Transformation Groups

FORMALITY THEOREM WITH COEFFICIENTS IN A MODULE

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Abstract. In this paper, X will denote a $\mathcal{C}^\infty$ manifold. In a very famous paper, Kontsevich [Ko] showed that the differential graded Lie algebra (DGLA) of polydifferential operators on X is formal. Calaque [C1] extended this theorem to any Lie algebroid. More precisely, given any Lie algebroid E over X, he defined the DGLA of E-polydifferential operators, $\Gamma(X,^E D^*_{\text{poly}})$, and showed that it is formal. Denote by $\Gamma(X,^E T^*_{\text{poly}})$ the DGLA of E-polyvector fields. Considering M, a module over E, we define $\Gamma(X,^E T^*_{\text{poly}}(M))$ the $\Gamma(X,^E T^*_{\text{poly}})$-module of E-polyvector fields with values in M. Similarly, we define the $\Gamma(X,^E D^*_{\text{poly}})$-module of E-polydifferential operators with values in M, $\Gamma(X,^E D^*_{\text{poly}}(M))$. We show that there is a quasi-isomorphism of L_∞-modules over $\Gamma(X,^E T^*_{\text{poly}})$ from $\Gamma(X,^E T^*_{\text{poly}}(M))$ to $\Gamma(X,^E D^*_{\text{poly}}(M))$. Our result extends Calaque 's (and Kontsevich's) result.

1. Introduction

In this paper, X will denote a $\mathcal{C}^\infty$-manifold and $\mathcal{O}_X$ will denote the sheaf of $\mathcal{C}^\infty$ functions. To X are associated two sheaves of differential graded Lie algebras (DGLAs) T^*_{poly} and D^*_{poly}. The first one, T^*_{poly} is the sheaf of DGLAs of polyvector fields on X with differential zero and Schouten bracket. The second one, D^*_{poly}, is the sheaf of DGLAs of polydifferential operators on X with Hochschild differential and Gerstenhaber bracket. Kontsevich showed that there is a quasi-isomorphism of L_∞-algebras from $\Gamma(X, T^*_{\text{poly}})$ to $\Gamma(X, D^*_{\text{poly}})$, that is to say, that $\Gamma(X, D^*_{\text{poly}})$ is formal. The aim of this paper is to introduce a module in the Kontsevitch formality theorem.

Let us now consider a $\mathcal{D}_X$-module M. Inspired by the expression of the Schouten bracket, we endow $T^*_{\text{poly}}(M) = T^*_{\text{poly}} \otimes_{\mathcal{O}_X} M$ with a T^*_{poly}-module structure. Similarly, we can endow $D^*_{\text{poly}}(M) = D^*_{\text{poly}} \otimes_{\mathcal{O}_X} M$ with a D^*_{poly}-module structure as follows: if $P \in D^p_{\text{poly}}$ and $Q \in D^q_{\text{poly}}(M)$,

$$P \cdot_G Q = P \bullet Q - (-1)^{pq} Q \bullet P,$$

with

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$\forall\, a_0, \ldots, a_{p+q} \in \mathcal{O}_X$,

$$(P \bullet Q)(a_0, \ldots, a_{p+q}) = \sum_{i=0}^{p} (-1)^{iq} P(a_0, \ldots, a_{i-1}, Q(a_i, \ldots, a_{i+q}), \ldots, a_{p+q}).$$

The formula makes sense because Q is a differential operator with coefficients in a $\mathcal{D}_X$-module M. The expression $Q \bullet P$ is defined in an analogous way. The differential on $D^*_{\text{poly}}(M)$ is given by the action of the multiplication μ, $\mu \cdot_G -$. Using Kontsevich's formality theorem, one may see $D^*_{\text{poly}}(M)$ as an L_∞-module over T^*_{poly} and we will prove that it is formal. We will work in the more general setting of Lie algebroids.

Let us now consider a Lie algebroid E. To E is associated a sheaf of E-differential operators, $D(E)$ ([R]). Lie algebroids generalize at the same time the sheaf of vector fields on a manifold (in this case $E = TX$ and $D(E) = \mathcal{D}_X$) and Lie algebras (in this case, $D(E)$ is the enveloping algebra). Lie algebroids have been extensively studied recently because many examples of Lie algebroids arise from geometry (Poisson manifolds, group actions, foliations ...). To E, one can associate the sheaf of DGLAs of E-polyvector fields ${}^E T^*_{\text{poly}} = \bigoplus_{k=-1}^{\infty} \wedge^{k+1} E$ with zero differential and a Schouten-type Lie bracket [C1]. Calaque has given an appropriate generalization of the notion of polydifferential operators. In [C1] he defines the DGLA of E-polydifferential operators, $\Gamma(X, {}^E D^*_{\text{poly}})$, and constructs an L_∞-quasi-isomorphism from $\Gamma(X, {}^E T^*_{\text{poly}})$ to $\Gamma(X, {}^E D^*_{\text{poly}})$.

Let us now consider a $D(E)$-module M. We can perform the construction described above and define the ${}^E T^*_{\text{poly}}$-module ${}^E T^*_{\text{poly}}(M)$ (the sheaf of the E-polyvectors with coefficients in M) and the ${}^E D^*_{\text{poly}}$-module ${}^E D^*_{\text{poly}}(M)$ (the sheaf of E-polydifferential operators with coefficients in M). By Calaque's result we know that $\Gamma(X, {}^E D^*_{\text{poly}}(M))$ is an L_∞-module over $\Gamma(X, {}^E T^*_{\text{poly}})$. The main result of the paper is the following theorem.

Theorem 10. *There is a quasi-isomorphism of L_∞-modules over $\Gamma(X, {}^E T_{\text{poly}})$ from $\Gamma(X, {}^E T_{\text{poly}}(M))$ to $\Gamma(X, {}^E D_{\text{poly}}(M))$.*

Our result extends Calaque's formality theorem ([C1], take $M = \mathcal{O}_X$) and Kontsevich's formality theorem ([Ko], take $M = \mathcal{O}_X$ and $E = TX$).

If X is a Poisson manifold, we know from Kontsevitch's work [Ko] that there is a star product on $O = \Gamma(\mathcal{O}_X)$. Let $\mathcal{M}$ be a $\mathcal{D}_X$-module and $M = \Gamma(X, \mathcal{M})$. Using the star product, we can endow $M[[h]]$ with an $O[[h]] \otimes O[[h]]^{\text{op}}$-module structure. If $\pi \in \Gamma(\Lambda^2 TX)$ is the bivector defining the Poisson structure on O, $h\pi$ defines a Poisson structure on the algebra $O[[h]]$. As a corollary of our theorem, we get an isomorphism from the Poisson cohomology of the Poisson algebra $O[[h]]$ with coefficients in $M[[h]]$ and the differential Hochschild cohomology of $O[[h]]$ with coefficients in $M[[h]]$.

Our proofs are analogous to that of [D1],[C1], [D2], [CDH]. We use Kontsevitch's formality theorem for $\mathbb{R}^d_{\text{formal}}$ and Fedosov-like globalization techniques.

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Notation. For a study of L_∞ structures, we refer to [AMM], [D2], [D3], [HS], [LS].

Let k be a field of characteristic zero and let V be a $\mathbb{Z}$-graded k-vector space

$$V = \bigoplus_{i\in\mathbb{Z}} V_i.$$

If x is in V_i, we set $|x| = i$. We will always assume that the gradation is bounded below. Recall the definition of the graded symmetric algebra and the graded wedge algebra:

$$S(V) = \frac{T(V)}{\langle x\otimes y - (-1)^{|x||y|} y\otimes x\rangle},$$
$$\Lambda(V) = \frac{T(V)}{\langle x\otimes y + (-1)^{|x||y|} y\otimes x\rangle}.$$

If i is in $\mathbb{Z}$, we will denote by $V[i]$ the graded vector space defined by $V[i]^n = V^{i+n}$.

Denote by $S^c(V)$ the cofree cocommutative coalgebra without counity cofreely cogenerated by V. As a vector space $S^c(V)$ is $S^+(V)$. Its comultiplication is given by

$$\Delta(x_1\dots x_n) = \sum_{\substack{I\sqcup J=[1,n]\\ I\neq\varnothing\\ J\neq\varnothing}} (-1)^{\epsilon(I,J)} x_I\otimes x_J.$$

where $\epsilon(I,J)$ is the number of inversions of odd elements when going from $x_I x_J$ to $x_1\dots x_n$. A coderivation Q on $S^c(V)$ is determined by its Taylor coefficients $Q^{[n]} : S^n(V)\to V$ (obtained by composing Q with the projection from $S(V)$ onto V).

An L_∞ algebra is a couple (L,Q) where L is a graded vector space and Q is a degree 1 two-nilpotent coderivation of $S^c(L[1]) = C(L)$. The coderivation Q is determined by its Taylor coefficients $(Q^{[n]})_{n\geqslant 1}$. Using an isomorphism between $S^n(L[1])$ and $\Lambda^n(L)[n]$, the Taylor coefficients may be seen as maps $\overline{Q}^{[n]} : \Lambda^n L\to L[2-n]$. A differential graded Lie algebra $(L, d, [\ ,\ ])$ (with differential d and Lie bracket $[\ ,\ ]$) gives rise to an L_∞-algebra determined by $\overline{Q}^{[1]} = d$, $\overline{Q}^{[2]} = [\ ,\ ]$ and $\overline{Q}^{[i]} = 0$ for $i\geqslant 2$.

Let L be a differential graded Lie algebra. We will say that it is a filtered DGLA if it is equipped with a complete descending filtration, $\dots\mathcal{F}^1L\subset\mathcal{F}^0L = L$ such that $L = \lim\limits_{n} L/\mathcal{F}^nL$. A Maurer Cartan element of L is an element x of $\mathcal{F}^1L^1$ such that $Q^{[1]}x + \frac{1}{2}Q^{[2]}(x^2) = 0$.

Let (L_1,Q_1) and (L_2,Q_2) be two L_∞-algebras. An L_∞-morphism F from (L_1,Q_1) to (L_2,Q_2) is a morphism of coalgebras $F : C(L_1)\to C(L_2)$ compatible with coderivations (this means that $F\circ Q_1 = Q_2\circ F$). As F is a morphism of coalgebras, it is determined by its Taylor coefficients $(F^{[n]} : S^n(L_1[1])\to L_2[1])_{n\geqslant 1}$ or $(\overline{F}^{[n]} : \Lambda^n(L_1)\to L_2[1-n])_{n\geqslant 1}$. The relation $F\circ Q_1 = Q_2\circ F$ boils down to saying that $F^{[n]}$ satisfy an infinite collection of equations.

Let (L_1,Q_1) and (L_2,Q_2) be two filtered DGLAs and let F be an L_∞-morphism from (L_1,Q_1) to (L_2,Q_2) compatible with these filtrations. If x is a Maurer Cartan element of L_1, then $\sum_{n\geqslant 1} F^{[n]}(x^n)/n!$ is a Maurer Cartan element of L_2.

Let L be an L_∞-algebra and M a graded vector space. We will consider the $C(L)$-comodule $S(L[1])\otimes M$ with the coaction

$$\mathfrak{a}(x_1\dots x_n\otimes v)=\sum_{\substack{I\sqcup J=[1,n]\\ I\neq\varnothing}}(-1)^{\epsilon(I,J)}x_I\otimes(x_J\otimes v),$$

where $\epsilon(I,J)$ is the number of inversions of odd elements when going from x_Ix_J to $x_1\dots x_n$. An L_∞-module is a couple (M,ϕ) where ϕ is a degree 1 two-nilpotent coderivation of the $C(L)$-comodule $S(L[1])\otimes M$. The coderivation ϕ is determined by its Taylor coefficients $\phi^{[n]}: S^n(L[1])\otimes M\to M[1]$ or $\overline{\phi}^{[n]}:\Lambda^n(L)\otimes M\to M[1-n]$. The map $\phi^{[0]}$ is a differential on M. A module M over a differential graded Lie algebra $(L,d,[\ ,\ ])$ is an L_∞-module with Taylor coefficients $\overline{\phi}^{[0]}=d$, $\overline{\phi}^{[1]}(X\otimes m)=X\cdot m$ ($X\in L$, $m\in M$) and $\overline{\phi}^{[n]}=0$ if $n>1$.

Let (M_1,ϕ_1) and (M_2,ϕ_2) be two L_∞-modules. An L_∞-morphism $\mathcal{V}$ from (M_1,ϕ_1) to (M_2,ϕ_2) is a (degree 0) morphism of comodules from $S(L[1])\otimes M_1$ to $S(L[1])\otimes M_2$ such that $\mathcal{V}\circ\phi_1=\phi_2\circ\mathcal{V}$. It is determined by its Taylor coefficients $(\mathcal{V}^{[n]}:S^n(L[1])\otimes M_1\to M_2)_{n\geqslant 0}$ or $(\overline{\mathcal{V}}^{[n]}:\Lambda^n(L)\otimes M_1\to M_2[-n])_{n\geqslant 0}$. The compatibility of $\mathcal{V}$ with coderivation is expressed by an infinite collection of equations satisfied by $\mathcal{V}^{[n]}$.

In this text, DGLA (resp., DGAA) will stand for differential graded Lie algebra (resp., differential graded associative algebra).

We assume Einstein convention for the summation over repeated indices.

If $\mathcal{F}$ is a sheaf over X, then $\Gamma(\mathcal{F})$ denotes its global sections. If $\mathcal{F}$ and $\mathcal{G}$ are two sheaves and if $\Theta:\mathcal{F}\to\mathcal{G}$ is a morphism of sheaves, then $\Theta(X)$ will denote the morphism from $\Gamma(\mathcal{F})$ to $\Gamma(\mathcal{G})$ contained in Θ.

2. Recollections

2.1. Lie algebroids: Definitions and first properties

Let X be a $\mathcal{C}^\infty$-manifold and let $\mathcal{O}_X$ be the sheaf of $\mathcal{C}^\infty$ functions on X. Let Θ_X be the $\mathcal{O}_X$-module of $\mathcal{C}^\infty$ vector fields on X.

Definition 1. A sheaf in $\mathbb{R}$-Lie algebras over X, E, is a sheaf of $\mathbb{R}$-vector spaces such that for any open subset U, $E(U)$ is equipped with the structure of a Lie algebra and the restriction morphisms are Lie algebra homomorphisms.

A morphism between two sheaves of Lie algebras E and F is an $\mathbb{R}_X$-module morphism which is a Lie algebra morphism on each open subset.

Definition 2. A Lie algebroid over X is a pair (E,ω) where:

- E is a locally free $\mathcal{O}_X$-module of finite constant rank, that is to say a vector bundle over X;
- E is a sheaf of $\mathbb{R}$-Lie algebras;
- $\omega:\ E\to\Theta_X$ is an $\mathcal{O}_X$-linear morphism of sheaves of $\mathbb{R}$-Lie algebras such that the following compatibility relation holds:

$$\forall\,(\xi,\zeta)\in E^2,\ \ \forall\, f\in\mathcal{O}_X,\ \ [\xi,f\zeta]=\omega(\xi)(f)\zeta+f[\xi,\zeta].$$

One calls ω the anchor map. When there is no ambiguity, we will drop the anchor map in the notation of the Lie algebroid.

For example, TX is a Lie algebroid over X and a finite-dimensional Lie algebra is a Lie algebroid over a point. Other examples arise from Poisson manifolds, foliations, Lie group actions (see [F] for example).

A Lie algebroid (E,ω) gives rise to the sheaf of E-differential operators generated by $\mathcal{O}_X$ and E which is denoted by $D(E)$.

Definition 3. $D(E)$ is the sheaf associated to the presheaf

$$U \mapsto T^+_{\mathbb{R}}(\mathcal{O}_X(U) \oplus E(U))/J_U,$$

where J_U is the two-sided ideal generated by the relations

$$\forall\,(f,g) \in \mathcal{O}_X(U),\ \ \forall\,(\xi,\zeta) \in E(U)^2, \qquad \begin{array}{l} (1)\ f\otimes g = fg, \\ (2)\ f\otimes\xi = f\xi, \\ (3)\ \xi\otimes\zeta - \zeta\otimes\xi = [\xi,\zeta], \\ (4)\ \xi\otimes f - f\otimes\xi = \omega(\xi)(f). \end{array}$$

If $E = TX$, $D(E)$ is the sheaf of differential operators on X, $\mathcal{D}_X$. If E is a finite-dimensional Lie algebra $\mathfrak{g}$, $D(E)$ is $U(\mathfrak{g})$, the enveloping algebra of $\mathfrak{g}$.

