Extremal projectors in the semi-classical case

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Abstract
Using extremal projectors, Zhelobenko solved extremal equations in a generic Verma module of a complex semi-simple Lie algebra. We will solve similar equations in the semi-classical case. Our proof will be geometric. In the appendix, we give a factorization for the extremal projector of the Virasoro algebra in the semi-classical case.

Résumé
En utilisant les projecteurs extrémaux, Zhelobenko a résolu des équations extrémales dans le cas d'un module de Verma générique d'une algèbre de Lie semi-simple complexe. Nous résolvons des équations similaires dans le cas semi-classique. Notre preuve sera géométrique. Dans l'appendice, nous donnons une factorisation du projecteur extrémal pour l'algèbre de Virasoro dans le cas semi-classique.

1 Introduction

Let $\mathfrak{g}$ be a complex semi-simple finite dimensional Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and Δ the root system associated to $\mathfrak{h}$. We will write Δ^+ (respectively Δ^-) for the set of positive (respectively negative) roots of Δ and put $\rho = \frac{1}{2}\sum_{\gamma\in\Delta^+}\gamma$. We will denote by $B = (\alpha_1, \ldots, \alpha_l)$ the set of simple roots. Let $\mathfrak{g}_\gamma$ be the root space associated to the root γ. We put

$$\mathfrak{n} = \underset{\gamma\in\Delta^+}{\oplus} \mathfrak{g}_\gamma, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \ , \mathfrak{n}_- = \underset{\gamma\in\Delta^+}{\oplus} \mathfrak{g}_{-\gamma}.$$

Let $R(\mathfrak{h})$ be the field of rational functions on $\mathfrak{h}^*$. One introduces the algebra $U'(\mathfrak{g}) = U(\mathfrak{g}) \underset{S(\mathfrak{h})}{\otimes} R(\mathfrak{h})$. Let us consider the generic Verma module $V = \dfrac{U'(\mathfrak{g})}{U'(\mathfrak{g})\mathfrak{n}}$. Zhelobenko ([Z1]) showed that $V^{\mathfrak{n}} = R(\mathfrak{h})1_+$ (where $1_+ = 1 + U'(\mathfrak{g})\mathfrak{n}$). The decomposition $V = \mathfrak{n}^- V \oplus R(\mathfrak{h})1_+$ defines a projector p onto $R(\mathfrak{h})1_+$ called the extremal projector. Inspired by a work of Asherova, Smirnov and Tolstoy ([A-S-T]), Zhelobenko ([Z1]) showed that p factorizes into elementary projectors. Let $(\gamma_1, \ldots, \gamma_m)$ be a normal ordering on the positive roots. Introduce the following notations:

$$\begin{array}{l} p_\alpha = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k! f_{\alpha,k}} e_{-\alpha}^k e_\alpha^k \\ f_{\alpha,0} = 1, \\ \text{if } k > 0, \ f_{\alpha,k} = (h_\alpha + \rho(h_\alpha) + 1) \ldots (h_\alpha + \rho(h_\alpha) + k) \end{array}$$

(e_δ being the root vector associated to the root δ and h_δ the coroot). We have $p = p_{\gamma_1} \ldots p_{\gamma_m}$ ([Z1]). Let $w = s_1 \ldots s_j$ be a reduced decomposition of $w \in W$ (with $s_k = s_{\beta_k}$, β_k a simple root). Put $w_i = s_1 \ldots s_i$. The roots $\gamma_i = w_{i-1}(\beta_i)$ $(w_0 = 1)$ are pairwise distinct and

$$\Delta_w = \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) < 0\} = \{\gamma_1, \ldots, \gamma_j\}.$$

Put $\mathfrak{n}_w = \underset{\alpha\in\Delta_w}{\oplus} \mathfrak{g}_\alpha$. In [Z2], Zhelobenko gives an explicit description of $V^{\mathfrak{n}_w}$. We will establish similar results for the symmetric algebra (the so-called semi-classical case).

