Operations for modules on Lie-Rinehart superalgebras

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Let k be a field of characteristic 0 and let A be a supercommutative associative k-superalgebra. Let $\mathcal{L}$ be a $k-A$-Lie-Rinehart superalgebra. From these data, one can construct a superalgebra of differential operators $\mathcal{V}(A,\mathcal{L})$ (generalizing the enveloping superalgebra of a Lie superalgebra). We will give a definition of Lie-Rinehart superalgebra morphisms allowing to generalize the notions of inverse image and direct image. We will prove that a Lie-Rinehart superalgebra morphism decomposes into a closed imbedding and a projection. Furthermore, we will see that, under some technical conditions, a closed imbedding decomposes into two closed imbeddings of different nature. The first one looks like a Lie superalgebra morphism. The second one looks like a supermanifold closed imbedding and satisfies a generalization of the Kashiwara's theorem. Then, as in the $\mathcal{D}$-module theory, we introduce a duality functor. Finally, we will prove that, in the closed imbedding case, the direct image and the duality functor commute.

1 Introduction

Let k be a commutative field of characteristic 0 and let A be an associative supercommutative k-superalgebra with unity. $(A,\mathcal{L},\sigma)$ is a $k-A$-Lie-Rinehart superalgebra or a $k-A$-Lie-superalgebra if it is endowed with a k-Lie superalgebra structure and an A-module structure satisfying the following compatibility relation: There exists a Lie superalgebra and an A-module morphism $\sigma: \mathcal{L} \to \mathrm{Der}_k(A)$ such that

$$\forall (D,\Delta) \in \mathcal{L}^2,\ \forall a \in A,\ [D,a\Delta] = \sigma(D)(a)\Delta + (-1)^{|a||D|}a[D,\Delta].$$

Lie-Rinehart superalgebras are the algebraic analog of the Lie-algebroid concept ([19]). They give rise to superalgebras of differential operators which generalize at the same time enveloping superalgebras and superalgebras of differential operators on a supermanifold. Let $(A,\mathcal{L},\sigma)$ be a $k-A$ Lie superalgebra and let $\mathcal{V}(A,\mathcal{L},\sigma)$ (or for short $\mathcal{V}(A,\mathcal{L})$) be the superalgebra of differential operators it defines. Most of the time we will need to assume that $\mathcal{L}$ is a finitely generated projective A-module with a rank. This allows us to define $\mathrm{Ber}(\mathcal{L}^*)$, the Berezinian module of $\mathcal{L}^* = \mathrm{Hom}_A(\mathcal{L}, A)$ which plays the same role as the differential forms of maximal degree for $\mathcal{D}$-modules. We already know ([7]) that there is a correspondence (analogous to Bernstein's correspondence) between left and right $\mathcal{V}(A, \mathcal{L})$-modules involving $\mathrm{Ber}(\mathcal{L}^*)$. In this article, we will adopt a definition of morphisms of Lie-Rinehart superalgebras which generalizes what happens in the Lie superalgebra case and in the algebraic smooth supermanifold case. This definition is compatible with the Lie algebroid morphisms defined by Almeida and Kumpera ([1]). We will prove that a Lie-Rinehart superalgebra morphism decomposes into a closed imbedding and a projection. Furthermore, we will see that, under some technical conditions, a closed imbedding decomposes into two closed imbeddings of different nature.

- The first one looks like a Lie superalgebra morphism.
- The second one looks like a supermanifold closed imbedding and satisfies a generalization of the Kashiwara's theorem.

Imbeddings of the second type were already studied by Levasseur ([17]) in the case analogous to $\{x\} \hookrightarrow X$.

We then generalize the notions of inverse image and direct image existing in the $\mathcal{D}$-module theory. We will introduce the duality functor and prove that this latter commutes with the direct image in the case of a closed imbedding. This will allow us to recover some duality property for induced representations of Lie superalgebras.

Notations :

For most of the definitions about supermathematics, we refer the reader to [15]. k will be a commutative field of characteristic zero. We will denote by $\bar{0}$ and $\bar{1}$ the elements of $\mathbb{Z}/2\mathbb{Z}$. We will call superspace a k-vector space graded over $\mathbb{Z}/2\mathbb{Z}$, $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Let V and W be two superspaces. If f is a morphism of degree i from V to W and if v is in V_j, we put $< v, f >= (-1)^{ij} f(v)$. If V is a superspace, one defines the superspace ΠV which, as a vector space, is equal to V but whose grading is $(\Pi V)_{\bar{0}} = V_{\bar{1}}$ and $(\Pi V)_{\bar{1}} = V_{\bar{0}}$. Let us introduce the map $\pi : \ V \to \Pi V$ which, as a morphism of vector spaces, equals the identity. It is of degree $\bar{1}$. The symmetric superalgebra of V will be denoted by $S(V)$.

Let A be an associative supercommutative superalgebra with unity and let M be an A-module. A basis of M is a family $(m_i)_{i \in I \amalg J} \in M_{\bar{0}}^I \times M_{\bar{1}}^J$ such that each element of M can be expressed in a unique way as a linear combination of the $(m_i)_{i \in I \amalg J}$. If I and J are finite, their cardinalities are independent of the basis of the A-module M. Then, the dimension of M over A is the element $\mid I \mid + \epsilon \mid J \mid$ of $\mathbb{Z}[\epsilon]/(\epsilon^2 - 1)$. If $(e_1, ..., e_n)$ is a basis of the A-module M, then the family $(e^1, ..., e^n)$ where $< e_i, e^j >= \delta_{i,j}$ is a basis of $Hom_A(M, A)$ called the dual basis of $(e_1, ..., e_n)$. Moreover, if M is an A-module, then ΠM has a natural A-module structure defined by:

$$\forall m \in M, \ \forall a \in A, \ \ a \cdot \pi m = (-1)^{|a|} \pi(a \cdot m).$$

One defines supermatrices by assigning each line and each column a parity. In general even lines and even columns are put first so that $\mathrm{Mat}(r + \epsilon s, A)$ will be the set of matrices with r even lines and columns and with s odd lines and columns. To get a correspondence between endomorphisms and matrices, one has to consider right A-modules. We will denote by $\mathrm{GL}(r+\epsilon s, A)$ the group of invertible even elements of $\mathrm{Mat}(r+\epsilon s, A)$.

We will only consider localization with respect to even multiplicative systems. Let S be a multiplicative system of $A_{\bar{0}}$, then M_S will denote the localized module with respect to S. If $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}$ is a prime ideal (respectively f an element of $A_{\bar{0}}$), then $M_{\mathfrak{p}}$ (respectively M_f) will denote the localization of M with respect to the multiplicative system $A_{\bar{0}} - \mathfrak{p}_{\bar{0}}$ (respectively $\{f^n \mid n \in \mathbb{N}\}$).

If J is an ideal of A, we put

$$\Gamma_J(M) = \cup_{n\in\mathbb{N}}\{m \in M \mid \exists n \in \mathbb{N} \mid J^n \cdot m = 0\} = \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_B(B/J^n, M).$$

The functor Γ_J is left exact. We denote by H^j_J its jth right derived functor.

Let $\mathcal{A}$ be an abelian category of objects graded over $\mathbb{Z}/2\mathbb{Z}$. We will adopt the following conventions for the complexes: we require that the differentials defining the complexes be odd whereas the morphisms between complexes have to be even. If $M^\bullet$ is a complex of objects of $\mathcal{A}$, $M^\bullet[1]$ will be the complex defined by $(M^\bullet[1])^i = M^{i+1}$. We will denote by $D(\mathcal{A})$ the derived category of $\mathcal{A}$. One can also define $D^-(\mathcal{A})$.

If B is an associative superalgebra with unity, then $\mathrm{Mod}^{\mathrm{l}}_{\mathrm{B}}$ (respectively $\mathrm{Mod}^{\mathrm{r}}_{\mathrm{B}}$) will be the category of graded left (respectively right) B-modules.

Let $\mathfrak{a}$ be a k-Lie superalgebra. We will write $U(\mathfrak{a})$ for its enveloping superalgebra and Δ for the coproduct in $U(\mathfrak{a})$. If M is a left $U(\mathfrak{a})$-module, then M^* will be the contragredient module. Let now $\mathfrak{g}$ be a Lie superalgebra and $\mathfrak{h}$ be a Lie subsuperalgebra. Let V (respectively W) be a left (respectively right) $U(\mathfrak{h})$-module. We will denote by $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}}(V)$ (respectively $\mathcal{IND}^{\mathfrak{g}}_{\mathfrak{h}}(W)$) be the left (respectively right) $U(\mathfrak{g})$-module $U(\mathfrak{g}) \underset{U(\mathfrak{h})}{\otimes} V$ (respectively $W \underset{U(\mathfrak{h})}{\otimes} U(\mathfrak{g})$). We define the coinduced space $\mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{h}}(V)$ from V as being $\mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), V)$. The transpose of right multiplication on $U(\mathfrak{g})$ endows $\mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{h}}(V)$ with a left $\mathfrak{g}$-module structure. We thus get the coinduced representation from V. If $V = k$ is the trivial module, we put $A = \mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{h}}(k)$. One defines a k-superalgebra structure on A as follows: for all (f, g) in A^2 and u in $U(\mathfrak{g})$,

$$< u, f \cdot g >= \sum_j < u'_j, f >< u''_j, g > (-1)^{|f||u''_j|}$$

where $\Delta(u) = \sum_j u'_j \otimes u''_j$. The superalgebra A is associative, supercommutative and with unity. It is a local superalgebra ([6]).

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2 Ring of differential operators defined by a $k - A$ Lie superalgebra

2.1 Recollections.

Let A be a supercommutative associative k-superalgebra with unity and let $\mathcal{L}$ be a k-Lie superalgebra that is also an A-module. Assume that we are given $\sigma : \mathcal{L} \to Der(A)$ a morphism of Lie superalgebras and of A-modules. Assume moreover that for all D and Δ belonging to $\mathcal{L}$ and all a in A, we have

$$[\Delta, aD] = a\,[\Delta, D]\,(-1)^{|a||\Delta|} + \sigma(\Delta)(a)D.$$

$(A, \mathcal{L}, \sigma)$ is called a $k - A$-Lie-Rinehart superalgebra or $k - A$-Lie superalgebra. Let $\mathcal{V}(A, \mathcal{L}, \sigma)$ (or $\mathcal{V}(A, \mathcal{L})$) be the superalgebra of differential operators generated $(A, \mathcal{L}, \sigma)$ ([9],[22]). It can be described as follows: $\mathcal{V}(A, \mathcal{L})$ is the k-superalgebra linearly generated by the elements of A, the elements of $\mathcal{L}$ and the following relations:

$$(\$) \qquad \begin{cases} a \cdot b = (ab) \\ D \cdot a - (-1)^{|a||D|} a \cdot D = \sigma(D)(a) \\ D \cdot \Delta - (-1)^{|\Delta||D|} \Delta \cdot D = [D, \Delta] \\ a \cdot D = (aD) \end{cases}$$

Let $\mathcal{V}(A, \mathcal{L})_n$ be the left A-submodule of $\mathcal{V}(A, \mathcal{L})$ generated by products of at most n elements of $\mathcal{L}$. We define thus a filtration on $\mathcal{V}(A, \mathcal{L})$. If $\mathcal{L}$ is A-projective, then the graded A- superalgebra $\mathrm{Gr}\mathcal{V}(A, \mathcal{L})$ (with respect to this filtration) is isomorphic to the symmetric superalgebra $S_A(\mathcal{L})$ ([22] p 198).

