A duality property for complex Lie algebroids

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Abstract
Interpreting Lie algebroid theory in terms of $\mathcal{D}$-modules, we define a duality functor for a Lie algebroid as well as a direct image functor for a morphism of Lie algebroids. Generalizing the work of Schneiders (see also the work of Schapira-Schneiders) and making assumptions analog to his, we show that the duality functor and the direct image functor commute. As an application, we extend to Lie algebroids some duality properties already known for Lie algebras.

1 Introduction

Let X be a complex analytic manifold of complex dimension x, $\mathcal{O}_X$ the sheaf of holomorphic functions on X, Θ_X the sheaf of holomorphic vector fields and $\mathcal{D}_X$ the sheaf of ring of differential operators on X. A complex Lie algebroid over X is a pair $(\mathcal{L}_X, \omega_X)$ where

- $\mathcal{L}_X$ is a sheaf of Lie algebras and a locally free $\mathcal{O}_X$-module of finite rank.

- ω is a map from $\mathcal{L}_X$ to Θ_X (called the anchor map)
- The following compatibility relation holds

$$\forall(\xi,\zeta)\in\mathcal{L}_X^2,\ \ \forall f\in\mathcal{O}_X,\ \ [\xi,f\zeta]=\omega(\xi)(f)\zeta+f[\xi,\zeta].$$

It is easy to see that Lie algebroids generalize at the same time Lie algebras and tangent bundles. The Lie algebroid $\mathcal{L}_X$ gives rise to a sheaf of rings of differential operators $\mathcal{D}(\mathcal{L}_X)$. Hence the idea of applying $\mathcal{D}$-module theory to Lie algebroids (as it was already done in [C2] for the affine case). If $\mathcal{L}_X=\Theta_X$ and $\omega=id$ then $\mathcal{D}(\mathcal{L}_X)$ is the sheaf of rings of differential operators on X. If X is a point, $\mathcal{L}_X$ is a Lie algebra and $\mathcal{D}(\mathcal{L}_X)$ is its enveloping algebra.

Set $\Omega_X=\Lambda^x\Theta_X^*$ where $\Theta_X^*=\mathcal{H}om_{\mathcal{O}_X}(\Theta_X,\mathcal{O}_X)$. It is well known that Ω_X is endowed with a right $\mathcal{D}_X$-module structure. Through the anchor map, we can equip Ω_X with a right $\mathcal{D}(\mathcal{L}_X)$-module structure. This allows to construct an equivalence of categories between complexes of left $\mathcal{D}(\mathcal{L}_X)$-modules and complexes of right $\mathcal{D}(\mathcal{L}_X)$-modules. Thus $\Omega_X\underset{\mathcal{O}_X}{\otimes}\mathcal{D}(\mathcal{L}_X)$ is endowed with a right $\mathcal{D}(\mathcal{L}_X)$-bimodule structure. Moreover, we define the following duality functor in the derived category of bounded complexes of right $\mathcal{D}(\mathcal{L}_X)$-modules with coherent cohomology:

$$\forall\mathcal{M}^\bullet\in D^b_{coh}\left(\mathcal{D}(\mathcal{L}_X)^{op}\right),\ \Delta_{\mathcal{L}_X}(\mathcal{M}^\bullet)=R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}\left(\mathcal{M}^\bullet,\Omega_X[x]\underset{\mathcal{O}_X}{\otimes}\mathcal{D}(\mathcal{L}_X)\right).$$

A Lie algebroid morphism Φ from $(\mathcal{L}_X,\omega_X)$ to $(\mathcal{L}_Y,\omega_Y)$ is the data of a pair (f,F) where f is an analytic map from X to Y and F is a $\mathcal{O}_X$-module map from $\mathcal{L}_X$ to $f^*\mathcal{L}_Y$ with some requirements (see section 2.4 for details). As in $\mathcal{D}$-module theory, one may define a transfer $(\mathcal{D}(\mathcal{L}_X)-f^{-1}\mathcal{D}(\mathcal{L}_Y)^{op})$ -bimodule $\mathcal{D}_{\mathcal{L}_X\to\mathcal{L}_Y}$. Hence the existence of the direct image functor :

$$\underline{\Phi}_!(\mathcal{M}^\bullet)=Rf_!\left(\mathcal{M}^\bullet\overset{L}{\underset{\mathcal{D}(\mathcal{L}_X)}{\otimes}}\mathcal{D}_{\mathcal{L}_X\to\mathcal{L}_Y}\right).$$

Schneiders showed ([S1], [S2]) that, under some appropriate hypotheses, the duality functor and the direct image functor commute in the case of $\mathcal{D}$-modules (and even for modules over rings of relative differential operators). The case of algebraic smooth varieties had already been treated by Bernstein for a proper morphism ([Be], [Bo], [Ho]). Moreover Mebkhout had already treated the absolute case (i.e Y consists in a point) in [Me1], [Me2]. Schneiders' work was generalized to elliptic pairs by Schapira and Schneiders ([S-S]). The aim of this article is to generalize Schneiders' result to Lie algebroids. As in $\mathcal{D}$-module theory ([S-S]), we introduce the notion of "good" modules (due to Kashiwara) which is a refinement of the property of having a good filtration in a neighborhood of any compact. Denote by $D^b_{good}(\mathcal{D}(\mathcal{L}_X)^{op})$ the derived category of bounded complexes with "good" cohomology. The interest of such a notion being that if f is proper on the support of an element $\mathcal{M}^\bullet$ of $D^b_{good}(\mathcal{D}(\mathcal{L}_X)^{op})$, then $\underline{\Phi}_!(\mathcal{M}^\bullet)$ is in $D^b_{good}(\mathcal{D}(\mathcal{L}_Y)^{op})$. We prove :