Let us consider the analytic manifold $(\mathfrak{g}/\mathfrak{n})^*$. We will endow it with the following coordinate system $\left((e_{-\alpha})_{\alpha\in\Delta_+}, (h_{\alpha_i})_{i\in[1,l]}\right)$. We will call U_δ the open subset of $(\mathfrak{g}/\mathfrak{n})^*$ defined by the equation $h_\delta \neq 0$. We define Φ_δ to be the following rational map of U_δ:

$$\forall \lambda \in U_\delta, \ \Phi_\delta(\lambda) = exp\left(\frac{e_{-\delta}(\lambda)}{h_\delta(\lambda)} e_\delta\right) \cdot \lambda$$

where the dot denotes natural action of $\mathfrak{n}$ on $(\mathfrak{g}/\mathfrak{n})^*$. By composition, Φ_δ defines an algebra morphism of $\mathcal{A}(U_\delta)$ which we call π_δ. We put

$$U_w = U_{\gamma_1} \cap \ldots \cap U_{\gamma_j}.$$

We will denote by $\mathcal{P}(U_w)$ (respectively $\mathcal{A}(U_w)$) the set of regular functions (respectively analytic functions) on U_w and we will write $\mathcal{P}(U_w)^{\mathfrak{n}_w}$ (respectively $\mathcal{A}(U_w)^{\mathfrak{n}_w}$) the set of invariant functions of $\mathcal{P}(U_w)$ (respectively $\mathcal{A}(U_w)$) under the action of $\mathfrak{n}_w$. We prove the following result:

Theorem *The algebra morphism $\pi_w = \pi_{\gamma_1} \circ \ldots \circ \pi_{\gamma_j}$ does not depend on the reduced expression of w. It establishes an isomorphism between*

$$\mathcal{C}_w = \{f \in \mathcal{A}(U_w) \mid \frac{\partial f}{\partial e_{-\gamma_1}} = \ldots \frac{\partial f}{\partial e_{-\gamma_j}} = 0\}$$

and $\mathcal{A}(U_w)^{\mathfrak{n}_w}$. Moreover π_w sends $\mathcal{C}_w \cap \mathcal{P}(U_w)$ onto $\mathcal{P}(U_w)^{\mathfrak{n}_w}$.

Let N_w be the connected simply connected group whose Lie algebra is $\mathfrak{n}_w$. The main ingredient of the proof will be the choice of a point in each N_w-orbit lying in U_w in accordance with the following proposition :

Proposition *Let λ be in U_w. The point $\Phi_{\gamma_j}\Phi_{\gamma_{j-1}} \ldots \Phi_{\gamma_1}(\lambda)$ is the unique point of the orbit $N_w \cdot \lambda$ whose coordinates $e_{-\gamma_1}, \ldots, e_{-\gamma_j}$ vanish.*

In the appendix, we shall give a factorization for the extremal projector of the Virasoro algebra in the semi-classical case. Note that the non commutative case is still open. It is very different from the semi-simple case because the Virasoro algebra does not admit any normal ordering.

Notations

Along all this article $\mathfrak{g}$ will denote a complex semi-simple finite dimensional Lie algebra and $\mathfrak{h}, \Delta, \Delta_+, \Delta_-, \mathfrak{n}, \mathfrak{n}_-, B = (\alpha_1, \ldots, \alpha_l)$ will be as above. Denote by W the Weyl group associated to these choices and $\overline{w}$ its longest element. Let γ be an element of Δ^+ and let h_γ be the unique element of $[\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}]$ such that $\gamma(h_\gamma) = 2$. If e_γ is in $\mathfrak{g}_\gamma$, then there exists a unique $e_{-\gamma}$ such that $(h_\gamma, e_\gamma, e_{-\gamma})$ is a sl(2)-triple. If α and β are two roots, we set $[e_\alpha, e_\beta] = C_{\alpha,\beta} e_{\alpha+\beta}$ with the convention that $C_{\alpha,\beta}$ is zero if $\alpha + \beta$ is not a root.