Examples

- The simplest example is obtained when σ is 0. Then, $\mathcal{L}$ is just an A-Lie superalgebra and $\mathcal{V}(A, \mathcal{L})$ is its enveloping superalgebra.
- Let $\mathcal{X}$ be a paracompact smooth supermanifold (over $\mathbb{R}$ or $\mathbb{C}$)([15]). Let X (respectively $O_{\mathcal{X}}$) be the underlying topological space (structural sheaf) of $\mathcal{X}$. We write $\mathcal{X} = (X, O_{\mathcal{X}})$. Put $A = O_{\mathcal{X}}(X)$, $\mathcal{L} = \mathrm{Der}\ O_{\mathcal{X}}(X)$ and $\sigma = id$. Then $\mathcal{V}(A, \mathcal{L})$ is the superalgebra of differential operators over X. Moreover, $\mathrm{Der}\ O_{\mathcal{X}}(X)$ is a finitely generated projective $O_{\mathcal{X}}(X)$-module (see [13] p. 31, [23] p 266 and [25] p 100).
- Let A be a k-Poisson superalgebra. Let D_A^{ev} be the A-module of Kähler differentials for A with a grading determined by $| d^{ev}a | = | a |$. Then D_A^{ev} is naturally endowed with a $k - A$-Lie superalgebra structure ([12]). Note that this structure depends on the Poisson bracket on A. There exists a similar construction for the differential forms of degree one on a Poisson supermanifold.

- Let $\mathfrak{g}$ be a k-Lie superalgebra and A be a supercommutative associative superalgebra with unity. Assume that there is a Lie superalgebra morphism $\sigma_0 : \mathfrak{g} \to \mathrm{Der(A)}$. On $\mathcal{L} = A \otimes \mathfrak{g}$, we put the following bracket : for all (a, a', X, X') in $A^2 \times \mathfrak{g}^2$,

$$\begin{aligned}[a \otimes X, a' \otimes X'] &= (-1)^{|X||a'|} aa' \otimes [X, X'] + a\sigma_0(X)(a') \otimes X' \\ &\quad -a'\sigma_0(X')(a) \otimes X(-1)^{(|a|+|X|)(|a'|+|X'|)}.\end{aligned}$$

  Moreover, we extend σ_0 to a A-module morphism σ from $\mathcal{L}$ to Der(A). We thus define a $k - A$-Lie superalgebra structure on $\mathcal{L}$ denoted $A\#\mathfrak{g}$ and called the crossed product of A and $\mathfrak{g}$. An example of this situation is the following. Let $\mathfrak{h}$ be a sub-superalgebra of $\mathfrak{g}$. Put $A=\mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(k)$ and take for $\sigma_0 : \mathfrak{g} \to \mathrm{Der}(A)$ the coinduced representation.

- Let $(A, \mathcal{L}_A, \sigma_A)$ be a $k - A$-Lie superalgebra and let $(B, \mathcal{L}_B, \sigma_B)$ be a $k - B$- Lie superalgebra. Put

$$\mathcal{L}_{A\otimes B} = B \underset{k}{\otimes} \mathcal{L}_A \oplus A \underset{k}{\otimes} \mathcal{L}_B.$$

  Then $\mathcal{L}_{A\otimes B}$ is a $A \otimes B$-module. We define the $A \otimes B$-module morphism $\sigma_{A\otimes B} : \mathcal{L}_{A\otimes B} \to Der(A \otimes B)$ as follows

$$\forall D \in \mathcal{L}_A,\ \forall a \in A,\ \forall b \in B,\ \ \sigma_{A\otimes B}(D)(a \otimes b) = \sigma_A(D)(a) \otimes b$$

$$\forall \Delta \in \mathcal{L}_B,\ \forall a \in A,\ \forall b \in B,\ \ \sigma_{A\otimes B}(\Delta)(a \otimes b) = a \otimes \sigma_B(\Delta)(b)(-1)^{|a||\Delta|}.$$

  On $\mathcal{L}_{A\otimes B}$, we put the Lie bracket extending those of $\mathcal{L}_A$ and $\mathcal{L}_B$ and such that:

  - $[\mathcal{L}_A, \mathcal{L}_B] = 0$

  - $(A \otimes B, \mathcal{L}_{A\otimes B}, \sigma_{A\otimes B})$ is a $k - A \otimes B$-Lie superalgebra.

3 Equivalence of category between left $\mathcal{V}(A, \mathcal{L})$-modules and right $\mathcal{V}(A, \mathcal{L})$-modules

The notations are the same as in the previous paragraph.

3.1 The Berezininan module.

Let us remark that a prime ideal $\mathfrak{p}$ of A is characterized by its intersection with $A_{\bar{0}}$, $A_{\bar{0}} \cap \mathfrak{p}$, which is a prime ideal of $A_{\bar{0}}$ (because $A_{\bar{1}}$ is included in $\mathfrak{p}$). One defines the ringed space, Spec(A), as follows ([8],[16]). The underlying topological space X is the set of prime ideals of $A_{\bar{0}}$ endowed with the Zariski topology. The structural sheaf O_X of Spec(A) is defined as in the non graded case. If $f \in A_{\bar{0}}$, let $D(f)$ be the open subset

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}(A_{\bar{0}})/f \notin \mathfrak{p}\}.$$

Then, $(D(f))_{f\in A_{\bar{0}}}$ form a basis for the Zariski topology on Spec$(A_{\bar{0}})$.

Case of a finite dimensional free module.
Let M be a free A-module of dimension $d_0+\epsilon d_1$. Put $n=d_0+d_1$. Let $(e_1,...,e_n)$ be a basis of M such that $(e_1,...,e_{d_0})$ are even and $(e_{d_0+1},...,e_n)$ are odd. Let us denote by $(e^1,...,e^n)$ the dual basis and let d be left multiplication by $\sum_{i=1}^{n}(-1)^{|e_i|+1}\pi e_i\otimes e^i$ in the superalgebra $S_A(\Pi M\oplus_A M^*)$. The endomorphism d does not depend on the choice of a basis.

Proposition 3.1.1 *The complex* $J(M)=\left(S_A(\Pi M\oplus M^*)=\underset{n\in\mathbb{N}}{\oplus}S^n(\Pi M)\underset{A}{\otimes}S(M^*),d\right)$ *has no cohomology except in degree d_0. The A-module $H^{d_0}(J(M))$ is free of dimension 1 or ϵ. More precisely, the element $\pi e_1...\pi e_{d_0}\otimes e^{d_0+1}...e^n$ is a cycle whose class is a basis of $H^{d_0}(J(M))$.*

A proof of the proposition 3.1.1 is given in [18] p. 172.

Definition 3.1.2 *The module $H^{d_0}(J(M))$ is called the Berezinian module of M and is denoted $Ber(M)$.*

Remark :
Note that any basis $[e]=(e_1,...,e_n)$ of M defines a basis of Ber(M), namely $\omega_{[e]}=\left[\pi e_1...\pi e_{d_0}\otimes e^{d_0+1}...e^n\right]$.

Case of a finitely generated projective A-module with a rank
Let M be a finitely generated projective A-module. Assume that the localized module with respect to any prime ideal, $M_{\mathfrak{p}}$, has dimension $d_0+\epsilon d_1$. Then M has a rank. We will write : $\mathrm{rk}\mathcal{L}=\mathrm{d}_0+\epsilon\mathrm{d}_1$ and $\mathrm{erk}\mathcal{L}=\mathrm{d}_0$. The proof of the existence of the Berezinian module of M relies on the following result ([4] p. 141). Let us denote by $\mathcal{I}(M)$ the set of all elements f of $A_{\bar{0}}$ such that M_f is a free A_f-module.

Lemma 3.1.3 *$(D(f))_{f\in\mathcal{I}(M)}$ is an open covering of Spec(A).*

We come now to the theorem which will allow us to define the Berezinian module of a finitely generated projective A-module with a rank (see [7] for the proofs).

Theorem 3.1.4 *Let M be a finitely generated projective A-module with a rank. There is a unique A-module (up to isomorphism) denoted $Ber(M)$ such that, for all f in $\mathcal{I}(M)$, $Ber(M)_f$ is canonically isomorphic to $Ber(M_f)$.*

Definition 3.1.5 *The module $Ber(M)$ constructed by the previous theorem is called the Berezinian module of M.*

3.2 Equivalence of categories.

In this paragraph, we recall that Bernstein's correspondence between left and right $\mathcal{D}$-modules extends to our context. For short, we put $\mathcal{V} = \mathcal{V}(A, \mathcal{L})$

Proposition 3.2.1 *If $\mathcal{L}$ is a finitely generated projective A-module with a rank, then $Ber(\mathcal{L}^*)$ is endowed with a natural right $\mathcal{V}$-module structure. In the case where $\mathcal{L}$ is free, the action of $\mathcal{L}$ on $Ber(\mathcal{L}^*)$ is induced by its adjoint action on $J(\mathcal{L}^*)$.*

Proof of the proposition 3.2.1 : see [7].

Corollary 3.2.2 *Assume that $\mathcal{L}$ is a finitely generated projective A-module with a rank.*

a) If M is a left $\mathcal{V}$-module, then $M \underset{A}{\otimes} Ber(\mathcal{L}^*)$ is endowed with a right $\mathcal{V}$-module structure determined by the following operations : for all m in M, ω in $Ber(\mathcal{L}^*)$, a in A and D in $\mathcal{L}$, we have

$$(m \otimes \omega) \cdot a = m \otimes \omega \cdot a = (-1)^{|a|(|m|+|\omega|)} a \cdot m \otimes \omega$$
$$(m \otimes \omega) \cdot D = -(-1)^{|D|(|m|+|\omega|)} D \cdot m \otimes \omega + m \otimes \omega \cdot D.$$

b) The functor $\Omega : M \mapsto \Omega(M) = M \underset{A}{\otimes} \mathrm{Ber}(\mathcal{L}^*)$ provides an equivalence between the category of left $\mathcal{V}$-modules and the category of right $\mathcal{V}$-modules.

