FROBENIUS AND QUASI-FROBENIUS LEFT HOPF ALGEBROIDS

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ABSTRACT. We study when left (op)Hopf algebroids in the sense of Takeuchi-Schauenburg give rise to a Frobenius or quasi-Frobenius extension. The case of Hopf algebroids in the sense of Böhm was treated by G. Böhm ([**?**]). Contrary to Hopf algebroids, (op)Hopf left algebroids don't necessarily have an antipode but their Hopf-Galois map is invertible. We make use of recent results about left Hopf algebroids ([**?**], [**?**], [**?**]). Our results are applied to the restricted enveloping algebra of a restricted Lie-Rinehart algebra.

1. INTRODUCTION

Left bialgebroids, called also $\times_A$-bialgebras ([**?**]), generalize k-bialgebras (k being a commutative ring included in the center) to the case where the basis is not necessarily commutative. A left bialgebroid $U = (U, s^\ell, t^\ell, \Delta, \mu, \epsilon)$ over A is the data of

– A k-algebra structure (U, μ) on U.
– Two morphisms of k-algebras $s^\ell : A \to U$ and $t^\ell : A^{op} \to U$ commuting.
– A comultiplication, which is a morphism of A^e-algebras, defined on U and taking values in the Takeuchi product $U_{t^\ell} \times_{A\, s^\ell} U \subset U_{t^\ell} \otimes_{A\, s^\ell} U$.
– a counit ϵ.

A left (respectively right) bialgebroid over A has two duals, a left one and a right one. Both duals are endowed with a right (respectively left) bialgebroid structures over A ([**?**]).

There exist two main generalizations of Hopf algebras to the non commutative setting:

– Hopf algebroids in the sense of Böhm for which an antipode is assumed to exist.
– Left Hopf algebroids in the sense of Takeuchi-Schauenburg (or $\times_A$ Hopf algebras) where one only assumes that the Hopf-Galois map is an isomorphism.

It has been shown recently ([**?**], [**?**]) that, under finiteness conditions, the right (left) dual of a left (op)Hopf algebroid are right (op)Hopf algebroids.

Hopf algebroids are left Hopf and opHopf algebroids but the converse is in general not true (see [**?**]). For example, (restricted) enveloping algebras of (restricted) Lie-Rinehart algebras are left Hopf and opHopf algebroids but are not, in general, Hopf algebroids.

More recently, Hopf algebras were studied from a more categorical point of view: bialgebras were viewed as bimonads ([**?**]) and left Hopf algebroids as Hopf monads ([**?**], [**?**]). Hopf categories were defined in [**?**]. Under certain conditions, the latter provide weak Hopf algebras (see [**?**] section 6) and thus Hopf algebroids.

Sweedler ([**?**]) introduced the notion of integral for Hopf algebras and the Larson Sweedler theorem for Hopf modules was proved in [**?**]. These results were generalized to Hopf algebroids ([**?**]), to Hopf bimonads ([**?**], [**?**]) and more recently to Hopf categories ([**?**]).

Recently, a Maschke type theorem for Hopf monoids relating the separability of the underlying monoid to the existence of a normalized integral was proved in [**?**]. This work covers the case of a Hopf monoid in braided monoidal categories, weak Hopf algebras, Hopf algebroids over a central basis and Hopf monads on autonomous monoidal categories ([**?**]).

Many authors have studied relations between Hopf algebras and Frobenius algebras: [**?**], [**?**], [**?**], etc... The following question arises: when is a left Hopf algebroid a (quasi-)Frobenius extension of its basis? In [**?**], it is shown that any finite dimensional weak Hopf algebra is quasi-Frobenius. These questions are treated for Hopf algebroids in [**?**] with answers involving integrals. In [**?**], weak Hopf algebras that are Frobenius are characterized by a criterion on their semi-simple base algebra A.

Let $(U, s^\ell, t^\ell, \Delta, \mu, \epsilon)$ be a left Hopf algebroid satisfying some projectiveness and finiteness assumptions. We study when the extension $t^\ell : A \to U$ is Frobenius ([**?**]) or quasi-Frobenius (in the sense of Muller [**?**]). We show:

- The extension $t^\ell : A \to U$ is Frobenius if and only if the A^{op}-module of its left integrals is a free A^{op}-module of rank one.
- The extension $t^\ell : A \to U$ is quasi-Frobenius if and only if the A^{op} module of its left integrals is a projective finitely generated A^{op}-module.

The main tool is the fundamental theorem for Hopf-modules in the setting of left Hopf algebroids ([**?**]) and the dual theory for left Hopf algebroids demonstrated by Schauenburg ([**?**]) and Kowalzig ([**?**]).

We apply our results to restricted enveloping algebras of restricted Lie-Rinehart algebras.

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Notations

Fix an (associative, unital, commutative) ground ring k. Unadorned tensor products will always be meant over k. All other algebras, modules etc. will have an underlying structure of a k-module. Secondly, fix an associative and unital k-algebra A, *i.e.*, a ring with a ring homomorphism $\eta_A : k \to Z(A)$ to its centre. Denote by $A^{\rm op}$ the opposite algebra and by $A^{\rm e} := A \otimes A^{\rm op}$ the enveloping algebra of A, and by A-$\mathbf{Mod}$ the category of left A-modules.

The notions of *A–ring* and *A–coring* are direct generalizations of the notions of algebra and coalgebra over a commutative ring. An A-ring (H, μ, η) is a monoid in the monoidal category (A^e-Mod, $\otimes_A, A$) of A^e-modules fulfilling the associativity and the unitarity conditions. It is well known (see [**?**]) that A–rings H correspond bijectively to k–algebra homomorphisms $\iota : A \longrightarrow H$. An A-ring H is endowed with an A^e-module structure:

$$\forall h \in H, \quad a, b \in H, \quad a \cdot h \cdot b = \iota(a) h \iota(b).$$

An A-coring C is a comonoid in the monoidal category of A^e-modules satisfying the coassociativity and the counitarity conditions. As usual, we adopt Sweedler's Σ–notation $\Delta(c) = c_{(1)} \otimes c_{(2)}$ or $\Delta(c) = c^{(1)} \otimes c^{(2)}$ for $c \in C$.

2. Preliminaries

We list here those preliminaries with respect to bialgebroids and their duals that are needed to make this article self contained; see, *e.g.*, [**?**] and references below for an overview on this subject.

2.1. **Bialgebroids.** For an A^e-ring U given by the k-algebra map $\eta : A^{\rm e} \to U$, consider the restrictions $s := \eta(- \otimes 1_U)$ and $t := \eta(1_U \otimes -)$, called *source* and *target* map, respectively. Thus an A^e-ring U carries two A-module structures from the left and two from the right, namely

$$a \triangleright u \triangleleft b := s(a)t(b)u, \qquad a \blacktriangleright u \blacktriangleleft b := ut(a)s(b), \qquad \forall\, a, b \in A, u \in U.$$

If we let $U_{\triangleleft}\otimes_{A}{}_{\triangleright}U$ be the corresponding tensor product of U (as an A^{e}-module) with itself, we define the *(left) Takeuchi-Sweedler product* as

$$U_{\triangleleft}\times_{A}{}_{\triangleright}U \ := \ \left\{\textstyle\sum_i u_i \otimes u'_i \in U_{\triangleleft}\otimes_{A}{}_{\triangleright}U \mid \sum_i (a \blacktriangleright u_i)\otimes u'_i = \sum_i u_i \otimes (u'_i \blacktriangleleft a), \ \forall a \in A\right\}. \quad (2.1)$$

By construction, $U_{\triangleleft}\times_{A}{}_{\triangleright}U$ is an A^{e}-submodule of $U_{\triangleleft}\otimes_{A}{}_{\triangleright}U$; it is also an A^{e}-ring via factorwise multiplication, with unit $1_U \otimes 1_U$ and $\eta_{U_{\triangleleft}\times_{A}{}_{\triangleright}U}(a\otimes\tilde{a}) := s(a)\otimes t(\tilde{a})$.

Symmetrically, one can consider the tensor product $U_{\blacktriangleleft}\otimes_{A}{}_{\blacktriangleright}U$ and define the *(right) Takeuchi-Sweedler product* as $U_{\blacktriangleleft}\times_{A}{}_{\blacktriangleright}U$, which is an A^{e}-ring inside $U_{\blacktriangleleft}\otimes_{A}{}_{\blacktriangleright}U$.

Definition 2.1.1. A *left bialgebroid* (U, A) is a k-module U with the structure of an A^{e}-ring (U, s^{ℓ}, t^{ℓ}) and an A-coring $(U, \Delta_{\ell}, \epsilon)$ subject to the following compatibility relations:

(i) the A^{e}-module structure on the A-coring U is that of ${}_{\triangleright}U_{\triangleleft}$;

(ii) the coproduct Δ_{ℓ} is a unital k-algebra morphism taking values in $U_{\triangleleft}\times_{A}{}_{\triangleright}U$;

(iii) for all $a, b \in A$, $u, u' \in U$, one has:

$$\epsilon(1_U) = 1_A, \quad \epsilon(a \triangleright u \triangleleft b) = a\epsilon(u)b, \quad \epsilon(uu') = \epsilon\big(u \blacktriangleleft \epsilon(u')\big) = \epsilon\big(\epsilon(u') \blacktriangleright u\big). \quad (2.2)$$

A *morphism* between left bialgebroids (U, A) and (U', A') is a pair (F, f) of maps $F : U \to U'$, $f : A \to A'$ that commute with all structure maps in an obvious way.

Remark 2.1.2. Szlachànyi has shown that left bialgebroids may be interpreted in terms of bimonads ([**?**]).