Theorem 4.3.1 *Let X and Y be two complex manifolds of complex dimension respectively x and y. Let $\Phi = (f, F)$ be a Lie algebroid morphism from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$. Let $\mathcal{M}^\bullet$ be an element of $D^b_{good}(\mathcal{D}(\mathcal{L}_X)^{op})$ such that f is proper on the support of $\mathcal{M}^\bullet$. There is a functorial isomorphism from $\underline{\Phi}_!\Delta_{\mathcal{L}_X}(\mathcal{M}^\bullet)$ to $\Delta_{\mathcal{L}_Y}\underline{\Phi}_!(\mathcal{M}^\bullet)$* in $D^b_{good}(\mathcal{D}(\mathcal{L}_Y)^{op})$. As a particular case

of this theorem, we get :

Proposition 4.3.5 *Let X and Y be two complex manifolds of complex dimension respectively x and y. Assume that $\mathcal{L}_X$ is a rank $d_{\mathcal{L}_X}$ Lie algebroid. Let $\Phi = (f, F)$ be a Lie algebroid morphism from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$ such that f is proper. Let $\mathcal{E}$ be a right $\mathcal{D}(\mathcal{L}_X)$-module which is a locally free $\mathcal{O}_X$-module of finite rank. Then we have the following isomorphism in $D^b_{good}(\mathcal{D}(\mathcal{L}_Y)^{op})$*

$$\underline{\Phi}_!\left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \Omega_X[x]) \underset{\mathcal{O}_X}{\otimes} \Lambda^{d_X}\mathcal{L}_X^*[-d_{\mathcal{L}_X}]\right) \simeq \Delta_{\mathcal{L}_Y}\underline{\Phi}_!(\mathcal{E}).$$

As a corollary of this proposition, we obtain :

Corollary 4.3.6 *Let X be a compact analytic manifold and let $\mathcal{L}_X$ be a rank $d_{\mathcal{L}_X}$ Lie algebroid. Let $\mathcal{N}$ be a left $\mathcal{D}(\mathcal{L}_X)$-module which a is locally free $\mathcal{O}_X$-module of finite rank. Then, for any i in $\mathbb{Z}$, $\mathrm{Ext}^i_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X, \mathcal{N})$ is finite dimensional and*

$$\mathrm{Ext}^{d_{\mathcal{L}_X}+x-i}_{\mathcal{D}(\mathcal{L}_X)}\left(\mathcal{O}_X, \mathcal{N}^* \underset{\mathcal{O}_X}{\otimes} \mathcal{H}om_{\mathcal{O}_X}\left(\Lambda^{d_{\mathcal{L}_X}}\mathcal{L}_X^* =, \Omega_X\right)\right) \simeq \left(\mathrm{Ext}^i_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X, \mathcal{N})\right)^*.$$

Hence, the corollary 4.3.6 generalizes at the same time Serre duality (case when $\mathcal{L}_X = 0$) for compact complex analytic manifolds and Poincaré duality for Lie algebras (case when X is a point).

Applying theorem 4.3.1, we get what we might call a generalized Zuckerman duality ([B-C])

Corollary 4.3.7 *Let X, Y, Z be three analytic manifolds of dimension respectively x, y and z. Let $\mathcal{L}_X$ (respectively $\mathcal{L}_Y$, $\mathcal{L}_Z$) be a Lie algebroid over X (respectively Y, Z) of rank $d_{\mathcal{L}_X}$ (respectively $d_{\mathcal{L}_Y}$, $d_{\mathcal{L}_Z}$). Let $\Phi = (f, F)$ (respectively $\Psi = (g, G)$) be a Lie algebroid morphism from $\mathcal{L}_X$ (respectively $\mathcal{L}_Z$) to $\mathcal{L}_Y$. We assume that f and g are proper. Let $\mathcal{M}$ (respectively $\mathcal{N}$) be a $\mathcal{D}(\mathcal{L}_X)$ (respectively $\mathcal{D}(\mathcal{L}_Z)$) -right module which is a locally free $\mathcal{O}_X$ (respectively $\mathcal{O}_Z$)-module of finite rank. Introduce the right $\mathcal{D}(\mathcal{L}_X)$ (respectively $\mathcal{D}(\mathcal{L}_Z)$)-modules $\widetilde{\mathcal{M}}$ (respectively $\widetilde{\mathcal{N}}$)*

$$\widetilde{\mathcal{M}} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \Omega_X) \underset{\mathcal{O}_X}{\otimes} \bigwedge^{d_{\mathcal{L}_X}} \mathcal{L}_X^*$$
$$\widetilde{\mathcal{N}} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{M}, \Omega_Z) \underset{\mathcal{O}_Z}{\otimes} \bigwedge^{d_{\mathcal{L}_Z}} \mathcal{L}_Z^*.$$

For all n in $\mathbb{Z}$, we have the following isomorphism

$$\mathcal{E}xt^{n-d_{\mathcal{L}_Z}+z}_{\mathcal{D}(\mathcal{L}_Y)} \left(\underline{\Phi}_!(\widetilde{\mathcal{M}}), \underline{\Psi}_!(\widetilde{\mathcal{N}})\right) \simeq \mathcal{E}xt^{n-d_{\mathcal{L}_X}+x}_{\mathcal{D}(\mathcal{L}_Y)} \left(\underline{\Psi}_!(\mathcal{N}), \underline{\Phi}_!(\mathcal{M})\right).$$

Corollary 4.3.7 was proved in [C1] in the case where $\mathcal{L}_Y = \mathfrak{g}$ is a Lie algebra and $\mathcal{L}_X = \mathfrak{h}$, $\mathcal{L}_Z = \mathfrak{t}$ are Lie subalgebras of $\mathfrak{g}$. The proof of [C1] is different from the one in this paper. It is inspired by a method used by M. Duflo ([D]). See also [B-C], [G], [C-S] and [D] for particular cases of this duality in the Lie algebra context.