Remarks

In the case where $\mathcal{L}$ is a free finite dimensional A-module, Fel'dman ([9] p. 127) exhibited an anti-involution of $\mathcal{V}$ (depending on the choice of a basis) which gives rise to the same equivalence between left and right $\mathcal{V}$-modules. The use of the Berezinian module is more canonical.

The Bernstein's equivalence was extended to the supermanifold case by Penkov ([20]).

4 Morphism of Lie-Rinehart superalgebras

Let $(A, \mathcal{L}_A, \sigma_A)$ and $(B, \mathcal{L}_B, \sigma_B)$ be two Lie-Rinehart superalgebras. When there is no ambiguity, we will omit σ_A and σ_B in the formulas.

Definition 4.0.3 *A Lie-Rinehart superalgebra morphism from $(A, \mathcal{L}_A, \sigma_A)$ to $(B, \mathcal{L}_B, \sigma_B)$ is the datum of two maps (ϕ, ψ) such that*

1) $\phi : B \to A$ is a k-superalgebra morphism.

2) $\psi : \mathcal{L}_A \to A \underset{B}{\otimes} \mathcal{L}_B$ is a A-module morphism satisfying the two following properties :

a) Let D be in $\mathcal{L}_A$. If $\psi(D) = \sum_i a_i \otimes \Delta_i$ with $a_i \in A$ and $\Delta_i \in \mathcal{L}_B$, then

$$\sum_i a_i \left(\phi \circ \sigma_B(\Delta_i)\right) = \sigma_A(D) \circ \phi.$$

b) Put on $\mathcal{V}_B$ the left $\mathcal{V}_B$-module structure given by left multiplication. Then the two following operations endow $A\underset{B}{\otimes}\mathcal{V}_B$ with a left $\mathcal{V}_A$- module structure:

$$\begin{array}{l} \forall D \in \mathcal{L}_A,\ \forall (a,g) \in A^2, \forall m \in \mathcal{V}_B, \\ (*) \quad a \cdot (g \otimes m) = ag \otimes m \\ (**) \ \ D \cdot (g \otimes m) = D(g) \otimes m + \sum_i g a_i \otimes \Delta_i \cdot m \end{array}$$

where $\psi(D) = \sum_i a_i \otimes \Delta_i$

Remarks :

1) This definition coincides with the definition given by Almeida and Kumpera in the Lie-algebroids context ([1]).

2) Using the properties of the tensor product and the compatibility relation a), one shows that the operation (**) is well defined. Among all the relations ($), the third relation is the only one which is not automatically satisfied by $A\underset{B}{\otimes}\mathcal{V}_B$.

3) The condition b) of the definition may be replaced by the following condition : Let D and D' be two elements of $\mathcal{L}_A$. If $\psi(D) = \sum_i a_i \otimes D_i$ and $\psi(D') = \sum_i a'_i \otimes D'_i$ with $(a_i, a'_i) \in A^2$ and $(D_i, D'_i) \in \mathcal{L}_B^2$, then

$$(\Delta) \qquad \psi([D,D']) = \sum_i D(a'_i) \otimes D'_i - \sum_i D'(a_i) \otimes D_i (-1)^{|D||D'|} + \sum_{i,j} (-1)^{|D||a'_j|} a'_j a_i \otimes \left[D_i, D'_j\right].$$

4) Our definition of Lie-Rinehart morphisms generalizes at the same time what happens in the Lie superalgebra case and in the algebraic smooth supervariety case.

Examples of Lie-Rinehart superalgebra morphisms

1) Assume that we have two crossed products $(A, A\otimes\mathfrak{g}, \sigma_0 : \mathfrak{g} \to \mathrm{Der}(A))$ and $(B, B\otimes \mathfrak{g}', \sigma'_0 :\ \mathfrak{g}' \to \mathrm{Der(B)})$. Assume that we have a Lie superalgebra morphism $f :\ \mathfrak{g} \to \mathfrak{g}'$ and a superalgebra morphism $\phi :\ B \to A$ such that

$$(\&) \quad \forall b \in B,\ \forall X \in \mathfrak{g},\ \phi[f(X)(b)] = X(\phi(b)).$$

Here, we omit σ_0 and σ'_0 for short. Let $\psi :\ A \otimes \mathfrak{g} \to A\underset{B}{\otimes}(B \otimes \mathfrak{g}')$ be the A-module morphism extending f. Then (ϕ, ψ) is a Lie-Rinehart superalgebra morphism from $A\otimes\mathfrak{g}$ to $B \otimes \mathfrak{g}'$.

We provide now an example of such a case. Let $\mathfrak{h}$ be a subsuperalgebra of $\mathfrak{g}$. Let $\mathfrak{h}'$ be a Lie subsuperalgebra of $\mathfrak{g}'$ containing $f(\mathfrak{h})$. We have a map

$$\begin{array}{rcl} \phi :\ \mathrm{Coind}_{\mathfrak{h}'}^{\mathfrak{g}'}(\mathrm{k}) & \to & \mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(\mathrm{k}) \\ \lambda & \mapsto & (u \mapsto < f(u), \lambda >) \end{array}$$

where f is seen as a superalgebra morphism from $U(\mathfrak{g})$ to $U(\mathfrak{g}')$. One can check that the relation & is satisfied. So that we have constructed a Lie-Rinehart superalgebra morphism from $\left(\mathfrak{g}, \mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(\mathrm{k})\right)$ to $\left(\mathfrak{g}', \mathrm{Coind}_{\mathfrak{h}'}^{\mathfrak{g}'}(\mathrm{k})\right)$.

2) Let $(A, \mathcal{L}_A, \sigma_A)$ and $(B, \mathcal{L}_B, \sigma_B)$ be two Lie-Rinehart superalgebras. Assume that $\mathcal{L}_B$ is a finite dimensional free B-module and that we can find a basis $(f_1, \ldots, f_n)$ of $\mathcal{L}_B$ such that $[f_i, f_j] = 0$. Let $\phi : B \to A$ be a superalgebra morphism and let a_i be elements of A such that $| a_i | = | f_i |$. Define $\psi : \mathcal{L}_A \to A \underset{B}{\otimes} \mathcal{L}_B$ defined by

$$\psi(D) = \sum_i D(a_i) \otimes f_i.$$

If the compatibility relation a) is satisfied, then (ϕ, ψ) is a Lie-Rinehart superalgebra morphism from $(A, \mathcal{L}_A)$ to $(B, \mathcal{L}_B)$. We will now give a particular case of this example which involves Poisson superalgebras. Let $A = k[X_1, \ldots, X_n]$ be a Poisson superalgebra such that $\{X_i, X_j\}$ belongs to k and let $B = k[Y_1, \ldots, Y_p, \tilde{Y}_1, \ldots, \tilde{Y}_p]$ with $| Y_i | = | \tilde{Y}_i |$ be the Poisson superalgebra whose bracket is given by $\{Y_i, \tilde{Y}_j\} = \delta_{i,j}$, $\{Y_i, Y_j\} = \{\tilde{Y}_i, \tilde{Y}_j\} = 0$. Let $\phi : B \to A$ be a superalgebra morphism. Define $\psi : D_A^{ev} \to A \underset{B}{\otimes} D_B^{ev}$ by

$$\psi(dX_j) = \sum_i -(-1)^{|Y_i||\tilde{Y}_i|}\{X_j, \phi(Y_i)\} \otimes d\tilde{Y}_i + \sum_i \{X_j, \phi(\tilde{Y}_i)\} \otimes dY_i.$$

Then (ϕ, ψ) is a Lie-Rinehart superalgebra morphism from (A, D_A^{ev}) to (B, D_B^{ev}).

5 Direct Image

This section is inspired by the affine case of the $\mathcal{D}$-module theory ([3]). Let $(A, \mathcal{L}_A, \sigma_A)$ and $(B, \mathcal{L}_B, \sigma_B)$ be two Lie-Rinehart superalgebras. We assume that $\mathcal{L}_A$ (respectively $\mathcal{L}_B$) is a finitely generated projective A-module (respectively B-module) with a rank. Let $\Phi = (\phi, \psi)$ be a Lie-Rinehart superalgebra morphism from $(A, \mathcal{L}_A, \sigma_A)$ to $(B, \mathcal{L}_B, \sigma_B)$. For short, we will write $\mathcal{V}(\mathcal{L}_A)$ or even $\mathcal{V}_A$ for $\mathcal{V}(A, \mathcal{L}_A, \sigma_A)$ when there is no ambiguity. We will adopt the same abbreviation for $\mathcal{V}(B, \mathcal{L}_B, \sigma_B)$.

By definition of a Lie-Rinehart superalgebra morphism, we know that $A \underset{B}{\otimes} \mathcal{V}_B$ is a left $\mathcal{V}_A$-module. It is also clearly endowed with a right $\mathcal{V}_B$-module structure commuting with the previous one. This $(\mathcal{V}_A, \mathcal{V}_B)$-bimodule will be denoted $\mathcal{V}_{\mathcal{L}_A \to \mathcal{L}_B}$ and will be called the tranfer module. Let M be a left $\mathcal{V}_B$-module. The equality

$$A \underset{B}{\otimes} M = \mathcal{V}_{\mathcal{L}_A \to \mathcal{L}_B} \underset{\mathcal{V}_B}{\otimes} M,$$

endows $A \underset{B}{\otimes} M$ with a left $\mathcal{V}_A$-module structure. $A \underset{B}{\otimes} M$ will be called the inverse image of M and will be denoted $\Phi^o(M)$.

Let M be a right $\mathcal{V}_B$-module. The inverse image of M is the right $\mathcal{V}_A$-module $\left(\Omega_A \circ \Phi^o \circ \Omega_B^{-1}\right)(M)$. The inverse image of $\mathcal{V}_B^r$, that is to say $\mathcal{V}_B$ considered as a right $\mathcal{V}_B$-module under right multiplication, will be denoted $\mathcal{V}_{\mathcal{L}_B \leftarrow \mathcal{L}_A}$. It is a $(\mathcal{V}_B, \mathcal{V}_A)$-bimodule, the left $\mathcal{V}_B$-module structure being given by left multiplication on $\mathcal{V}_B$. It is easily seen that $\mathcal{V}_{\mathcal{L}_B \leftarrow \mathcal{L}_A}$ is obtained from $\mathcal{V}_{\mathcal{L}_A \rightarrow \mathcal{L}_B}$ by turning the left (respectively right) action of $\mathcal{V}_A$ (respectively $\mathcal{V}_B$) into a right (respectively a left) one.