$D(E)$ is also endowed with a coassociative $\mathcal{O}_X$-linear coproduct $\Delta : D(E) \to D(E)\otimes_{\mathcal{O}_X} D(E)$ defined as follows (see [X, Example 3.1]):

$$\begin{aligned} \Delta(1) &= 1\otimes 1, \\ \forall\, u \in E,\ \ \Delta(u) &= u\otimes 1 + 1\otimes u, \\ \forall\,(P,Q) \in D(E)^2,\ \ \Delta(PQ) &= \Delta(P)\Delta(Q). \end{aligned}$$

Let M be a $D(E)$-module. The cohomology of E with coefficients in M is computed by the complex $(\mathrm{Hom}_{\mathcal{O}_X}(\wedge^* E, M), {}^E d_M)$ where ${}^E d_M$ is given by $\forall\,\phi \in \mathrm{Hom}_{\mathcal{O}_X}(\wedge^n E, M)$, $\forall\, u_0,\dots,u_n \in E$,

$$\begin{aligned} {}^E d_M \phi(u_0,\dots,u_n) = &\sum_{i=1}^{n} (-1)^i u_i \cdot \phi(u_1,\dots,\widehat{u_i},\dots,u_n) \\ &+ \sum_{i<j} (-1)^{i+j} \phi([u_i,u_j],u_0,\dots,\widehat{u_i},\dots,\widehat{u_j},\dots,u_n). \end{aligned}$$

Recall that $\mathcal{O}_X$ has a natural left $D(E)$-module structure defined by:

$$\forall\, f \in \mathcal{O}_X,\ \forall\, P \in D(E), \quad P\cdot f = \omega(P)(f).$$

If $M = \mathcal{O}_X$, we set ${}^E d_M = {}^E d$ and the complex above will be called the Lie cohomology complex of E.

If M is a $D(E)$-module, a tensor with coefficients in M is a section of $M \otimes (\otimes E^*) \otimes (\otimes E)$.

The notion of connections has been extended to Lie algebroids (see [F], for example). Let $\mathcal{B}$ be an $\mathcal{O}_X$-module. An E-connection on $\mathcal{B}$ is a linear operator

$$\nabla : \Gamma(\mathcal{B}) \to \Gamma({}^E\Omega^1(\mathcal{B})) = \Gamma(\mathrm{Hom}_{\mathcal{O}_X}(\Lambda^1 E, \mathcal{B}))$$

satisfying the following equation: for any $f \in \Gamma(\mathcal{O}_X)$ and any $v \in \Gamma(\mathcal{B})$,

$$\nabla(fv) = {}^E\!d(f)v + f\nabla(v).$$

If u is an element of E, the connection ∇ defines a map $\nabla_u : \mathcal{B} \to \mathcal{B}$.

Assume now that $\mathcal{B}$ is a bundle. If $(e_1, \ldots, e_d)$ is a local basis of E and $(b_1, \ldots, b_n)$ is a local basis of $\mathcal{B}$, one has

$$\nabla_{e_i}(b_j) = \Gamma^k_{i,j} b_k.$$

The connection ∇ is determined by its Christoffel symbol $\Gamma^k_{i,j}$.

Definition 4. The curvature R of a connection ∇ with values in $\mathcal{B}$ is the section R of the bundle $E^* \otimes E^* \otimes \mathcal{B}^* \otimes \mathcal{B}$ defined by: For any u, v in $\Gamma(E)$ and b in $\Gamma(\mathcal{B})$,

$$R(u, v)(b) = (\nabla_u \circ \nabla_v - \nabla_v \circ \nabla_u - \nabla_{[u,v]})(b).$$

The curvature tensor is locally determined by the $(R_{i,j})^l_k$ defined by

$$R(e_i, e_j)b_k = (R_{i,j})^l_k b_l.$$

For a connection ∇ on $\mathcal{B} = E$, one can define the torsion tensor.

Definition 5. The torsion of ∇ is a section of $E \otimes E^* \otimes E^*$ defined by: For any u,v in $\Gamma(E)$,

$$T(u, v) = \nabla_u(v) - \nabla_v(u) - [u, v].$$

Proposition 1. *A torsion-free connection on E exists.*

A proof of this proposition can be found in [C2].

Examples of $D(E)$-modules

Example 1. Flat connections provide examples of $D(E)$-modules.

Example 2. If E is a Lie algebroid with anchor map ω, then $Ker\omega$ is a left $D(E)$-module for the following operations: for all f in $\mathcal{O}_X$, for all ξ in E, and for all σ in $\mathrm{Ker}\,\omega$,

$$f \cdot \sigma = f\sigma, \quad \xi \cdot \sigma = [\xi, \sigma].$$

Example 3. If M and N are two left $D(E)$-modules, then (see [Bo] for the $\mathcal{D}_X$-module case and [Ch2]) $M \otimes_{\mathcal{O}_X} N$ and $\mathcal{H}om_{\mathcal{O}_X}(M, N)$, endowed with the two operations described below, are left $D(E)$-modules:

$$\begin{aligned}
&\forall\, m \in M,\ \forall\, n \in N,\ \forall\, a \in \mathcal{O}_X,\ \forall\, \xi \in E,\\
&\quad a \cdot (m \otimes n) \cdot a = a \cdot m \otimes n,\\
&\quad\quad \xi \cdot (m \otimes n) = \xi \cdot m \otimes n + m \otimes \xi \cdot n,
\end{aligned}$$

$$\begin{aligned}
&\forall\, \phi \in \mathcal{H}om_{\mathcal{O}_X}(M, N),\ \forall\, m \in M, \forall\, a \in \mathcal{O}_X,\ \forall\, \xi \in E,\\
&\quad (a \cdot \phi)(m) = a\phi(m),\\
&\quad (\xi \cdot \phi)(m) = \xi \cdot \phi(m) - \phi(\xi \cdot m).
\end{aligned}$$

Example 4. It is a well-known fact ([Bo], [Ka]) that the $\mathcal{O}_X$-module of differential forms of maximal degree, $\Omega_X^{\dim X}$, is endowed with a right $\mathcal{D}_X$-module structure. We may extend this result [Ch1] to $\wedge^d(E^*)$ where d is the rank of E. Indeed E acts on $\wedge^d(E^*)$ by the adjoint action. The action of an element ξ of E is called the Lie derivative of ξ and is denoted L_ξ. The $\mathcal{O}_X$-module $\wedge^d(E^*)$, endowed with the following operations:

$$\begin{aligned} &\forall\, \sigma \in \wedge^d(E^*),\ \forall\, \xi \in E,\ \forall\, f \in \mathcal{O}_X, \\ &\qquad \sigma \cdot a = a\sigma, \\ &\qquad \sigma \cdot \xi = -L_\xi(\sigma), \end{aligned}$$

is a right $D(E)$-module.

Example 5. If $\mathcal{M}$ and $\mathcal{N}$ are two right $D(E)$-modules, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$, endowed with the two following operations:

$$\begin{aligned} &\forall\, \phi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}),\ \forall\, m \in \mathcal{M}, \forall\, a \in \mathcal{O}_X,\ \forall\, \xi \in E, \\ &\qquad (a \cdot \phi)(m) = \phi(m) \cdot a, \\ &\qquad (\xi \cdot \phi)(m) = -\phi(m) \cdot \xi + \phi(m \cdot \xi), \end{aligned}$$

is a left $D(E)$-module [Ch2]. This was already known for D-modules. In particular, $\mathcal{H}om_{\mathcal{O}_X}(\wedge^d(E^*), \Omega_X^{\dim X})$ is a left $D(E)$-module which is used in [ELW] to define the modular class of E.

Example 6. If $\mathcal{M}$ is a right $D(E)$-module and $\mathcal{N}$ is a left $D(E)$-module, then $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N}$, endowed with the two following operations:

$$\begin{aligned} &\forall\, m \in \mathcal{M},\ \forall\, n \in \mathcal{N},\ \forall\, a \in \mathcal{O}_X,\ \forall\, \xi \in E, \\ &\qquad (m \otimes n) \cdot a = m \otimes a \cdot n = m \cdot a \otimes n, \\ &\qquad (m \otimes n) \cdot \xi = m \cdot \xi \otimes n - m \otimes \xi \cdot n, \end{aligned}$$

is a right $D(E)$-module (see [Bo] for D-modules and [Ch2]. Given any $D(E)$-module which is locally free of rank one, the functor $\mathcal{N} \mapsto \mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{N}$ establishes an equivalence of categories between left and right $D(E)$-modules. Its inverse functor is given by $\mathcal{M} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{M})$. This equivalence of categories is well-known for D-modules [Bo], [Ka] and was generalized to Lie algebroids in [Ch2]. In the case where $X = \mathbb{R}^d$ and $E = T\mathbb{R}^d$, this equivalence of categories is particularly simple because we may choose $dx^1\wedge\cdots\wedge dx^d$ as a basis of the $\mathcal{O}_{\mathbb{R}^d}$-module Ω_X^d. There exists a unique anti-isomorphism of $\mathcal{D}_{\mathbb{R}^d}$, σ, such that $\sigma(f) = f$ and $\sigma(\partial/\partial x^i) = -\partial/\partial x^i$. Any left $\mathcal{D}_{\mathbb{R}^d}$-module can be seen as a right $\mathcal{D}_{\mathbb{R}^d}$-module (and conversely) in the following way:

$$\forall\, P \in \mathcal{D}_{\mathbb{R}^d},\quad \forall\, m \in M, \quad m \cdot P = \sigma(P) \cdot m.$$

Example 7. Let $\mathcal{D}b_X$ be the sheaf of distributions over X. As $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module, $\mathcal{D}b_X$ is a right $\mathcal{D}_X$-module (by transposition).

Example 8. Let us recall our definition of a Lie algebroid morphism [Ch2] which coincides with that of Almeida and Kumpera [AK].

Definition 6. Let (E_X, ω_X) (resp., (E_Y, ω_Y)) be a Lie algebroid over X (resp., Y). A morphism Φ from (E_X, ω_X) to (E_Y, ω_Y) is a pair (f, F) such that:

- $f\colon : X \to Y$ is a $\mathcal{C}^\infty$-morphism.
- $F\colon E_X \to f^*E_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}E_Y$ such that the two following conditions are satisfied:
  (1) The diagram

$$\begin{array}{ccc} E_X & \xrightarrow{F} & f^*E_Y \\ {\scriptstyle \omega_X}\downarrow & & \downarrow{\scriptstyle f^*\omega_Y} \\ \Theta_X & \xrightarrow{Tf} & f^*\Theta_Y \end{array}$$

  commutes.
  (2) Let ξ and η be two elements of E_X^2. Put $F(\xi) = \sum_{i=1}^m a_i \otimes \xi_i$ and $F(\eta) = \sum_{j=1}^m b_j \otimes \eta_j$, then

$$F([\xi, \eta]) = \sum_{j=1}^n \omega_X(\xi)(b_j) \otimes \eta_j - \sum_{i=1}^n \omega_X(\eta)(a_i) \otimes \xi_i + \sum_{i,j} a_i b_j \otimes [\xi_i, \eta_j].$$

If $\Phi = (f, F)$ is Lie algebroid morphism from (E_X, ω_X) to (E_Y, ω_Y) and $\mathcal{M}$ is a $D(E_Y)$-module, then $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}$ endowed with the two following operations:

$$\begin{aligned} &\forall\, (a, b) \in \mathcal{O}_X^2,\ \ \forall\, \xi \in E_X,\ \ \forall\, m \in f^{-1}\mathcal{M}, \\ &\qquad a \cdot (b \otimes m) = ab \otimes m, \\ &\qquad \xi \cdot (b \otimes m) = \omega_X(\xi)(b) \otimes m + \sum_i b a_i \otimes \xi_i m, \end{aligned}$$

(where $F(\xi) = \sum_i a_i \otimes \xi_i$ with a_i in $\mathcal{O}_X$ and ξ_i in $f^{-1}E_Y$) is a left $D(E_X)$-module ([Ch2]).

Morphisms of Lie algebroids generalize at the same time Lie algebra morphisms and morphisms between $\mathcal{C}^\infty$-manifolds. Examples of Lie algebroid morphisms can be found in [Ch3]. The $D(E_X) \otimes f^{-1}D(E_Y)^{\mathrm{op}}$-module $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D(E)$ generalizes the transfer module for D-modules (see [Bo], [Ka], [Ch2]).

2.2. The sheaves of DGLAs ${}^E T_{\mathrm{poly}}$ and ${}^E D_{\mathrm{poly}}$

The sheaf of DGLAs of polyvectorfields can be extended to the Lie algebroids setting. The sheaf of DGLAs ${}^E T_{\mathrm{poly}}$ of E-polyvector fields is defined as follows ([C1]):

$${}^E T_{\mathrm{poly}} = \bigoplus_{k \geqslant -1} {}^E T^k_{\mathrm{poly}} = \bigoplus_{k \geqslant -1} \wedge^{k+1} E,$$

endowed with the zero differential and the Lie bracket $[\ ,\ ]_S$ uniquely defined by the following properties:

- $\forall\, f, g \in \mathcal{O}_X,\ \ [f, g]_S = 0,$
- $\forall\, \xi \in E,\ \forall\, f \in \mathcal{O}_X,\ \ [\xi, f]_S = \omega(\xi)(f),$
- $\forall\, \xi, \eta \in E,\ \ [\xi, \eta]_S = [\xi, \eta]_E,$
- $\forall\, u \in {}^E T^k_{\mathrm{poly}},\ v \in {}^E T^l_{\mathrm{poly}},\ w \in {}^E T_{\mathrm{poly}},$

  $$[u, v \wedge w]_S = [u, v]_S \wedge w + (-1)^{k(l+1)} v \wedge [u, w]_S.$$

In [C1], Calaque extended the sheaf of DGLAs of polydifferential operators to the Lie algebroid setting. Before recalling his construction, let us fix some notations.

Notation. Let $M_0, M_1, \dots, M_n$ be $D(E)$-modules. Denote by $\pi_i \colon D(E) \to \mathrm{End}\,(M_i)$ the maps defined by these actions. An element $P_0 \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} P_n$ of $D(E)^{\otimes n+1}$ defines a map

$$\pi_0(P_0) \otimes \ldots \otimes \pi_{n+1}(P_{n+1}) \colon M_0 \otimes_{\mathbb{R}_X} \cdots \otimes_{\mathbb{R}_X} M_n \to M_0 \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} M_n,$$
$$m_0 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} m_n \mapsto \pi_0(P_0)(m_0) \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \pi_n(P_n)(m_n).$$

In the sequel we will be in the following situation: $M_0, \dots, M_{i-1}, M_{i+1}, \dots, M_n$ are $D(E)$ endowed with left multiplication. If P is in $D(E)$, we will then write P for left multiplication with P, which amounts to omitting π_i. The $D(E)$-module M_i will be $\mathcal{O}_X$ (with its natural $D(E)$-module structure) and we will write ω (as the anchor map) for the map from $D(E)$ to $\mathrm{End}\,(\mathcal{O}_X)$.