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Notations :

We will follow the notations of [K-S] for sheaf theory.

Let $(\mathcal{L}_X, \omega)$ be a Lie algebroid over X and let $\mathcal{D}(\mathcal{L}_X)$ be the sheaf of rings it defines. We will denote by $D^b_{coh}(\mathcal{D}(\mathcal{L}_X)^{op})$ the derived category of bounded complexes of right $\mathcal{D}(\mathcal{L}_X)$-modules with coherent cohomology. We will write

$D^b\left(\mathcal{D}(\mathcal{L}_X)^{op} - \mathcal{D}(\mathcal{L}_X)^{op}\right)$ for the bounded derived category of right $\mathcal{D}(\mathcal{L}_X)$-bimodules. If $\mathcal{M}$ and $\mathcal{M}'$ are two right $\mathcal{D}(\mathcal{L}_X)$-modules, we put

$$\mathcal{E}xt^i_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{M}, \mathcal{M}') = \mathcal{H}^i\left(R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{M}, \mathcal{M}')\right).$$

If V is a $\mathbb{C}$-vector space, we will write $T_{\mathbb{C}}(V)$ for the tensor algebra of V.

2 Lie algebroids

2.1 Definitions and examples

Let X be a complex analytic manifold and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. Let Θ_X be the $\mathcal{O}_X$-module of holomorphic vector fields on X.

Definition 2.1.1 *A sheaf in Lie algebras , $\mathcal{L}_X$ is a sheaf of $\mathbb{C}$-vector spaces such that for any open subset U, $\mathcal{L}_X(U)$ is equipped with a Lie bracket compatible with restrictions.*

A morphism between two sheaves of Lie algebras $\mathcal{L}_X$ and $\mathcal{M}_X$ is a $\mathbb{C}_X$-module morphism which is a Lie algebra morphism on each open subset.

Definition 2.1.2 *A complex Lie algebroid over X is a pair $(\mathcal{L}_X, \omega)$ where*

- *$\mathcal{L}_X$ is a locally free $\mathcal{O}_X$-module of finite rank.*
- *$\mathcal{L}_X$ is a sheaf of Lie algebras.*
- *ω is a $\mathcal{O}_X$-module morphism and a sheaf of Lie algebras morphism from $\mathcal{L}_X$ to Θ_X such that the following compatibility relation holds*

$$\forall(\xi, \zeta) \in \mathcal{L}_X^2,\ \ \forall f \in \mathcal{O}_X,\ \ [\xi, f\zeta] = \omega(\xi)(f)\zeta + f[\xi, \zeta].$$

One calls ω the anchor map. When there is no ambiguity, we will drop the anchor map in the notation of the Lie algebroid. If $(\mathcal{L}_X, \omega)$ is a Lie algebroid over X, then $\left(\mathcal{L}_{X|U}, \omega_{|U}\right)$ is a Lie algebroid over U which will be denoted $\mathcal{L}_U$.

A Lie algebroid $(\mathcal{L}_X, \omega)$ gives rise to a sheaf of differential operators. We will denote by $\mathcal{D}(\mathcal{L}_X)$ the sheaf associated to the presheaf:

$$U \mapsto T_{\mathbb{C}}\left(\mathcal{O}_X(U) \oplus \mathcal{L}_X(U)\right)/J_U$$

where J_U is the two sided ideal generated by the relations

$$\forall (f,g) \in \mathcal{O}_X(U), \;\; \forall (\xi,\zeta) \in \mathcal{L}_X(U)^2$$

$$\begin{aligned} &1) f \cdot g = (fg) \\ &2) f \cdot D = (fD) \\ &3) \xi \cdot \zeta - \zeta \cdot \xi = [\xi,\zeta] \\ &4) \xi \cdot f - f \cdot \xi = \omega(\xi)(f) \end{aligned}$$

Note that $\mathcal{D}(\mathcal{L}_X)$ is a filtered sheaf of rings, the filtration $(\mathcal{F}_n\mathcal{D}(\mathcal{L}_X))_{n\in\mathbb{N}} = (\mathcal{D}(\mathcal{L}_X)_n)_{n\in\mathbb{N}}$ being defined as follows:

$$\begin{aligned} \mathcal{D}(\mathcal{L}_X)_0 &= \mathcal{O}_X \\ \mathcal{D}(\mathcal{L}_X)_n &= \mathcal{D}(\mathcal{L}_X)_{n-1} \cdot \mathcal{L}_X \end{aligned}$$

Examples of Lie algebroids

1) The Lie algebroid (Θ_X, id) gives rise to the usual ring of differential operators.

2) Let $\mathfrak{g}$ be a finite dimensional Lie algebra. It is a Lie algebroid over a point with trivial anchor map. The ring of differential operators defined by this case is the enveloping algebra of $\mathfrak{g}$.

3) Let S be an analytic manifold. A relative analytic manifold over S is an analytic manifold X endowed with a surjective analytic submersion $\epsilon_X : X \to S$. For short a relative manifold over X will be denoted $X|S$. The differential $T\epsilon_X : TX \to X \times_S TS$ is onto. Its kernel is thus a subbundle of TX. We denote it $T(X|S)$ and call it the relative tangent bundle of X with respect to S. Its holomorphic sections form the sheaf $\Theta_{X|S}$ of vertical holomorphic vector fields on $X|S$. It is a Lie algebroid with anchor map the natural inclusion $\Theta_{X|S} \to \Theta_X$. This Lie algebroid gives rise to the so called ring of relative differential operators on $X|S$ ([S1], [S-S]).