The ordering $(\gamma_1, \ldots, \gamma_m)$ on the positive roots is normal if any composite root is located between its components. Thus for all positive roots $\gamma_i, \gamma_j, \gamma_k$, the equality $\gamma_k = \gamma_i + \gamma_j$ implies $i \leq k \leq j$ or $j \leq k \leq i$. There is a one to one correspondence between normal orderings and reduced expression of $\overline{w}$ ([Z2]). Let us recall it. Denote by s_i the reflexion with respect to a simple root β_i. If $\overline{w} = s_1 \ldots s_m$, then $(\beta_1, s_1(\beta_2), \ldots, s_1 \ldots s_{i-1}(\beta_i), \ldots, s_1 \ldots s_{m-1}(\beta_m))$ are in normal ordering.

If V is a vector space, $S(V)$ will be the symmetric algebra of V. Lastly, if P is in $S(V)$, $S(V)_P$ will be the localization of $S(V)$ with respect to $\{P^n \mid n \in \mathbb{N}\}$.

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2 Extremal equations in $(\mathfrak{g}/\mathfrak{n})^*$

We consider $(\mathfrak{g}/\mathfrak{n})^*$ as an analytic manifold. We endow it with the following coordinate system $\left((e_{-\alpha})_{\alpha\in\Delta_+}, (h_{\alpha_i})_{i\in[1,l]}\right)$. If δ is a positive root, we will denote by U_δ the open subset of $(\mathfrak{g}/\mathfrak{n})^*$ defined by the equation $h_\delta \neq 0$. If U is an open subset for the Zariski topology, we will write $\mathcal{A}(U)$ for the algebra of analytic functions on U and $\mathcal{P}(U)$ for the algebra of regular functions on U. We will define Φ_δ to be the following rational map of U_δ

$$\forall \lambda \in U_\delta, \quad \Phi_\delta(\lambda) = exp\left(\frac{e_{-\delta}(\lambda)}{h_\delta(\lambda)} e_\delta\right) \cdot \lambda.$$

By composition, Φ_δ defines an algebra morphism of $\mathcal{A}(U_\delta)$ which we call π_δ. We will denote by X_δ the natural action of e_δ on $\mathcal{P}(U_\delta)$. Remark that X_δ is a derivation. If f is in $\mathcal{P}(U_\delta)$, we have

$$(*) \quad \pi_\delta(f) = \sum_{k=0}^{\infty} (-1)^k \frac{e_{-\delta}^k}{k! h_\delta^k} X_\delta^k \cdot f$$

where $e_{-\delta}$ denotes the multiplication by $e_{-\delta}$. The operator π_δ is the commutative analog of the Zhelobenko's elementary projector.

Let $w = s_1 \dots s_j$ be a reduced decomposition of $w \in W$ (with $s_k = s_{\beta_k}$, $\beta_k \in B$). Put $w_i = s_1 \dots s_i$. The roots $\gamma_i = w_{i-1}(\beta_i)$ ($w_0 = 1$) are pairwise distinct and

$$\Delta_w = \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) < 0\} = \{\gamma_1, \dots, \gamma_j\}.$$

An ordering in Δ_w is called normal if it coincides with the initial segment of some normal ordering in Δ^+ (that is compatible with one of the reduced expression of $\overline{w}$). Note that $(\gamma_1, \dots, \gamma_j)$ is a normal ordering of Δ_w. Put

$$U_w = \bigcap_{\delta \in \Delta_w} U_\delta.$$

We have

$$\mathcal{P}(U_w) = \left(\frac{S(\mathfrak{g})}{S(\mathfrak{g})\mathfrak{n}}\right)_{h_{\gamma_1} \dots h_{\gamma_j}} = S\left(\frac{\mathfrak{g}}{\mathfrak{n}}\right)_{h_{\gamma_1} \dots h_{\gamma_j}}.$$

We will denote by N_w the connected and simply connected group whose Lie algebra is $\mathfrak{n}_w = \bigoplus_{\alpha \in \Delta_w} \mathfrak{g}_\alpha$. We will start by proving the following proposition.

