

Definition 7.1.1 *Assume that* A *is a left and right noetherian* k*-algebra. An object* R *of* $D^b(A^e)$ *is called a dualizing complex if it satisfies the following conditions.*

a) R *is of finite injective dimension over* A *and* A^{op}*.*

b) The cohomology of R *is given by bimodules which are finitely generated on both sides.*

c) The natural morphisms $\Phi : A \to RHom_A(R, R)$ *and* $\Psi : A \to RHom_{A^{op}}(R, R)$ *are isomorphisms in* $D^b(A^e)$*.*

Remarks : ([Y1] and [Y3])

1) If R is a dualizing complex, then $RHom_A(-, R)$ defines a duality between the full subcategories of $D^b(A)$ and $D^b(A^{op})$, $D^b_f(A)$ and $D^b_f(A^{op})$, consisting of complexes with finitely generated cohomology.

2) A dualizing complex is only determined up to derived tensor product by a tilting complex ([Y3] theorem 4.5). This leads to the following notion introduced by M. Van den Bergh [VdB].

Definition 7.1.2 *Let A be a left and right noetherian k-algebra. A dualizing complex R is rigid if*

$$R \simeq RHom_{A^e}\left(A,_A R \otimes R_A\right)$$

in $D(A^e)$. The notations $_AR$ and R_A means that $RHom$ is taken over the left A-module structure and the right A-module structure of R respectively.

Remarks :

1) The rigid dualizing complex, if it exists, is unique up to unique isomorphism in $D^b(A^e)$ ([VdB1], [Y3] theorem 5.2).

2) The rigid dualizing complex generalizes the Grothendieck dualizing complex.

3) Recently, the notions of dualizing complex and of rigid dualizing complex as well as their basic properties were extended to non commutative ringed spaces ([Y-Z]).

We have computed the rigid dualizing complexes of an algebra of generalized differential operators and of a quantum enveloping algebra.

7.2 Rigid dualizing complex of an algebra of generalized differential operators ([9])

Let X be an affine algebraic manifold and let $\mathcal{O}_X$ be the sheaf of regular functions over X. Let Θ_X be the $\mathcal{O}_X$-module of regular vector fields over X. We put $L_X = \mathcal{L}_X(X)$ and $D(L_X) = \mathcal{D}(\mathcal{L}_X)(X)$.

Using proposition 3.1.1 c), we endow $\mu_{\mathcal{L}_X} = \mathcal{H}om_{\mathcal{O}_X}\left(\Lambda^{d_{\mathcal{L}_X}}\mathcal{L}_X^*, \Omega_X\right)$ with a left $\mathcal{D}(\mathcal{L}_X)$-module structure. Thus $\mathcal{D}(\mathcal{L}_X)\underset{\mathcal{O}_X}{\otimes}\mu_{\mathcal{L}_X}$ is endowed with a $\mathcal{D}(\mathcal{L}_X)\otimes\mathcal{D}(\mathcal{L}_X)^{op}$ -module structure determined by : for any $P, Q \in \mathcal{D}(\mathcal{L}_X)$, any $D \in \mathcal{L}_X$ and any $a \in \mathcal{O}_X$,

$$\begin{aligned}
&Q \cdot (P \otimes \mu) = QP \otimes \mu \\
&(P \otimes \mu) \cdot D = PD \otimes \mu - P \otimes D \cdot \mu \\
&(P \otimes \mu) \cdot a = Pa \otimes \mu.
\end{aligned}$$

Applying theorem 4.1.1, we compute the rigid dualizing complex of the algebra $D(L_X)$.

Theorem 7.2.1 *Let $(X, \mathcal{L}_X)$ be a Lie algebroid over an affine algebraic manifold. Put $x = dimX$, $d_{\mathcal{L}_X} = rank(\mathcal{L}_X)$, $G_X = \mathcal{O}_X(X)$, $L_X = \mathcal{L}_X(X)$,*

$\omega_X = \Omega_X(X)$, $D(L_X) = \mathcal{D}(\mathcal{L}_X)(X)$. The rigid dualizing complex of $D(L_X)$ is

$$R_{L_X} = D(L_X) \underset{G_X}{\otimes} Hom_{G_X}\left(\Lambda^{d_{L_X}} L_X^*, \omega_X\right)[x + d_{L_X}].$$

This theorem was proved by Yekutieli in the case of the enveloping algebra of a finite dimensional Lie algebra and in the case of the algebra differential operators over X. Our proof is analogous to the one of [Y4] in the case of $\mathcal{D}_X$.