4) Let $\mathfrak{g}$ be a Lie algebra. Assume that there is a Lie algebra morphism $\sigma : \mathfrak{g} \to \Theta_X$. Then $\mathcal{O}_X \otimes \mathfrak{g}$ has a natural Lie algebroid structure with anchor map ω defined by

$$\forall f \in \mathcal{O}_X, \; \forall \xi \in \mathfrak{g}, \;\; \omega(f \otimes \xi) = f\sigma(\xi).$$

The Lie algebra bracket on $\mathcal{O}_X \otimes \mathfrak{g}$ is given

$$[f\xi, g\eta] = f\sigma(\xi)(g)\eta - g\sigma(\eta)(f) + fg[\xi,\eta].$$

5) Let X be a Poisson analytic manifold. We will denote by $\{\ ,\ \}$ the Poisson bracket on $\mathcal{O}_X$. The $\mathcal{O}_X$-module of differential forms of degree 1, Ω^1_X, is endowed with a natural Lie algebroid structure with anchor map

$$\begin{array}{ccc} \Omega^1_X & \to & \Theta_X \\ fdg & \mapsto & f\{g,\bullet\}. \end{array}$$

Recall that the Lie bracket on Ω^1_X is given by

$$[fda, gdb] = fgd\{a,b\} + f\{a,g\}db - g\{b,f\}da.$$

6) Let $(\mathcal{L}_X, \omega_X)$ (respectively $(\mathcal{L}_Y, \omega_Y)$) be a Lie algebroid over an analytic manifold X (respectively Y). Let p_1 (respectively p_2) be the projection from $X \times Y$ over X (respectively Y). The $\mathcal{O}_{X\times Y}$-module

$$\mathcal{L}_{X\times Y} = \mathcal{O}_{X\times Y} \underset{p_1^{-1}\mathcal{O}_X}{\otimes} p_1^{-1}\mathcal{L}_X \oplus \mathcal{O}_{X\times Y} \underset{p_2^{-1}\mathcal{O}_Y}{\otimes} p_2^{-1}\mathcal{L}_Y$$

is endowed with a natural Lie algebroid structure over $X \times Y$. The anchor map of $\mathcal{L}_{X\times Y}$ is $\omega_{X\times Y} = \mathcal{L}_{X\times Y} \to \Theta_{X\times Y}$ determined by

$$\begin{array}{l} \forall (f,g) \in \mathcal{O}_X \times \mathcal{O}_Y,\ \ \forall (\xi,\zeta) \in \mathcal{L}_X \times \mathcal{L}_Y \\ \omega_{X\times Y}(\xi)(f \otimes g) = \omega_X(\xi)(f)g \\ \omega_{X\times Y}(\zeta)(f \otimes g) = f\omega_Y(\zeta)(g). \end{array}$$

On $\mathcal{L}_{X\times Y}$, we put the Lie bracket extending that of $\mathcal{L}_X$ and $\mathcal{L}_Y$ and satisfying the following relations $[p_1^{-1}\mathcal{L}_X, p_2^{-1}\mathcal{L}_Y] = 0$.

Similarly introduce

$$\mathcal{P}_{X\times Y} = \mathcal{O}_{X\times Y} \underset{p_1^{-1}\mathcal{O}_X}{\otimes} p_1^{-1}\Theta_X \oplus \mathcal{O}_{X\times Y} \underset{p_2^{-1}\mathcal{O}_Y}{\otimes} p_2^{-1}\mathcal{L}_Y.$$

Then $\mathcal{P}_{X\times Y}$ has a natural Lie algebroid structure over $X \times Y$ with anchor map $\overline{\omega}_{X\times Y}$.

Remark :

Let $\mathcal{L}_X$ be a Lie algebroid. As it is a locally free $\mathcal{O}_X$-module of finite rank, $\mathcal{D}(\mathcal{L}_X)$ has finite global homological dimension. Moreover, there exists an integer p such that every coherent $\mathcal{D}(\mathcal{L}_X)$-module $\mathcal{M}$ has locally a free resolution of length inferior or equal to p

$$0 \to \mathcal{D}(\mathcal{L}_V)^{l_p} \to \ldots \to \mathcal{D}(\mathcal{L}_V)^{l_0} \to \mathcal{M}_{|V}.$$

The proof of this statement in the $\mathcal{D}$-module case extends to Lie algebroids ([S3] p.14).

2.2 Left and right-modules

The following proposition is classical for $\mathcal{D}$-modules and is easy to generalize to Lie algebroids.

Proposition 2.2.1 *a) if $\mathcal{M}$ (respectively $\mathcal{N}$) is a right (respectively a left) $\mathcal{D}(\mathcal{L}_X)$-module, then $\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{N}$ endowed with the two following operations :*

$$\begin{array}{l} \forall a \in \mathcal{O}_X,\ \forall m \in \mathcal{M}, \forall n \in \mathcal{N}, \forall D \in \mathcal{L}_X \\ (m \otimes n) \cdot a = m \otimes a \cdot n = m \cdot a \otimes n \\ (m \otimes n) \cdot D = m \cdot D \otimes n - m \otimes D \cdot n \end{array}$$

is a right $\mathcal{D}(\mathcal{L}_X)$-module.

b) If $\mathcal{M}$ and $\mathcal{M}'$ are two right $\mathcal{D}(\mathcal{L}_X)$-modules, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}')$ endowed with the two following operations

$$\begin{array}{l} \forall \phi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}'),\ \forall m \in \mathcal{M}, \forall a \in \mathcal{O}_X, \forall D \in \mathcal{L}_X \\ (\phi \cdot a)(m) = \phi(m) \cdot a \\ (\phi \cdot D)(m) = \phi(m) \cdot D - \phi(m \cdot D) \end{array}$$

is a left $\mathcal{D}(\mathcal{L}_X)$-module.