Let $M^\bullet$ be an element of $D^-\left(\mathrm{Mod}^{\mathrm{r}}_{\mathcal{V}_\mathrm{A}}\right)$ (respectively $D^-\left(\mathrm{Mod}^{\mathrm{l}}_{\mathcal{V}_\mathrm{A}}\right)$). We define its derived direct image as being the element $M^\bullet \overset{L}{\underset{\mathcal{V}_A}{\otimes}} \mathcal{V}_{\mathcal{L}_A \rightarrow \mathcal{L}_B}$ (respectively $\mathcal{V}_{\mathcal{L}_B \leftarrow \mathcal{L}_A} \overset{L}{\underset{\mathcal{V}_A}{\otimes}} M^\bullet$) of $D^-\left(\mathrm{Mod}^{\mathrm{r}}_{\mathcal{V}_\mathrm{B}}\right)$ (respectively $D^-\left(\mathrm{Mod}^{\mathrm{l}}_{\mathcal{V}_\mathrm{B}}\right)$).

Sometimes, to avoid ambiguity, we will denote by Φ_+^l the direct image for left $\mathcal{V}_A$-modules and Φ_+^r the direct image for right $\mathcal{V}_A$-modules

The direct image behaves well under composition of morphisms as in the D-module case ([3])

6 Decomposition of a Lie-Rinehart superalgebra morphism

The assumptions are the same as in the previous section. We introduce the Lie-Rinehart superalgebra $(A \otimes B, \mathcal{L}_{A\otimes B}, \sigma_{A\otimes B})$ as in section 2.1. We decompose (ϕ, ψ) into two maps

$$(\phi, \psi) = (s, S) \circ (u, U)\,.$$

Let us first descibe $\mathcal{S} = (s, S)$:

$$\begin{array}{rccc} s: & B & \rightarrow & A \otimes B \\ & b & \mapsto & 1 \otimes b \end{array}$$

and $S: \ \mathcal{L}_{A\otimes B} \rightarrow (A \otimes B) \underset{B}{\otimes} \mathcal{L}_B \simeq A \underset{k}{\otimes} \mathcal{L}_B$ is the $A \otimes B$-module morphism given by:

$$\forall D \in \mathcal{L}_A, \ \ S(1 \otimes D) = 0 \ \text{ and } \ \forall \Delta \in \mathcal{L}_B, \ \ S(1 \otimes \Delta) = 1 \otimes \Delta.$$

Let us now describe $\mathcal{U} = (u, U)$. The algebra morphism $u: \ A \otimes B \rightarrow A$ is defined by $u(a \otimes b) = a\phi(b)$ and $U: \ \mathcal{L}_A \rightarrow A \underset{A\otimes B}{\otimes} \mathcal{L}_{A\otimes B} \simeq \mathcal{L}_A \oplus A \underset{B}{\otimes} \mathcal{L}_B$ is the A-module morphism determined by : for all D in $\mathcal{L}_A$,

$$U(D) = D + \psi(D).$$

One checks easily that $\mathcal{S}$ and $\mathcal{U}$ are Lie-Rinehart superalgebra morphisms.

Definition 6.0.4 *A Lie-Rinehart superalgebra morphism (ϕ, ψ) is a closed imbedding if ϕ is onto.*

$\mathcal{U}$ is a closed imbedding.

The study of $\mathcal{S}$ is very close to the study of a projection in the D-module theory (see [3] p. 246), that is why we won't reproduce it here.

7 Study of a closed imbedding

7.1 Decomposition of a closed imbedding.

In this paragraph, we will study the closed imbedding case. Let $(A, \mathcal{L}_A, \sigma_A)$ and $(B, \mathcal{L}_B, \sigma_B)$ be two Lie-Rinehart superalgebras and let $\mathcal{U} = (u, U) : (A, \mathcal{L}_A, \sigma_A) \to (B, \mathcal{L}_B, \sigma_B)$ be a closed imbedding. Then we can assume that $A = B/J$ and that u is the natural projection. Put

$$\mathcal{L}_B(J) = \{D \in \mathcal{L}_B \;/\; D(J) \subset J\}.$$

Along all this section, we will make the following hypothesis (assumption $\mathcal{A}$) : We will assume that we can find homogeneous generators $\underline{x} = (x_1, \ldots, x_n)$ of J and homogeneous elements $(\partial_1, \ldots, \partial_n)$ of $\mathcal{L}_B$ such that $\partial_i(x_j) = \delta_{i,j}$.

Introduce

$$\Theta_{B,\underline{x}} = \{D \in \mathcal{L}_B \;/\; D(x_i) = 0\}.$$

Then,

$$\mathcal{L}_B = \underset{i}{\oplus} B\partial_i \oplus \Theta_{B,\underline{x}}.$$

If $\underline{x'} = (x'_1, \ldots, x'_n)$ is another system of generators of J satisfying the assumption $\mathcal{A}$ then, by a reasoning on the rank, it is easy to that $\underline{x}$ and $\underline{x'}$ have the same number r of even elements and the same number s of odd elements. We will always assume that the even elements come first. We have the following isomorphism of left $\mathcal{V}(B, \Theta_{B,\underline{x}})$-modules

$$\mathcal{V}_B \simeq \underset{(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^r\times\{0,1\}^s}{\oplus} \mathcal{V}(\Theta_{B,\underline{x}}) \otimes \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}.$$

The natural injection $\Theta_{B,\underline{x}} \hookrightarrow \mathcal{L}_B(J)$ gives rise to an A-module isomorphism $\chi_{\underline{x}} : A \underset{B}{\otimes} \Theta_{B,\underline{x}} \to \mathcal{L}_B(J)/J\mathcal{L}_B$.

Lemma 7.1.1 *We have the following A-module isomorphism*

$$A \underset{B}{\otimes} \mathcal{L}_B \simeq \oplus_i A\partial_i \oplus A \underset{B}{\otimes} \Theta_{B,\underline{x}}$$

and

$$U(\mathcal{L}_A) \subset A \underset{B}{\otimes} \Theta_{B,\underline{x}}.$$

Proof of the lemma 7.1.1:
Let D be in $\mathcal{L}_A$. Put $U(D) = \sum a_i \otimes D_i$ with $a_i \in A$ and $D_i \in \mathcal{L}_B$. We have

$$\sum_i a_i \otimes D_i = \sum_{i,j} a_i u\left(D_i\left(x_j\right)\right)\partial_j + \sum_i a_i \otimes \left(D_i - \sum_j D_i(x_j)\partial_j\right).$$

But

$$\sum_i a_i u\left(D_i\left(x_j\right)\right) = D\left(u(x_i)\right) = 0.$$

So $\sum a_i \otimes D_i \in A \underset{B}{\otimes} \Theta_{B,\underline{x}}$ and this for any choice of $\underline{x}$. This finishes the proof of the lemma.

The morphism σ_B induces a B-module morphism from $\Theta_{B,\underline{x}}$ to $\mathrm{Der}(A)$. Then $A \underset{B}{\otimes} \Theta_{B,\underline{x}}$, endowed with the following bracket : for all (a,a') in A^2 and all (D,D') in $\Theta_{B,\underline{x}}$,

$$[a \otimes D, a' \otimes D'] = aD(a') \otimes D' - (-1)^{(|a|+|D|)(|a'|+|D'|)} a'D'(a) \otimes D + (-1)^{|D||a'|} aa' \otimes [D, D']$$

is a $k-A$-Lie superalgebra. Moreover, the Lie bracket on $\mathcal{L}_B(J)$ induces a Lie bracket on $\mathcal{L}_B(J)/J\mathcal{L}_B$. Thus $\mathcal{L}_B(J)/J\mathcal{L}_B$ is a $k-A$-Lie superalgebra. The morphism $\left(id, \chi_{\underline{x}}\right)$ is a Lie-Rinehart superalgebra isomorphism.

Proposition 7.1.2 *The morphism $U_1 : \mathcal{L}_A \to \mathcal{L}_B(J)/J\mathcal{L}_B$ described by*

$$\begin{array}{ccccc} \mathcal{L}_A & \to & A \underset{B}{\otimes} \Theta_{B,\underline{x}} & \overset{\chi_{\underline{x}}}{\to} & \mathcal{L}_B(J)/J\mathcal{L}_B \\ D & \mapsto & \sum_i a_i \otimes \left(D_i - \sum_j D_i(x_j)\partial_j\right) & \mapsto & \chi_{\underline{x}}\left(\sum_i a_i \otimes \left(D_i - \sum_j D_i(x_j)\partial_j\right)\right). \end{array}$$

does not depend on the choice of $\underline{x}$. So that $\mathcal{U}_1 = (id, U_1)$ is a Lie-Rinehart superalgebra morphism.

The proof of the proposition 7.1.2 is left to the reader.

Remark :

Using the universal property of the ring of differential operators $\mathcal{V}(A, \mathcal{L}_A)$ ([9] p. 125), one can see that $\mathcal{U}_1$ gives rise to a superalgebra morphism from $\mathcal{V}(A, \mathcal{L}_A)$ to $\mathcal{V}(A, \mathcal{L}_B(J)/J\mathcal{L}_B)$.

We have the following isomorphisms of left $\mathcal{V}(A, \mathcal{L}_A)$-modules

$$\begin{array}{rcl} \mathcal{V}_{\mathcal{L}_A \to \mathcal{L}_B} & \simeq & \bigoplus_{(\alpha_1,\dots,\alpha_n) \in \mathbb{N}^r \times \{0,1\}^s} \mathcal{V}\left(\Theta_{B,\underline{x}}\right) / J\mathcal{V}\left(\Theta_{B,\underline{x}}\right) \otimes \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} \\ & \simeq & \bigoplus_{(\alpha_1,\dots,\alpha_n) \in \mathbb{N}^r \times \{0,1\}^s} \mathcal{V}\left(\mathcal{L}_B(J)/J\mathcal{L}_B\right) \otimes \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}. \end{array}$$

Proposition 7.1.3 *Let $\mathcal{U}_2 = (u_2, U_2) : (A, \mathcal{L}_B(J)/J\mathcal{L}_B) \to (B, \mathcal{L}_B)$ be the morphism defined by : $u_2 = u$ and $U_2 \circ \chi_{\underline{x}} : A \underset{B}{\otimes} \Theta_{B,\underline{x}} \to A \otimes_B \mathcal{L}_B$ is the natural injection. $\mathcal{U}_2$ does not depend on $\underline{x}$. $\mathcal{U}$ can be factorized as follows*

$$\mathcal{U} = \mathcal{U}_2 \circ \mathcal{U}_1.$$

The proof of the proposition 7.1.3 is left to the reader.