As for any ring, we can define the categories U-$\mathbf{Mod}$ and $\mathbf{Mod}$-U of left and right modules over U. Note that U-$\mathbf{Mod}$ forms a monoidal category but $\mathbf{Mod}$-U usually does not. However, in both cases there is a forgetful functor U-$\mathbf{Mod} \to A^{\mathrm{e}}$-$\mathbf{Mod}$, resp. $\mathbf{Mod}$-$U \to A^{\mathrm{e}}$-$\mathbf{Mod}$ given by the formulas : for $m \in M,\ n \in N,\ a, b \in A$

$$a \triangleright m \triangleleft b := s^{\ell}(a)t^{\ell}(b)m, \qquad a \blacktriangleright m \blacktriangleleft b := ns^{\ell}(b)t^{\ell}(a)$$

For example, the base algebra A itself is a left U-module via the left action

$$u(a) := \epsilon(u \blacktriangleleft a) = \epsilon(a \blacktriangleright u), \quad \forall u \in U, \quad \forall a \in A, \quad (2.3)$$

but in general there is no right U-action on A.

Dually, one can introduce the categories U-$\mathbf{Comod}$ and $\mathbf{Comod}$-U of left resp. right U-comodules, both of which are monoidal; here again, one has forgetful functors U-$\mathbf{Comod} \to A^{\mathrm{e}}$-$\mathbf{Mod}$ and $\mathbf{Comod}$-$U \to A^{\mathrm{e}}$-$\mathbf{Mod}$. More precisely (see, *e.g.*, [**?**]), a (say) left comodule is a left comodule of the coring underlying U, *i.e.*, a left A-module M and a left A-module map $\Delta_M : M \to U_{\triangleleft} \otimes_A M, \quad m \mapsto m_{(-1)} \otimes_A m_{(0)}$, satisfying the usual coassociativity and counitality axioms. On any $M \in U$-$\mathbf{Comod}$ there is an induced *right* A-action given by

$$ma := \epsilon(m_{(-1)} \blacktriangleleft a)m_{(0)}, \quad (2.4)$$

and Δ_M is then an A^{e}-module morphism $M \to U_{\triangleleft} \times_A M$, where $U_{\triangleleft} \times_A M$ is the A^{e}-submodule of $U_{\triangleleft} \otimes_A M$ whose elements $\sum_i u_i \otimes_A m_i$ fulfil

$$\textstyle\sum_i a \blacktriangleright u_i \otimes_A m_i = \sum_i u_i t^l(a) \otimes_A m_i = \sum_i u_i \otimes_A m_i \cdot a, \quad \forall a \in A. \quad (2.5)$$

The following identity is easy to check

$$\Delta_M(amb) = s^l(a)m_{(-1)}s^l(b) \otimes_A m_{(0)}.$$

Coinvariant elements of a comodule will play an important role in the sequel:

Definition 2.1.3. Let $(U, A, s^{\ell}, t^{\ell}, \Delta, \epsilon)$ be a left bialgebroid over A.

(i) Let (M, Δ_M) be a left U-comodule. An element m in M is coinvariant if $\Delta_M(m) = 1 \otimes m$. The set of coinvariant elements will be denoted M^{cov}. It is endowed with a natural A^{op}-module structure via t^{ℓ}.

(ii) Let (N, Δ_N) be a right U-comodule. An element n in N is coinvariant if $\Delta_N(n) = n \otimes 1$. The set of coinvariant elements will be denoted N^{cov}. It is endowed with a natural A-module structure via s^{ℓ}.

Examples 2.1.4. (*i*) If U is a left bialgebroid, then ${}_{s^\ell}U_{s^\ell} = {}_{\triangleright}U_{\blacktriangleleft}$ is a left U-comodule and the $t^\ell(A)$-module of its coinvariant elements is $t^\ell(A)$.
(*ii*) ${}_{t^\ell}U_{t^\ell} = {}_{\blacktriangleright}U_{\triangleleft}$ is a right U-comodule and the $s^\ell(A)$-module of its coinvariant elements is $s^\ell(A)$.

The notion of a *right bialgebroid* is obtained from that of *left bialgebroid* exchanging the role of $\triangleright, \triangleleft$ and $\blacktriangleright, \blacktriangleleft$. Then one starts with the A^{e}-module structure given by $\blacktriangleright$ and $\blacktriangleleft$ instead of $\triangleright$ and $\triangleleft$ and the coproduct takes values in $U_{\blacktriangleleft} \times_A {}_{\blacktriangleright}U$ instead of $U_{\triangleleft} \times_A {}_{\triangleright}U$. We refer to [**?**] for details.

Remark 2.1.5. The *opposite* of a left bialgebroid $(U, A, s^\ell, t^\ell, \Delta_\ell, \epsilon)$ yields a *right* bialgebroid $(U^{\mathrm{op}}, A, t^\ell, s^\ell, \Delta_\ell, \epsilon)$. The *coopposite* of a left bialgebroid is the *left* bialgebroid given by $(U, A^{\mathrm{op}}, t^\ell, s^\ell, \Delta_\ell^{\mathrm{coop}}, \epsilon)$.

Left and right comodules over a right bialgebroid W are also well defined.

2.2. **Dual bialgebroids.** Let (U, A) be a left bialgebroid, $M, M' \in U$-$\mathbf{Mod}$ be left U-modules. Define

$$\mathrm{Hom}_{A^{\mathrm{op}}}(M, M') := \mathrm{Hom}_{A^{\mathrm{op}}}(M_{\triangleleft}, M'_{\triangleleft}), \quad \mathrm{Hom}_A(M, M') := \mathrm{Hom}_A({}_{\triangleright}M, {}_{\triangleright}M'),$$
$$\mathrm{Hom}_{A^{\mathrm{op}}}(N, N') := \mathrm{Hom}_{A^{\mathrm{op}}}(N_{\blacktriangleleft}, N'_{\blacktriangleleft}), \quad \mathrm{Hom}_A(N, N') := \mathrm{Hom}_A({}_{\blacktriangleright}N, {}_{\blacktriangleright}N').$$

In particular, for $M' := A$, we set $M_* := \mathrm{Hom}_A(M, A)$ and $M^* := \mathrm{Hom}_{A^{\mathrm{op}}}(M, A)$, called, respectively, the *left* and *right* dual of M.

If $M = U$, the two duals U^* (the right dual) and U_* (the left dual) are endowed with an A^{e}-ring structure, and even a right bialgebroid structure under finiteness and projectiveness conditions ([**?**]).

The case of U^*:
For $a \in A$, let us introduce the two elements $s_r^*(a)$ and $t_r^*(a)$ of U^* defined by

$$\forall u \in U, \quad < t_r^*(a), u >= a < \epsilon, u >, \quad < s_r^*(a), u >=< \epsilon, us^\ell(a) > . \tag{2.6}$$

Endowed with the following multiplication, U^* is an associative k-algebra with unit ϵ: For all $\phi, \phi' \in U^*$ and all $u \in U$

$$\left\langle u\,,\, \phi\,\phi' \right\rangle \;=\; \left\langle s^\ell\big(\left\langle u_{(1)}\,,\, \phi \right\rangle\big)\, u_{(2)}\,,\, \phi' \right\rangle \tag{2.7}$$

Then $s_r^* : A \to U^*$ and $t_r^* : A^{op} \to U^*$ are algebra morphisms and define an A^e-ring structure on U^*:

$$\phi \blacktriangleleft a = \phi s_r^*(a) \quad \text{and} \quad a \blacktriangleright \phi = \phi t_r^*(a).$$

The product on U^* can be written :

$$\left\langle u\,,\, \phi\,\phi' \right\rangle \;= \left\langle u_{(2)}\,,\, t_r^*\big(\left\langle u_{(1)}\,,\, \phi \right\rangle\big)\, \phi' \right\rangle \tag{2.8}$$

If $U_{\triangleleft}$ is a finite projective A^{op}-module, the following formula defines a coproduct on U^* :

$$\left\langle u\,u'\,,\, \phi \right\rangle \;=\; \left\langle u\,t_\ell\big(\left\langle u'\,,\, \phi_{(2)} \right\rangle\big)\,,\, \phi_{(1)} \right\rangle \;=\; \left\langle u\,,\, \phi_{(1)}\, s_r^*\big(\left\langle u'\,,\, \phi_{(2)} \right\rangle\big) \right\rangle$$

Lastly we have a counit $\eta \in U^*$

$$\left\langle 1\,,\, \phi \right\rangle \;=\; \eta(\phi). \tag{2.9}$$

Thus $(U^*, A, s_r^*, t_r^*, \Delta, \eta)$ is a right bialgebroid.

The case of U_*: If ${}_{\triangleright}U$ is a finite projective A-module, U_* is endowed with the right bialgebroid structure over A such that $(U_{coop})_* = (U^*)_{coop}$.

In a similar way, if W is a right bialgebroid, then its left dual ${}_*W$ and its right dual ${}^*W$ are endowed with an A^e-ring structure. Under finiteness and projectiveness conditions, they are left bialgebroids . Moreover the left bialgebroids ${}_*(U^*)$ and ${}^*(U_*)$ are canonically isomorphic to U. The formulas above also describe the left bialgebroid structure on ${}^*W$ and ${}_*W$ ($\phi \in W$ and $u \in_* W$, $\psi \in W$ and $u \in^* W$). See for example [**?**] for a detailled exposition.

Remark 2.2.1. Under the appropriate finiteness conditions, the right bialgebroids ${}^*(U^{op}_{coop})$ and $(U_*)^{op}_{coop}$ are isomorphic.

2.2.2. The module-comodule correspondence
The classical bialgebra module-comodule correspondence extends to bialgebroids.