The following theorem is now a consequence of the previous proposition.

Theorem 2.2.2 *Let $\mathcal{E}$ be right $\mathcal{D}(\mathcal{L}_X)$-module which is a locally free $\mathcal{O}_X$-module of rank one. The functor $\mathcal{N}^\bullet \mapsto \mathcal{E} \underset{\mathcal{O}_X}{\otimes} \mathcal{N}^\bullet$ establishes an equivalence of categories between complexes of left and complexes of right $\mathcal{D}(\mathcal{L}_X)$-modules. Its inverse functor is given by $\mathcal{M} \mapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M})$.*

It is a well known fact due to Bernstein that Ω_X is endowed with a right $\mathcal{D}_X$-module structure (see [S3] p.9, [Bo] p. 226 or [Sc]). Hence theorem 2.2.2 applies in particular if $\mathcal{E} = \Omega_X^x$. Put

$$\mathcal{L}_X^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_X, \mathcal{O}_X)$$

and let $d_{\mathcal{L}_X}$ be the rank of $\mathcal{L}_X$. Then one may take $\mathcal{E} = \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*)$. Indeed, $\mathcal{L}_X$ acts on $\Lambda(\mathcal{L}_X^*)$ by the adjoint action. The action of an element D of $\mathcal{L}_X$

on $\Lambda(\mathcal{L}_X^*)$ is called the Lie derivative of D and is denoted L_D. $\Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*)$, endowed with the following two operations,

$$\begin{aligned}
&\forall \sigma \in \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*),\ \forall D \in \mathcal{L}_X, \forall a \in \mathcal{O}_X \\
&\sigma \cdot a = a \cdot \sigma \\
&\sigma \cdot D = -L_D(\sigma)
\end{aligned}$$

is a right $\mathcal{D}(\mathcal{L}_X)$-module (see [C1]).

2.3 Resolution of $\mathcal{O}_X$ as a left $\mathcal{D}(\mathcal{L}_X)$-module.

Let $(\mathcal{L}_X, \omega_X)$ be a Lie algebroid over X. Set $\mathcal{L}_X^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_X, \mathcal{O}_X)$. Consider the graded left $\mathcal{D}(\mathcal{L}_X)$-module $\mathcal{D}(\mathcal{L}_X) \underset{\mathcal{O}_X}{\otimes} \wedge^\bullet(\mathcal{L}_X) = \underset{n}{\oplus} \mathcal{D}(\mathcal{L}_X) \underset{\mathcal{O}_X}{\otimes} \bigwedge^n \mathcal{L}_X$ where $\mathcal{D}(\mathcal{L}_X)$ acts by left multiplication. One can prove ([R] p 200) that there exists an endomorphism of degree -1 on this module such that: for all v in $\mathcal{D}(\mathcal{L}_X)$ and all ξ_i in $\mathcal{L}_X$, we have

$$\begin{aligned}
&d(v \otimes \xi_1 \wedge ... \wedge \xi_n) = \sum_{i=1}^{n} (-1)^{i-1} v\xi_i \otimes \xi_1 \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \xi_n \\
&+ \sum_{k<i} (-1)^{i+k} v \otimes [\xi_k, \xi_i] \wedge \xi_1 \wedge ... \wedge \widehat{\xi_k} \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \xi_n
\end{aligned}$$

where the notation $\hat{x}$ means that x is omitted. Moreover, we have

$$\forall v \in \mathcal{D}(\mathcal{L}_X), \ \ d(v \otimes 1) = \omega_X(v)(1).$$

Theorem 2.3.1 *Let $(\mathcal{L}_X, \omega_X)$ be Lie algebroid. The complex $P^\bullet$ defined by*

$$\forall n \in \mathbb{Z}, \ \ P^{-n} = \mathcal{D}(\mathcal{L}_X) \underset{\mathcal{O}_X}{\otimes} \Lambda^n(\mathcal{L}_X)$$

and the differential above is a resolution of $\mathcal{O}_X$ by locally free left $\mathcal{D}(\mathcal{L}_X)$-modules.

Proof of the theorem 2.3.1 : See [R] p 202.

Using this resolution, one gets the theorem ([C1])

Theorem 2.3.2 *Let $(\mathcal{L}_X, \omega_X)$ be a rank $d_{\mathcal{L}_X}$ Lie algebroid. Let $\mathcal{N}^\bullet$ be a complex of left $\mathcal{D}(\mathcal{L}_X)$-modules. Then*

$$R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X, \mathcal{N}) \simeq \left(\mathcal{N} \underset{\mathcal{O}_X}{\otimes} \Lambda^{d_{\mathcal{L}_X}} \mathcal{L}_X^*[d_{\mathcal{L}_X}] \right) \overset{L}{\underset{\mathcal{D}(\mathcal{L}_X)}{\otimes}} \mathcal{O}_X.$$

2.4 Lie algebroid morphisms

Definition 2.4.1 *Let $(\mathcal{L}_X, \omega_X)$ (respectively $(\mathcal{L}_Y, \omega_Y)$) be a Lie algebroid over X (respectively Y). A morphism Φ from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$ is a pair (f, F) such that*

- *$f : X \to Y$ is a holomorphic map*
- *$F : \mathcal{L}_X \to f^*\mathcal{L}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{L}_Y$ such that the two following conditions are satisfied:*

  1) The diagram

$$\begin{array}{ccc} \mathcal{L}_X & \xrightarrow{\psi} & f^*\mathcal{L}_Y \\ \omega_X \downarrow & & \downarrow f^*\omega_Y \\ \Theta_X & \xrightarrow[f^*]{} & \Theta_Y \end{array}$$

  commutes.