Calaque defines the sheaf of DGLAs ${}^E D^*_{\mathrm{poly}}$ of E-polydifferential operators as follows:

$${}^E D^*_{\mathrm{poly}} = \bigoplus_{k \geqslant -1} {}^E D^k_{\mathrm{poly}},$$

where

$${}^E D^{-1}_{\mathrm{poly}} = \mathcal{O}_X,$$
$${}^E D^k_{\mathrm{poly}} = D(E)^{\otimes^{k+1}_{\mathcal{O}_X}} \quad \text{if} \quad k \geqslant 0.$$

Before defining the Lie bracket over ${}^E D^*_{\mathrm{poly}}$, we need to introduce the bilinear product of degree 0,

$$\bullet : {}^E D^*_{\mathrm{poly}} \otimes {}^E D^*_{\mathrm{poly}} \to {}^E D^*_{\mathrm{poly}}.$$

Let P (resp., Q) be an homogeneous element of ${}^E D^*_{\mathrm{poly}}$ of positive degree $|P|$ (resp., $|Q|$), and let f (resp., g) be an element of ${}^E D^{-1}_{\mathrm{poly}} = \mathcal{O}_X$. We have

$$P \bullet Q = \sum_{i=0}^{|P|} (-1)^{i|Q|} (\mathrm{id}^{\otimes^i} \otimes \Delta^{(|Q|)} \otimes \mathrm{id}^{\otimes |P|-i})(P) \cdot (1^{\otimes^i} \otimes_{\mathbb{R}} Q \otimes_{\mathbb{R}} 1^{\otimes |P|-i}),$$
$$P \bullet f = \sum_{i=0}^{|P|} (-1)^{i} (\mathrm{id}^{\otimes^i} \otimes \omega \otimes \mathrm{id}^{\otimes |P|-i})(P) \cdot (1^{\otimes^i} \otimes_{\mathbb{R}} f \otimes_{\mathbb{R}} 1^{\otimes |P|-i}),$$
$$f \bullet g = 0,$$
$$f \bullet P = 0.$$

The Lie bracket between $P_1 \in {}^E D^{k_1}_{\mathrm{poly}}$ and $P_2 \in {}^E D^{k_2}_{\mathrm{poly}}$ is

$$[P_1, P_2] = P_1 \bullet P_2 - (-1)^{k_1 k_2} P_2 \bullet P_1.$$

The differential on ${}^E D_{\mathrm{poly}}$ is $\partial = [1 \otimes 1, -]$.

Calaque has proved the following theorem ([C1]) which generalizes Kontsevitch's result ([Ko]).

Theorem 2. *There exists a quasi-isomorphism of L_∞-algebras, Υ, from $\Gamma({}^E T^*_{\mathrm{poly}})$ to $\Gamma({}^E D^*_{\mathrm{poly}})$. In other words, $\Gamma({}^E D^*_{\mathrm{poly}})$ is formal.*

3. Main results

Let E be a Lie algebroid over a manifold X and let $D(E)$ be the sheaf of E-differential operators. We will denote by M a left $D(E)$-module.

3.1. The ${}^ET^*_{\mathrm{poly}}$-module ${}^ET^*_{\mathrm{poly}}(M)$

We introduce the complex ${}^ET^*_{\mathrm{poly}}(M)$ of E-polyvector fields with values in M,

$$ {}^ET^*_{\mathrm{poly}}(M) = \bigoplus_{k\geqslant -1} {}^ET^k_{\mathrm{poly}}(M) = \bigoplus_{k\geqslant -1} \Lambda^{k+1}E\otimes M $$

with differential zero. If m is in M, we will identify m with $1\otimes m$.

Proposition 3. *${}^ET^*_{\mathrm{poly}}(M)$ is endowed with a ${}^ET^*_{\mathrm{poly}}$-module structure described as follows: for all $u=\xi_1\wedge\cdots\wedge\xi_{k+1}\in {}^ET^k_{\mathrm{poly}}$, $v\in^E T^l_{\mathrm{poly}}$ (with $k,l\geqslant 0$), $f\in\mathcal{O}_X$, $m\in M$,*

- $f\cdot_S m=0$;
- $(\xi_1\wedge\cdots\wedge\xi_{k+1})\cdot_S m=\sum_{i=1}^{k+1}(-1)^{k+1-i}\xi_1\wedge\cdots\wedge\widehat{\xi_i}\wedge\cdots\wedge\xi_{k+1}\otimes\xi_i\cdot m$;
- $f\cdot_S(v\otimes m)=[f,v]_S\otimes m$;
- $u\cdot_S(v\otimes m)=[u,v]_S\otimes m+(-1)^{k(l+1)}v\wedge u\cdot_S m$.

When there is no ambiguity, we will drop the subscript S in the notation of the action of ${}^ET^*_{\mathrm{poly}}$ over ${}^ET^*_{\mathrm{poly}}(M)$.

Proof of the proposition. It is easy to check that the actions above are well defined. Let a be in ${}^ET^s_{\mathrm{poly}}$. We need to verify that the following relation holds:

$$ u\cdot(v\cdot(a\otimes m))-(-1)^{kl}v\cdot(u\cdot(a\otimes m))=[u,v]\cdot(a\otimes m). $$

A straightforward computation shows that it is enough to check this relation for $a=1$, which we will assume. We will need the two following lemmas.

Lemma 4. *If $a\in {}^ET^*_{\mathrm{poly}}$, $u\in {}^ET^k_{\mathrm{poly}}$, $v\in {}^ET^l_{\mathrm{poly}}$ $(k,l\geqslant -1)$, one has*

$$ u\cdot(v\wedge a\otimes m)=[u,v]\wedge a\otimes m+(-1)^{k(l+1)}v\wedge u\cdot(a\otimes m). $$

Proof of the lemma. It is a straightforward computation. □

Lemma 5. *Let $a\in {}^ET^*_{\mathrm{poly}}$, $m\in M$, $k,l\geqslant 0$, $u\in {}^ET^k_{\mathrm{poly}}$, $v\in {}^ET^l_{\mathrm{poly}}$. One has the following relation*

$$ (u\wedge v)\cdot(a\otimes m)=u\wedge(v\cdot(a\otimes m))+(-1)^{(k+1)(l+1)}v\wedge(u\cdot(a\otimes m)). $$

Proof of the lemma. An easy computation shows that we may assume $a=1$. The proof of the lemma goes by induction over k. The case $k=0$ is obvious so that we assume $k\geqslant 1$. Set $u=\xi_1\wedge\cdots\wedge\xi_{k+1}$ and $u'=\xi_2\wedge\cdots\wedge\xi_{k+1}$ so that $u=\xi_1\wedge u'$. Using the induction hypothesis and the case $k=0$, we get the following sequence of equalities:

$$ \begin{aligned} (u\wedge v)\cdot m &= (-1)^{l+k+1}(u'\wedge v)\otimes\xi_1\cdot m+\xi_1\wedge((u'\wedge v)\cdot m)\\ &= (-1)^{l+k+1+k(l+1)}v\wedge u'\otimes\xi_1\cdot m+\xi_1\wedge u'\wedge(v\cdot m)\\ &\quad +(-1)^{k(l+1)}\xi_1\wedge v\wedge(u'\cdot m)\\ &= u\wedge(v\cdot m)+(-1)^{(k+1)(l+1)}v\wedge(u\cdot m). \quad\square \end{aligned} $$

We will show the relation

$$u\cdot(v\cdot m)-(-1)^{kl}v\cdot(u\cdot m)=[u,v]\cdot m$$

by induction on l.

First case: $l=-1$.

In this case, v is a function on X which will be denoted f. We proceed by induction over k. The cases $k=-1$ or $k=0$ are obvious so that we assume $k\geqslant 1$. We set $u=\xi_1\wedge\cdots\wedge\xi_{k+1}$ and $u'=\xi_2\wedge\cdots\wedge\xi_{k+1}$.

Using the two previous lemmas and the induction hypothesis, we get the following sequence of equalities:

$$\begin{aligned}
&u\cdot(f\cdot m)-(-1)^k f\cdot(u\cdot m)\\
&\qquad=-(-1)^k f\cdot(u\cdot m)\\
&\qquad=-(-1)^k f\cdot(\xi_1\wedge(u'\cdot m)+(-1)^k u'\otimes\xi_1\cdot m)\\
&\qquad=-(-1)^k[f,\xi_1](u'\cdot m)+(-1)^k\xi_1\wedge(f\cdot(u'\cdot m))-[f,u']\otimes\xi_1\cdot m\\
&\qquad=-(-1)^k[f,\xi_1](u'\cdot m)+(-1)^k\xi_1\wedge([f,u']\cdot m)-[f,u']\otimes\xi_1\cdot m.
\end{aligned}$$

On the other hand,

$$[f,u]=[f,\xi_1]u'-\xi_1\wedge[f,u'],$$

hence,

$$[f,u]\cdot m=[f,\xi_1]u'\cdot m-\xi_1\wedge([f,u']\cdot m)-(-1)^{k+1}[f,u']\otimes\xi_1\cdot m.$$

The case $l=-1$ follows.

Second case: $l=0$.

In this case, v is an element of E which will be denoted η. We proceed by induction over k. The cases $k=-1$ or $k=0$ are obvious so that we assume $k\geqslant 1$. We set $u=\xi_1\wedge\cdots\wedge\xi_{k+1}$ and $u'=\xi_2\wedge\cdots\wedge\xi_{k+1}$.

Using the two previous lemmas, we get the following sequence of equalities:

$$\begin{aligned}
u\cdot(\eta\cdot m)-\eta\cdot(u\cdot m)&=\xi_1\wedge(u'\cdot(\eta\cdot m))+(-1)^k u'\otimes\xi_1\cdot(\eta\cdot m)\\
&\quad-\eta\cdot(\xi_1\wedge(u'\cdot m)+(-1)^k u'\otimes\xi_1\cdot m)\\
&=\xi_1\wedge([u',\eta]\cdot m)+(-1)^k u'\otimes[\xi_1,\eta]\cdot m\\
&\quad-[\eta,\xi_1]\wedge(u'\cdot m)-(-1)^k[\eta,u']\otimes\xi_1\cdot m.
\end{aligned}$$

On the other hand,

$$[u,\eta]=-[\eta,\xi_1]\wedge u'-\xi_1\wedge[\eta,u'],$$

hence,

$$[u,\eta]\cdot m=-[\eta,\xi_1]\wedge(u'\cdot m)-(-1)^k u'\otimes[\eta,\xi_1]\cdot m-(-1)^k[\eta,u']\otimes\xi_1\cdot m-\xi_1\wedge[\eta,u']\cdot m.$$

Third case: $l \geqslant 1$.

We proceed by induction. We set $v = \eta_1 \wedge \cdots \wedge \eta_{k+1}$ and $u' = \eta_2 \wedge \cdots \wedge \eta_{k+1}$. Using the previous lemmas and the induction hypothesis, we get the following sequences of equalities:

$$\begin{aligned} & u \cdot (v \cdot m) - (-1)^{kl} v \cdot (u \cdot m) \\ &\qquad = u \cdot (\eta_1 \wedge (v' \cdot m) + (-1)^l v' \otimes \eta_1 \cdot m) - (-1)^{kl} \eta_1 \wedge (v' \cdot (u \cdot m)) \\ &\qquad\quad - (-1)^{lk+l} v' \wedge (\eta_1 \cdot (u \cdot m)) \\ &\qquad = (-1)^{k(l+1)} v' \wedge ([u, \eta_1] \cdot m) + (-1)^k \eta_1 \wedge [u, v'] \cdot m + [u, \eta_1] \wedge (v' \cdot m) \\ &\qquad\quad + (-1)^l [u, v'] \otimes \eta_1 \cdot m. \end{aligned}$$

On the other hand,

$$[u, v] = [u, \eta_1] \wedge v' + (-1)^k \eta_1 \wedge [u, v'],$$

hence,

$$\begin{aligned} [u, v] \cdot m = {} & [u, \eta_1] \wedge (v' \cdot m) + (-1)^{l(k+1)} v' \wedge ([u, \eta_1] \cdot m) + (-1)^k \eta_1 \wedge ([u, v'] \cdot m) \\ & + (-1)^l [u, v'] \otimes \eta_1 \cdot m. \end{aligned}$$

The case $l \geqslant 1$ follows. $\square$

3.2. The ${}^E D^*_{\mathrm{poly}}$-module ${}^E D^*_{\mathrm{poly}}(M)$

Let M be a $D(E)$-module. Denote by π the map from $D(E)$ to $\mathrm{End}\,(M)$ determined by the left $D(E)$-module structure on M. We will use the same notation as in Section 2.2. We will also use the map

$$\begin{aligned} \tau_i \colon \Big(\bigotimes\nolimits^{i}_{\mathcal{O}_X} D(E)\Big) \otimes_{\mathcal{O}_X} M \otimes_{\mathcal{O}_X} \Big(\bigotimes\nolimits^{q+1-i}_{\mathcal{O}_X} D(E)\Big) &\to \Big(\bigotimes\nolimits^{q+1}_{\mathcal{O}_X} D(E)\Big) \otimes_{\mathcal{O}_X} M, \\ Q_1 \otimes \cdots \otimes Q_i \otimes m \otimes Q_{i+1} \otimes \cdots \otimes Q_{q+1} &\mapsto Q_1 \otimes \cdots \otimes Q_{q+1} \otimes m. \end{aligned}$$

Let us introduce the complex ${}^E D_{\mathrm{poly}}(M)$ of E-polydifferential operators with values in M as follows:

$${}^E D_{\mathrm{poly}}(M) = \bigoplus\nolimits_{k \geqslant -1} {}^E D^k_{\mathrm{poly}}(M),$$

where

$$\begin{aligned} & {}^E D^{-1}_{\mathrm{poly}}(M) = M, \\ & {}^E D^{k}_{\mathrm{poly}}(M) = D(E)^{\otimes^{k+1}_{\mathcal{O}_X}} \otimes_{\mathcal{O}_X} M \quad \text{if} \;\; k \geqslant 0. \end{aligned}$$

Let us define two maps denoted in the same way

$$\begin{aligned} \bullet : {}^E D^*_{\mathrm{poly}} \otimes {}^E D^*_{\mathrm{poly}}(M) \to {}^E D^*_{\mathrm{poly}}(M), \\ \bullet : {}^E D^*_{\mathrm{poly}}(M) \otimes {}^E D^*_{\mathrm{poly}} \to {}^E D^*_{\mathrm{poly}}(M). \end{aligned}$$

If P and Q are homogeneous elements of ${}^E D^*_{\text{poly}}$ of nonnegative degree, respectively, $|P|$ and $|Q|$, f is an element of ${}^E D^{-1}_{\text{poly}}$ and m is in M, then

$$\begin{aligned} P \bullet (Q \otimes m) &= \sum_{i=0}^{|P|} (-1)^{i|Q|} \tau_{i+|Q|+1} \big[(\mathrm{id}^{\otimes^i} \otimes \Delta^{(|Q|+1)} \otimes \mathrm{id}^{\otimes |P|-i})(P) \\ &\qquad \cdot (1^{\otimes^i} \otimes_{\mathbb{R}} (Q \otimes m) \otimes_{\mathbb{R}} 1^{\otimes |P|-i}) \big], \\ P \bullet m &= \sum_{i=0}^{|P|} (-1)^i \tau_i \big[(\mathrm{id}^{\otimes^i} \otimes \pi \otimes \mathrm{id}^{\otimes |P|-i})(P) \cdot (1^{\otimes^i} \otimes_{\mathbb{R}} m \otimes_{\mathbb{R}} 1^{\otimes |P|-i}) \big], \\ f \bullet m &= 0, \\ f \bullet (Q \otimes m) &= 0, \\ (Q \otimes m) \bullet P &= Q \bullet P \otimes m, \\ m \bullet P &= 0, \\ m \bullet f &= 0. \end{aligned}$$

Note that second, third, and fourth equations could be recovered from the first one. The differential, ∂_M, on ${}^E D^*_{\text{poly}}(M)$ is given by: For all $Q \otimes m$ in ${}^E D^*_{\text{poly}}(M)$,

$$\begin{aligned} \partial_M (Q \otimes m) &= (1 \otimes 1) \bullet (Q \otimes m) - (-1)^{|Q|} (Q \otimes m) \bullet (1 \otimes 1) \\ &= \partial(Q) \otimes m, \end{aligned}$$

where $1 \otimes 1 \in {}^E D^1_{\text{poly}}$.

Theorem 6. *${}^E D^*_{\text{poly}}(M)$ is endowed with an ${}^E D^*_{\text{poly}}$-module structure as follows:*

$$\begin{aligned} &\forall\, P \in {}^E D^p_{\text{poly}},\ \forall\, (Q \otimes m) \in {}^E D^q_{\text{poly}}(M), \\ &\qquad P \cdot_G (Q \otimes m) = P \bullet (Q \otimes m) - (-1)^{pq} (Q \otimes m) \bullet P. \end{aligned}$$

Proof of the theorem. Let $P \in {}^E D^p_{\text{poly}}$, $Q \in {}^E D^q_{\text{poly}}$, $\lambda \in {}^E D^r_{\text{poly}}(M)$. Introduce the following quantity:

$$A(P, Q, \lambda) = (P \bullet Q) \bullet \lambda - P \bullet (Q \bullet \lambda).$$

The theorem follows from the lemma below.