Let M be a right $\mathcal{V}(\mathcal{L}_B)$-module. Then it is clear that

$$M_J = \Gamma_J^1(M) = \{m \in M \ / \ m \cdot J = 0\}$$

is a right $\mathcal{V}(A, \mathcal{L}_B(J)/J\mathcal{L}_B)$-module.

We can now state the analog of the Kashiwara's theorem.

Theorem 7.1.4 *We assume that $\mathcal{A}$ is satisfied and, for short, we put $\mathcal{M}_A = \mathcal{L}_B(J)/J\mathcal{L}_B$. We consider the map $(u_2, U_2) : (A, \mathcal{M}_A) \to (B, \mathcal{L}_B)$.*

1) Let N be a right $\mathcal{V}(\mathcal{M}_A)$-module. The morphism

$$\begin{array}{ccc} N & \to & \Gamma_J^1\left(N \underset{\mathcal{V}(\mathcal{M}_A)}{\otimes} \mathcal{V}_{\mathcal{M}_A \to \mathcal{L}_B}\right) \\ n & \mapsto & n \otimes \bar{1} \end{array}$$

is an isomorphism of right $\mathcal{V}(\mathcal{M}_A)$-modules.

2) Let M be a right $\mathcal{V}(\mathcal{L}_B)$-module such that $\Gamma_J(M) = M$. Then the morphism

$$\begin{array}{ccc} \left(M_J \underset{\mathcal{V}(\mathcal{M}_A)}{\otimes} \mathcal{V}_{\mathcal{M}_A \to \mathcal{L}_B}\right) & \to & M \\ m \otimes \bar{v} & \mapsto & m \cdot \bar{v} \end{array}$$

is an isomorphism of right $\mathcal{V}(\mathcal{L}_B)$-modules.

For the proof of the theorem 7.1.4 , see [3] p 261, [17].

Corollary 7.1.5 *Under the same assumptions and notations as in the theorem, there is an equivalence of categories between the category of right $\mathcal{V}(\mathcal{L}_B)$-modules such that $\Gamma_J(M) = M$ and right $\mathcal{V}(\mathcal{M}_A)$-modules.*

Remarks :

1) This theorem was proved by Levasseur ([17]) in the case where B is a regular local ring and J is the maximal ideal of A.

2) Let $(x_1, \ldots, x_n)$ be a set of homogeneous generators of J. If there exists $(\partial_1, \ldots, \partial_n)$ in $\mathcal{L}_B$ such that the matrix $X = (\partial_i(x_j))_{i,j}$ is invertible in B, then one can find d_i such that $d_i(x_j) = \delta_{i,j}$.

Conclusion :

Assuming that $\mathcal{A}$ is satisfied, we have decomposed $\mathcal{U}$ into $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $\mathcal{U}_2$ satisfies the Kashiwara's theorem and $\mathcal{U}_1 = (id, U_1)$ rather looks like a Lie superalgebra morphism.

7.2 Kashiwara's theorem for left modules.

We keep the same notations as in the previous paragraph.

Proposition 7.2.1 *We assume that we are in the case where $\mathcal{A}$ is satisfied. We are considering the Lie-Rinehart superalgebra morphism $\mathcal{U}_2 : (A, \mathcal{M}_A) \to (B, \mathcal{L}_B)$. Let $(x_1, ..., x_n)$ be an homogeneous set of generators of J and let $(\partial_1, \ldots, \partial_n)$ be elements of $\mathcal{L}_B$ such that $\partial_i(x_j) = \delta_{i,j}$. Put $\bar{x}_i = x_i + J^2$. As $(x_1, \ldots, x_n)$ is regular for B (see appendix*

2), $(\bar{x}_1, \ldots, \bar{x}_n)$ is a basis for the A-module J/J^2. Denote by $\omega_{\underline{x}}$ the basis of $\mathrm{Ber}(J/J^2)$ *it defines. Let M be a left $\mathcal{V}(\mathcal{L}_B)$-module. Then* $\mathrm{Ber}(J/J^2)\underset{A}{\otimes}{}_J M$ *is endowed with a left $\mathcal{V}(\mathcal{M}_A)$-module structure as follows. Using the isomorphism $\chi_{\underline{x}} : A\underset{B}{\otimes}\Theta_{B,\underline{x}} \to \mathcal{M}_A$, the two natural operations*

$$\begin{array}{l}\forall D \in \Theta_{B,\underline{x}},\ \forall m \in {}_J M,\ \forall a \in A,\\ \chi_{\underline{x}}(\bar{1} \otimes D) \cdot \left(\omega_{\underline{x}} \otimes m\right) = \omega_{\underline{x}} \otimes D \cdot m(-1)^{n|D|}\\ a \cdot \left(\omega_{\underline{x}} \otimes m\right) = \omega_{\underline{x}} \otimes a \cdot m(-1)^{n|a|}\end{array}$$

define an action of $(A, \mathcal{L}_B(J)/J\mathcal{L}_B)$ on $\mathrm{Ber}(J/J^2)\underset{A}{\otimes}{}_J M$.

Proof of the proposition 7.2.1:
We have to check that the definition of the two operations does not depend on the choice of the system $(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n)$. So let us consider another such system $(x'_1, \ldots, x'_n, \partial'_1, \ldots, \partial'_n)$. There exist $(a_{i,j}) \in B^{n^2}$ and $(b_{i,j}) \in B^{n^2}$ such that

$$x'_j = \sum_i x_i a_{i,j} \ \text{ and } \ x_j = \sum_i x'_i b_{i,j}.$$

Denote by $\bar{a}_{i,j} = a_{i,j} + J \in A$. Then the matrices $X = (\bar{a}_{i,j})$ and $Y = (\bar{b}_{i,j})$ (with coefficients in A) are inverse from each other. We have

$$\omega_{\underline{x'}} = Ber(X)\omega_{\underline{x}}.$$

We have to verify that for all m in ${}_J M$, the following equality holds

$$\omega_{\underline{x'}} \otimes \left(D - \sum_i D(x'_i)\partial'_i\right)(Ber(Y) \cdot m) = \omega_{\underline{x}} \otimes D \cdot m.$$

A computation shows that this amounts to prove that

$$\partial'_j \left(D(x'_j)\right) \cdot m = \sum_i b_{j,i} D(a_{i,j})(-1)^{|D||x_i|} \cdot m.$$

The proposition follows then from the formula

$$\mathrm{str}\left(\mathrm{D(X)X^{-1}}\right) = \mathrm{D}\left(\mathrm{ber(X)}\right)\mathrm{ber(X^{-1})}$$

whose proof is given in the appendix 1.

Remark :
If A is noetherian, using the appendix 3, we have the following isomorphism of left $\mathcal{V}_B$-modules

$$\mathcal{V}_{\mathcal{L}_B \leftarrow \mathcal{M}_A} \simeq \mathcal{V}_B/\mathcal{V}_B J \underset{A}{\otimes} \mathrm{Ber}\left(J/J^2\right)^*.$$

We can now state the analog of the theorem 7.1.4 for left modules.

Theorem 7.2.2 *We keep the same assumptions and notations as in the previous proposition. Moreover, we assume that A is noetherian.*

1) Let N be a left $\mathcal{V}(\mathcal{M}_A)$-module. The map

$$\begin{array}{rcl} \alpha: \ N & \to & \mathrm{Ber}(J/J^2)\underset{A}{\otimes}\Gamma_J^1\left(\mathcal{V}_{\mathcal{L}_B\leftarrow\mathcal{M}_A}\underset{\mathcal{V}(\mathcal{M}_A)}{\otimes}N\right) \\ n & \to & \omega_{\underline{x}}\otimes\left(\left(\bar{1}\otimes\omega_{\underline{x}}^{-1}\right)\otimes n\right) \end{array}$$

(where $\bar{1}\in\mathcal{V}_B/\mathcal{V}_BJ$) is a left $\mathcal{V}(\mathcal{M}_A)$-module isomorphism.

2) Let M be a left $\mathcal{V}(\mathcal{L}_B)$-module such that $\Gamma_J(M)=M$. The map

$$\begin{array}{c} \beta: \ \mathcal{V}_{\mathcal{L}_B\leftarrow\mathcal{M}_A}\underset{\mathcal{V}(\mathcal{M}_A)}{\otimes}\left(\mathrm{Ber}(J/J^2)\underset{A}{\otimes}\Gamma_J^1(M)\right) \ \to \ M \\ \left(\bar{v}\otimes\omega_{\underline{x}}^{-1}\right)\otimes\left(\omega_{\underline{x}}\otimes m\right)\to\bar{v}\cdot m \end{array}$$

(where $\bar{v}\in\mathcal{V}_B/\mathcal{V}_BJ$) is a left $\mathcal{V}(\mathcal{L}_B)$-module isomorphism.

Proof of the theorem 7.2.2 :
Let $(x_1,\dots,x_n,\partial_1,\dots,\partial_n)$ such that $\partial_i(x_j)=\delta_{i,j}$. We have

$$\mathcal{L}_B^*=\oplus_iB\partial_i^*\oplus\Theta_{B,\underline{x}}^*.$$

As $[\Theta_{B,\underline{x}},\partial_i]\subset\Theta_{B,\underline{x}}$, if we write $\mathrm{Ber}(\mathcal{L}_B^*)=\omega_{[\partial_i^*]}\otimes\mathrm{Ber}(\Theta_{B,\underline{x}}^*)$, we have

$$\begin{array}{c} \forall D\in\Theta_{B,\underline{x}},\forall\sigma\in\mathrm{Ber}(\Theta_{B,\underline{x}}^*), \\ \left(\omega_{[\partial_i^*]}\otimes\sigma\right)\cdot D=\omega_{[\partial_i^*]}\otimes\sigma\cdot D. \end{array}$$

With this remark, one can check that α is a left $\mathcal{V}(\mathcal{M}_A)$-module morphism. Clearly, β is a left $\mathcal{V}_B$-module morphism. For the rest of the proof, we refer the reader to [3] p 261.

Corollary 7.2.3 *Under the same hypothesis as in the previous theorem, there is an equivalence of category between the category of left $\mathcal{V}(\mathcal{L}_B)$-modules such that $\Gamma_J(M)=M$ and the category of left $\mathcal{V}(\mathcal{M}_A)$-modules.*

7.3 An application of Kashiwara's theorem.

Let $(B,\mathcal{L}_B)$ be a $k-B$-Lie superalgebra. Let J be an ideal of B. Put $A=B/J$ and $\mathcal{M}_A=\mathcal{L}_B(J)/J\mathcal{L}_B$. Put $\mathcal{V}_B=\mathcal{V}(B,\mathcal{L}_B)$ and $\mathcal{V}_{\mathcal{M}_A}=\mathcal{V}(A,\mathcal{M}_A)$. We have a natural imbedding $\mathcal{U}_2:\ (A,\mathcal{M}_A)\to(B,\mathcal{L}_B)$ for which the tranfer module is $\mathcal{V}_B/J\mathcal{V}_B$.