  2) $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} (f^{-1}\mathcal{D}_Y)$ endowed with the two following operations

$$\forall (a, b) \in \mathcal{O}_X^2,\ \forall \xi \in \mathcal{L}_X,\ \forall v \in f^{-1}\mathcal{D}_Y$$

$$\begin{aligned} a \cdot (b \otimes v) &= ab \otimes v \\ \xi \cdot (b \otimes v) &= \omega_X(\xi)(b) \otimes v + \textstyle\sum_i ba_i \otimes \xi_i v \end{aligned}$$

  (where $\psi(\xi) = \sum_i a_i \otimes \xi_i$ with a_i in $\mathcal{O}_X$ and ξ_i in $f^{-1}\mathcal{L}_Y$) is a left $\mathcal{D}(\mathcal{L}_X)$-module.

Note that condition 2) is equivalent to the following more explicit condition. Let ξ and η be two elements of $\mathcal{L}_X^2$. Put $F(\xi) = \sum_{i=1}^{m} a_i \otimes \xi_i$ and $F(\eta) = \sum_{j=1}^{m} b_j \otimes \eta_j$, then

$$F([\xi, \eta]) = \sum_{j=1}^{n} \omega_X(\xi)(b_j) \otimes \eta_j - \sum_{i=1}^{n} \omega_X(\eta)(a_i) \otimes \xi_i + \sum_{i,j} a_i b_j \otimes [\xi_i, \eta_j].$$

Our definition coincides with that of Almeida and Kumpera ([A-K]) and generalizes the point of view of [C2].

Example of Lie algebroid morphisms

1) Our definition generalizes at the same time Lie algebra morphisms and morphisms between complex analytic manifolds.

2) Let $X|S$ and $Y|S$ be two relative analytic manifolds over S. A morphism $f : X|S \to Y|S$ of relative analytic manifolds is an analytic map $f : X \to Y$ such that $\epsilon_Y \circ f = \epsilon_X$. One can check easily that a morphism of relative analytic manifolds gives rise to a Lie algebroid morphism $\Theta_{X|S} \to \Theta_{Y|S}$.

3) Let X be a Poisson analytic manifold and let (Ω^1_X, ω_X) be the Lie algebroid it defines. Let Y be an analytic manifold and let $f : X \to Y$ be an analytic map. The composition

$$\Omega^1_X \xrightarrow{\omega_X} \Theta_X \xrightarrow{Tf} f^*\Theta_Y$$

defines a Lie algebroid morphism from (Ω^1_X, ω_X) to (Θ_Y, id).

4) Let G and G' be two complex Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ and let $\chi : G \to G'$ be a Lie group morphism. We will also denote by χ the differential of χ at the unity. Let X (respectively X') be an analytic manifold with action of G (respectively G'). Let $f : X \to X'$ be an equivariant map in the sense that

$$\forall g \in G,\ \forall x \in X,\ \ f(g \cdot x) = \chi(g) \cdot f(x).$$

Let

$$\begin{array}{rcl} F : \mathcal{O}_X \otimes \mathfrak{g} & \to & \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_{X'}} f^{-1}(\mathcal{O}_{X'} \otimes \mathfrak{g}) \\ f \otimes \xi & \mapsto & f \otimes 1 \otimes \chi(\xi) \end{array}$$

Then (f, F) is a Lie algebroid morphism from $\mathcal{O}_X \otimes \mathfrak{g}$ to $\mathcal{O}_{X'} \otimes \mathfrak{g}'$.

3 Some operations for modules on Lie algebroids

3.1 Direct images

Let $\Phi = (f, F)$ be a Lie algebroid morphism from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$. Then $\mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{D}(\mathcal{L}_Y)$ has a $(\mathcal{D}(\mathcal{L}_X) - f^{-1}\mathcal{D}(\mathcal{L}_Y)^{op})$- bimodule structure. It is called the transfer module of Φ and is denoted $\mathcal{D}_{(\mathcal{L}_X,\omega_X)\to(\mathcal{L}_Y,\omega_Y)}$ or $\mathcal{D}_{\mathcal{L}_X\to\mathcal{L}_Y}$ if there is no ambiguity on the anchor maps.

Let $\mathcal{M}^\bullet$ be an object of $D^b(\mathcal{D}(\mathcal{L}_X)^{op})$. Set

$$\underline{\Phi}_!(\mathcal{M}^\bullet) = Rf_!\left(\mathcal{M}^\bullet \underset{\mathcal{D}(\mathcal{L}_X)}{\overset{L}{\otimes}} \mathcal{D}_{\mathcal{L}_X\to\mathcal{L}_Y}\right).$$

Then $\underline{\Phi}_!(\mathcal{M}^\bullet)$ is in $D^b(\mathcal{D}(\mathcal{L}_Y)^{op})$. It is called the direct image of $\mathcal{M}^\bullet$ by Φ. If $\Phi = (f, Tf)$, we recover the $\mathcal{D}$-module construction (see [S3] for example). Then $\mathcal{D}_{\Theta_X\to\Theta_Y}$ is denoted $\mathcal{D}_{X\to Y}$ and $\underline{\Phi}_!$ is denoted $\underline{f}_!$.