Lemma 7. *The following equality holds:*

$$A(P, Q, \lambda) = (-1)^{qr} A(P, \lambda, Q).$$

This lemma is well-known in the case where $E = TX$ and $M = \mathcal{O}_X$ (see, e.g., the paper of Keller in [BCKT]). In the general case, it follows from a straightforward but tedious computation. □

3.3. The Hochschid–Kostant–Rosenberg theorem

Theorem 8. *The map U^M_{HKR} from $({}^ET^*_{\mathrm{poly}}(M),0)$ to $({}^ED^*_{\mathrm{poly}}(M),\partial_M)$ defined by: for all $v_1,\dots,v_n$ in E and all m in M,*

$$U^M_{\mathrm{HKR}}(v_0\wedge\cdots\wedge v_n\otimes m)=\frac{1}{(n+1)!}\sum_{\sigma\in S_{n+1}}\epsilon(\sigma)v_{\sigma(0)}\otimes\cdots\otimes v_{\sigma(n)}\otimes m,$$
$$U^M_{\mathrm{HKR}}(m)=m,$$

is a quasi-isomorphism.

The first one to have proved such a statement in the affine case (for $E=TX$ and $M=\mathcal{O}_X$) seems to be J. Vey [V]. A proof for the tangent bundle of any manifold (and $M=\mathcal{O}_X$) can be found in [Ko]. This theorem is proved in [C1] for any Lie algebroid and $M=\mathcal{O}_X$.

Proof of the theorem. This theorem will be a consequence of the proof of Theorem 10 and of the following well-known result.

Lemma 9. *T be a finite-dimensional $\mathbb{R}$-vector space. Consider the complex $\Lambda^*T=\bigoplus_{p\in\mathbb{N}}\Lambda^pT$ with zero differential and the complex $\bigoplus_{p\in\mathbb{N}}(\bigotimes^p S(E))$ with the differential*

$$\partial=\mathrm{id}^{\otimes p}\otimes 1+(-1)^{p-1}1\otimes\mathrm{id}^{\otimes p}+(-1)^{p-1}\sum_{i=0}^{n}(-1)^i\mathrm{id}^{\otimes i}\otimes\Delta\otimes\mathrm{id}^{\otimes n-i}.$$

The $\mathbb{R}$-linear map Θ from Λ^*T to $\bigoplus_{p\in\mathbb{N}}\bigotimes^p S(T)$ defined by: For all $v_1,\dots,v_p$ in T,

$$\Theta(v_0\wedge\cdots\wedge v_p)=\frac{1}{(p+1)!}\sum_{\sigma\in S_{p+1}}\epsilon(\sigma)v_{\sigma(0)}\otimes\cdots\otimes v_{\sigma(p)},$$
$$\Theta(1)=1,$$

is a quasi-isomorphism.

3.4. Main statement

We have seen that $\Gamma({}^ED^*_{\mathrm{poly}}(M))$ is a module over the DGLA $\Gamma({}^ED^*_{\mathrm{poly}})$. As we know ([C1]) that there is a L_∞-morphism from $\Gamma({}^ET^*_{\mathrm{poly}})$ to $\Gamma({}^ED^*_{\mathrm{poly}})$, we deduce that $\Gamma({}^ED^*_{\mathrm{poly}}(M))$ is naturally endowed with the structure of an L_∞-module over the DGLA $\Gamma({}^ET^*_{\mathrm{poly}})$. We can now state the main result of this paper.

Theorem 10. *There is a quasi-isomorphism of L_∞-modules over $\Gamma({}^ET^*_{\mathrm{poly}})$ from $\Gamma({}^ET^*_{\mathrm{poly}}(M))$ to $\Gamma({}^ED^*_{\mathrm{poly}}(M))$ that induces U^M_{HKR} in cohomology.*

Our result extends Calaque's result ([C1], take $M=\mathcal{O}_X$) and Kontsevitch's result ([Ko], take $M=\mathcal{O}_X$ and $E=TX$).

4. Proof

The proof is analogous to that of [D1], [C1], [D2], [CDH].

4.1. Fedosov resolutions

As before, E will denote a Lie algebroid and M will be a $D(E)$-module.

Following Fedosov [Fe] and Dolgushev [D1], Calaque introduced ([C1], see also [CDH]), the locally free $\mathcal{O}_X$-modules $\mathcal{W}=\widehat{S}(E^*)$, $\mathcal{T}^*$ and $\mathcal{D}^*$. Let us recall their definition.

- $\mathcal{W}=\widehat{S}(E^*)$ is the locally free $\mathcal{O}_X$-module whose sections are functions that are formal in the fiber. An element s of $\Gamma(U,\mathcal{W})$ can be locally written

$$s=\sum_{l=0}^{\infty} s_{i_1,\dots,i_l} y^{i_1}\cdots y^{i_l},$$

  where $y^1,\dots,y^d$ are coordinates in the fiber of E and $s_{i_1,\dots,i_l}$ are coefficients of a symmetric covariant E-tensor.
- $\mathcal{T}^*=\mathcal{W}\otimes_{\mathcal{O}_X}\Lambda^{*+1}E$ is the graded locally free $\mathcal{O}_X$-module of formal fiberwise polyvector fields on E with shifted degree. A homogeneous section of degree k of $\mathcal{T}^*$ can be locally written

$$\sum_{l=0}^{\infty} v^{j_0,\dots,j_k}_{i_1,\dots,i_l} y^{i_1}\cdots y^{i_l}\frac{\partial}{\partial y^{j_0}}\wedge\cdots\wedge\frac{\partial}{\partial y^{j_k}},$$

  where $v^{j_0,\dots,j_k}_{i_1,\dots,i_l}$ are components of an E-tensor symmetric covariant in the indices $i_1,\dots,i_l$, contravariant antisymmetric in the indices $j_0,\dots,j_k$.
- $\mathcal{D}^*=\widehat{S}(E^*)\otimes_{\mathcal{O}_X}T^{*+1}(S(E))$ is the graded locally free $\mathcal{O}_X$-module of formal fiberwise E-polydifferential operators with shifted degree. An homogeneous section of degree k of $\mathcal{D}^*$ can be locally written

$$\sum_{l=0}^{\infty} P^{\alpha_0,\dots,\alpha_k}_{i_1,\dots,i_l}(x) y^{i_1}\cdots y^{i_l}\frac{\partial^{|\alpha_0|}}{\partial y^{\alpha_0}}\otimes\cdots\otimes\frac{\partial^{|\alpha_k|}}{\partial y^{\alpha_k}},$$

  where the α_i's are multi-indices, the $P^{\alpha_0,\dots,\alpha_k}_{i_1,\dots,i_l}(x)$ are components of an E-tensor with obvious symmetry.

  We will need to introduce the $\mathcal{O}_X$-modules $\mathcal{D}^*(M)$ and $\mathcal{T}^*(M)$.
- $\mathcal{T}^*(M)$ is the graded $\mathcal{O}_X$-module of formal fiberwise polyvector fields on E with values in M with shifted degree. A homogeneous section of degree k of $\mathcal{T}^*(M)$ can be locally written

$$\sum_{l=0}^{\infty} m^{j_0,\dots,j_k}_{i_1,\dots,i_l} y^{i_1}\cdots y^{i_l}\frac{\partial}{\partial y^{j_0}}\wedge\cdots\wedge\frac{\partial}{\partial y^{j_k}},$$

  where $m^{j_0,\dots,j_k}_{i_1,\dots,i_l}$ are components of an E-tensor with values in M symmetric covariant in the indices $i_1,\dots,i_l$, contravariant antisymmetric in the indices $j_0,\dots,j_k$.

- $\mathcal{D}^*(M)$ is the graded $\mathcal{O}_X$-modules of formal fiberwise E-polydifferential operators with values in M (with shifted degree). A homogeneous section of degree k of $\mathcal{D}^*(M)$ can be locally written

$$\sum_{l=0}^{\infty}\mu_{i_1,\ldots,i_l}^{\alpha_0,\ldots,\alpha_k}(x)y^{i_1}\cdots y^{i_l}\frac{\partial^{|\alpha_0|}}{\partial y^{\alpha_0}}\otimes\cdots\otimes\frac{\partial^{|\alpha_k|}}{\partial y^{\alpha_k}},$$

where the α_i's are multi-indices, the $\mu_{i_1,\ldots,i_l}^{\alpha_0,\ldots,\alpha_k}(x)$ are coefficients of an E-tensor with values in M with obvious symmetry.

Remark 1. One has the obvious equality $\mathcal{T}^*(\mathcal{O}_X)=\mathcal{T}^*$ and $\mathcal{D}^*(\mathcal{O}_X)=\mathcal{D}^*$.

Notation. Let $\mathbb{R}^d_{\text{formal}}$ be the formal completion of $\mathbb{R}^d$ at the origin. The ring of functions on $\mathbb{R}^d_{\text{formal}}$ is $\mathbb{R}[[y^1,\ldots,y^d]]$ and the Lie–Rinehart algebra of vector fields is $\text{Der}(\mathbb{R}[[y^1,\ldots,y^d]])$. Denote by $T^*_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ and $D^*_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ the DGLAs of polyvector fields and polydifferential operators on $\mathbb{R}^d_{\text{formal}}$, respectively. If $t_1\in D^{k_1-1}_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ and $t_2\in D^{k_2-1}_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$, one defines their cup-product $t_1\sqcup t_2\in D^{k_1+k_2-1}_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ by:

$$\begin{aligned}&\forall\, a_1,\ldots,a_{k_1+k_2}\in\mathbb{R}[[y^1,\ldots,y^d]],\\ &\qquad (t_1\sqcup t_2)(a_1,\ldots,a_{k_1+k_2})=t_1(a_1,\ldots,a_{k_1})t_2(a_{k_1+1},\ldots,a_{k_1+k_2})\end{aligned}$$

The cup-product endows $D^*_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ with the structure of a DGAA.

Remark 2. The fiberwise product endows $\mathcal{W}$ with the structure of bundle of commutative algebra. $\mathcal{T}^*$ is a differential Lie algebra with zero differential and Lie bracket induced by the fiberwise Schouten bracket on $T_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$. Similarly, the fiberwise Schouten bracket allows us to endow $\mathcal{T}^*(M)$ with a $\mathcal{T}^*$-module structure. We can make the same type of remark for $\mathcal{D}$, $\mathcal{D}(M)$ and the Gerstenhaber bracket.

Let $\mathcal{B}$ be any of the $\mathcal{O}_X$-modules introduced above. We will need to tensor $\mathcal{B}$ by $\Lambda^*(E^*)$. We set ${}^E\Omega(\mathcal{B})=\Lambda^*(E^*)\otimes\mathcal{B}$.

Structures on ${}^E\Omega(\mathcal{B})$

- ${}^E\Omega(\mathcal{W})$ is a bundle of graded commutative algebras with grading given by exterior degree of E-forms.
- The Schouten bracket on $T^*_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ induces a structure of sheaf of graded Lie algebras over ${}^E\Omega^*(\mathcal{T})$. The grading is the sum of the exterior degree and the degree of an E-polyvector. The fiberwise Schouten bracket also endows ${}^E\Omega^*(\mathcal{T}(M))$ with a structure of module over the graded Lie algebra ${}^E\Omega^*(\mathcal{T})$. These structures will be respectively denoted by $[\ ,\ ]_S$ and $\cdot_S$. By fiberwise exterior product on $T^*_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$, ${}^E\Omega^*(\mathcal{T})$ also carries a structure of sheaf of graded commutative algebras and ${}^E\Omega^*(\mathcal{T}(M))$ becomes a module over the sheaf of graded commutative algebras ${}^E\Omega^*(\mathcal{T})$. These structures will both be denoted by a $\wedge$. Thus ${}^E\Omega^*(\mathcal{T}(M))$ is a module over the sheaf of Gerstenhaber algebras ${}^E\Omega(\mathcal{T})$.

- Using the fiberwise Gerstenhaber bracket, we see that ${}^E\Omega^*(\mathcal{D})$ is a sheaf of differential graded Lie algebras and ${}^E\Omega(\mathcal{D}(M))$ is a module over the sheaf of DGLAs ${}^E\Omega(\mathcal{D})$. These two structures will be denoted $[\ ,\ ]_G$ and $\cdot_G$. The grading is the sum of the exterior degree and the degree of the E-polydifferential operator. Cuproduct in the space $D^*_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ endows ${}^E\Omega(\mathcal{D})$ with the structure of a sheaf of DGAAs and ${}^E\Omega(\mathcal{D}(M))$ with the structure of a module over the sheaf of DGAAs ${}^E\Omega(\mathcal{D})$.

${}^E\Omega(\mathcal{W})$, ${}^E\Omega(\mathcal{T}(M))$, and ${}^E\Omega(\mathcal{D}(M))$ are equipped with a decreasing filtration given by the order of the monomials in the fiber coordinates y^i.

In the sequel, we will denote by ξ^i the variable y^i considered as an element of $\Lambda^1(E^*)$. Introduce the 2-nilpotent derivation $\delta : {}^E\Omega^*(\mathcal{W}) \to {}^E\Omega^{*+1}(\mathcal{W})$ of the sheaf of superalgebras ${}^E\Omega^*(\mathcal{W})$ defined by $\delta = \xi^i\partial/\partial y^i$. Using $\cdot_S$ and $\cdot_G$, δ extends to a 2-nilpotent differential of $\mathcal{T}(M)$ and $\mathcal{D}(M)$.

Proposition 11. *Let $\mathcal{B}$ be any of the sheaves $\mathcal{W}$, $\mathcal{T}(M)$, or $\mathcal{D}(M)$. Then*

$$H^{\geqslant 1}({}^E\Omega(\mathcal{B}), \delta) = 0.$$

Furthermore, we have the following isomorphisms of sheaves of graded $\mathcal{O}_X$-modules:

$$\begin{aligned} H^0({}^E\Omega(\mathcal{W}), \delta) &= \mathcal{O}_X, \\ H^0({}^E\Omega(\mathcal{T}(M)), \delta) &= {}^E T_{\mathrm{poly}}(M), \\ H^0({}^E\Omega(\mathcal{D}^*(M)), \delta) &= \bigotimes^{*+1} S(E) \otimes_{\mathcal{O}_X} M. \end{aligned}$$

This proposition is known for $\mathcal{W}$ and $M = \mathcal{O}_X$. It is due to Dolgushev ([D1]) for $E = TX$ and to Calaque ([C1]) for any Lie algebroid. Our proof is totally analogous to that of Dolgushev.

Proof of the proposition. Let us consider the operator $\kappa : {}^E\Omega^*(\mathcal{B}) \to {}^E\Omega^{*-1}(\mathcal{B})$ defined by

$$\forall\, \sigma \in \Omega^{>0}(\mathcal{T}(M)), \quad \kappa(\sigma) = y^m \frac{\partial}{\partial \xi^m} \int_0^1 \sigma(x, ty, t\xi) \frac{dt}{t}, \ \ \kappa|_{\mathcal{T}(M)} = 0.$$

It satisfies the relation

$$\delta\kappa + \kappa\delta + \mathcal{H} = \mathrm{id},$$

where

$$\forall\, u \in {}^E\Omega^*(\mathcal{B}), \quad \mathcal{H}(u) = u|_{y^i = \xi^i = 0}.$$

The proposition follows. □

Remark 3. We will keep using the operator κ in our proofs. Note that κ has the two following properties:

- $\kappa^2 = 0$;
- κ increases the filtration in the variables y^i's by one.