Definition 7.3.1 *Let M be a $\mathcal{V}_{\mathcal{M}_A}$-module. We put*

$$\mathrm{Coind}_{\mathcal{V}_{\mathcal{M}_A}}^{\mathcal{V}_B}(M)=\mathrm{Hom}_{\mathcal{V}_{\mathcal{M}_A}}\left(\mathcal{V}_B/J\mathcal{V}_B,M\right).$$

$\mathrm{Coind}_{\mathcal{V}_{\mathcal{M}_A}}^{\mathcal{V}_B}(M)$ *is naturally endowed with a left $\mathcal{V}_B$-module structure as follows. For all φ in $\mathrm{Coind}_{\mathcal{V}_{\mathcal{M}_A}}^{\mathcal{V}_B}(M)$, $\bar{u}$ in $\mathcal{V}_B/J\mathcal{V}_B$ and all v in $\mathcal{V}_B$,*

$$<\bar{u},v\cdot\varphi>=<\overline{uv},\varphi>.$$

From now on, we assume that we can find homogeneous generators $\underline{x} = (x_1, \ldots, x_n)$ in J and $(\partial_1, \ldots, \partial_n)$ in $\mathcal{L}_B$ such that $\partial_i(x_j) = \delta_{i,j}$. We assume that $(x_1, \ldots, x_r)$ are even whereas $(x_{r+1}, \ldots, x_n)$ are odd. Under this assumption, we have the following characterization of coinduced modules which is due to Levasseur ([17]). Denote by $\mathcal{S}$ the category of left $\mathcal{V}_B$-modules which are Hausdorff and complete for the J-adic topology.

Proposition 7.3.2 *Let M be a $\mathcal{V}_{\mathcal{M}_A}$-module.* $\mathrm{Coind}_{\mathcal{V}_{\mathcal{M}_A}}^{\mathcal{V}_B}(M)$ *belongs to $\mathcal{S}$.*

Theorem 7.3.3 *There exists an equivalence of categories between $\mathcal{S}$ and the category of left $\mathcal{V}_{\mathcal{M}_A}$-modules given by the following functors :*

$$\begin{array}{rcl} M \in \mathrm{Mod}^l_{\mathcal{V}_{\mathcal{M}_A}} & \mapsto & \mathrm{Coind}_{\mathcal{V}_{\mathcal{M}_A}}^{\mathcal{V}_B}(M) \in \mathcal{S} \\ W \in \mathcal{S} & \mapsto & W/JW \in \mathrm{Mod}^l_{\mathcal{V}_{\mathcal{M}_A}} \end{array}$$

Let W be a $\mathcal{V}_B$-module. $\mathcal{V}_B$ being a projective B-module, it is flat. So it is the same to derive Γ_J in the category of $\mathcal{V}_B$-modules and in the category of B-modules. So that the $H^j_J(W)$ are naturally endowed with a $\mathcal{V}_B$-module structure.

Proposition 7.3.4 *Let M be a $\mathcal{V}_{\mathcal{M}_A}$-module. Under the same assumptions and notations as before, $H^r_J\left(\mathrm{Coind}_{\mathcal{V}_{\mathcal{M}_A}}^{\mathcal{V}_B}(W)\right)$ and $\mathcal{U}_{2+}(M)$ are canonically isomorphic as left $\mathcal{V}_B$-modules.*

Proof of the proposition 7.3.4
Using Kashiwara's theorem for left modules, we get

$$H^r_J(W) \simeq \mathcal{U}_{2+}\left(\mathrm{Ber}\left(J/J^2\right) \underset{A}{\otimes} \mathrm{Ext}^r_B\left(B/J, W\right)\right).$$

But we have the following well known A-module isomorphism :

$$\mathrm{Ext}^r_B\left(B/J, W\right) \simeq \mathrm{Ber}(J/J^2)^* \underset{A}{\otimes} W/JW$$

(see [2] theorem 4.5 p. 13). Using the notations of the appendix 2 for the definition of $K_{\underline{x}} = (S(\Pi L), d)$, if we let $\Theta_{B,\underline{x}}$ act trivially on the l_i's, then the components of the Koszul complex are $\Theta_{B,\underline{x}}$-modules and d is a $\Theta_{B,\underline{x}}$- morphism. Then, one can see that the map

$$\begin{array}{rcl} M = W/JW & \rightarrow & \mathrm{Ext}^r_B(B/J, W) \\ m & \mapsto & \omega_{\underline{x}}^{-1} \otimes m \end{array}$$

is a $\Theta_{B,\underline{x}}$-module isomorphism. This allows to finish the proof of the proposition.

Remarks :

1) One can show that if $i \neq r$, $H^i_J(W)$ equals 0 (see [9]).

2) Assume that we are in the following situation : Let $\mathfrak{g}$ be a Lie superalgebra and $\mathfrak{h}$ be a Lie subsuperalgebra of finite codimension. We put $B = \mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(k)$ and $\mathcal{V}_B = \mathcal{V}(B, B\#\mathfrak{g})$. J will be the maximal ideal of B. If M is an $\mathfrak{h}$-module and r the dimension of $(\mathfrak{g}/\mathfrak{h})_{\bar{0}}$.

Then, applying the proposition 7.3.4, we get that the left $\mathcal{V}_B$-modules $H^r_J\left(\mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{h}}(M)\right)$ and $\mathcal{U}_{2+}(M)$ are isomorphic. Then, using an induction criterion ([17], [8]), it is not hard to see that

$$\mathcal{U}_{2+}(M) = \mathcal{V}_B/\mathcal{V}_B\mathfrak{M} \underset{U(\mathfrak{h})}{\otimes} (M \otimes \mathrm{Ber}\,(\mathfrak{g}/\mathfrak{h}))\,.$$

So that

$$H^r_J\left(\mathrm{Coind}^{\mathcal{V}_B}_{\mathcal{V}_{\mathcal{M}_A}}(W)\right) \simeq \mathcal{V}_B/\mathcal{V}_B\mathfrak{M} \underset{U(\mathfrak{h})}{\otimes} (M \otimes \mathrm{Ber}\,(\mathfrak{g}/\mathfrak{h}))\,.$$

We recover the result of [8] (theorem 3.1.1) but, in [8], the isomorphism we got was not canonical whereas here it is.

8 The duality property

8.1 The duality functor.

In this paragraph, we will assume that A is noetherian. $(A, \mathcal{L}_A, \sigma_A)$ will be a Lie-Rinehart superalgebra such that $\mathcal{L}_A$ is a finitely generated projective A-module with a rank $rk(\mathcal{L}_A) = d_0 + \epsilon d_1$. We will put for simplicity $\mathcal{V}_A = \mathcal{V}(A, \mathcal{L}_A, \sigma_A)$. We will denote by $D^-_c\left(\mathrm{Mod}^l_{\mathcal{V}_A}\right)$ the derived category of bounded above complexes of finitely generated $\mathcal{V}_A$-modules.

Definition 8.1.1 *Let $M^\bullet$ be a bounded above complex of left $\mathcal{V}_A$-modules. We put*

$$D_A(M^\bullet) = \mathrm{RHom}\,(M^\bullet, \mathcal{V}_A) \underset{A}{\otimes} \mathrm{Ber}(\mathcal{L}_A)[d_0].$$

$D_A(M^\bullet)$ is a bounded below complex of left $\mathcal{V}_A$-modules.

Proposition 8.1.2 *Let M be a left $\mathcal{V}_A$-module which is a finitely generated projective A-module. Then $D_A(M)$ is isomorphic to $Hom_A(M, A)$ in $D^+(\mathrm{Mod}^l_{\mathcal{V}_A})$.*

Remark :
This property is known for $\mathcal{D}$-modules ([11] p. 106).

This proposition was already proved for $M = A$ in [7] (theorem 5.4.1). We leave it to the reader to adjust the proof to the case where M is only a finitely generated projective A-module.

8.2 A duality property.

In this paragraph, we will assume that A and B are noetherian supercommutative associative superalgebras with unity. $(A, \mathcal{L}_A, \sigma_A)$ (respectively $(B, \mathcal{L}_B, \sigma_B)$) will be a Lie-Rinehart superalgebra such that $\mathcal{L}_A$ (respectively $\mathcal{L}_B$) is a finitely generated projective A-module (respectively B-module) with a rank. We will put for simplicity $\mathcal{V}_A = \mathcal{V}(A, \mathcal{L}_A, \sigma_A)$ and $\mathcal{V}_B = \mathcal{V}(B, \mathcal{L}_B, \sigma_B)$.

Theorem 8.2.1 *Let $\mathcal{U} = (u, U) : (A, \mathcal{L}_A) \to (B, \mathcal{L}_B)$ be an imbedding. Put $A = B/J$. We assume that the hypothesis $\mathcal{A}$ is satisfied and we introduce as before $\mathcal{M}_A = \mathcal{L}_B(J)/J\mathcal{L}_B$. Then*

$$\Omega^{-1}_{\mathcal{M}_A/\mathcal{L}_A} = \mathrm{Hom}_A\left(\mathrm{Ber}(\mathcal{L}^*_A), \mathrm{Ber}(\mathcal{M}^*_A)\right)[erk(\mathcal{M}_A) - erk(\mathcal{L}_A)]$$

is a complex of left $\mathcal{V}_A$-modules. Let $M^\bullet$ be bounded above complex of finitely generated $\mathcal{V}_A$-modules. $D_B \circ \mathcal{U}_+(M^\bullet \otimes \Omega^{-1}_{\mathcal{M}_A/\mathcal{L}_A})$ and $\mathcal{U}_+ \circ D_A(M^\bullet)$ are functorially isomorphic in $D^+\left(\mathrm{Mod}^l_{\mathcal{V}_B}\right)$

Proof of the theorem 8.2.1
We adjust the proof for $\mathcal{D}$-modules ([3] p 261) to our case. Let us first prove the theorem in the case of the left module $\mathcal{V}_A$. Let $\underline{x} = (x_1, \ldots, x_n)$ be homogeneous generators of J such that there exists $(\partial_1, \ldots, \partial_n)$ in $\mathcal{L}^n_B$ satisfying $\partial_i(x_j) = \delta_{i,j}$ for all i and j in $[1, n]$. We assume that $(x_1, \ldots, x_r)$ are even and $(x_{r+1}, \ldots, x_n)$ are odd.