Proposition 2.2.3. *1) Let (U, A) be a left bialgebroid.*

(i) *There exists a functor* $\mathbf{Comod}\text{-}U \to \mathbf{Mod}\text{-}U_*$*; namely, if M is a right U-comodule with coaction $m \mapsto m_{(0)} \otimes_A m_{(1)}$, then*
$$M \otimes_A U_* \to M, \quad m \otimes_k \psi \mapsto m_{(0)}\psi(m_{(1)}), \tag{2.10}$$
defines a right module structure over the A^{e}-ring U_*. If ${}_{\triangleright}U$ is finitely generated A-projective (so that U_* is a right bialgebroid), this functor is monoidal and has a quasi-inverse $\mathbf{Mod}\text{-}U_* \to \mathbf{Comod}\text{-}U$ *such that there is an equivalence* $\mathbf{Comod}\text{-}U \simeq \mathbf{Mod}\text{-}U_*$ *of categories.*

(ii) *Likewise, there exists a functor* $U\text{-}\mathbf{Comod} \to \mathbf{Mod}\text{-}U^*$*; namely, if N is a left U-comodule with coaction $n \mapsto n_{(-1)} \otimes_A n_{(0)}$, then*
$$N \otimes_A U^* \to N, \quad n \otimes_k \phi \mapsto \phi(n_{(-1)})n_{(0)}, \tag{2.11}$$
defines a right module structure over the A^{e}-ring U^*. If $U_{\triangleleft}$ is finitely generated A-projective (so that U^* is a right bialgebroid), this functor is monoidal and has a quasi-inverse $\mathbf{Mod}\text{-}U^* \to U\text{-}\mathbf{Comod}$ *such that there is an equivalence* $U\text{-}\mathbf{Comod} \simeq \mathbf{Mod}\text{-}U^*$ *of categories.*

2) Similar statements holds in the case of right bialgebroids.

The case $1)(ii)$ of the above Proposition **??** can also be found in [**?**, §5]. An explicit proof and a description of all involved functors is given in [**?**, §3.1].

2.3. **Left Hopf and opHopf algebroids.** For any left bialgebroid U, define the *Hopf-Galois maps*
$$\begin{aligned} \alpha_\ell : {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} &\to U_{\triangleleft} \otimes_A {}_{\triangleright}U, & u \otimes_{A^{\mathrm{op}}} v &\mapsto u_{(1)} \otimes_A u_{(2)}v, \\ \alpha_r : U_{\blacktriangleleft} \otimes^A {}_{\triangleright}U &\to U_{\triangleleft} \otimes_A {}_{\triangleright}U, & u \otimes^A v &\mapsto u_{(1)}v \otimes_A u_{(2)}. \end{aligned}$$
and for a *right* bialgebroid W the Hopf-Galois maps
$$\beta_\ell : W_{\triangleleft} \otimes_B {}_{\blacktriangleright}W \to W_{\blacktriangleleft} \otimes_B {}_{\blacktriangleright}W, \quad w \otimes y \mapsto yw^{(1)} \otimes w^{(2)},$$
$$\beta_r : {}_{\blacktriangleright}W \underset{B^{op}}{\otimes} W_{\triangleleft} \to W_{\blacktriangleleft} \underset{B}{\otimes} {}_{\blacktriangleright}W, \quad w \otimes y \mapsto w^{(1)} \otimes y\, w^{(2)}.$$
These maps give rise to the following definition ([**?**]):

Definition 2.3.1. 1) A left bialgebroid U is called a *left Hopf algebroid* or $\times_A$ *Hopf algebra* if α_ℓ is a bijection. Likewise, it is called a *left opHopf algebroid* if α_r is a bijection. In either case, we adopt for all $u \in U$ the following (Sweedler-like) notation
$$u_+ \otimes_{A^{\mathrm{op}}} u_- := \alpha_\ell^{-1}(u \otimes_A 1), \qquad u_{[+]} \otimes^A u_{[-]} := \alpha_r^{-1}(1 \otimes_A u), \tag{2.12}$$
and call both maps $u \mapsto u_+ \otimes_{A^{\mathrm{op}}} u_-$ and $u \mapsto u_{[+]} \otimes^A u_{[-]}$ *translation maps.*

2) Let W be a *right* B-bialgebroid. Then W is called a *right Hopf algebroid (=RHB)*, respectively a *right opHopf algebroid (=RopHB)* if the map β_r, resp. β_ℓ, is a bijection. If $w \in W$, one sets $\beta_r^{-1}(1 \otimes w) = w^- \otimes w^+$ and the translation map is $w \mapsto w^- \otimes_{B^{\mathrm{op}}} w^+$.

Remarks 2.3.2. Let $(U, A, s^\ell, t^\ell, \Delta, \epsilon)$ be a left bialgebroid.

(i) In case $A = k$ is central in U, one can show that α_ℓ is invertible if and only if U is a Hopf algebra, and the translation map reads $u_+ \otimes u_- := u_{(1)} \otimes S(u_{(2)})$, where S is the antipode of U. On the other hand, U is a Hopf algebra with invertible antipode if and only if both α_ℓ and α_r are invertible, and then $u_{[+]} \otimes u_{[-]} := u_{(2)} \otimes S^{-1}(u_{(1)})$.

(ii) The underlying left bialgebroid in a Hopf algebroid with bijective antipode is both a left Hopf and opHopf algebroid (but not necessarily vice versa [**?**]); see [**?**] [Prop. 4.2] for the details of this construction.

(iii) Definition **??** extends to the bimonad framework ([**?**], [**?**]) to give left Hopf bimonads.

Remark 2.3.3. The right bialgebroid $(W, A, s^r, t^r, \Delta, \epsilon)$ is a right (op)Hopf algebroid if and only if the left bialgebroid W^{op}_{coop} is a (op)Hopf algebroid. This remark will allow us not to treat the case of right bialgebroids in detail.

The following proposition collects some properties we will need of the translation maps [**?**]:

Proposition 2.3.4. *Let U be a left bialgebroid.*

(i) *If U is a left Hopf algebroid, the following relations hold:*

$$\begin{aligned}
u_+ \otimes_{A^{\mathrm{op}}} u_- &\in U \times_{A^{\mathrm{op}}} U, && (2.13)\\
u_{+(1)} \otimes_A u_{+(2)} \otimes_{A^{\mathrm{op}}} u_- &= u_{(1)} \otimes_A u_{(2)+} \otimes_{A^{\mathrm{op}}} u_{(2)-}, && (2.14)\\
u_+ \otimes_{A^{\mathrm{op}}} u_{-(1)} \otimes_A u_{-(2)} &= u_{++} \otimes_{A^{\mathrm{op}}} u_- \otimes_A u_{+-}, && (2.15)\\
(uv)_+ \otimes_{A^{\mathrm{op}}} (uv)_- &= u_+ v_+ \otimes_{A^{\mathrm{op}}} v_- u_-, && (2.16)\\
u_+ u_- &= s^\ell(\varepsilon(u)), && (2.17)\\
(s^\ell(a) t^\ell(b))_+ \otimes_{A^{\mathrm{op}}} (s^\ell(a) t^\ell(b))_- &= s^\ell(a) \otimes_{A^{\mathrm{op}}} s^\ell(b), && (2.18)
\end{aligned}$$

*where in (**??**) we mean the Takeuchi-Sweedler product*

$$U \times_{A^{\mathrm{op}}} U := \big\{ \textstyle\sum_i u_i \otimes v_i \in {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} \mid \sum_i u_i \triangleleft a \otimes v_i = \sum_i u_i \otimes a \blacktriangleright v_i, \ \forall a \in A \big\}.$$

(ii) *Analogously, if U is a right Hopf algebroid, one has:*

$$u_{[+]} \otimes^A u_{[-]} \in U \times^A U, \qquad (2.19)$$

*where in (**??**) we mean the Sweedler-Takeuchi product*

$$U \times^A U := \big\{ \textstyle\sum_i u_i \otimes v_i \in U_{\blacktriangleleft} \otimes^A {}_{\triangleright}U \mid \sum_i a \triangleright u_i \otimes v_i = \sum_i u_i \otimes v_i \blacktriangleleft a, \quad \forall a \in A \big\}.$$

$u_{[+]} \otimes^A u_{[-]}$ satisfies properties similar to those satisfied by $u_+ \otimes_{A^{\mathrm{op}}} u_-$.

The following theorem, originally due to [**?**] was improved in [**?**]. It asserts that, if U is a left Hopf and opHopf algebroid such that $U_\triangleleft$ (respectively ${}_\triangleright U$) is a projective A^{op}-module (respectively A-module), there is an equivalence of categories between U-**Comod** and **Comod**-U.

Theorem 2.3.5. *Let (U, A) be a left bialgebroid.*

(i) *Let (U, A) be additionally a left Hopf algebroid such that $U_\triangleleft$ is projective. Then there exists a (strict) monoidal functor F : **Comod**-$U \to U$-**Comod**; namely, if M is a right U-comodule with coaction $m \mapsto m_{(0)} \otimes_A m_{(1)}$, then*

$$\lambda_M : M \to U_\triangleleft \otimes_A M, \quad m \mapsto m_{(1)-} \otimes_A m_{(0)} \epsilon(m_{(1)+}), \qquad (2.20)$$

defines a left comodule structure on M over U.

(ii) *Let (U, A) be a left (op)Hopf algebroid such that ${}_\triangleright U$ is projective. Then there exists a (strict) monoidal functor G : U-**Comod** $\to$ **Comod**-U; namely, if N is a left U-comodule with coaction $n \mapsto n_{(-1)} \otimes_A n_{(0)}$, then*

$$\rho_N : N \to N \otimes_A {}_\triangleright U, \quad n \mapsto \epsilon(n_{(-1)[+]}) n_{(0)} \otimes_A n_{(-1)[-]}, \qquad (2.21)$$

defines a right comodule structure on N over U.

(iii) *If U is both a left Hopf and opHopf algebroid and if both $U_{\triangleleft}$ and ${}_{\triangleright}U$ are projective, then the functors mentioned in (i) and (ii) are quasi-inverse to each other and we have an equivalence*

$$U\text{-}\mathbf{Comod} \simeq \mathbf{Comod}\text{-}U$$

of monoidal categories.

Remark 2.3.6. The equivalence of categories of Theorem **??** preserves coinvariant elements.