Proposition 2.1 *Let λ be in U_w. The point $\Phi_{\gamma_j}\Phi_{\gamma_{j-1}} \dots \Phi_{\gamma_1}(\lambda)$ is the unique point of the orbit $N_w \cdot \lambda$ whose coordinates $e_{-\gamma_1}, \dots, e_{-\gamma_j}$ vanish. In particular $\Phi_{\gamma_j}\Phi_{\gamma_{j-1}} \dots \Phi_{\gamma_1}$ does not depend on the normal ordering on Δ_w.*

Proof of the proposition 2.1:
Complete $(\gamma_1, \dots, \gamma_j)$ into a normal ordering on the positive roots $(\gamma_1, \dots, \gamma_m)$. $\mathfrak{g}/\mathfrak{n}$ is endowed with the basis $(e_{-\gamma_1}, \dots, e_{-\gamma_m}, h_{\alpha_1}, \dots, h_{\alpha_l})$. Let $\left(e^*_{-\gamma_1}, \dots, e^*_{-\gamma_m}, h^*_{\alpha_1}, \dots, h^*_{\alpha_l}\right)$ be the dual basis. We will often identify the point $a_{\gamma_1} e^*_{-\gamma_1} + \dots + a_{\gamma_m} e^*_{-\gamma_m} + b_1 h^*_{\alpha_1} + \dots + b_l h^*_{\alpha_l}$ with its coordinates $(a_{\gamma_1}, \dots, a_{\gamma_m}, b_1, \dots, b_l)$. Let us see that there is a unique point in $N_w \cdot \lambda$ whose coordinates $e_{-\gamma_1}, \dots, e_{-\gamma_j}$ vanish. Assume that there are two such points $f = (0, \dots, 0, a_{\gamma_{j+1}}, \dots, a_{\gamma_m}, b_1, \dots, b_l)$ and $f' = (0, \dots, 0, a'_{\gamma_{j+1}}, \dots, a'_{\gamma_m}, b'_1, \dots, b'_l)$. Then there exist complex numbers $(t_1, \dots, t_j)$ such that $exp(t_1 e_{\gamma_1} + \dots + t_j e_{\gamma_j}) \cdot f = f'$. One can show easily the following equalities

$$\begin{aligned} e_{\gamma_l} \cdot e^*_{-\gamma_k} &= -C_{\gamma_l, -\gamma_k} e^*_{-\gamma_l - \gamma_k} \\ e_{\gamma_l} \cdot h^*_{-\alpha_i} &= -h^*_{-\alpha_i}(h_{\gamma_l}) e^*_{-\gamma_l}. \end{aligned}$$

From these equalities, one deduces easily that the term in $e^*_{-\gamma_1}$ of $exp(t_1 e_{\gamma_1} + \dots + t_j e_{\gamma_j}) \cdot (0, \dots, 0, a_{\gamma_{j+1}}, \dots, a_{\gamma_m}, b_1, \dots, b_l)$ is $-t_1 f(h_{\gamma_1})$. As f is in U_w, we get $t_1 = 0$. We reproduce the same reasoning to show that $t_2, t_3, \ldots, t_j$ are zero. So that we have proved that the two points f and f' coincide. It is not difficult to deduce from the normal ordering property that Φ_{γ_i} sends the point $(x_{\gamma_1}, \ldots, x_{\gamma_m}, y_1, \ldots, y_l)$ to a point $(x'_{\gamma_1}, \ldots, x'_{\gamma_{i-1}}, 0, x'_{\gamma_{i+1}}, \ldots, x'_{\gamma_m}, y_1, \ldots, y_l)$ and that it sends the point $(0, \ldots, 0, x_{\gamma_i}, \ldots, x_{\gamma_m}, y_1, \ldots, y_l)$ to a point $(0, \ldots, 0, x'_{\gamma_{i+1}}, \ldots, x'_{\gamma_m}, y_1, \ldots, y_l)$. So that $\Phi_{\gamma_j}\Phi_{\gamma_{j-1}} \ldots \Phi_{\gamma_1}(\lambda)$ is the unique point of $N_w \cdot \lambda$ whose coordinates $e_{-\gamma_1}, \ldots, e_{-\gamma_j}$ vanish. This finishes the proof of the proposition 2.1.