7.3 Rigid dualizing complex of a quantum enveloping algebra ([9])

For basic definitions and results on quantum enveloping algebras, we refer the reader to [C-P].

Let $\mathfrak{g}$ be a finite dimensional complex semi-simple Lie algebra and let $A = (a_{i,j})_{(i,j)\in[1,n]^2}$ be its Cartan matrix. The matrix A is not always symmetric but is always symmetrizable. This implies that there exist coprime positive integers $d_1, \ldots, d_n$ such that $(d_i a_{i,j})$ is symmetric positive definite. The d_i's are uniquely determined. Let us introduce the following subset of $\mathbb{C}$:

$$\mathbb{C}_{\mathfrak{g}} = \{\epsilon \in \mathbb{C}^* \mid \epsilon^{2d_i} \neq 1, \ \ \forall i \in [1,n]\}.$$

Let q be an indeterminate. Following Jimbo, we consider the $\mathbb{C}(q)$- quantum algebra $U_q(\mathfrak{g})$ ([C-P] p. 280) as well as its non restricted specialization $U_\epsilon(\mathfrak{g})$ for ϵ in $\mathbb{C}_{\mathfrak{g}}$ ([C-P] p. 289). Using a filtration introduced by De Concini and Kac ([deC-K]) on $U_q(\mathfrak{g})$ and $U_\epsilon(\mathfrak{g})$ (for ϵ in $\mathbb{C}_{\mathfrak{g}}$), we proved :

Proposition 7.3.1 *Let $\mathfrak{g}$ be a finite dimensional complex semi-simple Lie algebra. We have the following isomorphisms :*

$$Ext^i_{U_q(\mathfrak{g})}(\mathbb{C}(q), U_q(\mathfrak{g})) = 0 \quad \text{for} \quad i \neq dim\mathfrak{g}$$
$$Ext^{dim\mathfrak{g}}_{U_q(\mathfrak{g})}(\mathbb{C}(q), U_q(\mathfrak{g})) = \mathbb{C}(q).$$

Endow $\mathbb{C}(q)$ with the trivial representation and $Ext^{dim\mathfrak{g}}_{U_q(\mathfrak{g})}(\mathbb{C}(q), U_q(\mathfrak{g}))$ with right multiplication. The last isomorphism is a right $U_q(\mathfrak{g})$-modules isomorphism. If ϵ is in $\mathbb{C}_{\mathfrak{g}}$, the proposition is still true if we remplace $U_q(\mathfrak{g})$ by $U_\epsilon(\mathfrak{g})$ and $\mathbb{C}(q)$ by $\mathbb{C}$.

As corollaries of proposition 7.3.1, we get duality properties analogous to 6.1.1, 6.2.1, 6.2.2 in the quantum groups setting. Proposition 7.3.1 also allowed us to compute the rigid dualizing complexes of $U_q(\mathfrak{g})$ (for generic q) and $U_\epsilon(\mathfrak{g})$ (ϵ in $\mathbb{C}_{\mathfrak{g}}$).

Theorem 7.3.2 *Let $\mathfrak{g}$ be a finite dimensional complex semi-simple Lie algebra. Assume that ϵ is in $\mathbb{C}_{\mathfrak{g}}$. The rigid dualizing complexes of $U_q(\mathfrak{g})$ and $U_\epsilon(\mathfrak{g})$ are $U_q(\mathfrak{g})[dim\mathfrak{g}]$ and $U_\epsilon(\mathfrak{g})[dim\mathfrak{g}]$ respectively.*

Theorem 7.3.2 answers a Yekutieli's question ([Y4]).

8 Extremal equations in the semi-classical case ([6])

Let $\mathfrak{g}$ be a finite dimensional complex semi-simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, Δ the root system associated to $\mathfrak{h}$ and $B = (\alpha_1, \ldots, \alpha_n)$ a simple root system of Δ. We will denote by Δ^+ the set of positive roots, W the Weyl group and s_i the symmetry with respect to the simple root α_i.