Applying Theorem **??** to the situation of Theorem **??**, the functor F can be transformed into a functor between the module categories over the left and the right dual algebra of U and this functor in turn induces an algebra morphism between these dual algebras. Thus, the functor F comes from an algebra morphism $S_* : U_* \to U^*$ and the functor G comes from an algebra morphism $S^* : U^* \to U_*$. The morphism S^* an S_* are studied in [**?**]:

Theorem 2.3.7. *Let (U, A) be a left bialgebroid.*

(i) *If (U, A) is moreover a left Hopf algebroid, the map $S^* : U^* \to U_*$ is defined by*

$$\forall \psi \in U_*, \quad \forall u \in U, \qquad S^*(\phi)(u) := \epsilon_U\big(u_+ t^\ell(\phi(u_-))\big)$$

is a morphism of A^e-rings with augmentation; if, in addition, both ${}_{\triangleright}U$ and $U_{\triangleleft}$ are finitely generated A-projective, then (S^*, id_A) is a morphism of right bialgebroids.

(ii) *If (U, A) is a left opHopf algebroid instead, the map $S_* : U_* \to U^*$*

$$\forall \psi \in U_*, \quad \forall u \in U, \qquad S_*(\psi)(u) := \epsilon\big(u_{[+]} s^\ell(\psi(u_{[-]}))\big)$$

is a morphism of A^e-rings with augmentation; if, in addition, both ${}_{\triangleright}U$ and $U_{\triangleleft}$ are finitely generated A-projective, then (S_*, id_A) is a morphism of right bialgebroids.

(iii) *If (U, A) is simultaneously both a left Hopf and opHopf left algebroid, $S^* : U^* \to U_*$ is an isomorphism and $S_* = (S^*)^{-1}$.*

The maps S^* and S_* have even more properties.

Proposition 2.3.8. *U_* is endowed with the following left U-action :*

$$\forall (u, v) \in U^2, \quad \forall \psi \in U_*, \quad < u \rightharpoonup \psi, v >=< \psi, vu > .$$

U^* is endowed with the following left U-action :

$$\forall (u, v) \in U^2, \quad \forall \phi \in U^*, \quad < u \bullet \phi, v >= u_+ \left[< \phi, u_- v >\right]. \tag{2.22}$$

The map S^* sends $(U^*, \bullet)$ to $(U_*, \rightharpoonup)$.

The proof of the proposition **??** is straightforward.

The following recent result will play a key role in our study. It was proved by categorical arguments in [**?**]. Kowalzig ([**?**]) gave an explicit formula for the translation map of the dual.

Theorem 2.3.9. ([**?**], [**?**]) *1) If U is a left Hopf algebroid, then U^* (respectively U_*) is a right Hopf algebroid.*

2) If U is a left opHopf algebroid, then U^* (respectively U_*) is a right opHopf algebroid.

2.4. **Left and right integrals.** Left and right integrals were defined for Hopf algebras in [**?**] and were generalized to bialgebroids in [**?**]. They were also defined in the more abstract context of bimonads on a monoidal category ([**?**]) and that of Hopf categories ([**?**]). Let us recall their definition in our framework:

Definition 2.4.1. Let $(U, A, s^\ell, t^\ell, m, \Delta, \epsilon)$ be a left bialgebroid. A left integral of U is an element u_0 of U such that

$$\forall u \in U, \quad uu_0 = s^\ell(< \epsilon, u >)u_0.$$

The set of left integrals of U will be denoted $\int_U^\ell$.

Let $(W, A, s^r, t^r, m, \Delta, \epsilon)$ be a right bialgebroid. A right integral of W is an element w_0 of W such that

$$\forall w \in U, \quad w_0 w = w_0 s^r(< \epsilon, \psi >).$$

The set of left integrals of W will be denoted $\int_W^r$.

Remark 2.4.2. The left integrals of U are the same as the left integrals of U_{coop}. The right integrals of W are the same as the right integrals of W_{coop}.

Indeed, let $u_0 \in \int_U^\ell$. Forall $u \in U$, one has $uu_0 = s^\ell\epsilon(u)u_0$. In particular, $t^\ell\epsilon(u)u_0 = s^\ell\epsilon(u)u_0$. The remark follows.

Proposition 2.4.3. *Let U be a left opHopf algebroid. An element l is in $\int_U^\ell$ if and only if it satisfies the following property :*

$$\forall u \in U, \quad ul_{[+]} \otimes l_{[-]} = l_{[+]} \otimes l_{[-]}u.$$

Remark 2.4.4. In the case of Hopf algebroids, this proposition follows from the scholium 2.8 of [**?**]. The proof uses the properties of $u_{[+]} \otimes u_{[-]}$ and is left to the reader.

3. Hopf-modules

Left-left Hopf modules are the objects of study of the fundamental theorem for Hopf modules ([**?**]). The latter states that, if H is a k-Hopf algebra, there is an equivalence of categories between left-left Hopf modules and k-vecteor spaces. Left-left Hopf modules can be defined in the case of Hopf algebroids (in the sense of Böhm) ([**?**]), in the framework of bimonads over a monoidal category ([**?**]) and in the context of Hopf categories ([**?**]). In all these cases, the Larson-Sweedler theorem for Hopf modules was proved. We will use only a part of this theorem that follows from a flat descent argument.

Definition 3.1. *1) Let $(U, A, s^\ell, t^\ell, \Delta, \epsilon)$ be a left bialgebroid over the k-algebra A.*

We will say that M is endowed with a left-left Hopf U-module structure if

- *(i) M is endowed with a left U-module structure.*
- *(ii) M is endowed with a left U-comodule structure denoted Δ_M.*
- *(iii) These two structures are linked by the following relation: For all $m \in M$ and all $u \in U$,*

$$u_{(1)}m_{(-1)} \otimes u_{(2)}m_{(0)} = \Delta_M(u \cdot m).$$

(iv) $a \cdot m = s^\ell(a)m$. In the left hand side, $a \cdot m$ is the left A-module structure coming from the left U-comodule structure.

2) Let $(W, B, s^r, t^r, \Delta, \partial)$ be a right bialgebroid over the k-algebra A. We will say that M is endowed with a right-right Hopf W-module structure if

- *(i) M is endowed with a right W-module structure.*
- *(ii) M is endowed with a right W-comodule structure denoted Δ_M.*
- *(iii) These two structures are linked by the following relation : for all $m \in M$, $w \in W$ and $b \in B$*

$$m_{(0)}w_{(1)} \otimes m_{(1)}w_{(2)} = \Delta_M(m \cdot w).$$

(iv) $m \cdot b = ms^r(b)$.

Example 3.2. *If P is a right A-module, then ${}_{\blacktriangleright}U \otimes_{A^{op}} P$ is a left left Hopf U-module as follows: For all $(u, v) \in U^2$ and all $x \in P$,*

$$u \cdot (v \otimes x) = uv \otimes x \quad \textit{and} \quad \Delta_{U \otimes N}(v \otimes x) = v_{(1)} \otimes v_{(2)} \otimes x.$$

It will follow from the fundamental theorem for Hopf modules that, if U is a left Hopf algebroid and under flatness conditions, all left-left Hopf U-modules are of this type (up to isomorphisms).

Example 3.3. *If N is a left U-module, then $U_{\triangleleft} \otimes_A N$ is a left -left Hopf U-module as follows: For all $(u, v) \in U^2$ and all $n \in N$,*

$$u \cdot (v \otimes n) = u_{(1)} v \otimes u_{(2)} n \quad \textit{and} \quad \Delta_{U \otimes N}(v \otimes n) = v_{(1)} \otimes v_{(2)} \otimes n.$$

Examples **??** and **??** are linked as explained in the following proposition which proof is left to the reader.

Proposition 3.4. 1) Let N be a left U-module. The map

$$\begin{array}{rcl} \delta_N :_{\blacktriangleright} U \otimes_{A^{op}} N_{\triangleleft} & \to & U_{\triangleleft} \otimes_{A} {}_{\triangleright} N \\ u \otimes n & \mapsto & u_{(1)} \otimes u_{(2)} n \end{array}$$

is a morphism of left-left Hopf U-modules from Examples **??** to **??**.

2) If U is a left Hopf algebroid it is an isomorphism.

In the study of integrals for Hopf algebras, a technic is to apply the fundamental theorem to the Hopf module U^*. In the case of a Hopf algebras, U^* and ${}_*U$ coincide. In [**?**] (Proposition 4.4.) ${}_*U$ is endowed with a left -left Hopf U-module in the case where U is a Hopf algebroid. This structure is then transferred to U_* using the antipode. We will endow U^* with a left left Hopf U-module structure and we will transfer this structure to U_* using the map S^*.

Proposition 3.5. Let $(U, s^\ell, t^\ell, \Delta, \epsilon)$ be a left-Hopf left bialgebroid over A such that $U_{\triangleleft}$ is a finitely generated and projective right A-module. We set $U^* = Hom_{A^{op}}(U, A)$. Let $(e_1, \ldots, e_n) \in U_{\triangleleft}^n$ and $(e_1^*, \ldots, e_n^*) \in U^{*n}$ be a dual basis (([**?**] p. 203) of the projective A^{op}module $U_{\triangleleft}$.

- (i) We endow U^* with the U-action of equation **??**.
- (ii) We endow U^* with the left U-comodule structure determined par right multiplication on U^*(see Theorem **??**)

$$\Delta(\phi) = \sum e_{i \triangleleft} \otimes_{\blacktriangleright} \phi e_i^*. \tag{3.1}$$

- (iii) With the two structures above, U^* is a left-left Hopf U-module.

Proof:

Assertion (i) is proved in [**?**].

Assertion (ii) is well known (see [**?**] for details).