As a consequence of the previous proposition, we may write Φ_w for the operator $\Phi_{\gamma_j}\Phi_{\gamma_{j-1}} \ldots \Phi_{\gamma_1}$. The algebra homomorphism defined by Φ_w on $\mathcal{A}(U_w)$ will be denoted by π_w. Using the proposition 2.1, we will give a geometric proof of the following result.

Theorem 2.2 *1) If $\overline{\mathfrak{n}}_w$ denotes the linear hull of $(e_{-\alpha})_{\alpha \in \Delta_w}$, one has $Ker\pi_w = \overline{\mathfrak{n}}_w \mathcal{A}(U_w)$.*

2) The operator π_w is the projector onto $\mathcal{A}(U_w)^{\mathfrak{n}_w}$ with kernel $\overline{\mathfrak{n}}_w \mathcal{A}(U_w)$ and its restriction to $\mathcal{P}(U_w)$ is the projector onto $\mathcal{P}(U_w)^{\mathfrak{n}_w}$ with kernel $\overline{\mathfrak{n}}_w \mathcal{P}(U_w)$.

3) The operator π_w establishes an isomorphism Π_w between

$$\mathcal{C}_w = \{f \in \mathcal{A}(U_w) \mid \frac{\partial f}{\partial e_{-\gamma_1}} = \ldots \frac{\partial f}{\partial e_{-\gamma_j}} = 0\}$$

and $\mathcal{A}(U_w)^{\mathfrak{n}_w}$. Moreover Π_w sends $\mathcal{C}_w \cap \mathcal{P}(U_w)$ onto $\mathcal{P}(U_w)^{\mathfrak{n}_w}$. If f is in $\mathcal{A}(U_w)^{\mathfrak{n}_w}$, $\Pi_w^{-1}(f)$ is the restriction of f to the subvariety of equations $e_{-\gamma_1} = \ldots = e_{-\gamma_j} = 0$.

Proof of the theorem 2.2:
From the previous proposition, the inclusion $\overline{\mathfrak{n}}_w \mathcal{A}(U_w) \subset Ker\pi_w$ is clear. Moreover, a standard reasoning shows that

$$\mathcal{A}(U_w) = \mathcal{C}_w \oplus \overline{\mathfrak{n}}_w \mathcal{A}(U_w).$$

Then one sees easily that $Ker\pi_w \cap \mathcal{C}_w = \{0\}$. So that we have $\overline{\mathfrak{n}}_w \mathcal{A}(U_w) = Ker\pi_w$.

Let us now show that $\mathrm{Im}\pi_w = \mathcal{A}(U_w)^{\mathfrak{n}_w}$ and that π_w is a projector. Let α be in Δ_w. For any f in $\mathcal{A}(U_w)$ and any λ in U_w, we have

$$(e_\alpha \circ \pi_w)(f)(\lambda) = \frac{d}{dt} f\left(\Phi_{\gamma_j} \ldots \Phi_{\gamma_1} exp(-te_\alpha)\lambda\right)_{|t=0}.$$