Let ∇ be a torsion-free connection on E. Let $(e_1, \ldots, e_n)$ be a local basis of E. Denote by $\Gamma^k_{i,j}$ the Christoffel symbol of ∇ with respect to this basis. As is explained in previous works ([D1], [C1], [D2], [CDH]) such a connection allows us to define a connection on $\mathcal{W}$ (still denoted ∇) as follows:

$$\nabla = {}^E\!d + \Gamma \cdot \quad \text{with} \quad \Gamma = -\xi^i \Gamma^k_{i,j} y^j \frac{\partial}{\partial y^k}.$$

It also allows us to define a connection on $\mathcal{T}(M)$ and $\mathcal{D}(M)$ given by

$$\nabla_M = {}^E\!d_M + \Gamma \cdot .$$

For example, if

$$\sigma = \sum_{l=0}^{\infty} m^{j_0,\ldots,j_k}_{i_1,\ldots,i_l} y^{i_1} \cdots y^{i_l} \frac{\partial}{\partial y^{j_0}} \wedge \cdots \wedge \frac{\partial}{\partial y^{j_k}}$$

is a local section of $\mathcal{T}(M)$, one has

$$\begin{aligned} \nabla_M(\sigma) = & \sum_{l=0}^{\infty} {}^E\!d_M(m^{j_0,\ldots,j_k}_{i_1,\ldots,i_l}) y^{i_1} \cdots y^{i_l} \frac{\partial}{\partial y^{j_0}} \wedge \cdots \wedge \frac{\partial}{\partial y^{j_k}} \\ & + \sum_{l=0}^{\infty} m^{j_0,\ldots,j_k}_{i_1,\ldots,i_l} \Gamma \cdot_S (y^{i_1} \cdots y^{i_l} \frac{\partial}{\partial y^{j_0}} \wedge \cdots \wedge \frac{\partial}{\partial y^{j_k}}). \end{aligned}$$

Since ∇ is torsion-free, one has $\nabla_M \delta + \delta \nabla_M = 0$. The curvature tensor allows us to define the following element of ${}^E\Omega^2(\mathcal{T}^0)$:

$$R = -\tfrac{1}{2} \xi^i \xi^j (R_{ij})^l_k(x) y^k \frac{\partial}{\partial y^l}.$$

A computation shows $\nabla_M^2 = R\cdot : {}^E\Omega^*(\mathcal{B}) \to {}^E\Omega^{*+2}(\mathcal{B})$.

Theorem 12. *Let $\mathcal{B}$ be any of the sheaves $\mathcal{T}(M)$ and $\mathcal{D}(M)$. There exists a section*

$$A = \sum_{s=2}^{\infty} \xi^k A^j_{k,i_1,\ldots,i_s}(x) y^{i_1} \cdots y^{i_s} \frac{\partial}{\partial y^j}$$

of the sheaf ${}^E\Omega^1(\mathcal{T}^0)$ such that the operator $D_M \colon {}^E\Omega^*(\mathcal{B}) \to {}^E\Omega^{*+1}(\mathcal{B})$

$$D_M = \nabla_M - \delta + A\cdot$$

is 2-nilpotent and is compatible with the DG-algebraic structures on ${}^E\Omega^*(\mathcal{B})$.

The theorem was proved for $\mathcal{B} = \mathcal{W}, \mathcal{T}$, and $\mathcal{D}$ in [D1] for $E = TX$ and in [C1] for any algebroid. Our proof is inspired by that of [D1] (see also [C1]).

Proof of the theorem. A computation shows that D_M is two-nilpotent if the following condition holds:

$$R + \nabla A - \delta A + \tfrac{1}{2}[A, A]_S = 0. \tag{1}$$

The following equation

$$A = \kappa R + \kappa(\nabla(A) + \tfrac{1}{2}[A, A]_S) \tag{2}$$

has a unique solution (computed by induction on the order in the fiber coordinates y^i's). It is shown in [D1] that the solution of equation (2) satisfies (1). We won't reproduce the proof here.

If α is in ${}^E\Omega(\mathcal{T})$ and μ is in ${}^E\Omega(\mathcal{T}(M))$, we have the relations

$$D(\alpha \wedge \mu) = D(\alpha) \wedge \mu + (-1)^{|\alpha|+1}\alpha \wedge D(\mu),$$
$$D(\alpha \cdot_S \mu) = D(\alpha) \cdot_S \mu + (-1)^{|\alpha|}\alpha \cdot_S D(\mu),$$

where $|\alpha|$ denotes the degree of α in the graded Lie algebra ${}^E\Omega(\mathcal{T})$. Similarly, if α is in ${}^E\Omega(\mathcal{D})$ and μ is in ${}^E\Omega(\mathcal{D}(M))$, we have the relations

$$D(\alpha \sqcup \mu) = D(\alpha) \sqcup \mu + (-1)^{|\alpha|+1}\alpha \sqcup D(\mu),$$
$$D(\alpha \cdot_G \mu) = D(\alpha) \cdot_G \mu + (-1)^{|\alpha|}\alpha \cdot_G D(\mu),$$

where $|\alpha|$ denotes the degree of α in the graded Lie algebra ${}^E\Omega(\mathcal{D})$. $\square$

One can compute the cohomology of the Fedosov differential D.

Theorem 13. *Let $\mathcal{B}$ be any of the sheaves ${}^E\Omega(\mathcal{W})$, ${}^E\Omega(\mathcal{T}(M))$, or ${}^E\Omega(\mathcal{D}(M))$. Then*

$$H^{\geqslant 1}(\mathcal{B}, D) = 0.$$

Furthermore, we have the following isomorphisms of sheaves of graded commutative algebras

$$H^0({}^E\Omega(\mathcal{W}), D) \simeq \mathcal{O}_X,$$
$$H^0({}^E\Omega(\mathcal{T}), D) \simeq \wedge^{*+1}E,$$

and the following isomorphism of sheaves of DGAAs (over $\mathbb{R}$)

$$H^0({}^E\Omega(\mathcal{D}), D) \simeq \bigotimes{}^{*+1}S(E).$$

Using the identification above, $H^0({}^E\Omega(\mathcal{T}(M)), D)$ and $\wedge^{*+1}E \otimes_{\mathcal{O}_X} M$ are isomorphic as $H^0({}^E\Omega(\mathcal{T}), D) \simeq \wedge^{*+1}E$-modules. Furthermore, $H^0({}^E\Omega(\mathcal{D}(M)), D)$ and $\bigotimes^{*+1} S(E) \otimes_{\mathcal{O}_X} M$ are isomorphic as $H^0({}^E\Omega(\mathcal{D}), D) \simeq \bigotimes^{*+1} S(E)$-modules.

This theorem is already known for $M = \mathcal{O}_X$: see [D1] for the case where $E = TX$ and [C1], [C2] for any Lie algebroid. The proof of the theorem is very similar to the proof in the case where $M = \mathcal{O}_X$. That is why we give only a sketch of it and refer to [CDH] and [C2] for details.

Proof of the theorem. The first assertion of the theorem follows from a spectral sequence argument using the filtration on $\mathcal{B}$ given by the order on the y^i's (see [CDH, Theorem 2.4] for details).

Let $u \in \mathcal{B} \cap \operatorname{Ker} \delta$. One can show (solving the equation by induction on the order in the fiber coordinates y^i's) that there exists a unique $\lambda(u) \in \mathcal{B} \cap \operatorname{Ker} D$ such that

$$\lambda(u) = u + \kappa(\nabla \lambda(u) + A \cdot \lambda(u)).$$

Thus, we have defined a map $\lambda : \operatorname{Ker} \delta \cap \mathcal{B} \to \operatorname{Ker} D \cap \mathcal{B}$. One can show that λ is bijective and that $\lambda^{-1} = \mathcal{H}$. The following relations (easy to establish) allows us to finish the proof of the theorem:

- If $\alpha, \beta \in {}^E\Omega(\mathcal{W})$, then $\mathcal{H}(\alpha\beta) = \mathcal{H}(\alpha)\mathcal{H}(\beta)$.
- If $\alpha \in {}^E\Omega(\mathcal{T})$ and $\mu \in {}^E\Omega(\mathcal{T}(M))$, then $\mathcal{H}(\alpha \wedge \mu) = \mathcal{H}(\alpha) \wedge \mathcal{H}(\mu)$.
- If $\alpha \in {}^E\Omega(\mathcal{D})$ and $\mu \in {}^E\Omega(\mathcal{D}(M))$, then $\mathcal{H}(\alpha \sqcup \mu) = \mathcal{H}(\alpha) \sqcup \mathcal{H}(\mu)$. □

As D is compatible with the action $\cdot_S$ of ${}^E\Omega^*(\mathcal{T})$ over ${}^E\Omega^*(\mathcal{T}(M))$ and hence with the Schouten bracket on ${}^E\Omega^*(\mathcal{T})$, $H^*({}^E\Omega(\mathcal{T}), D)$ is a graded Lie algebra and $H^*({}^E\Omega(\mathcal{T}(M)), D)$ is a module over the graded Lie algebra $H^*({}^E\Omega^*(\mathcal{T}), D)$. So, it is natural to wonder whether the isomorphisms of the previous proposition respect this structure.

Proposition 14. *The map $\mathcal{H} : \mathcal{T}^* \cap \operatorname{Ker} D \to \mathcal{T}^* \cap \operatorname{Ker} \delta \simeq {}^ET^*_{\text{poly}}$ is an isomorphism of graded Lie algebras.*

The map $\mathcal{H} : \mathcal{T}^*(M) \cap \operatorname{Ker} D \to {}^ET^*_{\text{poly}}(M)$ is an isomorphism of modules over the graded Lie algebras $\mathcal{T}^* \cap \operatorname{Ker} D \simeq {}^ET^*_{\text{poly}}$.

Proof of the proposition. The first assertion of the proposition is proved in [C1], [C2]. Let us now prove the second assertion. Denote by π the map from $D(E)$ to $\operatorname{End}(M)$ defined by the action of $D(E)$ on M.

Let m be an element of M and let $u = \sum_{i=1}^d u_i(x)e_i \in {}^ET^0_{\text{poly}}$. Using the definition of λ, one finds easily:

$$\lambda(m) = m + \sum_{i=1}^d y^i \pi(e_i) \cdot m \mod |y|,$$

$$\lambda(u) = \sum_{i=1}^d u_i \frac{\partial}{\partial y^i} \mod |y|.$$

Hence,

$$\lambda(u) \cdot \lambda(m) = \sum_{i=1}^d u_i \pi(e_i) \cdot m \mod |y|$$

and

$$\mathcal{H}(\lambda(u) \cdot \lambda(m)) = u \cdot m = \mathcal{H}(\lambda(u)) \cdot \mathcal{H}(\lambda(m)).$$

The end of the proof follows from the definition of the action of ${}^ET_{\text{poly}}$ on ${}^ET_{\text{poly}}(M)$ and the previous theorem. □

The morphism μ'_M

Let us first recall the construction of μ' ([CDH]). $\mathcal{T}^0$ is the sheaf of Lie algebras over the sheaf of algebras $\mathcal{T}^{-1} = \widehat{S}(E^*)$ and we have $\mathcal{D}^0 = D(\mathcal{T}^0)$. The morphism of Lie algebras $\lambda = \mathcal{H}^{-1} : E \to \mathcal{T}^0 \cap \operatorname{Ker} D$ induces a morphism of sheaves of algebras $\mu : D(E) \to \mathcal{D}^0$ that takes values in $\operatorname{Ker} D \cap \mathcal{D}^0$. We will denote by μ' the only morphism of sheaves of DGAAs from ${}^E D^*_{\text{poly}}$ to $\mathcal{D}^*$ defined by

$$\mu'|_{{}^E D^0_{\text{poly}}} = \mu, \quad \mu'|_{\mathcal{O}_X} = \lambda.$$

Let $\mu'_M : {}^E D^*_{\text{poly}}(M) \to \mathcal{D}^*(M)$ the morphism defined by:

$$\forall\, P_0, \dots, P_n \in D(E), \;\; \forall\, m \in M,$$
$$\mu'_M(m) = \lambda(m),$$
$$\mu'_M(P_0 \otimes \cdots \otimes P_n \otimes m) = \mu(P_0) \otimes \cdots \otimes \mu(P_n) \otimes \lambda(m).$$

Note that $\mu' = \mu'_{\mathcal{O}_X}$.

Proposition 15.

(a) *μ is an isomorphism of sheaves of algebras from $D(E)$ to $\mathcal{D}^0 \cap \operatorname{Ker} D$. It is also a morphism of sheaves of bialgebroids.*

(b) *μ' is an isomorphism of sheaves of DGLAs from ${}^E D^*_{\text{poly}}$ to $\mathcal{D}^* \cap \operatorname{Ker} D$. It is also an isomorphism of sheaves of DGAAs.*

(c) *$\mu'_M : {}^E D^*_{\text{poly}}(M) \to \mathcal{D}^*(M) \cap \operatorname{Ker} D$ is an isomorphism of modules over the sheaf of DGLAs ${}^E D^*_{\text{poly}} \simeq \mathcal{D}^* \cap \operatorname{Ker} D$. It is also an isomorphism of modules over the sheaf of DGAAs ${}^E D^*_{\text{poly}} \simeq \mathcal{D}^* \cap \operatorname{Ker} D$.*

Proof of the proposition. Parts (a) and (b) are shown in [CDH]. The proof of (c) is analogous. Using the definition of μ and μ'_M, one can easily show the following:

$$\forall\, P \in D(E), \;\; \forall\, m \in M, \;\; \mu'_M(P \cdot m) = \mu(P) \cdot \mu'_M(m).$$

As, moreover, μ is an isomorphism of bialgebroids ([CDH]), μ'_M is a morphism of modules over the sheaf of DGLAs ${}^E D^*_{\text{poly}} \simeq \mathcal{D}^* \cap \operatorname{Ker} D$. μ'_M is clearly a morphism of modules over the sheaf of DGAAs ${}^E D^*_{\text{poly}} \simeq \mathcal{D}^* \cap \operatorname{Ker} D$. The fact that μ'_M is an isomorphism follows from (a) and Theorem 13. $\square$

4.2. Kontsevitch's result

Recall that $\mathbb{R}^d_{\text{formal}}$ is the formal completion of $\mathbb{R}^d$ at the origin. The ring of functions on $\mathbb{R}^d_{\text{formal}}$ is $\mathbb{R}[[y^1, \dots, y^d]]$ and the Lie-Rinehart algebra of vector fields is $\operatorname{Der}(\mathbb{R}[[y^1, \dots, y^d]])$. Denote by $T^*_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ and $D^*_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ the DGLAs of polyvector fields and polydifferential operators on $\mathbb{R}^d_{\text{formal}}$ respectively.

Theorem 16. *There exists a quasi-isomorphism U of L_∞-algebras from the DGLA $T^*_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ to the DGLA $D^*_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ such that:*

(1) *The first structure map* $U^{[1]}$ *is the quasi-isomorphism* U_{HKR}.
(2) U *is* $\mathrm{GL}_d(\mathbb{R})$*-equivariant.*
(3) *If* $n>1$*, then for any vector fields* $v_1,\dots,v_n \in T^0_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$,

$$U^{[n]}(v_1,\dots,v_n)=0.$$

(4) *If* $n>1$*, then for any vector field* v *linear in the coordinates* y^i *and polyvector fields* $\chi_2,\dots,\chi_n \in T^*(\mathbb{R}^d_{\mathrm{formal}})$,

$$U^{[n]}(v,\chi_2,\dots,\chi_n)=0.$$

Moreover, Kontsevitch gives an explicit expression for $U^{[n]}$ ([Ko], see also [AMM] or [BCKT] for a detailed exposition) which involves admissible graphs.

Definition 7. Let n and m be two integers. An admissible graph Γ of type (n,m) is a labeled oriented graph satisfying the following properties. Let V_Γ be the set of vertices of Γ and let E_Γ be the set of edges of Γ:

(1) $V_\Gamma=\{1,\dots,n\}\sqcup\{\bar{1},\dots,\bar{m}\}$. Elements of $\{1,\dots,n\}$ are called first-type vertices and elements of $\{\bar{1},\dots,\bar{m}\}$ second-type vertices.
(2) Every edge of Γ starts from a first-type vertex.
(3) There is no loop.
(4) Two edges can't have the same source and the same target.