Let us now check assertion (iii). As $U_{\triangleleft}$ is a projective finitely generated A^{op}-module, we may identify $U_{\triangleleft} \otimes_{A} {}_{\blacktriangleright} U^*$ with $\mathrm{Hom}_{A^{op}}(U_{\triangleleft}, U_{\triangleleft})$ as follows:

$$\begin{array}{rcl} U_{\triangleleft} \otimes_{A} {}_{\blacktriangleright} U^* & \to & \mathrm{Hom}_{A^{op}}(U_{\triangleleft}, U_{\triangleleft}) \\ u \otimes \phi & \mapsto & \left[v \mapsto t^\ell \left(< \phi, v > \right) u \right] \end{array}$$

On one hand,

$$\begin{array}{rcl} \Delta_M(u \bullet \phi)(v) & = & t^\ell(< (u \bullet \phi) e_i^*, v >) e_i \\ & \underset{??}{=} & t^\ell \left[< s^\ell(< v_{(1)}, u \bullet \phi >) v_{(2)}, e_i^* > \right] e_i \\ & = & s^\ell(< v_{(1)}, u \bullet \phi >) v_{(2)}. \end{array}$$

On the other hand, let us compute $< u_{(1)} \bullet \phi_{(-1)} \otimes u_{(2)} \bullet \phi_{(0)}, v >$.

Before starting our computation, let us remark the following relation:

$$t^\ell\left[u_{+(2)}(a)\right]u_{+(1)} \underset{\mathbf{??}}{=} t^\ell\left[\epsilon\left(u_{+(2)}s^\ell(a)\right)\right]u_{+(1)} \underset{\mathbf{??}}{=} t^\ell\left[\epsilon\left(u_{+(2)}\right)\right]u_{+(1)}t^\ell(a) = u_+t^\ell(a) \tag{3.2}$$

$$\begin{array}{rcl} < u_{(1)}\bullet\phi_{(-1)}\otimes u_{(2)}\bullet\phi_{(0)}, v > & = & t^\ell(< u_{(2)}\bullet\phi_{(0)}, v >)u_{(1)}\bullet\phi_{(-1)} \\ & = & t^\ell\left[< u_{(2)}\bullet\phi e_i^*, v >\right]u_{(1)}e_i \\ & \underset{\mathbf{??}}{=} & t^\ell\left[< u_{(2)+}\left(< \phi e_i^*, u_{(2)-}v >\right)\right]u_{(1)}e_i \\ & \underset{\mathbf{??}}{=} & t^\ell\left[< u_{+(2)}\left(< \phi e_i^*, u_-v >\right)\right]u_{+(1)}e_i \\ & \underset{\mathbf{??}}{=} & u_+t^\ell\left[< \phi e_i^*, u_-v >\right]e_i \\ & \underset{\mathbf{??}}{=} & u_+t^\ell\left[< s^\ell(< \phi, u_{-(1)}v_{(1)} >)u_{-(2)}v_{(2)}, e_i^* >\right]e_i \\ & = & u_+s^\ell(< \phi, u_{-(1)}v_{(1)} >)u_{-(2)}v_{(2)} \\ & \underset{\mathbf{??}}{=} & u_{++}s^\ell(< \phi, u_-v_{(1)} >)u_{+-}v_{(2)} \\ & \underset{\mathbf{??}}{=} & \left[u_+s^\ell(< \phi, u_-v_{(1)} >)\right]_+\left[u_+s^\ell(< \phi, u_-v_{(1)} >)\right]_-v_{(2)} \\ & \underset{\mathbf{??}}{=} & s^\ell\epsilon\left[u_+s^\ell(< \phi, u_-v_{(1)} >)\right]v_{(2)} \\ & \underset{\mathbf{??}}{=} & s^\ell\left[u_+\left(< \phi, u_-v_{(1)} >\right)\right]v_{(2)} \\ & = & s^\ell\left[< u\bullet\phi, v_{(1)} >\right]v_{(2)}.\Box \end{array}$$

Remark 3.6. *Let U be a left Hopf algebroid. We know from* [**?**], [**?**] *that $(U^*)^{op}_{coop}$ is a left Hopf algebroid. By Proposition* **??** *and Remark* **??**, *$U^{op}_{coop} = \left[(U^*)^{op}_{coop}\right]^*$ is a left left Hopf $(U^*)^{op}_{coop}$-module. Thus U is a right right Hopf U^*-module. We will adopt the following convention: An element $u \in U$ (respectively $\phi \in U^*$) will be denoted $\check{u}$ if considered as element of U^{op}_{coop} (respectively $\check{\phi} \in (U^*)^{op}_{coop}$) . The structure on U is defined as follows: For all $u, v \in U$ and all $\phi \in U^*$*

$$\check{u}\check{v} = \check{vu}$$
$$\Delta(u) = u^{(0)} \otimes u^{(1)} \in U_{\triangleleft}\otimes_A {}_{\blacktriangleright}U^* \quad \text{if} \quad \check{\Delta}(\check{u}) = \check{u}^{(1)} \otimes \check{u}^{(0)} \in (U^*)^{op}_{coop^{\triangleleft}}\otimes_{A^{op}} {}_{\blacktriangleright}U^{op}_{coop}$$

Corollary 3.7. Let U be a left Hopf and opHopf algebroid over A such that $U_{\triangleleft}$ is a finitely generated and projective right A-module. Then U_* endowed with

– the left U-module structure

$$\forall u \in U, \quad \forall \psi \in U_*, \quad \forall v \in U, \quad < u \rightharpoonup \psi, v >=< \psi, vu >$$

– the left U-comodule structure defined by the right U^*-module structure

$$\forall \psi \in U_*, \quad \forall \phi \in U^*, \quad \psi \cdot \phi = \psi S^*(\phi).$$

is a left left Hopf U-module.

Proof:

The isomorphism $S^* : U^* \to U_*$ ([**?**]) transfers the structure of Theorem **??** onto the structure of Corollary **??**. □

We will make use of a flat descent theorem for corings from Brzezinski ([**?**]) .

Theorem 3.8. Let U be a left Hopf algebroid.

1) Let M be a left-left Hopf U-module.

The set of covariant elements $M^{cov} = \{m \in M, \quad \Delta_M(m) = 1 \otimes m\}$ is endowed with a right A-module denoted $\triangleleft$ as follows: For all $m \in M^{cov}$ and all $a \in A$,

$$m \cdot_{\triangleleft} a = t^\ell(a)m.$$

2) The map γ_M

$$\begin{array}{rcl} \gamma_M :_{\blacktriangleright} U \otimes M^{cov}{}_{\triangleleft} & \to & M \\ u \otimes m & \mapsto & um \end{array}$$

is an epimorphism of left-left Hopf U-modules. If the left A-module ${}_{\blacktriangleright}U$ is flat, the map γ_M is an isomorphism of left-left Hopf U-modules.

Proof : 1) is obvious.

2) It follows from [**?**] that left-left Hopf U-modules are left comodules over the coring $\mathcal{W} = (U_{\triangleleft}\otimes_{A}{}_{\triangleright}U, \Delta\otimes id, \epsilon\otimes id)$ where the $U-U$-bimodule structure is given by

$$\forall(u,x,y,v)\in U^4, \quad u\cdot(x\otimes y)\cdot v = u_{(1)}x\otimes x_{(2)}yv.$$

The coring $\mathcal{W}$ is studied in [**?**]. It was shown to possess a grouplike element and to be Galois if and only if U is a left Hopf algebroid. Thus, one can apply theorem 5.6 of [**?**]. □

Remark 3.9. *Theorem* **??** *holds for right right Hopf-modules with appropriate hypothesis.*

Corollary 3.10. Let U be a left Hopf algebroid.
Let M be a left-left Hopf U-module different from $\{0\}$, then $M^{cov}\neq\{0\}$.

Applying the corollary to the left-left Hopf U-module U^*, we get the following proposition.

Corollary 3.11. Let U be a left Hopf algebroid such that the A^{op}- module $U_{\triangleleft}$ is finitely generated projective. The right bialgebroid U^* admits a right integral.

Proof:

1) Let $\phi\in U^*$. The left A-module structure on the U-comodule U^* is $a\cdot\phi = \phi t_r^*(a)$. If we set $\Delta\phi = \phi^{(-1)}\otimes\phi^{(0)}\in U_{\triangleleft}\otimes_{A}{}_{\blacktriangleright}U^*$, then for all $\psi\in U^*$, one has (equation **??**)

$$\phi\psi = \phi^{(0)}t_r^*(<\psi,\phi^{(-1)}>).$$

Consequently, $(U^*)^{cov} = \{\phi\in U^*, \forall\psi\in U^*, \quad \phi\psi = \epsilon(\phi)\cdot\psi = \phi t_r^*(<1,\psi>))\}\neq\varnothing$. If $\phi\in(U^*)^{cov}$, then $\phi s_r^*(<\psi,1>) \underset{??}{=} \phi t_r^*(<\psi,1>) = \phi\psi$. Thus $\phi\in\int_{U^*}^r$ and the first assertion is proved.

Remark 3.12. *This proposition was proved for Hopf categories in* [**?**] *(Proposition 9.5) and* [**?**] *(Proposition 4.10).*

4. FROBENIUS EXTENSION

A monomorphism of k-algebras $s: A\to U$ defines an A^e-module structure on U :

$$\forall(a,b)\in A^2, \quad \forall u\in U, \quad a\cdot u\cdot b = s(a)us(b).$$

As usual, $a\cdot u\cdot b$ will be denoted $a\triangleright u\blacktriangleleft b$. Recall that an A^e-module structure on U defines an A^e-module structure on U_* as follows :

$$\forall\psi\in U_*, \quad \forall a\in A, \quad \forall v\in U, \quad a\blacktriangleright\psi = s(a)\rightharpoonup\psi, \quad <\psi\blacktriangleleft a, v> = <\psi, v> a.$$

Definition 4.1. *(*[**?**]*) A monomorphism of k-algebras $s: A\to U$ is called a Frobenius extension if*

(*i*) *${}_{\triangleright}U$ is finitely generated and projective*

(*ii*) *Endow U_* with the left U-module structure given by the transpose of the right multiplication*

$$\forall\psi\in U_*, \quad \forall(u,v)\in U^2, \quad (v\rightharpoonup\psi)(u) = \psi(uv).$$

The $U\otimes A^{op}$-modules ${}_{U}U_{\blacktriangleleft}$ and $U_{*\blacktriangleleft}$ are isomorphic

Remarks 4.2. (*i*) *The second property holds if and only if there exist a Frobenius system $(\theta, \sum x_i\otimes y_i)$ where $\theta: U\to A$ is a A^e-module map and $x_i\otimes y_i\in U_{\blacktriangleleft}\otimes_{A}{}_{\triangleright}U$ such that*

$$\forall u\in U, \quad \sum_i s\circ\theta(ux_i)y_i = u = \sum_i x_i s\circ\theta(y_i u).$$

If $\chi: U\to U_*$ is an isomorphism of $U\otimes A^{op}$-module, one has $\theta = \chi(1)$.