But for any t, $\Phi_{\gamma_j} \dots \Phi_{\gamma_1} exp(-te_\alpha)\lambda$ is the unique point of $N_w \cdot \lambda$ whose coordinates $e_{-\gamma_1}, \dots, e_{-\gamma_j}$ vanish. So that $e_\alpha \circ \pi_w = 0$. We have thus proved the inclusion $Im\pi_w \subset \mathcal{A}(U_w)^{\mathfrak{n}_w}$. Now it is clear that π_w is a projector : check that $\pi_w \circ \pi_w = \pi_w$ on coordinates using the formula (*). The reverse inclusion $\mathcal{A}(U_w)^{\mathfrak{n}_w} \subset Im\pi_w$ will be a consequence of the following lemma.

Lemma 2.3 *Let k be in $[1,j]$ and let f be in $\mathcal{A}(U_w)$. If $X_{\gamma_k} f = 0$, then $\pi_{\gamma_k} f = f$.*

Proof of the lemma 2.3 :
We first remark that $(\pi_{\gamma_k}(e_{-\gamma_1}), \dots, \pi_{\gamma_k}(e_{-\gamma_{k-1}}), e_{-\gamma_k}, \pi_{\gamma_k}(e_{-\gamma_{k+1}}), \dots, \pi_{\gamma_k}(e_{-\gamma_m}), h_{\alpha_1}, \dots, h_{\alpha_l})$ is a coordinate system in U_w. Indeed, one may see by induction that for any $i \leq k-1$ (respectively $i \geq k+1$), $e_{-\gamma_i}$ may be expressed as a regular function of $(\pi_{\gamma_k}(e_{-\gamma_1}), \dots, \pi_{\gamma_k}(e_{-\gamma_i}), h_{\alpha_1}, \dots, h_{\alpha_l})$ (respectively $(\pi_{\gamma_k}(e_{-\gamma_i}), \dots, \pi_{\gamma_k}(e_{-\gamma_m}), h_{\alpha_1}, \dots, h_{\alpha_l})$). We put $(\epsilon_1, \dots, \epsilon_{m+l}) = \Big(\pi_{\gamma_k}(e_{-\gamma_1}), \dots, \pi_{\gamma_k}(e_{-\gamma_{k-1}}), e_{-\gamma_k}, \pi_{\gamma_k}(e_{-\gamma_{k+1}}), \dots, \pi_{\gamma_k}(e_{-\gamma_m}), h_{\alpha_1}, \dots, h_{\alpha_l}\Big)$. In these coordinates, we have $X_{\gamma_k} = h_{\gamma_k} \dfrac{\partial}{\partial \epsilon_k}$. So that if $X_{\gamma_k} f = 0$, then f does not depend on ϵ_k and it becomes clear that there exists g such that $f = \pi_{\gamma_k} g$. As π_{γ_k} is a projector, we have $\pi_{\gamma_k} f = \pi_{\gamma_k}\pi_{\gamma_k} g = \pi_{\gamma_k} g$, which finishes the proof of the lemma.
It is clear from the proof that π_w sends $\mathcal{C}_w \cap \mathcal{P}(U_w)$ onto $\mathcal{P}(U_w)^{\mathfrak{n}_w}$.

In particular $\pi_{\overline{w}|\mathcal{P}(U_{\overline{w}})}$ is the projector onto $S(\mathfrak{h})_{h_{\gamma_1} \dots h_{\gamma_m}}$ with kernel $\mathfrak{n}_- \mathcal{P}(U_{\overline{w}})$. By analogy to Asherova, Tolstoy, Smirnov and Zhelobenko's work, we will call it the extremal projector.

The proposition 2.1 gives a geometric interpretation of the projector π_w.