We will write $G_{n,m}$ for the set of admissible graphs with n first-type vertices and m second-type vertices. Let Γ be an element of $G_{n,m}$. We will denote by E_Γ the set of its edges. If γ is in E_Γ, then $s(\gamma)$ will be its source and $t(\gamma)$ its target. Let us introduce the following notation: If k is a vertex of first-type

$$(k,*)=\{\gamma\in E_\Gamma \mid s(\gamma)=k\}=\{e_k^1,\dots,e_k^{s_k}\}.$$

Similarly, the subset $(*,k)$ of E_Γ is defined for any vertex of Γ.

Let $\alpha_1,\dots,\alpha_n$ be n polyvector fields such that for any $j\in[1,n]$, α_j is an s_j polyvector field. We will associate to such $\alpha_1,\dots,\alpha_n$ an m polydifferential operator $B_\Gamma(\alpha_1,\dots,\alpha_n)$. Write

$$\alpha_j=\sum_{i_1,\dots,i_{s_j}}\alpha^{i_1,\dots,i_{s_j}}\partial_{i_1}\wedge\cdots\wedge\partial_{i_{s_j}} \text{ with } \partial_k=\frac{\partial}{\partial y^k}.$$

If $I: E_\Gamma\to\{1,\dots,d\}$ is a map from E_Γ to $\{1,\dots,d\}$, we set

$$D_{I(x)}=\prod_{e\in(*,x)}\partial_{I(e)}$$
$$\alpha_k^I=\alpha_k^{I(e_k^1),\dots,I(e_k^{s_k})}.$$

$B_\Gamma(\alpha_1\otimes\cdots\otimes\alpha_n)$ is the m-differential operator defined by: For any functions $f_1,\dots,f_m$,

$$B_\Gamma(\alpha_1\otimes\cdots\otimes\alpha_n)(f_1,\dots,f_m)=\sum_{I:E_\Gamma\to\{1,\dots,d\}}\prod_{k=1}^{k=n}D_{I(k)}\alpha_k^I\prod_{l=1}^{l=m}D_{I(\bar{l})}f_l.$$

If $\alpha_1, \dots, \alpha_n$ are any graded elements of T_{poly}, one has

$$U^{[n]}(\alpha_1, \dots, \alpha_n) = \sum_{\Gamma \in G_{n,m}} W_\Gamma B_\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n),$$

where the sum is taken over the graph Γ in $G_{n,m}$ such that $B_\Gamma(\alpha_1 \otimes \cdots \otimes \alpha_n)$ is defined and the relation $m - 2 + 2n = \sum_{i=1}^n s_k$ is satisfied. The coefficient W_Γ can be different from zero only if $|E_\Gamma| = 2n + m - 2$. Let us now describe it.

Let $\mathcal{H}$ be the Poincaré half-plane:

$$\mathcal{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}.$$

Introduce

$$\text{Conf}_{n,m} = \{(p_1, \dots, p_n, q_{\bar{1}}, \dots, q_{\bar{m}}) \in \mathcal{H}^n \times \mathbb{R}^m \mid p_i \neq p_j,\ q_{\bar{i}} \neq q_{\bar{j}}\}.$$

The group $G = \{z \mapsto az + b \mid (a, b) \in \mathbb{R}^{+*} \times \mathbb{R}\}$ acts freely on $\text{Conf}_{n,m}$. The quotient $C_{n,m} = \text{Conf}_{n,m}/G$ is a manifold of dimension $2n + m - 2$. As $\text{Conf}_{n,m}$ is naturally oriented and the action of G preserves this orientation, $C_{n,m}$ inherits a natural orientation. $C_{n,m}$ has several connected components, we will use one of them, $C^+_{n,m}$, defined by

$$C^+_{n,m} = \{(p_1, \dots, p_n, q_{\bar{1}}, \dots, q_{\bar{m}}) \mid q_{\bar{1}} < \cdots < q_{\bar{m}}\}.$$

If $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n\} \sqcup \{\bar{1}, \dots, \bar{m}\}$ (with $i \neq j$), one defines a function

$$\begin{aligned} \theta_{i,j} : C_{n,m} &\to \mathbb{R}/2\pi\mathbb{Z}, \\ (z_k)_{k \in [1,n] \sqcup [1,\bar{m}]} &\mapsto \frac{1}{2\pi} \text{Arg} \frac{z_j - z_i}{z_j - \overline{z_i}}. \end{aligned}$$

Let Γ be an element of $G_{n,m}$. We order E_Γ with the lexicographic order and define the closed form

$$\omega_\Gamma = \bigwedge_{\gamma \in E_\Gamma} d\theta_{s(\gamma), t(\gamma)}.$$

One then puts

$$W_\Gamma = \int_{C^+_{n,m}} \omega_\Gamma.$$

This integral is absolutely convergent as the integrand extends to a differential form on a compactification of $C^+_{n,m}$, $\overline{C^+_{n,m}}$, which is a manifold with corners of dimension $2n + m - 2$ ([Ko], see also [AMM] and [BCKT]).

Lemma 17. *Let n be a nonzero integer. For any polyvector fields $\gamma_1, \dots, \gamma_n$, one has*

$$U^{[n+1]}\left(\frac{\partial}{\partial y^i}, \gamma_1, \dots, \gamma_n\right) = 0.$$

Proof of the lemma. We will prove that for any Γ in $G_{n+1,m}$ having a contribution in $U^{[n+1]}$, one has $W_\Gamma = 0$. For such a Γ, there is no edge going to the vertex 1 and there is exactly one edge starting from the vertex 1 and going to a vertex i_0 which might be of first or second-type. We will denote by Γ' the element of $G_{n,m}$ obtained from Γ by removing the vertex 1 and the edge going from 1 to i_0.

First case: i_0 is of first-type

Using the action of G, we put p_{i_0} in i. If j is in $[1, n+1] - \{i_0\}$, we will write $z_j = a_j + ib_j$ for the affix of p_j and if k is in $[1, m]$, we will write t_k for the coordinate of q_k. One has

$$\omega_\Gamma = \frac{1}{2\pi} d\mathrm{Arg}\left(\frac{i - z_1}{i - \overline{z_1}}\right) \wedge \omega_{\Gamma'},$$

and $\omega_{\Gamma'}$ is a differential form of degree $2(n+1)+m-3$ in the $2(n-1)+m$ variables $da_2, db_2, \ldots, \widehat{da_{i_0}}, \widehat{db_{i_0}}, \ldots, da_{n+1}, db_{n+1}, dt_1, \ldots, dt_m$. Hence $\omega_{\Gamma'} = 0$ and $\omega_\Gamma = 0$.

Second case: i_0 is of second-type

We treat the case where $i_0 \neq \bar{m}$. The case where $i_0 = \bar{m}$ is treated analogously. Using the action of G, we put q_{i_0} in 0 and q_{i_0+1} in 1. One has

$$\omega_\Gamma = \frac{1}{\pi} d\mathrm{Arg}(z_1) \wedge \omega_{\Gamma'},$$

and $\omega_{\Gamma'}$ is a differential form of degree $2(n+1)+m-3$ in the $2n+m-2$ variables $a_2, b_2, \ldots, a_{n+1}, b_{n+1}, q_1, \ldots, \widehat{q_{i_0}}, \widehat{q_{i_0+1}}, \ldots, q_m$. Hence $\omega_{\Gamma'} = 0$ and $\omega_\Gamma = 0$. □

4.3. Proof of the formality theorem

The proof will follow [C2]. Before starting the proof, let's recall the following well-known fact of sheaf theory: If $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ are complexes of c-soft sheaves and if Θ is a quasi-isomorphism from $\mathcal{C}_1^*$ to $\mathcal{C}_2^*$, then $\Gamma(\Theta)$ is a quasi-isomorphism from $\Gamma(\mathcal{C}_1^*)$ to $\Gamma(\mathcal{C}_2^*)$.

We will adopt the following notation:

$\lambda_T^M : {}^ET^*_{\mathrm{poly}}(M) \to {}^E\Omega(\mathcal{T}(M))$ is the inverse of the map $\mathcal{H}$.
$\lambda_D^M : {}^ED^*_{\mathrm{poly}}(M) \to {}^E\Omega(\mathcal{D}(M))$ is the map μ'_M.

We set $\lambda_D^{\mathcal{O}_X} = \lambda_D$ and $\lambda_T^{\mathcal{O}_X} = \lambda_T$. From Kontsevitch's work (theorem 16, we know that there exists a fiberwise quasi-isomorphism $\mathcal{U}$ of L_∞-algebras from ${}^E\Omega(\mathcal{T})$ to ${}^E\Omega(\mathcal{D})$ whose Taylor coefficients will be denoted $\mathcal{U}^{[n]} : S^n({}^E\Omega(\mathcal{T})[1]) \to {}^E\Omega(\mathcal{D})$ (first we construct $\mathcal{U}$ on an open subset trivializing E and then glue the L_∞-morphisms). Using the explicit expression of $U^{[n]}$ ([Ko], [AMM]), one sees easily that $\mathcal{U}^{[n]}$ still make sense if we replace the last argument by an element of ${}^E\Omega(\mathcal{T}(M))$. Thus we define $\mathcal{V}^{[n]} : S^n({}^E\Omega(\mathcal{T})[1]) \otimes {}^E\Omega(\mathcal{T}(M)) \to {}^E\Omega(\mathcal{D}(M))$ by

$$\forall \gamma_1, \ldots, \gamma_n \in {}^E\Omega(\mathcal{T})[1], \ \forall \nu \in {}^E\Omega(\mathcal{T}(M)),$$
$$\mathcal{V}^{[n]}(\gamma_1, \ldots, \gamma_n, \nu) = \mathcal{U}^{[n+1]}(\gamma_1, \ldots, \gamma_n, \nu).$$

Thus we get the following diagram:

$$\begin{array}{ccc} ({}^E\Omega(\mathcal{T}), 0, [\ ,\ ]_S) & \xrightarrow{\ \mathcal{U}\ } & ({}^E\Omega(\mathcal{D}), \partial, [\ ,\ ]_G) \\ {\scriptstyle \cdot_S}\big\downarrow{\scriptstyle L_\infty\text{-mod}} & & {\scriptstyle \cdot_G}\big\downarrow{\scriptstyle L_\infty\text{-mod}} \\ ({}^E\Omega(\mathcal{T}(M)), 0, \cdot_S) & \xrightarrow{\ \mathcal{V}\ } & ({}^E\Omega(\mathcal{D}(M)), \partial_M, \cdot_G)\ . \end{array}$$

Let V be an open subset on which $E|_V$ is trivial. The differential ${}^Ed$ (resp., ${}^Ed_M$) is defined on ${}^E\Omega(\mathcal{T})|_V$ and ${}^E\Omega(\mathcal{D})|_V$ (resp., ${}^E\Omega(\mathcal{T}(M))|_V$ and ${}^E\Omega(\mathcal{D}(M))|_V$). As the quasi-isomorphisms of the previous diagram are fiberwise, we can add the differentials ${}^Ed$ and ${}^Ed_M$, in the previous quasi-isomorphism. We get a morphism of L_∞-algebras

$$\overline{\mathcal{U}}:\ \ ({}^E\Omega(\mathcal{T})|_V, {}^Ed, [\ ,\ ]_S) \to ({}^E\Omega(\mathcal{D})|_V, {}^Ed + \partial, [\ ,\ ]_G)$$

and a morphism of L_∞-modules over ${}^E\Omega(\mathcal{T})|_V$,

$$\overline{\mathcal{V}}:\ \ ({}^E\Omega(\mathcal{T}(M))|_V, {}^Ed_M, \cdot_S) \to ({}^E\Omega(\mathcal{D}(M))|_V, {}^Ed_M + \partial_M, \cdot_G).$$

We endow $\mathcal{B} = \mathcal{T}(M)|_V$ or $\mathcal{D}(M)|_V$ with the filtration

$$F^p({}^E\Omega(\mathcal{B})) = \bigoplus_{k\geqslant p} {}^E\Omega^k(\mathcal{B}).$$

A spectral sequence argument shows that $\overline{\mathcal{U}}$ and $\overline{\mathcal{V}}$ are quasi-isomorphisms (see [C2] and [CDH] for details). Thus, we have the following diagram where the horizontal arrows are quasi-isomorphisms

$$\begin{array}{ccc} ({}^E\Omega(\mathcal{T})|_V, {}^Ed, [\ ,\ ]_S) & \xrightarrow{\ \overline{\mathcal{U}}\ } & ({}^E\Omega(\mathcal{D})|_V, {}^Ed + \partial, [\ ,\ ]_G) \\ {\scriptstyle \cdot_S}\big\downarrow{\scriptstyle L_\infty\text{-mod}} & & {\scriptstyle \cdot_G}\big\downarrow{\scriptstyle L_\infty\text{-mod}} \\ ({}^E\Omega(\mathcal{T}(M))|_V, {}^Ed_M, \cdot_S) & \xrightarrow{\ \overline{\mathcal{V}}\ } & ({}^E\Omega(\mathcal{D}(M))|_V, {}^Ed_M + \partial_M, \cdot_G)\ . \end{array}$$

On V, the Fedosov differential can be written $D_M = {}^Ed_M + B$ with

$$B = \sum_{p=0}^{\infty} \xi^i B_{i,j_1,\ldots,j_p}(x) y^{j_1} \cdots y^{j_p} \frac{\partial}{\partial y^k}.$$

We set $D = D_{\mathcal{O}_X}$. The element B of ${}^E\Omega^1(\mathcal{T}^0)|_V$ is a Maurer Cartan element of the (filtered) sheaf of DGLAs $({}^E\Omega(\mathcal{T})|_V, {}^Ed, [\ ,\ ]_S)$. This means that $({}^E\Omega(\mathcal{T}(M))|_V, D_M, \cdot_S)$ is obtained from $({}^E\Omega(\mathcal{T}(M))|_V, {}^Ed_M, \cdot_S)$ via the twisting procedure by the Maurer Cartan element B ([D2]). We know that $\sum_{n\geqslant 1} \mathcal{U}^{[n]}(B^n)/n!$ is a Maurer Cartan section of $({}^E\Omega(\mathcal{D})|_V, {}^Ed + \partial, \cdot_G)$. But, due to property (3) of U, $\sum_{n\geqslant 1} \mathcal{U}^{[n]}(B^n)/n! = B$. Twisting $\overline{\mathcal{U}}$ and $\overline{\mathcal{V}}$ by the Maurer Cartan element

B ([D2]), we get the following diagram where the horizontal arrows are quasi-isomorphism

$$\begin{array}{ccc} ({}^E\Omega(\mathcal{T})|_V, D, [\ ,\ ]_S) & \xrightarrow{\overline{\mathcal{U}}^B} & ({}^E\Omega(\mathcal{D})|_V, D+\partial, [\ ,\ ]_G) \\ \cdot_S \Big\downarrow L_\infty\text{-mod} & & \cdot_G \Big\downarrow L_\infty\text{-mod} \\ ({}^E\Omega(\mathcal{T}(M))|_V, D_M, \cdot_S) & \xrightarrow{\overline{\mathcal{V}}^B} & ({}^E\Omega(\mathcal{D}(M))|_V, D_M+\partial_M, \cdot_G)\ . \end{array}$$

$\overline{\mathcal{U}}^B$ and $\overline{\mathcal{V}}^B$ do not depend on the choice of the trivialization of $E|_V$ and hence are well-defined morphisms of L_∞-algebras and L_∞-modules, respectively. Indeed, the only term in B that depends on the coordinates is $\Gamma = -\xi^i\Gamma^k_{i,j}y^j\partial/\partial y^k$ and it is linear in the fiber coordinates y^i so that it does neither contribute to $\overline{\mathcal{U}}^B$ nor to $\overline{\mathcal{V}}^B$ thanks to property (4) of U (see [D1], [C1],[D2], [CDH] for details). Hence $\overline{\mathcal{U}}^B$ and $\overline{\mathcal{V}}^B$ are defined globally and we get the following diagram:

$$\begin{array}{ccc} ({}^E\Omega(\mathcal{T}), D, [\ ,\ ]_S) & \xrightarrow{\overline{\mathcal{U}}^B} & ({}^E\Omega(\mathcal{D}), D+\partial, [\ ,\ ]_G) \\ \cdot_S \Big\downarrow L_\infty\text{-mod} & & \cdot_G \Big\downarrow L_\infty\text{-mod} \\ ({}^E\Omega(\mathcal{T}(M)), D_M, \cdot_S) & \xrightarrow{\overline{\mathcal{V}}^B} & ({}^E\Omega(\mathcal{D}(M)), D_M+\partial_M, \cdot_G)\ . \end{array}$$

The following lemma shows that the map $\lambda^M_D(X)$ (and hence $\lambda_D(X)$) is a quasi-isomorphism from $\left[\Gamma({}^E D_{\mathrm{poly}}(M)), \partial_M\right]$ to $\left[\Gamma({}^E\Omega(\mathcal{D}(M))), D_M + \partial_M\right]$.