(*ii*) *Let $t_0\in U$ be the element such that $\chi(t_0) = \epsilon$. Then $t_0\in\int_U^{\ell}$.*

(*iii*) *If $A = k$ is a field, the k-algebra U is Frobenius if and only if the monomorphism $k \to U$ is a Frobenius extension.*
(*iv*) *Morita (*[**?**]*) showed that the monomorphism $s : A \to U$ is a Frobenius extension if and only if the restriction functor is a Frobenius functor.*

Proposition 4.3. Let $(U, A, s^\ell, t^\ell, \Delta^\ell, \epsilon)$ be a left Hopf algebroid such that the A^{op}-module $U^*{}_{\triangleleft}$ is flat. The extension $t^\ell : A^{op} \to U$ is Frobenius if and only if

(*i*) $U_{\triangleleft}$ is a finitely projective A^{op}-module
(*ii*) $_{\blacktriangleright}\left(\int_U^\ell\right)$ is a free A-module of rank 1.

Proof:
We start the proof by preliminary remarks :
We have seen in Proposition **??** that there exists a correspondence between left U-module structures and right U^*-comodule structures. The left U-module structure $\rightharpoonup$ on U^* ($u \rightharpoonup \phi =< \phi, -u >$) corresponds to the right U^*-comodule structure on U^* given by the coproduct. Left multiplication endows U with a left U-module. It defines a right U^*-comodule structure which coinvariant elements are $\int_{U_{coop}}^\ell = \int_U^\ell$ (Proposition **??**).

Assume that $t^\ell : A^{op} \to U$ is Frobenius. Then $_U(U_{coop})_{s_{coop}}$ is isomorphic to $_U(U_{coop})_{*\,s_{coop *}}$. In other words, $_UU_{t^\ell}$ is isomorphic to $_UU^*{}_{t^{\ell *}}$. Then, using our preliminary remarks, considered as a right U^*-comodule, U is isomorphic to the right U^*-comodule U^*. Thus the A^{op}-module $U^{cov} = \int_U^\ell$ is isomorphic to $(U^*)^{cov} = t^{\ell *}(A)$ (see Examples **??**) and condition (ii) is satisfied.

Assume that $_{\blacktriangleright}\left(\int_U^\ell\right)$ is a free A-module of rank 1. The fundamental theorem applied to the right-right Hopf U^*-module U (see Remark **??**) gives an isomorphism of right U^*-modules and of right U^*-comodules

$$_{\blacktriangleright}\int_U^\ell \otimes_{A^{op}} U^*{}_{\triangleleft} = U.$$

It follows that U is isomorphic to U^* as right U^*module and as right U^*-comodules (that is left U-modules). Using our preliminary remark, $_UU$ is isomorphic to $_UU^*$. Moreover (see Proposition **??** for the notation) $u \bullet t^{\ell *}(a) = ut^\ell(a)$. Indeed, u considered as an element of U^{op}_{coop} will be denoted $\check{u}$ and $\phi \in U^*$, considered as an element of $(U^*)^{op}_{coop}$, will be denoted $\check{\phi}$. By definition of the left action of $(U^*)^{op}_{coop}$ on $U^{op}_{coop} = \left[(U^*)^{op}_{coop}\right]^*$, we have: For all u in U and a in A,

$$\check{t^{\ell *}}(a) \bullet \check{u} =< \check{u}, \check{s^{\ell *}}(a) - > .$$

Thus

$$< u \bullet t^{\ell *}(a), \phi >=< u, \phi s^{\ell *}(a) >=< ut^\ell(a), \phi >$$

so that $u \bullet t^{\ell *}(a) = ut^\ell(a)$.

Now, the assertion follows from $U^* = (U_{coop})_{*coop}$. We have proved that the extension $t^\ell : A^{op} \to U$ is Frobenius. □.

Remark 4.4. (*i*) *In* [**?**] *(Theorem 1), it is shown that a A-Hopf algebra (with A commutative) satisfying the two conditions of the theorem is Frobenius.*
(*ii*) *In* [**?**]*, M.C. Iovanov and L. Kadison investigate when a weak Hopf algebra (introduced in* [**?**]*) is Frobenius (see Remark* **??***).*

In [**?**], a collection of conditions equivalent to the Frobenius condition is given in the setting of Hopf algebroids. In the following theorem, we generalize them to the setting of left Hopf and opHopf algebroid.

Theorem 4.5. Let $(U, A, s^\ell, t^\ell, \Delta, \epsilon)$ be a left Hopf and opHopf algebroid such that $U_\triangleleft$ and ${}_\triangleright U$ are finitely generated projective. The following assertions are equivalent:

- 1. $\left(\int_{U_*}^r\right)_\triangleleft$ is a free A^{op}-module of rank 1.
- 2. The extension $s^\ell : A \rightarrow_\triangleright U$ is Frobenius.
- 3. $\left(\int_U^\ell\right)_\blacktriangleleft$ is a free A^{op}-module of rank 1.
- 4. The extension $s_*^r : A^{op} \rightarrow_\triangleright (U_*)^{op}_{coop}$ is Frobenius.
- 5. There exists a right integral $\psi_0 \in \int_{U_*}^r$ such that the map
$$\begin{array}{ccc} U_\blacktriangleleft & \rightarrow & (U_*)_\blacktriangleleft \\ u & \mapsto & u \rightharpoonup \psi_0 \end{array}$$
is an isomorphism of $U \otimes A^{op}$-module.
- 6. There exists a $t_0 \in \int_U^\ell$ such that the map
$$\begin{array}{ccc} {}_\triangleright(U_*) & \rightarrow & {}_\triangleright U \\ \psi & \mapsto & t_0 \leftharpoonup \psi = t^\ell(<\psi, t_{0(2)}>)t_{0(1)} \end{array}$$
is an isomorphism of right U_*-modules.
- 7. The extension $s_r^* : A \rightarrow (U^*)^{op}_{coop}$ is Frobenius.
- 8. The extension $t^\ell : A^{op} \rightarrow U$ is Frobenius.
- 9. The extension $t_*^r : A \rightarrow (U_*)^{op}_{coop}$ is Frobenius.
- 10. The extension $t_r^* : A \rightarrow (U^*)^{op}_{coop}$ is Frobenius.
- 11. There exists a right integral $\phi_0 \in \int_{U^*}^r$ such that the map
$$\begin{array}{ccc} {}_\blacktriangleright U & \rightarrow & {}_\blacktriangleright(U^*) \\ u & \mapsto & u \rightharpoonup \phi_0 \end{array}$$
is an isomorphism of $U \otimes A$-module.
- 12. There exists a $t_0 \in \int_U^\ell$ such that the map
$$\begin{array}{ccc} {}_\blacktriangleright(U^*) & \rightarrow & {}_\blacktriangleright U \\ \phi & \mapsto & t_0 \leftharpoonup \phi = s^\ell\left(<\phi, t_{0(1)}>\right) t_{0(2)} \end{array}$$
is an isomorphism of right U^*-modules.

Proof:
Let us first make the following remark:
If the A-module ${}_\triangleright U$ is finitely generated projective, then the A^{op}-module $(U^*)_\triangleleft$ is finitely generated projective (by the definition of multiplication in U^*). Consequently as, under our hypothesis, the right bialgebroids U^* and U_* are isomorphic, the A^{op}-module $(U_*)_\triangleleft$ is isomorphic.
Similarly, if the A^{op}-module $U_\triangleleft$ is finitely generated projective, then the A-module ${}_\triangleright(U_*)$ is finitely generated projective and ${}_\triangleright(U^*)$ is finitely generated projective.

1. $\Rightarrow$ 2. By application of the Theorem **??** to the left-left Hopf U-module U_* (see Corollary **??**), we get an isomorphism from ${}_U U_\blacktriangleleft$ to ${}_U U_{*\blacktriangleleft}$, which proves that $s^\ell : A \rightarrow U$ is Frobenius.

2. $\Rightarrow$ 3. follows from Proposition **??**. Recall that if $t_0 \in \int_U^\ell$, then $t_0 t^\ell(a) = t_0 s^\ell(a)$.

3. $\Rightarrow$ 4. is true: Indeed, it is 1. $\Rightarrow$ 2. applied to the left bialgebroid $(U_*)^{op}_{coop}$.

4. $\Rightarrow$ 1. is 2. $\Rightarrow$ 3. applied to the left bialgebroid $(U_*)^{op}_{coop}$.

5. $\Rightarrow$ 2. by definition of a Frobenius extension and 1. $\Rightarrow$ 5. by Theorem **??** applied to the left left Hopf module U_*.

5. is equivalent to 6. because one goes from one to other replacing U by $(U_*)^{op}_{coop}$.

Thus, conditions 1., 2., 3., 4., 5. and 6. are equivalent.

7. is equivalent to 4.: As U is a left Hopf and opHopf algebroid, the map $S^* : U^* \rightarrow U_*$ is an isomorphism of right bialgebroids.

8. is equivalent to 1. If t_0 is a left integral for U, then it is a left integral for U_{coop}. Moreover, for any $a \in A$, one has $t_0 s^\ell(a) = t_0 t^\ell(a)$ so that the A-module ${}_\blacktriangleright \int_U^\ell$ is free of dimension 1 if and only if the A^{op}-module $\left(\int_U^\ell\right)_\blacktriangleleft$ is so.