Proposition 3.1.1 *Let Φ (respectively Ψ) be a Lie algebroid morphism from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$ (respectively from $(\mathcal{L}_Y, \omega_Y)$ to $(\mathcal{L}_Z, \omega_Z)$), then*

$$\underline{\Psi}_! \circ \underline{\Phi}_! = \underline{\Psi \circ \Phi}_!.$$

The proof of proposition 3.1.1 is analog to the $\mathcal{D}$-module case (see [Bo] p. 251).

A right coherent $\mathcal{D}(\mathcal{L}_X)$-module $\mathcal{M}$ is generated by a coherent $\mathcal{O}_X$-module if $\mathcal{M}$ has a coherent $\mathcal{O}_X$-submodule $\mathcal{C}$ for which the natural morphism $\mathcal{C} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}(\mathcal{L}_X) \to \mathcal{M}$ is an epimorphism.

We recall here a definition due to Kashiwara (see [S-S]).

Definition 3.1.2 *A right coherent $\mathcal{D}(\mathcal{L}_X)$-module is good if, for any compact subset K of X, there exists an open neighborhood U of K such that $\mathcal{M}_{|U}$ has a filtration $(\mathcal{M}_k)_{k\in[1,n]}$ by coherent right $\mathcal{D}(\mathcal{L}_U)$-submodules such that each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$ is generated by a coherent $\mathcal{O}_U$-module.*

Note that if X is a smooth algebraic variety, all the coherent $\mathcal{D}(\mathcal{L}_X)$-modules are good. Good $\mathcal{D}(\mathcal{L}_X)$-modules form a thick subcategory of the category of coherent $\mathcal{D}(\mathcal{L}_X)$-modules. The associated full subcategory of $D^b(\mathcal{D}(\mathcal{L}_X)^{op})$ consisting in objects with good cohomology is denoted by $D^b_{good}(\mathcal{D}(\mathcal{L}_Y)^{op})$

Theorem 3.1.3 *Assume that $\mathcal{M}^\bullet$ is in $D^b_{good}(\mathcal{D}(\mathcal{L}_X)^{op})$ and that f is proper on Supp($\mathcal{M}$), then $\underline{\Phi}_!(\mathcal{M})$ is in $D^b_{good}(\mathcal{D}(\mathcal{L}_Y)^{op})$.*

The proof of Schneiders ([S3] p. 38) in the case of $\mathcal{D}$-modules extends without any change to our situation. The particular case where f is projective and $\mathcal{M}$ has a global good filtration was treated in [Ka].

To any right $\mathcal{D}(\mathcal{L}_X)$-module, one can associate canonically the right $(\mathcal{D}(\mathcal{L}_X) - \mathcal{D}(\mathcal{L}_X))$-bimodule $\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}(\mathcal{L}_X)$ (the first right $\mathcal{D}(\mathcal{L}_X)$-module structure is given by right multiplication, the second one is obtained by proposition 2.2.1) . As particular cases, we define the two following complexes of right $(\mathcal{D}(\mathcal{L}_X) - \mathcal{D}(\mathcal{L}_X))$-bimodules

$$\mathcal{K}_{\mathcal{L}_X} = \Omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}(\mathcal{L}_X)[x]$$

and if $\mathcal{L}_X$ is of rank $d_{\mathcal{L}_X}$

$$\mathcal{H}_{\mathcal{L}_X} = \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}^*_X) \underset{\mathcal{O}_X}{\otimes} \mathcal{D}(\mathcal{L}_X)[d_{\mathcal{L}_X}].$$

Define

$$\mathcal{D}^2_{\mathcal{L}_X \to \mathcal{L}_Y} = \mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_Y} \underset{\mathbb{C}}{\otimes} \mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_Y}$$

as a left $\mathcal{D}(\mathcal{L}_X) - \mathcal{D}(\mathcal{L}_X)$ bimodule.

Let $\mathcal{N}^\bullet$ be an element of $D^b(\mathcal{D}(\mathcal{L}_X)^{op} - \mathcal{D}(\mathcal{L}_X)^{op})$, the bounded derived category of complexes of right $(\mathcal{D}(\mathcal{L}_X) - \mathcal{D}(\mathcal{L}_X))$ -bimodules. We define its direct image by Φ:

$$\underline{\underline{\Phi}}_!(\mathcal{N}^\bullet) = Rf_!\left(\mathcal{N}^\bullet \overset{L}{\underset{\mathcal{D}(\mathcal{L}_X)\otimes\mathcal{D}(\mathcal{L}_X)}{\otimes}} \mathcal{D}^2_{\mathcal{L}_X \to \mathcal{L}_Y}\right).$$

$\underline{\underline{\Phi}}_!(\mathcal{N}^\bullet)$ is in $D^b(\mathcal{D}(\mathcal{L}_Y)^{op} - \mathcal{D}(\mathcal{L}_Y)^{op})$.

Proposition 3.1.4 *Let $\mathcal{M}$ be a right $\mathcal{D}(\mathcal{L}_X)$-module and let $\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}(\mathcal{L}_X)$ be the $(\mathcal{D}(\mathcal{L}_X) - \mathcal{D}(\mathcal{L}_X))$-bimodule which is canonically associated to it. Let*

$\Phi = (f, F)$ *be a Lie algebroid morphism from* $(\mathcal{L}_X, \omega_X)$ *to* $(\mathcal{L}_Y, \omega_Y)$. *One has a canonical isomorphism in* $D^b\left(\mathcal{D}(\mathcal{L}_Y)^{op} - \mathcal{D}(\mathcal{L}_Y)^{op}\right)$

$$\underline{\underline{\Phi}}_!\left(\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}(\mathcal{L}_X)\right) \to \underline{\underline{\Phi}}_!(\mathcal{M}) \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}(\mathcal{L}_Y).$$

For the proof of this proposition, we refer to [S-S] p. 33.