Lemma 18. *The natural inclusion*

$$\iota\colon [\Gamma(\mathcal{D}^*(M) \cap \operatorname{Ker} D_M), \partial_M] \hookrightarrow [\Gamma(\Omega^*(\mathcal{D}(M))), D_M + \partial_M]$$

is a quasi-isomorphism.

Proof of the lemma. Consider a decomposition of $\operatorname{Ker}(D_M + \partial_M)$ of the form

$$Y \oplus \operatorname{Im}(D_M + \partial_M) = \operatorname{Ker}(D_M + \partial_M).$$

One may construct a map $V : \operatorname{Ker}(D_M + \partial_M) \to \Gamma(\Omega(\mathcal{D}(M)))$ such that:

(i) for any x in $\operatorname{Ker}(D_M+\partial_M)$, $x-(D_M+\partial_M)(V(x)) \in \Gamma(\mathcal{D}(M)\cap \operatorname{Ker} D_M)$.
(ii) If $x \in \operatorname{Im}(D_M + \partial_M)$, $V(x)$ is a preimage of x by $D_M + \partial_M$.

It is enough to construct $V(x)$ for x in Y. Write $x = x_r + \cdots + x_0$ with $x_i \in \Gamma(\Omega^i(\mathcal{D}(M)))$. The equality $(D_M+\partial_M)(x) = 0$ implies $D_M(x_r) = 0$ (because ∂_M preserves the exterior degree). Then using the exactness of D_M, we construct a map $V_r : Y \to \Gamma(\Omega^{\leqslant r-1}(\mathcal{D}(M)))$ such that for any x in Y, $x - (D_M + \partial_M)V_r(x)$ has maximal exterior degree inferior or equal to $r - 1$. Going on like this, we construct V.

We may now exhibit an inverse to $H^i(\iota)$. With obvious notations, we have

$$H^i(\iota)^{-1}([\mu]) = [\mu - (D_M + \partial_M)V(\mu)].$$

This finishes the proof of the lemma. □

As $\lambda_D^M(X)$ is a quasi-isomorphism of L_∞-modules over $\Gamma({}^E D^*_{\rm poly})$, there exists a quasi-isomorphism of L_∞-modules over $\Gamma({}^E D^*_{\rm poly})$,

$$\alpha_D^M\colon \left[\Gamma({}^E\Omega(\mathcal{D}(M))), D_M + \partial_M\right] \to \left[\Gamma({}^E D^*_{\rm poly}(M)), \partial_M\right],$$

such that $H^i(\alpha_D^{M[1]}) = H^i(\lambda_D^M)^{-1}$ (see [AMM] for the case of L_∞ algebras). The morphism $\mathcal{V}_M = \alpha_D^M \circ \overline{\mathcal{V}}^B(X) \circ \lambda_T^M(X)$ is a quasi-isomorphism of L_∞-modules over $\Gamma({}^E T^*_{\rm poly})$ from $\Gamma({}^E T^*_{\rm poly}(M))$ to $\Gamma({}^E D^*_{\rm poly}(M))$. One checks easily that $\mathcal{V}_M^{[0]}$ induces $U^M_{\rm HKR}$ in cohomology.

Inverting λ_D into a quasi-isomorphism of L_∞ algebras provides Calaque's quasi-isomorphism of L_∞ algebras Υ from $\Gamma({}^E T^*_{\rm poly})$ to $\Gamma({}^E D^*_{\rm poly})$ ([C2]). This finishes the proof of the Theorem 10. □

4.4. Local expression of $\mathcal{V}_M$ in the case of the tangent bundle of $\mathbb{R}^d$

In this section we assume that $X = \mathbb{R}^d$ and $E = T\mathbb{R}^d$. We choose the connection whose Christoffel symbols are 0. Thus, we have

$$\nabla\left(f\frac{\partial}{\partial x^i}\right) = df\frac{\partial}{\partial x^i}.$$

In this case $A = 0$ and $D = d_E - \delta$. If u is in ${}^E T_{\rm poly}(M)$ or ${}^E D_{\rm poly}(M)$, a computation shows that

$$\lambda(u) = \sum_{\alpha_1,\dots,\alpha_d} \frac{(y^1)^{\alpha_1}}{\alpha_1!}\cdots\frac{(y^d)^{\alpha_d}}{\alpha_d!}\left[\left(\frac{\partial}{\partial x^1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x^d}\right)^{\alpha_d}\right]\cdot u.$$

For example,

$$\begin{aligned}
&\lambda_T(\gamma^{j_1,\dots,j_p}\frac{\partial}{\partial x^{j_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{j_p}})\\
&\quad= \sum_{\alpha_1,\dots,\alpha_d} \frac{(y^1)^{\alpha_1}}{\alpha_1!}\cdots\frac{(y^d)^{\alpha_d}}{\alpha_d!}\left[\left(\frac{\partial}{\partial x^1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x^d}\right)^{\alpha_d}(\gamma^{j_1,\dots,j_p})\right]\frac{\partial}{\partial y^{j_1}}\wedge\cdots\wedge\frac{\partial}{\partial y^{j_p}}\\
&\quad= \lambda_T(\gamma^{j_1,\dots,j_p})\frac{\partial}{\partial y^{j_1}}\wedge\cdots\wedge\frac{\partial}{\partial y^{j_p}}.
\end{aligned}$$

From Lemma 17, we see that $\overline{\mathcal{V}}^B = \overline{\mathcal{V}}$. If a is in $\mathcal{O}_{\mathbb{R}^d}$, one has

$$\frac{\partial}{\partial y^i}\lambda(a) = \lambda(\frac{\partial a}{\partial x^i}).$$

Then it is easy to see that in this special case $\overline{\mathcal{V}}\circ\lambda_T$ takes its values in $\mathcal{D}_{\rm poly}\cap{\rm Ker}\, D$.

B_Γ makes sense if we change the last argument by a polydifferential operator with coefficients in M and it is not hard to see that

$$\mathcal{V}_M^{[n]} = \sum_{\Gamma\in G_{n+1,m}} W_\Gamma B_\Gamma.$$

5. Applications

In this section we set $O=\Gamma(\mathcal{O}_X)$. Let E be a Lie algebroid, $\mathcal{M}$ a $D(E)$-module and $M=\Gamma(\mathcal{M})$. We denote by $\mathcal{V}_{\mathcal{M}}$ the quasi-isomorphism of L_∞-modules over $\Gamma({}^ET^*_{\rm poly})[[h]]$ from $\Gamma({}^ET^*_{\rm poly}(\mathcal{M}))[[h]]$ to $\Gamma({}^ED^*_{\rm poly}(\mathcal{M}))[[h]]$ given by Theorem 10. Then $\mathcal{V}_{\mathcal{O}_X}=\Upsilon$ is the L_∞-quasi-isomorphism of DGLAs from $\Gamma({}^ET^*_{\rm poly})[[h]]$ to $\Gamma({}^ED^*_{\rm poly})[[h]]$ constructed by Calaque ([C1]). Let π_h be a Maurer Cartan element of $\Gamma({}^ET^*_{\rm poly})[[h]]$. This means that

$$\pi_h\in\Gamma({}^ET^1_{\rm poly})[[h]] \text{ and } [\pi_h,\pi_h]_S=0.$$

Then it is well-known that $\sum_{n\geqslant 1}(1/n!)\Upsilon^{[n]}(\pi_h,\dots,\pi_h)$ is a Maurer Cartan element of $\Gamma({}^ED^*_{\rm poly})[[h]]$ (see [AMM, p. 63]). We set

$$\Pi_h=1\otimes 1+\sum_{n\geqslant 1}\frac{1}{n!}\Upsilon^{[n]}(\pi_h,\dots,\pi_h).$$

As $\Gamma({}^ET^*_{\rm poly}(\mathcal{M}))[[h]]$ is a module over the DGLA $\Gamma({}^ET^*_{\rm poly})[[h]]$, the map

$$\begin{aligned}\pi_h\cdot_S- : \Gamma({}^ET^k_{\rm poly}(\mathcal{M}))[[h]] &\to \Gamma({}^ET^{k+1}_{\rm poly}(\mathcal{M}))[[h]],\\ y&\mapsto \pi_h\cdot_S y,\end{aligned}$$

is a differential over $\Gamma({}^ET^*_{\rm poly}(\mathcal{M}))[[h]]$ (see [D2, Prop. 3 of Sect. 2.3]). Similarly, $\Pi_h\cdot_G-$ defines a differential on $\Gamma({}^ED^*_{\rm poly}(\mathcal{M}))[[h]]$.

Proposition 19. *The map*

$$\begin{aligned}(\mathcal{V}_{\mathcal{M}})'_\pi:\ (\Gamma({}^ET^*_{\rm poly}(\mathcal{M}))[[h]],\pi_h\cdot_S-) &\to (\Gamma({}^ED^*_{\rm poly}(\mathcal{M}))[[h]],\Pi_h\cdot_G-),\\ y&\mapsto\sum_{p=0}^{\infty}\frac{1}{p!}\mathcal{V}^{[p]}_{\mathcal{M}}(\pi_h,\dots,\pi_h,y)\,,\end{aligned}$$

is a quasi-isomorphism.

Proof of the proposition. The proposition follows from proposition 3 of paragraph 2.3 of [D2] and the definition of the L_∞-module structure of $\Gamma({}^ED^*_{\rm poly}(\mathcal{M}))$ over $\Gamma({}^ET^*_{\rm poly})$. □

If E is a Lie algebroid equipped with an E-bivector $\pi\in\Gamma(\Lambda^2E)$ satisfying $[\pi,\pi]=0$, it is called a Poisson Lie algebroid. If $E=TX$, we recover Poisson manifolds. Then, one can construct a Lie algebroid structure on E^* in the following way. Let $\pi^\sharp$ be the bundle map from E^* to E associated to π and $\omega_*=\omega\circ\pi^\sharp$: $E^*\to TX$. Define a Lie bracket on E^* by

$$\forall\,\theta,\eta\in E^*,\quad [\theta,\eta]=L_{\pi^\sharp\theta}(\eta)-L_{\pi^\sharp\eta}(\theta)-d\pi(\theta,\eta),$$

where L denotes the Lie derivative. Then E^*, endowed with the bracket above and the anchor ω_*, is a Lie algebroid ([KM], [MX]) and E is a Lie bialgebroid.

The differential of the Lie cohomology complex of E^* is $d_* = [\pi, -] : \Gamma(\Lambda^k E) \to \Gamma(\Lambda^{k+1} E)$.

Assume that we are in the case where E is a Poisson Lie algebroid with Poisson bivector π. Then, in the proposition above one may take $\pi_h = h\pi$ and Calaque ([C1]) shows that Π_h is a twistor for the standard Hopf algebroid $U(\Gamma(E))[[h]]$ (see [X]).

From now on we assume that $E = TX$ and that π is a Poisson bracket on X. Then the twistor Π_h defines a star product on $O[[h]]$ (see [X]) in the following way:

$$\forall\,(f, g) \in O, \;\; \Pi_h(f, g) = f *_h g.$$

Set

$$f *_h g = fg + \sum_{i=1}^{\infty} B_i(f, g) h^i.$$

Proposition 20. *$M[[h]]$ can be endowed with an $O[[h]] \otimes O[[h]]^{\mathrm{op}}$-module structure as follows: For all a in O and all m in M,*

$$a * m = a \cdot m + \sum_{i=1}^{\infty} h^i B_i(a, -) \cdot m, \quad m * a = a \cdot m + \sum_{i=1}^{\infty} h^i B_i(-, a) \cdot m.$$

Proof of the proposition. The proof of the proposition is a straightforward verification using the associativity of the star product. □

Applying the exact functor $N \mapsto N[[h]]$, we get an injection

$$\Gamma({}^E D^k_{\mathrm{poly}}(\mathcal{M}))[[h]] \hookrightarrow \mathrm{Hom}_{\mathbb{R}[[h]]}(O[[h]]^{\otimes^{k+1}_{\mathbb{R}[[h]]}}, M[[h]]).$$

The image of $\Gamma({}^E D^*_{\mathrm{poly}}(\mathcal{M}))[[h]]$ in $\mathrm{Hom}_{\mathbb{R}[[h]]}(O[[h]]^{\otimes^{*+1}_{\mathbb{R}[[h]]}}, M[[h]])$ will be denoted $\mathrm{Homdiff}_{\mathbb{R}[[h]]}(O[[h]]^{\otimes^{*+1}_{\mathbb{R}[[h]]}}, M[[h]])$.

Recall that the Hochschild cohomology of $O[[h]]$ with values in the bimodule $M[[h]]$, $HH^*(O[[h]], M[[h]])$, is the cohomology of the complex

$$(\mathrm{Hom}_{\mathbb{R}[[h]]}(O[[h]]^{\otimes^{*}_{\mathbb{R}[[h]]}}, M[[h]]), \beta)$$

where, with obvious notations,

$$\begin{aligned}\beta(\lambda)(a_1, \ldots, a_{n+1}) = {} & a_1 * \lambda(a_2, \ldots, a_{n+1}) \\ & + \sum_{0<i<n+1} (-1)^i \lambda(a_1, \ldots, a_i * a_{i+1}, \ldots, a_{n+1}) \\ & + (-1)^{n+1} \lambda(a_1, \ldots, a_n) * a_{n+1}.\end{aligned}$$

Denote by $HH^*_{\mathrm{md}}(O[[h]], M[[h]])$ the cohomology of the complex $(\mathrm{Homdiff}_{\mathbb{R}[[h]]}(O[[h]]^{\otimes^*}, M[[h]]), \beta)$.

The complex $(\Gamma({}^E T^*_{\mathrm{poly}}(\mathcal{M}))[[h]], \pi_h \cdot_S)$ computes the Lichnerowicz–Poisson cohomology of the $\mathbb{R}[[h]]$-Poisson algebra (defined by the bivector π_h) $O[[h]]$ with coefficients in $M[[h]]$,

$$H^i_{\mathrm{Poisson}}(O[[h]], M[[h]])$$

([Li],[Hu]). The complex $(\Gamma({}^E D^*_{\mathrm{poly}}(\mathcal{M}))[[h]], \Pi_h \cdot_G)$ computes $HH^*_{\mathrm{md}}(O[[h]], M[[h]])$. We get the following corollary.

Corollary 21. *One has an isomorphism*

$$H^i_{\text{Poisson}}(O[[h]], M[[h]]) \simeq HH^i_{\text{md}}(O[[h]], M[[h]]).$$

The exterior product, which will be denoted by $\wedge$, endows $H^*(\Gamma({}^E T^*_{\text{poly}}), [\pi_h, \cdot])$ with an associative supercommutative algebra structure. It also endows $H^*(\Gamma({}^E T^*_{\text{poly}}(\mathcal{M})), \pi_h \cdot_S)$ with a $[H^*(\Gamma({}^E T^*_{\text{poly}}), [\pi_h, \cdot]), \wedge]$-module structure.

To simplify the notation, from now on, we write Π instead of Π_h. D^*_{poly} is endowed with an associative graded product, $\sqcup_\Pi$, compatible with the differential $[\Pi, \cdot]$ defined by

$$\forall\, t_1 \in \Gamma(D^{k_1-1}_{\text{poly}}),\ \forall\, t_2 \in \Gamma(D^{k_2-1}_{\text{poly}}),\ \forall\, a_1, \dots, a_{k_1+k_2} \in O,$$
$$(t_1 \sqcup_\Pi t_2)(a_1, \dots, a_{k_1+k_2}) = t_1(a_1, \dots, a_{k_1}) \star_h t_2(a_{k_1+1}, \dots, a_{k_1+k_2}).$$

Thus, $[H^*(\Gamma(D_{\text{poly}}), [\Pi, \cdot]), \sqcup_\Pi]$ is an associative graded algebra. Notice that $t_1 \sqcup_\Pi t_2$ is also defined if $t_2 \in \Gamma(D^{k_2-1}_{\text{poly}}(\mathcal{M}))$. Thus, $[H^*(\Gamma(D_{\text{poly}}(\mathcal{M})), \Pi \cdot_G), \sqcup_\Pi]$ is a $[H^*(\Gamma(D_{\text{poly}}), [\Pi, \cdot]), \sqcup_\Pi]$-module.