Condition 9. (respectively 10., 11., 12.) is obtained from condition 7. (respectively 4., 5., 6.) replacing U by U_{coop}. □

5. Quasi-Frobenius extension

Definition 5.1. ([**?**]) *Recall that an A^e-module structure on U defines an A^e-module structure on U_* as follows :*

$$\forall \psi \in U_*, \quad \forall a \in A, \quad \forall v \in U, \quad a \blacktriangleright \psi = s(a) \rightharpoonup \psi, \quad <\psi \blacktriangleleft a, v> = <\psi, v> a.$$

Endow U_* with the left U-module structure given by the transpose of the right multiplication

$$\forall \psi \in U_*, \quad \forall (u,v) \in U^2, \quad (v \rightharpoonup \psi)(u) = \psi(uv).$$

A monomorphism of k-algebras $s : A \to U$ is called left quasi-Frobenius if

(*i*) *${}_\triangleright U$ is finitely generated and projective*

(*ii*) *The $U \otimes A^{op}$-module ${}_U U_\blacktriangleleft$ is a direct summand in a finite direct sum of copies of $U_{*\blacktriangleleft}$.*

Remarks 5.2. (*i*) *Quasi-Frobenius functors were introduced in* [**?**]. *The monomorphism $s : A \to U$ is a quasi-Frobenius extension if and only is the restriction functor is a quasi Frobenius functor.*

(*ii*) *A finitely generated projective Hopf algebra over a commutative ring is quasi-Frobenius (*[**?**]*).*

(*iii*) *In* [**?**], *it is shown that weak Hopf algebras are quasi-Frobenius.*

A counterexample of a Hopf algebroid U (such that ${}_\triangleright U$ is finitely generated projective) which is not quasi-Frobenius is exhibited in [**?**] (Lemma 5.3). In the same article, conditions are given for a Hopf algebroid to be quasi-Frobenius. In the following proposition, we extend the results obtained in [**?**] and characterize the left Hopf algebroids that are a quasi-Frobenius extension of their basis.

Proposition 5.3. Let (U, A, s^ℓ, t^ℓ) be a left Hopf algebroid such that the A^{op}-module $U^*{}_\triangleleft$ is flat. The extension $t^\ell : A^{op} \to U$ is quasi-Frobenius if and only if

(*i*) $U_\triangleleft$ is a finitely projective A^{op}-module

(*ii*) ${}_\blacktriangleright \left(\int_U^\ell\right)$ is a projective A-module.

Proof: The proof is similar to that of Proposition **??**.

Assume that $t^\ell : A^{op} \to U$ is quasi-Frobenius. Then ${}_U (U_{coop})_{s_{coop}}$ is a direct summand of a finite direct sum of copies of ${}_U (U_{coop})_{*_{s_{coop *}}}$. In other words, ${}_U U_{t^\ell}$ is a direct summand of a finite direct sum of copies ${}_U U^*{}_{t^\ell *}$. Then, using the preliminary remark in the proof of Proposition **??**, considered as a right U^*-comodule, U is a direct summand of a finite direct sum of copies of the right U^*-comodule U^*. Thus $U^{cov} = \int_U^\ell$ is a direct summand of a finite direct sum of copies of $(U^*)^{cov} = t^{\ell *}(A)$ (see Examples **??**) and condition (ii) is satisfied.

Assume that ${}_\blacktriangleright \left(\int_U^\ell\right)$ is a finitely generated projective A-module. Theorem **??** applied to the right-right Hopf U^*-module U (see Remark **??**) gives an isomorphism of right U^*-modules and of right U^*-comodules

$${}_\blacktriangleright \int_U^\ell \otimes_{A^{op}} U^*{}_\triangleleft = U.$$

It follows that U is a direct summand of a finite direct sum of copies of U^* as right U^*module and as right U^*-comodules (that is left U-modules). Using the preliminary remark in the proof of Theorem **??**, ${}_UU$ is a direct summand of a finite direct sum of copies of ${}_UU^*$. Moreover $u \cdot t^{\ell *}(a) = ut^\ell(a)$.

Now, the assertion follows from $U^* = (U_{coop})_{*coop}$. We have proved that the extension $t^\ell : A^{op} \to U$ is left quasi-Frobenius. □.

6. Application to restricted Lie-Rinehart algebras.

In this section, we apply our theory to the restricted enveloping algebra of a restricted Lie-Rinehart algebra. We will assume that k is a field of characteristic p.

Definition 6.1. *([**?**]) Let A be a commutative k-algebra with unity. A restricted Lie-Rinehart algebra $(A, L, (-)^{[p]}, \omega)$ over A is a Lie-Rinehart over A ([**?**]) such that*

(i) $(L, (-)^{[p]})$ is a restricted Lie algebra over k;
(ii) the anchor map $\omega : L \to \mathrm{Der}(A)$ is a restricted Lie algebra morphism;
(iii) For all $a \in A$ and all $X \in L$, the following relation holds

$$(aX)^{[p]} = a^p X^{[p]} + \omega((aX)^{p-1})(a)X.$$

Examples 6.2. *(i) If A is a commutative k algebra, then $(A, \mathrm{Der}(A), (-)^p, id)$ is a restricted Lie-Rinehart algebra ([**?**] lemma 1).*
*(ii) In [**?**], it is shown that weakly restricted Poisson algebras give rise to restricted Lie-Rinehart algebras.*
(iii) The restricted crossed product: Assume that $\mathfrak{g}$ is a restricted Lie algebra and that there exists a morphism of restricted Lie-algebras $\sigma : \mathfrak{g} \to Der(A)$. Then, there exists a unique structure of restricted Lie-Rinehart algebra on $A \otimes \mathfrak{g}$ (extending that of $\mathfrak{g}$) with anchor $\omega : A \otimes \mathfrak{g} \to Der(A)$, $\omega(a \otimes X) = a\sigma(X)$ and such that: For all $X, Y \in \mathfrak{g}$ and all $a, b \in A$

$$[a \otimes X, b \otimes Y] = a\sigma(X)(b) \otimes Y - b\sigma(Y)(a) \otimes X + ab \otimes [X, Y]$$

The enveloping algebra $U_A(L)$ of a Lie-Rinehart (A, L, ω) is defined in [**?**] by a universal property. It is explicitely constructed as follows :

$$U_A(L) = \frac{T_k^+(A \oplus L)}{I}$$

where I is the two sided ideal generated by the following relations : For all $a, b \in A$ and all $D, D' \in L$,

$$\begin{array}{ll} (i) & a \otimes b - ab \\ (ii) & a \otimes D - aD \\ (iii) & D \otimes D' - D' \otimes D - [D, D'] \\ (iv) & D \otimes a - a \otimes D - \omega(D)(a) \end{array}$$

If L is a projective A-module, a Poincaré-Birkhoff-Witt theorem is established in [**?**].

The restricted universal enveloping algebra of a restricted Lie-Rinehart algebra is defined as follows ([**?**]):

Definition 6.3. *Let A be a commutative k-algebra and let $(A, L, (-)^{[-]}, \omega)$ be a restricted Lie-Rinehart algebra. The restricted universal enveloping algebra is a universal triple $(U'_A(L), \iota_A, \iota_L)$ with an associative algebra map $\iota_A : A \to U'_A(L)$ and a restricted Lie algebra map $\iota_L : L \to U'_A(L)$ such that for all $D \in L$ and $a \in A$*

$$\begin{array}{l} \iota_A(a)\iota_L(D) = \iota_L(aD) \\ \iota_L(D)\iota_A(a) - \iota_A(a)\iota_L(D) = \iota_R(\omega(D)(a)) \end{array}$$

One has $U'_A(L) = \dfrac{U_A(L)}{< D^p - D^{[p]}, \quad D \in L >}$.

Remark 6.4. *Let $\mathfrak{p}$ be a prime ideal of A. Then $(L_\mathfrak{p}, A_\mathfrak{p})$ is endowed with a unique structure of restricted Lie-Rinehart algebra over $A_\mathfrak{p}$ extending that of $(L, A, (-)^{[-]})$([**?**]). Moreover, $U'_A(L)_\mathfrak{p} = U'_{A_\mathfrak{p}}(L_\mathfrak{p})$.*

An appropriate version of the Poincaré-Birkhoff-Witt theorem holds for $U'_A(L)$.

Theorem 6.5. ([**?**], see also [**?**]) Let (A, L, ω) be a restricted Lie-Rinehart algebra in characteristic p. If L is a free A-module with ordered basis $(e_i)_{i\in I}$, then $U'_A(L)$ is a free A-module with basis

$$\{\iota_L(e_{i_1})^{\alpha_1} \ldots \{\iota_L(e_{i_l})^{\alpha_l} \mid l \geqslant 0, \quad i_1 < \cdots < i_l, \quad 1 \leqslant \alpha_i < p\}$$

From now on, when there is no ambiguity, we will write $D \in L$ for its image $\iota_L(D)$ in $U'_A(L)$.

Let (L, A) be a Lie-Rinehart algebra. It is well known that its enveloping algebra is endowed with a standard left bialgebroid structure for which it is left Hopf and opHopf ([**?**]). If (L, A) is a restricted Lie-Rinehart algebra, its restricted enveloping algebra $U'_A(L)$ is also endowed with a standard left bialgebroid structure as follows:

(*i*) $s^\ell = t^\ell = \iota_A$

(*ii*) The coproduct Δ is defined by

$$\forall a \in A, \quad \Delta(a) = a \otimes 1, \qquad \forall D \in L, \quad \Delta(D) = D \otimes 1 + 1 \otimes D$$

(*iii*) $\epsilon(D) = 0$ and $\epsilon(a) = a$.

Moreover, for this structure, $U'_A(L)$ is left Hopf and

$$\forall D \in L, \quad D_+ \otimes D_- = D \otimes 1 - 1 \otimes D$$
$$\forall a \in A, \quad a_+ \otimes a_- = a \otimes 1$$

As $U'_A(L)$ is cocommutative, it is also opHopf.