If w is in W and if $w = s_{i_1} \ldots s_{i_j}$ is a reduced expression of w, then the roots $(\gamma_1, \ldots, \gamma_j) = \left(\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k}), \ldots, s_{i_1} \ldots s_{i_{j-1}}(\alpha_{i_j})\right)$ are pairwise distinct and

$$\Delta_w = \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) < 0\} = \{\gamma_1, \ldots, \gamma_j\}.$$

If γ is a positive root, $\mathfrak{g}_\gamma$ will be the root space associated to the root γ. Let h_γ be the unique element of $[\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}]$ such that $\gamma(h_\gamma) = 2$. If e_γ is in $\mathfrak{g}_\gamma$, there exists a unique element $e_{-\gamma}$ in $\mathfrak{g}_{-\gamma}$ such that $(h_\gamma, e_\gamma, e_{-\gamma})$ is a sl_2-triple. We put

$$\mathfrak{n} = \underset{\gamma \in \Delta^+}{\oplus} \mathfrak{g}_\gamma, \quad \mathfrak{n}_w = \underset{\gamma \in \Delta_w}{\oplus} \mathfrak{g}_\gamma.$$

Let $R(\mathfrak{h})$ be the field of rational functions on $\mathfrak{h}^*$. Introduce the algebra $U'(\mathfrak{g}) = U(\mathfrak{g}) \underset{S(\mathfrak{h})}{\otimes} R(\mathfrak{h})$ and consider the generic Verma module $V = \dfrac{U'(\mathfrak{g})}{U'(\mathfrak{g})\mathfrak{n}}$. In [Z], Zhelobenko gave an explicit description for $V^{\mathfrak{n}_w}$. We have established similar results for the symmetric algebra.

Consider the complex analytic manifold $(\mathfrak{g}/\mathfrak{n})^*$. We endow it with the following coordinate system $\left((e_{-\alpha})_{\alpha \in \Delta_+}, (h_{\alpha_i})_{i \in [1,n]}\right)$. If U is an open subset of $(\mathfrak{g}/\mathfrak{n})^*$, we will denote by $\mathcal{P}(U)$ (respectively $\mathcal{A}(U)$) the set of regular (respectively analytic) functions on U and by $\mathcal{P}(U)^{\mathfrak{n}_w}$ (respectively $\mathcal{A}(U)^{\mathfrak{n}_w}$) the subset of functions of $\mathcal{P}(U)$ (respectively $\mathcal{A}(U)$) invariant under the action $\mathfrak{n}_w$.

Let U_γ be the open subset of $(\mathfrak{g}/\mathfrak{n})^*$ defined by the equation $h_\gamma \neq 0$. Denote by Φ_γ the map of U_γ into itself defined by :

$$\forall \lambda \in U_\gamma, \quad \Phi_\gamma(\lambda) = exp\left(\frac{e_\gamma(\lambda)}{h_\gamma(\lambda)} e_\gamma\right) \cdot \lambda.$$

where $\cdot$ is the natural action of $\mathfrak{n}$ on $(\mathfrak{g}/\mathfrak{n})^*$. The composition with Φ_γ defines an algebra morphism of $\mathcal{A}(U_\gamma)$ which will be denoted by π_γ.

Theorem 8.0.1:
Let w be an element of W. Put $\Delta_w = (\gamma_1, \ldots, \gamma_j)$ and $U_w = U_{\gamma_1} \bigcap \ldots \bigcap U_{\gamma_j}$. The algebra morphism $\pi_w = \pi_{\gamma_1} \circ \ldots \circ \pi_{\gamma_j}$ does not depend on the reduced expression of w. It establishes an isomorphism between

$$\mathcal{C}_w = \{f \in \mathcal{A}(U_w) \mid \frac{\partial f}{\partial e_{-\gamma_1}} = \ldots = \frac{\partial f}{\partial e_{-\gamma_j}} = 0\}$$

and $\mathcal{A}(U_w)^{\mathfrak{n}_w}$. Moreover π_w sends $\mathcal{C}_w \bigcap \mathcal{P}(U_w)$ onto $\mathcal{P}(U_w)^{\mathfrak{n}_w}$.

Let N_w be the connected and simply connected Lie group with Lie algebra $\mathfrak{n}_w$. My proof relies on the following proposition :

Proposition 8.0.2 :
Let λ be in U_w. The point $\Phi_{\gamma_j} \ldots \Phi_{\gamma_1}(\lambda)$ is the unique point of the orbit $N_w \cdot \lambda$ whose coordinates $e_{-\gamma_1}, \ldots, e_{-\gamma_j}$ vanish.

9 Research plan

Problem 1 : develop a theory of holonomic modules for $\mathcal{D}(\mathcal{L}_X)$-modules ($\mathcal{L}_X$ being a Lie algebroid over X).

Problem 2 : P. Schapira asked me the following question : In the definition of a Lie algebroid, we do not assume anymore that $\mathcal{L}_X$ is a locally free $\mathcal{O}_X$-module but only a coherent $\mathcal{O}_X$-module. Is it possible to develop a theory of operations for $\mathcal{D}(\mathcal{L}_X)$-modules (as in the sections 3 et 4) in this more general setting?

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