If $X = \mathbb{R}^d$ and $E = T\mathbb{R}^d$, Kontsevich has proved ([Ko], see [MT] for a detailed proof) that the algebras $[H^*(\Gamma(T^*_{\text{poly}}), [\pi_h, \cdot]), \wedge]$ and $[H^*(\Gamma(D_{\text{poly}}), [\Pi, \cdot]), \sqcup_\Pi]$ are isomorphic. We will extend this result to our case.

Remark 4. In [CFT], a star product $*$ is constructed on any manifold X so that the algebras $[H^0(\Gamma(T^*_{\text{poly}}), [\pi_h, \cdot]), \wedge]$ and $[H^0(\Gamma(D_{\text{poly}}), [*, \cdot]), \sqcup_\Pi]$ are isomorphic.

Theorem 22. *Assume that $X = \mathbb{R}^d$ and $E = T\mathbb{R}^d$. The $[H^*(\Gamma(T^*_{\text{poly}}), [\pi_h, \cdot]), \wedge] \simeq [H^*(\Gamma(D_{\text{poly}}), [\Pi, \cdot]), \sqcup_\Pi]$-modules*

$$[H^*(\Gamma(T^*_{\text{poly}}(\mathcal{M})), \pi_h \cdot_S), \wedge] \quad \text{and} \quad [H^*(\Gamma(D_{\text{poly}}(\mathcal{M})), \Pi \cdot_G), \sqcup_\Pi]$$

are isomorphic.

Proof of Theorem 22. In this proof we keep the notations of the proof of the formality theorem (Section 4.3). We could reproduce the proof of [MT] using the explicit expression we found for $\mathcal{V}_\mathcal{M}$ in Section 4.4. We will use the decomposition $\mathcal{V}_\mathcal{M} = \lambda_D^{-1} \circ \overline{\mathcal{V}} \circ \lambda_T$ and use the results of [MT]. Put

$$\overline{\Pi} = \sum_{n \geqslant 1} \mathcal{U}^{[n]}(\lambda_T(\pi_h), \dots, \lambda_T(\pi_h)).$$

Lemma 23. *Let k_1 and k_2 be in $\mathbb{N}$. If $\tau_1 \in \Gamma(\mathcal{T}^{k_1-1}_{\text{poly}})$, $\tau_2 \in \Gamma(\mathcal{T}^{k_2-1}_{\text{poly}}(\mathcal{M}))$ and $m = k_1 + k_2$, then one has*

$$\begin{aligned}
&\overline{\mathcal{V}}'_{\lambda_T(\pi_h)}(\tau_1 \wedge \tau_2) - \overline{\mathcal{U}}'_{\lambda_T(\pi_h)}(\tau_1) \sqcup_{\overline{\Pi}} \overline{\mathcal{V}}'_{\lambda_T(\pi_h)}(\tau_2)\\
&\quad = \sum_{n \geqslant 0} \frac{h^n}{n!} \sum_{\Delta \in G_{n+2,m-1}} a_\Delta \overline{\Pi} \cdot_G B_\Delta(\lambda_T(\pi) \otimes \cdots \otimes \lambda_T(\pi) \otimes \tau_1 \otimes \tau_2)\\
&\qquad + \sum_{n \geqslant 0} \frac{h^n}{n!} \sum_{\Delta \in G_{n+1,m}} b_\Delta (-1)^{(k_1-1)k_2} B_\Delta(\lambda_T(\pi) \otimes \cdots \otimes \lambda_T(\pi) \otimes [\lambda_T(\pi), \tau_1] \otimes \tau_2)\\
&\quad = \sum_{n \geqslant 0} \frac{h^n}{n!} \sum_{\Delta \in G_{n+1,m}} b_\Delta (-1)^{k_1(k_2-1)} B_\Delta(\lambda_T(\pi) \otimes \cdots \otimes \lambda_T(\pi) \otimes \tau_1 \otimes \lambda_T(\pi) \cdot_S \tau_2)
\end{aligned}$$

where a_Δ and b_Δ are real.

Proof of Lemma 23. Lemma 23 is proved for $\mathcal{M} = \mathcal{O}_X$ in [MT]. Actually, the formula of Lemma 23 is slightly different from that of [MT]. To get it, one has to reproduce the proof of [MT] and make play to the vertices $n-1$ and n the role played by the vertices 1 and 2. Hence Lemma 23 holds for τ_2 in $\Gamma(\mathcal{T}_{\rm poly}^{k_2-1})\otimes_O M$. We will now prove that it is true for τ_2 in $\Gamma(\mathcal{T}_{\rm poly}^{k_2-1}(\mathcal{M}))$. If we apply it to $(f_1,\dots,f_m)$ in $\mathbb{R}[[y^1,\dots,y^d]]^m$, the relation of Lemma 23 can be written $\sum_{n\geqslant 0} h^n F_n = \sum_{n\geqslant 0} h^n G_n$ where the F_n's and the G_n's are maps from $\Gamma(\mathcal{T}_{\rm poly}^{k_2-1})\otimes_O M$ to $M[[y^1,\dots,y^d]]$. Let I be the ideal of $O[[y^1,\dots,y^d]]$ generated by $y^1,\dots,y^d$. The F_n's and the G_n's are continuous for the I-adic topology. This is a consequence of the following two remarks.

- Let $\gamma_1,\dots,\gamma_p$ be elements of $\Gamma(\mathcal{T}_{\rm poly})$ and let $(g_1,\dots,g_m)$ be elements of $O[[y^1,\dots,y^d]]$. Let Γ be an admissible graph of type $(p+1,m)$. The map

$$\begin{aligned}\Gamma(\mathcal{T}_{\rm poly}^{k_2-1})\otimes_O M &\to M[[y^1,\dots,y^d]],\\ \mu &\mapsto B_\Gamma(\gamma_1,\dots,\gamma_p,\mu)(g_1,\dots,g_m),\end{aligned}$$

  is continuous for the I-adic topology as it sends $I^N\Gamma(\mathcal{T}_{\rm poly}^{k_2-1})\otimes_O M$ to $I^{N-p}M[[y^1,\dots,y^d]]$.
- Let Γ be an admissible graph of type $(p,2)$ and let g be an element of $O[[y^1,\dots,y^d]]$. The map

$$\begin{aligned}O[[y^1,\dots,y^d]]\otimes_O M &\to M[[y^1,\dots,y^d]],\\ \mu &\mapsto B_\Gamma(\lambda_T(\pi),\dots,\lambda_T(\pi))(f,\mu),\end{aligned}$$

  is continuous for the I-adic topology as it sends $I^N O[[y^1,\dots,y^d]]\otimes_O M$ to $I^{N-p}M[[y^1,\dots,y^d]]$.

This finishes the proof of the lemma 23. □

Now, we go back to the proof of Theorem 22.

Let t_1 be in $\Gamma(T_{\rm poly}^{k_1-1})[[h]]\cap \mathrm{Ker}\,[\pi_h,]$ and let t_2 be in $\Gamma(\mathcal{T}_{\rm poly}^{k_2-1}(\mathcal{M}))[[h]]\cap \mathrm{Ker}\,(\pi_h\cdot_S)$. We apply Lemma 23 to $\tau_1=\lambda_T(t_1)$ and $\tau_2=\lambda_T^{\mathcal{M}}(t_2)$. We get

$$\begin{aligned}&\overline{\mathcal{V}}'_{\lambda_T(\pi_h)}(\lambda_T(t_1)\wedge\lambda_T^{\mathcal{M}}(t_2)) - \overline{\mathcal{U}}'_{\lambda_T(\pi_h)}(\lambda_T(t_1))\sqcup_{\overline{\Pi}}\overline{\mathcal{V}}'_{\lambda_T(\pi_h)}(\lambda_T^{\mathcal{M}}(t_2))\\ &\quad=\sum_{n\geqslant 0}\frac{h^n}{n!}\sum_{\Delta\in G_{n+2,m-1}} a_\Delta\overline{\Pi}\cdot_G B_\Delta(\lambda_T(\pi)\otimes\cdots\otimes\lambda_T(\pi)\otimes\lambda_T(t_1)\otimes\lambda_T^{\mathcal{M}}(t_2)).\end{aligned}$$

Apply $(\lambda_D^{\mathcal{M}})^{-1}$ and use the following facts:

- $\lambda_D^{-1}(\overline{\Pi})=\Pi$.
- With obvious notations, one has

$$\lambda_D(\sigma_1)\sqcup_{\overline{\Pi}}\lambda_D^{\mathcal{M}}(\sigma_2)=\lambda_D^{\mathcal{M}}(\sigma_1\sqcup_\Pi\sigma_2).$$

- $B_\Delta(\lambda_T(\pi),\dots,\lambda_T(\pi),\lambda_T(t_1),\lambda_T^{\mathcal{M}}(t_2))=\lambda_D^{\mathcal{M}}(B_\Delta(\pi,\dots,\pi,t_1,t_2)).$

We get

$$(\mathcal{V}_{\mathcal{M}})'_\pi(t_1 \wedge t_2) - \mathcal{U}'_\pi(t_1) \sqcup_\Pi (\mathcal{V}_{\mathcal{M}})'_\pi(t_2) \\ = \sum_{n \geqslant 0} \frac{h^n}{n!} \sum_{\Delta \in G_{n+2,m-1}} a_\Delta \Pi \cdot_G B_\Delta(\pi \otimes \cdots \otimes \pi \otimes t_1 \otimes t_2).$$

The right-hand side is a coboundary for the Hochschild cohomology complex. This finishes the proof of Theorem 22. □

Remark 5. Assume that X is the dual of a real Lie algebra endowed with its Kirillov–Kostant–Souriau Poisson structure denoted by π. Recall that if ξ and η are elements of $\mathfrak{g}$ considered as linear forms on $\mathfrak{g}^*$, then

$$\pi(\xi, \eta) = [\xi, \eta].$$

If $M = \mathcal{O}_X$, the isomorphism given by Theorem 22 has been studied. If $i = 0$, it gives Duflo's isomorphism ([Du], [Ko]). By analyzing which graphs contributes to $(\mathcal{V}_{\mathcal{O}_X})'_\pi = \Upsilon'_\pi$, Pevsner and Torossian [PT] have shown that Duflo's isomorphism extends to an isomorphism from $H^*_{\text{Poisson}}(\mathfrak{g}, S(\mathfrak{g}))$ to $H^*(\mathfrak{g}, U(\mathfrak{g}))$.

References

[AK] R. Almeida, A. Kumpera, *Structure produit dans la catégorie des algébroïdes de Lie*, Ann. Acad. Brasil. Cienc. **53** (1981), 247–250.

[AMM] D. Arnal, D. Manchon, M. Masmoudi, *Choix des signes pour la formalité de M. Kontsevich*, Pacific J. Math. **203** (2002), no. 1, 23–66.

[Bo] A. Borel, *Algebraic D-Modules*, Academic Press, New York, 1987.

[BCKT] A. Bruguières, A. Cattaneo, B. Keller, C. Torossian, *Déformation, Quantification, Théorie de Lie*, Panoramas et Synthèse, Soc. Math. France, Paris, 2005.

[C1] D. Calaque, *Formality for Lie algebroids*, Comm. Math. Phys. **257** (2005), no. 3, 563–578.

[C2] D.Calaque, *Théorèmes de Formalité pour les Algébroïdes de Lie et Quantification des r-Matrices Dynamiques*, Thèse de l'IRMA.

[CDH] D. Calaque, V. Dolgushev, G. Halbout, *Formality theorem for Hochschild chains in the Lie algebroid setting*, J. Reine Angew. Math. **612** (2007), 81–127.

[CFT] A. S. Cattaneo, G. Felder, L.Tomassini, *From local to global deformation quantization of Poisson manifolds*, Duke Math. J. **115** (2002), no. 2, 329–352.

[Ch1] S. Chemla, *Poincaré duality for k−A-Lie superalgebras*, Bull. Soc. Math. France **122** (1994), 371–397.

[Ch2] S. Chemla, *A duality property for complex Lie algebroids*, Math. Z. **232** (1999), 367–388.

[Ch3] S. Chemla, *An inverse image functor for Lie algebroids*, J. Algebra **269** (2003), 109–135.

[D1] V. Dolgushev, *Covariant and equivariant formality theorem*, Adv. Math. **191**, 1 (2005) 147–177 (math.QA/0307212).

[D2] V. Dolgushev, *A formality theorem for Hochschild chains*, Adv. Math. **200** (2006), no. 1, 51–101.

[D3] V. Dolgushev, *A Proof of Tsygan's Formality Conjecture for Arbitrary Smooth Manifolds*, PhD Thesis, `math QA/0504420`.

[Du] M. Duflo, *Opérateurs différentiels bi-invariants sur un groupe de Lie*, Ann. Sci. Ecole Norm. Sup. **10** (1977), 107–144.

[ELW] S. Evens, J.-H. Lu, A. Weinstein, *Transverse measures, the modular class and a cohomology pairing for Lie algebroids*, Quart. J. Math. **50** (1999), 171–220.

[Fe] B. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differential Geom. **40** (1994), 213–238.

[F] R. L. Fernandes, *Lie algebroids, holonomy and characteristic class*, Adv. Math. **170** (2002), no. 1, 119–179.

[HS] V. Hinich, V. Schechtman, *Homotopy Lie algebras*, I. M Gelfand Seminar, Adv. Soviet. Math. **16** (1993), no. 2, 1–28.

[HKR] G. Hochschild, B. Kostant, A. Rosenberg, *Differential forms on regular affine algebras*, Trans. Amer. Math. Soc. **102** (1962), 383–408.

[Hu] J. Huebschmann, *Poisson cohomology and quantization*, J. Reine Angew. Math. **408** (1990), 57–113.

[Ka] M. Kashiwara, *D-Modules and Microlocal Calculus*, Translations of Mathematical Monographs, Vol. 217, American Mathematical Society, Providence, RI,

[KS] M. Kashiwara, P. Schapira, *Sheaves on Manifolds*, Grundlehren der Mathematischen Wissenschaften, A Series of Comprehensive Studies in Mathematics, Springer-Verlag, New York, 1994. Russian transl.: М. Касивара, П. Шапира, *Пучки на многообразиях*, Мир, М., 1997 2003.

[Ko] M. Kontsevitch, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216.

[KM] Y. Kosmann-Schwarzbach, F. Magri, *Poisson Nijenhuis structures*, Ann. Inst. H. Poincaré, Série A, **53** (1990), 35–81.

[LS] T. Lada, J. Stasheff, *Introduction to SH Lie algebras for physicists*, Internat. J. Theoret. Phys. **32** (1993), 1087–1103.

[Li] A. Lichnerowicz, *Les variétes de Poissons et leurs algèbres associées*, J. Differential Geom. **12** (1977), 253–300.

[MX] K. C. H Mackenzie, P. Xu, *Lie bialgebroids and Poisson groupoids*, Duke Math. J. **73** (1994), 415–452.

[MT] D. Manchon, C. Torossian, *Cohomologie tangente et cup-produit pour la quantification de Kontsevitch*, Ann. Math. Blaise Pascal **10** (2003), no. 1, 75–106.

[PT] M. Pevsner, C. Torossian, *Isomorphisme de Duflo et cohomologie tangentielle*, J. Geom. Phys. **51** (2004), no. 4, 486–505.

[R] G. S.Rinehart, *Differential form on general commutative algebra*, Trans. Amer. Math. Soc **108** (1963), 195–222.

[V] J. Vey, *Déformation du crochet de Poisson sur une variété symplectique*, Comment. Math. Helv. **50** (1975), 421–454.

[X] P. Xu, *Quantum groupoids*, Comm. Math. Phys. **206** (2001), 539–581.