Set $J'_A(L) = (U'_A(L))_*$ the restricted jet bialgebroid of (L, A). A priori, $J'_A(L)$ is a right bialgebroid. But as it is a commutative algebra, it can be seen as a left bialgebroid. Thus, both $U'_A(L)$ and $J'_A(L)$ are left Hopf and opHopf algebroids.

Proposition 6.6. Assume that L is a free finitely generated A-module with basis $\underline{e} = (e_1, \ldots, e_n)$. Introduce $\lambda_i \in J'_A(L)$ defined by

$$\forall \alpha_1, \ldots, \alpha_r \in [0, p-1], \quad <\lambda_i, e_1^{\alpha_1} \ldots e_n^{\alpha_n}>= \delta_{\alpha_1,0} \ldots \delta_{\alpha_i,1} \ldots \delta_{\alpha_n,0}.$$

One has ${\lambda_i}^p = 0$.

$$J'_A(L) = k[\lambda_1, \ldots, \lambda_n].$$

1) $\omega_{\underline{e}} = {\lambda_1}^{p-1} \ldots {\lambda_n}^{p-1}$ belongs to $\int^\ell_{J'_A(L)}$.

2) $\triangleright \int^\ell_{J'_A(L)}$ and $\blacktriangleright \int^\ell_{J'_A(L)}$ are free A-module of dimension 1 with basis $\omega_{\underline{e}}$.

Proof: 1) Let $\mu = \sum s^r_*(a_{\alpha_1,\ldots\alpha_r})\lambda_1^{\alpha_1} \ldots \lambda_n^{\alpha_n}$.

$$\mu\omega_{\underline{e}} = s^r_*(a_{0,\ldots,0})\omega_{\underline{e}} = s^r_*(<\mu, 1>)\omega_{\underline{e}}.$$

2) Let $\omega = \sum s^r_*(\omega_{\alpha_1,\ldots\alpha_r})\lambda_1^{\alpha_1} \ldots \lambda_n^{\alpha_n}$ be an element of $\int^\ell_{J'_A(L)}$. For all $i \in [1, n]$, one has $\lambda_i\omega = 0$. It is then easy to see that $\omega = s^r_*(\omega_{p-1,\ldots,p-1})\lambda_1^{p-1} \ldots \lambda_n^{p-1}$. It is easy to check that $\omega_{\underline{e}}$ is free. Thus it forms a basis of $\int^\ell_{J'_A(L)}$.

The second assertion follows from the equality $s^r_*(a)\omega_{\underline{e}} = t^r_*(a)\omega_{\underline{e}}$, which is due to the fact that $\omega_{\underline{e}}$ is a left integral. □

Remark 6.7. *With the same assumptions on L, the same proof shows that $\int^\ell_{S'_A(L)} = Ae_1^{p-1} \ldots e_n^{p-1}$.*

Corollary 6.8. Assume that L is a free finitely generated A-module. The left bialgebroids $U'_A(L)$ and $J'_A(L)$ satisfy all the equivalent conditions of the Theorem **??**

Remark 6.9. *It is shown in* [**?**] *that the restricted enveloping algebra of a finite dimensional restricted Lie algebra (over a field) is Frobenius.*

Proposition 6.10. Assume that L is a finitely generated projective A-module with a rank. Then $\int^{\ell}_{U'_A(L)}$ and $\int^{\ell}_{J'_A(L)}$ are projective A-module of rank one. Thus, $s^{\ell} : A \to U'_A(L)$, $s^r_* : A \to J'_A(L)$ and $t^r_* : A \to J'_A(L)$ are left quasi-Frobenius extensions.

Proof: Let L_1 be a finitely generated module such that $L \bigoplus L_1 = F$ is finitely generated free. We endow F with the following restricted Lie-Rinehart structure:

- (*i*) The Lie bracket on F extends that on L. Moreover, for all $X \in L$ and all $(Y, Y') \in L_1^2$, $[X, Y] = [Y, Y'] = 0$.
- (*ii*) The anchor $\omega_F : F \to Der_k(A)$ extends that of L. Moreover, for all Y in L_1, one has $\omega_F(Y) = 0$.
- (*iii*) The p-operation extends that of L. Moreover, for all $Y \in L_1$, one has $Y^{[p]} = 0$.

In other term, F is the direct sum of L with the abelian restricted Lie-Rinehart algebra L_1 and $U'_A(F) = U'_A(L) \otimes S'_A(L_1)$. From Theorem **??**, $\int^{\ell}_{U'_A(F)}$ is a free A-module of dimension 1. From Remark **??**, we knows that $\int^{\ell}_{S'_A(L_1)}$ is ${S'_A}^{r(p-1)}(L_1)$ where r is the rank of L_1. We now show that

$$\int^{\ell}_{U'_A(L)} \otimes_A \int^{\ell}_{S'_A(L_1)} = \int^{\ell}_{U'_A(F)} .$$

The inclusion $\int^{\ell}_{U'_A(L)} \otimes_A \int^{\ell}_{S'_A(L_1)} \subset \int^{\ell}_{U'_A(F)}$ is easy to check. The equality follows from a localization argument (Remark **??**) and Theorem **??**).

Thus, $\int^{\ell}_{U'_A(L)} = Hom_A\left(\int^{\ell}_{S'_A(L_1)}, \int^{\ell}_{U'_A(F)}\right)$, which is a projective A-module.

Let $\mathfrak{p}$ be a prime ideal of A and $(L_{\mathfrak{p}}, A_{\mathfrak{p}})$ the restricted Lie-Rinehart algebra over $A_{\mathfrak{p}}$ extending $(L, A, (-)^{[-]})$ (Remark **??**). As $L_{\mathfrak{p}}$ is a free finite $A_{\mathfrak{p}}$-module, by application of Theorem **??**, the $A_{\mathfrak{p}}$-module, $\int^{\ell}_{U'_{A_{\mathfrak{p}}}(L_{\mathfrak{p}})}$ is free of dimension one. Thus $\left(\int^{\ell}_{U'_A(L)}\right)_{\mathfrak{p}}$ is a projective $A_{\mathfrak{p}}$-module of rank one by Lemma **??**.

One shows similarly that $s^r_* : A \to J'_A(L)$ and $t^r_* : A \to J'_A(L)$ are a left quasi-Frobenius extensions.□

Let $\mathfrak{p}$ be a prime ideal of A. Localization of the A-module (for left multiplication by elements of A) $U_A(L)$ and localization of the A-module (for right multiplication by elements of A) $U_A(L)$ are isomorphic to $U_{A_{\mathfrak{p}}}(L_{\mathfrak{p}})$.

Lemma 6.11. Let L be a restricted Lie-Rinehart algebra which is a finitely generated A-module and let $\mathfrak{p}$ be a prime ideal of A. Then

$$\left(\int^{\ell}_{U'_A(L)}\right)_{\mathfrak{p}} = \int^{\ell}_{U'_{A_{\mathfrak{p}}}(L_{\mathfrak{p}})}$$
$$\left(\int^{\ell}_{J'_A(L)}\right)_{\mathfrak{p}} = \int^{\ell}_{J'_{A_{\mathfrak{p}}}(L_{\mathfrak{p}})}$$

Proof:

We only prove the first equality. The proof of the second one is similar. Any element of $U'_{A_{\mathfrak{p}}}(L_{\mathfrak{p}})$ can be written $\frac{1}{\sigma} \times \frac{u}{1}$ or $\frac{v}{1} \times \frac{1}{\tau}$ with $u, v \in U'_A(L)$ and $\sigma, \tau \in A - \mathfrak{p}$.

First we prove the inclusion $\left(\int^{\ell}_{U'_A(L)}\right)_{\mathfrak{p}} \subset \int^{\ell}_{U'_{A_{\mathfrak{p}}}(L_{\mathfrak{p}})}$.

If u_0 is in $\int^{\ell}_{U'_A(L)}$ and $\sigma \in A - \mathfrak{p}$, then

$$\frac{1}{\tau} \times vu_0 \times \frac{1}{\sigma} = \frac{1}{\tau} \times \epsilon(v)u_0\frac{1}{\sigma} = \epsilon\left(\frac{1}{\tau}v\right)u_0\frac{1}{\sigma}.$$

Thus $u_0 \times \dfrac{1}{\sigma} \in \int^{\ell}_{U'_{A_\mathfrak{p}}(L_\mathfrak{p})}$.

Second we prove the inclusion $\int^{\ell}_{U'_{A_\mathfrak{p}}(L_\mathfrak{p})} \subset \left(\int^{\ell}_{U'_A(L)}\right)_\mathfrak{p}$. Let $u_0 \in U'_A(L)$ and $\sigma_0 \in A-\mathfrak{p}$ such that $u_0 \times \dfrac{1}{\sigma_0} \in \int^{\ell}_{U'_{A_\mathfrak{p}}(L_\mathfrak{p})}$. Then for any $v \in U'_A(L)$,

$$\frac{v}{1}u_0 \times \frac{1}{\sigma_0} = \frac{\epsilon(v)}{1}u_0 \times \frac{1}{\sigma_0} = \frac{\epsilon(v)u_0}{1} \times \frac{1}{\sigma_0}$$

As $U'_A(L)$ is finitely generated as an A-module, there exists $\tau \in A-\mathfrak{p}$ such that

$$\forall v \in U'_A(L), \quad vu_0\tau = \epsilon(v)u_0\tau$$

Thus $u_0\tau$ is in $\int^{\ell}_{U'_A(L)}$ and $u_0 \times \dfrac{1}{\sigma_0}$ is in $\left(\int^{\ell}_{U'_A(L)}\right)_\mathfrak{p}$.

Remark 6.12. *If the anchor is 0, $U'_A(L)$ is a projective finitely generated A-Hopf algebra and Theorem* **??** *was already known (*[**?**]*)*

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