Using the appendix 3, we have the following isomorphism of left $\mathcal{V}_B$-modules

$$\mathcal{V}_{\mathcal{L}_B \leftarrow \mathcal{L}_A} \simeq \mathcal{V}_B/\mathcal{V}_B J \underset{A}{\otimes} \Omega_{\mathcal{M}_A/\mathcal{L}_A} \underset{A}{\otimes} \mathrm{Ber}(J/J^2)^*.$$

where, on the right hand side, $\mathcal{V}_B$ acts by left multiplication. Hence

$$(D_B \circ \mathcal{U}_+)\left(\mathcal{V}_A \otimes \Omega^{-1}_{\mathcal{M}_A/\mathcal{L}_A}\right) = \mathrm{RHom}_{\mathcal{V}_B}\left(\mathcal{V}_B/\mathcal{V}_B J \underset{A}{\otimes} \mathrm{Ber}(J/J^2)^*, \mathcal{V}_B\right) \underset{B}{\otimes} \mathrm{Ber}(\mathcal{L}_B)[r + erk(\mathcal{L}_A)].$$

As $\mathcal{V}_B$ is a projective B-module, we have

$$(D_B \circ \mathcal{U}_+)(\mathcal{V}_A \otimes \Omega^{-1}_{\mathcal{M}_A/\mathcal{L}_A}) \simeq \mathrm{RHom}_B\left(\mathrm{Ber}(J/J^2)^*, \mathcal{V}_B\right) \underset{B}{\otimes} \mathrm{Ber}(\mathcal{L}_B)[r + erk(\mathcal{L}_A)].$$

As $\underline{x}$ is a regular sequence, using results and notations of the appendix 2, we get

$$\begin{aligned}(D_B \circ \mathcal{U}_+)\left(\mathcal{V}_A \otimes \Omega^{-1}_{\mathcal{M}_A/\mathcal{L}_A}\right) &\simeq \mathrm{Hom}_B\left(K^{\underline{x}}, \mathcal{V}_B\right) \underset{B}{\otimes} \mathrm{Ber}(\mathcal{L}_B)[erk(\mathcal{L}_A)] \\ &\simeq K_{\underline{x}} \underset{B}{\otimes} \mathcal{V}_B \underset{B}{\otimes} \mathrm{Ber}(\mathcal{L}_B)[erk(\mathcal{L}_A)].\end{aligned}$$

So, using again the property of the Koszul complex (see appendix 2), we see that this complex of $\mathcal{V}_B$-modules is quasi-isomorphic to $\mathcal{V}_B/J\mathcal{V}_B \underset{B}{\otimes} \mathrm{Ber}(\mathcal{L}_B)[erk(\mathcal{L}_A)]$ (where $\mathcal{V}_B$ acts by right multiplication turned to a left action). This quasi-isomorphism does not depend on the choice of $(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n)$. Moreover $\mathcal{V}_A \otimes \Omega^{-1}_{\mathcal{M}_A/\mathcal{L}_A}$ is a right $\mathcal{V}_A$-module by right multiplication on $\mathcal{V}_A$. Along all the previous isomorphisms, one may keep track of this action as follows. On $\mathcal{V}_B/\mathcal{V}_B J \underset{A}{\otimes} \mathrm{Ber}(J/J^2)^*$, this right action becomes

$$\forall \bar{v} \in \mathcal{V}_B/\mathcal{V}_B J,\ \forall \Delta \in \Theta_{B,\underline{x}},\ \left(\bar{v} \otimes \omega_{\underline{x}}\right) \cdot \bar{1} \otimes \Delta = \overline{v\Delta} \otimes \omega_{\underline{x}}.$$

Then we let $\Theta_{B,\underline{x}}$ act on $K^{\underline{x}}$ as in the proof of the proposition 7.3.4. Finally one can see that our computations transform this right $\mathcal{V}_A$-action into the left $\mathcal{V}_A$ action on $\mathcal{V}_B/J\mathcal{V}_B \underset{B}{\otimes} \mathrm{Ber}(\mathcal{L}_B)$ given by the definition 4.0.3 b).

On another hand, we have the following isomorphisms of complexes of left $\mathcal{V}_B$-modules.

$$\begin{array}{rcl}(\mathcal{U}_+ \circ D_A)(\mathcal{V}_A) & = & \mathcal{U}_+ \left(\mathcal{V}_A^r \underset{A}{\otimes} \mathrm{Ber}(\mathcal{L}_A)\right)[erk(\mathcal{L}_A)] \\ & \simeq & \mathcal{V}_A^r \underset{\mathcal{V}_A}{\otimes} \mathcal{V}_B/J\mathcal{V}_B \underset{B}{\otimes} \mathrm{Ber}(\mathcal{L}_B)[erk(\mathcal{L}_A)] \\ & \simeq & \mathcal{V}_B/J\mathcal{V}_B \underset{B}{\otimes} \mathrm{Ber}(\mathcal{L}_B)[erk(\mathcal{L}_A)]\end{array}$$

Right multiplication on $\mathcal{V}_A$ transforms as before.

So, we have constructed a quasi-isomorphism between the complexes of $\mathcal{V}_B$-modules $(D_B \circ \mathcal{U}_+)(\mathcal{V}_A)$ and $(\mathcal{U}_+ \circ D_A)(\mathcal{V}_A)$ commuting with right multiplication by $\mathcal{V}_A$ (that is to say with the endomorphisms of $\mathcal{V}_A$ considered as a left $\mathcal{V}_A$-module).

To extend our construction to any bounded above complex of finitely generated $\mathcal{V}_A$-modules, we proceed as in [3] p. 280.

Remarks

1) This duality theorem is well known for $\mathcal{D}$-modules (see [3] p. 178).

2) Assume that we are in the Lie superalgebra case. We have a finite dimensional Lie superalgebra $\mathfrak{g}$ and $\mathfrak{h}$ a subsuperalgebra of $\mathfrak{g}$. Let h_0 be the even dimension of $\mathfrak{h}$. The tranfer modules are

$$\mathcal{V}_{\mathfrak{h}\to\mathfrak{g}} = U(\mathfrak{g}) \ \text{ and} \mathcal{V}_{\mathfrak{g}\leftarrow\mathfrak{h}} = \mathrm{Ber}(\mathfrak{h}^*) \otimes U(\mathfrak{g}) \otimes \mathrm{Ber}(\mathfrak{g}).$$

Using an induction criterion ([17],[8] p. 380), it is not hard to see that, if V is a $\mathfrak{h}$-module,

$$\mathrm{Ber}(\mathfrak{h}^*) \otimes U(\mathfrak{g}) \otimes \mathrm{Ber}(\mathfrak{g}) \simeq U(\mathfrak{g}) \underset{U(\mathfrak{h})}{\otimes} \left(V \otimes \mathrm{Ber}\left(\mathfrak{g}/\mathfrak{h}\right)\right).$$

So that, if V is finite dimensional, the duality theorem gives

$$\mathrm{Ext}^i_{U(\mathfrak{g})}\left(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(V), U(\mathfrak{g})\right) = \begin{cases} \mathcal{IND}_{\mathfrak{h}}^{\mathfrak{g}}(V^* \otimes \mathrm{Ber}(\mathfrak{h}^*)) & \text{if } i = h_0 \\ 0 & \text{if } i \neq h_0. \end{cases}$$

We recover a duality result proved independantly by Brown and Levasseur ([5]) and by Kempf ([14]) when $\mathfrak{g}$ is semi-simple and V is a Verma module. This duality theorem was extended in [7] but, in [7], only $\mathfrak{h}$ is supposed to be finite dimensional.

3) We have :

Theorem 8.2.2 *The notations and the assumptions are the same as in the theorem 8.2.1. Put $\mathcal{L}_A = h_0 + \epsilon h_1$. Let M be a left $\mathcal{V}_A$-module which is a finitely generated projective A-module. On $\widetilde{\mathcal{V}}_B = \mathrm{Ber}(\mathcal{L}_B^*) \underset{B}{\otimes} \mathcal{V}_B$, there are two right $\mathcal{V}_B$-module structures which commute. The first one is obtain by right multiplication. The second one is obtained by transformation of left multiplication into a right $\mathcal{V}_B$-module structure. We have*

$$\mathrm{Ext}^i_{\mathcal{V}_B}\left(\mathcal{U}_+^r\left(M \underset{A}{\otimes} \mathrm{Ber}\left(\mathcal{L}_B(J)/J\mathcal{L}_B\right)^*\right), \widetilde{\mathcal{V}}_B\right) = \begin{cases} 0 & \textit{if } i \neq h_0 \\ \mathcal{U}_+^r\left(M^* \underset{A}{\otimes} \mathrm{Ber}(\mathcal{L}_A)^*\right) & \textit{if } i = h_0. \end{cases}$$

where the Ext is taken over right $\mathcal{V}_B$-modules and affects the second right module structure on $\widetilde{\mathcal{V}}_B$.

The proof of the theorem 8.2.2 is left to the reader.

Example
Put B=$\mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(k)$, $\mathcal{L}_B = B\#\mathfrak{g}$. Let $\mathfrak{M}$ be the maximal ideal of B. We have $\mathcal{L}_B(\mathfrak{M})/\mathfrak{M}\mathcal{L}_B = \mathfrak{h}$. Let $\mathfrak{h}'$ be a subsuperalgebra of $\mathfrak{h}$. Put $\dim\mathfrak{h}' = h_0 + \epsilon h_1$ and $\mathcal{V} = \mathcal{V}(B, B\#\mathfrak{g})$

$$\mathrm{Ext}^i_{\mathcal{V}_B}\left((M \otimes \mathrm{Ber}(\mathfrak{h}^*)) \underset{U(\mathfrak{h}')}{\otimes} \mathcal{V}/\mathfrak{M}\mathcal{V}, \widetilde{\mathcal{V}}\right) = \begin{cases} 0 & \text{if } i \neq h_0 \\ (M^* \otimes \mathrm{Ber}(\mathfrak{h}')^*) \underset{U(\mathfrak{h}')}{\otimes} \mathcal{V}/\mathfrak{M}\mathcal{V} & \text{if } i = h_0. \end{cases}$$

9 Appendices

9.1 Appendix 1.

In this appendix, we will prove a formula linking the supertrace ([15] p. 14) and the Berezinian.