3 Appendix : Extremal projector for the Virasoro algebra in the semi-classical case

In this section, we shall give a factorization of the Virasoro algebra extremal projector in the semi-classical case. Note that the non commutative case is still open. It is very different from the semi-simple case because the Virasoro algebra does not admit any normal ordering. Recall that the Virasoro algebra Vir is the infinite dimensional Lie algebra generated by $\{e_i \mid i \in \mathbb{Z}\} \cup \{c\}$

with commutation rules

$$[e_i, e_j] = (j - i)\ e_{i+j} + \frac{(j^3 - j)}{12}\delta_{i+j,0}c, \quad [e_i, c] = 0.$$

Vir admits the following triangular decomposition

$$Vir = Vir_+ \oplus Vir_0 \oplus Vir_-$$

where

$$Vir_+ = \underset{i\geq 1}{\oplus}\mathbb{C}e_i,\ Vir_0 = \mathbb{C}e_0 \oplus \mathbb{C}c,\ Vir_- = \underset{i\leq -1}{\oplus}\mathbb{C}e_i.$$

We will also use the notation

$$Vir_{r,+} = \underset{i\geq r}{\oplus}\mathbb{C}e_i \ \text{ and } \ Vir_{r,-} = \underset{i\leq -r}{\oplus}\mathbb{C}e_i.$$

$Vir_{r,+}$ and $Vir_{r,-}$ are Lie subalgebras of Vir.

Let $R(Vir_0)$ be the field of fractions of $S(Vir_0)$. We introduce the algebra

$$S'(Vir) = S(Vir) \underset{S(Vir_0)}{\otimes} R(Vir_0) = S'\left(Vir/Vir_-\right).$$

There is a natural action of Vir_- on $S'\left(Vir/Vir_-\right)$. Through this action, for any negative i, e_i defines a derivation X_i of $S'\left(\frac{Vir}{Vir_-}\right)$. Set

$$T_r = \left(\frac{S'(Vir)}{S'(Vir)Vir_-}\right)^{Vir_{r,-}}$$

The result and the proof of the following lemma is left to the reader.

Lemma 3.1

$$T_r = \underset{k_1,\ldots,k_{r-1}\in\mathbb{N}}{\oplus} R(Vir_0)e_1^{k_1}\ldots e_{r-1}^{k_{r-1}}$$

As a consequence of the lemma 3.1, we have the following decomposition

$$S'\left(Vir/Vir_-\right) = T_r \oplus Vir_{r,+}S'\left(Vir/Vir_-\right).$$

The proof of the next lemma is an easy computation.

Lemma 3.2 *For any* $i > 1$*, the operator* $\pi_i = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\left(2ie_0 + \frac{(i^3-i)c}{12}\right)^k} e_i^k X_{-i}^k$ *is an algebra morphism and satisfies the relations*

$$X_{-i} \circ \pi_i = 0 \text{ and } \pi_i \circ e_i = 0$$

(where e_i *denotes multiplication by* e_i*).*

It is not hard to see that the operator $\Pi_r = \prod_{i=r}^{\infty} \pi_i$ is well defined. Actually $\prod_{i=r}^{\infty} \pi_i(e_1^{a_1} \dots e_k^{a_k}) = \prod_{r \le i \le k} \pi_i(e_1^{a_1} \dots e_k^{a_k}) = 0$ (by the lemma 3.2).

Theorem 3.3 *The operator* Π_r *satisfies the relations*

$$\forall i \geq r,\ X_{-i} \circ \Pi_k = 0,\ \Pi_k \circ e_i = 0.$$

It is the projector onto T_r *with kernel* $Vir_{r,+}S'\left(\frac{Vir}{Vir_-}\right)$.

In particular, Π_1 is the extremal projector.

Proof of the theorem 3.3
The relations of a) are easy to check and they prove that Π_r is a projector. To prove that the kernel of Π_r is $Vir_{r,+}$, we proceed as in the semi-simple case. The inclusion $Im\Pi_r \subset T_r$ is a consequence of a). To prove the reverse inclusion, remark that if x is in T_r, then $\Pi_r x = x$, so that x is in $Im\Pi_r$.

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