Proposition 9.1.1 *Let C be a supercommutative associative superalgebra with unity. Let X be an element of $GL(r + \epsilon s, C)$ and D be in $Der(C)$. We denote by $D(X)$ the matrix defined by*

$$D(X)_{i,j} = (-1)^{|row\ i||D|} D(X_{i,j}).$$

We have

$$\mathrm{str}\left(D(X)X^{-1}\right) = D(\mathrm{Ber}(X))\mathrm{Ber}X^{-1}$$

Proof of the proposition 9.1.1 :

Let us first prove the non graded case We write $X = \begin{pmatrix} L_1 \\ \dots \\ L_n \end{pmatrix}$ where L_i is the ith line of X. We have

$$D(det(X)) = \sum_i det \begin{pmatrix} L_1 \\ \dots \\ D(L_i) \\ \dots \\ L_n \end{pmatrix}.$$

We then develop the first term following the first line, the second term following the second line and so on. The result follows then from the expression of A^{-1} in terms of the cofactor matrix and $\det(A)$.

Let us now treat the graded case. Let X and Y be two elements of $\mathrm{GL}(r+\epsilon s, C)$. It is easy to check the relation

$$D(XY) = D(X)Y + XD(Y).$$

We decompose X into four blocks following the parity of the lines and the columns. We put

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We express X as a product of three matrices (see [15] p 16)

$$X = X_+ X_0 X_-$$

where

$$X_+ \in \left\{ \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \right\}, \quad X_0 \in \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\} \text{ and } X_- \in \left\{ \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \right\}.$$

Then, thanks to the previous remark, it is enough to prove the proposition for X_+, X_0 and X_-. And, for these three latter matrices, the formula is obvious if one uses the explicit expression of the Berezinian character ([15] p 15).

9.2 Appendix 2.

In this appendix we will recall some material about regular sequences in the supercase ([21]). All along this paragraph, B will be a supercommutative associative superalgebra with unity.

Definition 9.2.1 *Let M be a B-module. A sequence $(x_1, \ldots, x_n)$ of homogeneous elements of B is regular for M if, for all i in $[1, n]$, the annihilator of x_i in $M/x_1M + \ldots + x_{i-1}M$, $Ann_{M/x_1M+\ldots+x_{i-1}M}(x_i)$, is equal to $\left(x_i - (-1)^{|x_i|}x_i\right) M/x_1M + \ldots + x_{i-1}M$.*

Assume that $(x_1, \ldots, x_r)$ are even and that $(x_{r+1}, \ldots, x_n)$ are odd. Let L be the free B-module with dimension $r + \epsilon(n-r)$ and with basis $(l_1, \ldots, l_n)$ with $(l_1, \ldots, l_r)$ even and $(l_{r+1}, \ldots, l_n)$ odd. On $S_B(\Pi L)$, we put the differential equal to $\sum_i x_i \dfrac{\partial}{\partial(\pi l_i)}$. We thus define the Koszul complex associated to the sequence $\underline{x} = (x_1, \ldots, x_n)$. Denote it by $K_{\underline{x}}$. Denote by $K^{\underline{x}}$ the dual complex. Lastly let J be the ideal generated by $(x_1, \ldots, x_n)$. We have the following proposition.

Proposition 9.2.2 *If $(x_1, \ldots, x_n)$ is regular, then*
a) $H_0(K_{\underline{x}}) = B/J$ and $H_i(K_{\underline{x}}) = 0$ if $i > 0$.
b) $H^r(K^{\underline{x}}) = \mathrm{Ber}\,(J/J^2)^*$ and $H^i(K^{\underline{x}}) = 0$ if $i \neq r$.

We will now see an example of regular sequence.

Proposition 9.2.3 *Let $(B, \mathcal{L}_B)$ be a $k-B$ Lie-Rinehart superalgebra. Let M be a $\mathcal{V}(\mathcal{L}_B)$-module which is noetherian as a B-module. Assume that we have a sequence $(x_1, \ldots, x_n)$ of homogeneous elements such that there exists $(\partial_1, \ldots, \partial_n)$ in $\mathcal{L}_B$ such that $\partial_i(x_j) = \delta_{i,j}$. Then $(x_1, \ldots, x_n)$ is regular in M.*

This proposition is already proved in [24]. We give another proof.

Proof of the proposition 9.2.3 :
We will proceed by induction on n.

For $n = 1$:

If $\mid x_1 \mid = \bar{0}$, we want to prove that $\mathrm{Ann}_{x_1}(M) = 0$. Assume that there exists m in $M - \{0\}$ such that $x_1 m = 0$. Consider the sequence of increasing graded B-submodules

$$I_n = \{a \in M / x_1^n a = 0\}.$$

For all n in $\mathbb{N}$, $\partial_1^n m \in I_{n+1} - I_n$. So that the sequence I_n does not become stationnary. This contredicts the noetherianity hypothesis. Hence $\mathrm{Ann}_{x_1}(M) = 0$.

If $\mid x_1 \mid = \bar{1}$, it is easy to prove that $\mathrm{Ann}_{x_1}(M) = x_1 M$.

Let $B < \partial_1 >$ be the Lie-Rinehart subsuperalgebra of $\mathcal{L}_B$ generated by B and ∂_1. For the case $n = 1$, we have just used the fact that M was a $B < \partial_1 >$-module.

Assume now that we have proved that $(x_1, \ldots, x_{i-1})$ is regular, then as $M/x_1 M + \ldots + x_{i-1} M$ is a $B < \partial_i >$-module, we may use the case $n = 1$, to finish the proof of the proposition.

9.3 Appendix 3.

Let $(A, \mathcal{L}_A, \sigma_A)$ (respectively $(B, \mathcal{L}_B, \sigma_B)$) be a Lie-Rinehart superalgebra such that $\mathcal{L}_A$ (respectively $\mathcal{L}_B$) is a finitely generated projective A-module (respectively B-module) with a rank.

Proposition 9.3.1 *Let $\mathcal{U} = (u, U) : (A, \mathcal{L}_A) \to (B, \mathcal{L}_B)$ be a closed imbedding. Put $A = B/J$. We assume that the hypothesis $\mathcal{A}$ is satisfied and that A is noetherian. Introduce as before $\mathcal{M}_A = \mathcal{L}_B(J)/J\mathcal{L}_B$. Put*

$$\Omega^{-1}_{\mathcal{M}_A/\mathcal{L}_B} = \left(A \underset{B}{\otimes} \mathrm{Ber}(\mathcal{L}_B^*) \right) \underset{A}{\otimes} \mathrm{Ber}(\mathcal{M}_A).$$

$\Omega^{-1}_{\mathcal{M}_A/\mathcal{L}_B}$ is a free A-module canonically isomorphic to $\mathrm{Ber}(J/J^2)$.

Remark :
The noetherianity condition is here to insure that $\mathcal{L}_B(J)/J\mathcal{L}_B$ is a finitely generated A-module.

Proof of the proposition 9.3.1 :
Using the isomorphism

$$\Omega^{-1}_{\mathcal{M}_A/\mathcal{L}_B} = \mathrm{Ber}(\oplus_i A\partial_i^*) \underset{A}{\otimes} \mathrm{Ber}(A\otimes_B\Theta_{B,\underline{x}})^* \underset{A}{\otimes} \mathrm{Ber}(\mathcal{M}_A) \text{ where } <\partial_i, \partial_j^*>= \delta_{i,j},$$

one can see easily that $\Omega^{-1}_{\mathcal{M}_A/\mathcal{L}_B}$ is isomorphic to $\mathrm{Ber}(\oplus_i A\partial_i^*)$. Denote by $\Omega_{\underline{x}}$ the basis of $\Omega^{-1}_{\mathcal{M}_A/\mathcal{L}_B}$ defined by the ∂_i's. We want to investigate how $\Omega_{\underline{x}}$ behaves under a change of coordinates. Let us consider another system $(x'_1, \ldots, x'_n, \partial'_1, \ldots, \partial'_n)$ such that $\partial'_i(x'_j) = \delta_{i,j}$. There exist $(a_{i,j}) \in B^{n^2}$ and $(b_{i,j}) \in B^{n^2}$ such that

$$x'_j = \sum_i x_i a_{i,j} \text{ and } x_j = \sum_i x'_i b_{i,j}.$$

Denote by $\bar{a}_{i,j} = a_{i,j} + J \in A$. Then the matrices $X = (\bar{a}_{i,j})$ and $Y = (\bar{b}_{i,j})$ (with coefficients in A) are inverse from each other.

By a localization argument, we put ourselves in the case where $\mathcal{L}_B(J)/J\mathcal{L}_B$ is a finitely generated free A-module and let $(e_1, ..., e_p)$ be a basis of $\mathcal{L}_B(J)/J\mathcal{L}_B$. Then, $(\partial_1, \ldots, \partial_n, \chi_{\underline{x}}^{-1}(e_1), \ldots, \chi_{\underline{x}}^{-1}(e_p))$ and $(\partial'_1, \ldots, \partial'_n, \chi_{\underline{x'}}^{-1}(e_1), \ldots, \chi_{\underline{x'}}^{-1}(e_p))$ are two basis of $A\underset{B}{\otimes}\mathcal{L}_B$. We will denote by $(\partial_1^*, \ldots, \partial_n^*, \chi_{\underline{x}}^{-1}(e_1)^*, \ldots, \chi_{\underline{x}}^{-1}(e_p)^*)$ and $(\partial_1'^*, \ldots, \partial_n'^*, \chi_{\underline{x'}}^{-1}(e_1)^*, \ldots, \chi_{\underline{x'}}^{-1}(e_p)^*)$ their dual basis. One can prove that :

$$\begin{array}{l} \partial_k'^* = \sum_{i=1}^n \partial_j^* \cdot \bar{a}_{j,k} \\ \chi_{\underline{x'}}^{-1}(e_j)^* = \chi_{\underline{x}}^{-1}(e_j)^* + \sum_i \partial_i^* \cdot < \partial_i, \chi_{\underline{x'}}^{-1}(e_j)^* > . \end{array}$$

In the latter equalities, we consider $A\otimes_B\mathcal{L}_B^*$ as a right A-module which allows us to use matrices (see [18] p.168). The transfer matrix from $(\partial_1^*, \ldots, \partial_n^*, \chi_{\underline{x}}^{-1}(e_1)^*, \ldots, \chi_{\underline{x}}^{-1}(e_p)^*)$ to $(\partial_1'^*, \ldots, \partial_n'^*, \chi_{\underline{x'}}^{-1}(e_1)^*, \ldots, \chi_{\underline{x'}}^{-1}(e_p)^*)$ is of the form

$$\begin{pmatrix} (a_{i,j}) & * \\ (0) & I \end{pmatrix}.$$

So its Berezinian equals the Berezinian of X. Whence

$$\Omega_{\underline{x'}} = Ber(X)\Omega_{\underline{x}}.$$

It is clear that this equality remains true in the case where $\mathcal{L}_B(J)/J\mathcal{L}_B$ is only a projective A-module. This finishes the proof of the proposition 9.3.1.

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