3.2 Duality functor

Let $\mathcal{M}^\bullet$ be an element of $D^b_{coh}\left(\mathcal{D}(\mathcal{L}_X)^{op}\right)$, then we set

$$\underline{D}_{\mathcal{L}_X}(\mathcal{M}^\bullet) = R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{M}^\bullet, \mathcal{H}_{\mathcal{L}_X})$$

and if the rank of $\mathcal{L}_X$ is $d_{\mathcal{L}_X}$,

$$\underline{\Delta}_{\mathcal{L}_X}(\mathcal{M}^\bullet) = R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{M}^\bullet, \mathcal{K}_{\mathcal{L}_X}).$$

There is an involution exchanging the two right module structures of $\mathcal{K}_{\mathcal{L}_X}$ and $\mathcal{H}_{\mathcal{L}_X}$ (see [S-S] lemma 2.3) so that it does not matter which one we use for the $R\mathcal{H}om$. The natural arrow $\mathcal{M}^\bullet \mapsto \underline{D}_{\mathcal{L}_X}\left(\underline{D}_{\mathcal{L}_X}(\mathcal{M}^\bullet)\right)$ (respectively $\mathcal{M}^\bullet \mapsto \underline{\Delta}_{\mathcal{L}_X}\left(\underline{\Delta}_{\mathcal{L}_X}(\mathcal{M}^\bullet)\right)$) is an isomorphism. That is why $\underline{D}_{\mathcal{L}_X}$ and $\underline{\Delta}_{\mathcal{L}_X}$ are called duality functors.

Proposition 3.2.1 *Assume that* $\mathcal{L}_X$ *is of rank* $d_{\mathcal{L}_X}$. *If* $\mathcal{M}$ *is a right* $\mathcal{D}(\mathcal{L}_X)$*-module which is locally free of finite rank as* $\mathcal{O}_X$*-module, then*

$$\underline{D}_{\mathcal{L}_X}(\mathcal{M}) = \mathcal{H}om\left(\mathcal{M}, \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*)\right) \underset{\mathcal{O}_X}{\otimes} \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*)$$
$$\underline{\Delta}_{\mathcal{L}_X}(\mathcal{M}) = \mathcal{H}om(\mathcal{M}, \Omega_X) \underset{\mathcal{O}_X}{\otimes} \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*)[x - d_{\mathcal{L}_X}]$$

This proposition is well known for $\mathcal{D}$-modules (see [Ho] p. 93).

Proof of the proposition 3.2.1

Put $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_X}\left(\Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*), \mathcal{M}\right)$. We know from theorem 2.2.2 that $\mathcal{N}$ is a left $\mathcal{D}(\mathcal{L}_X)$-module. We have the following isomorphism of right $\mathcal{D}(\mathcal{L}_X)$-modules

$$R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}\left(\mathcal{M}, \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*) \underset{\mathcal{O}_X}{\otimes} \mathcal{D}(\mathcal{L}_X)\right) \simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{N}, \mathcal{D}(\mathcal{L}_X))$$

where the first $R\mathcal{H}om$ is taken over right $\mathcal{D}(\mathcal{L}_X)$- modules and the second one is taken over left $\mathcal{D}(\mathcal{L}_X)$. We need to show that $R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{N}, \mathcal{D}(\mathcal{L}_X))$ and $\mathcal{N}^* \underset{\mathcal{O}_X}{\otimes} \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*)[-d_{\mathcal{L}_X}]$ are quasi-isomorphic as complexes of right $\mathcal{D}(\mathcal{L}_X)$-modules.

Lemma 3.2.2 *Let $\mathcal{N}$ be a left $\mathcal{D}(\mathcal{L}_X)$-module which is a locally free $\mathcal{O}_X$-module of finite rank, then $\mathcal{D}(\mathcal{L}_X) \underset{\mathcal{O}_X}{\otimes} \mathcal{N}$ is a locally free $\mathcal{D}(\mathcal{L}_X)$-module of finite rank.*

Proof of the lemma 3.2.2:

Let U be an open subset of X such that $\mathcal{N}_{|U}$ is a free $\mathcal{O}_U$-module with basis $(v_1, \ldots, v_n)$. Let us denote by ${}^l\mathcal{D}(\mathcal{L}_U) \underset{\mathcal{O}_U}{\otimes} \mathcal{N}_{|U}$ the left $\mathcal{D}(\mathcal{L}_U)$-module given by left multiplication. Let us consider the $\mathcal{D}(\mathcal{L}_U)$-module morphism χ

$$\begin{array}{rcl} {}^l\mathcal{D}(\mathcal{L}_U) \underset{\mathcal{O}_U}{\otimes} \mathcal{N}_{|U} & \to & \mathcal{D}(\mathcal{L}_U) \underset{\mathcal{O}_U}{\otimes} \mathcal{N}_{|U} \\ \sum_i P_i \otimes v_i & \mapsto & \sum_i P_i \cdot (1 \otimes v_i). \end{array}$$

$\mathcal{D}(\mathcal{L}_U) \underset{\mathcal{O}_U}{\otimes} \mathcal{N}_{|U}$ is filtered as follows:

$$\mathcal{F}_n \left(\mathcal{D}(\mathcal{L}_U) \underset{\mathcal{O}_U}{\otimes} \mathcal{N}_{|U} \right)_n = \mathcal{D}(\mathcal{L}_U)_n \underset{\mathcal{O}_U}{\otimes} \mathcal{N}_{|U}.$$

It is a filtration over $\mathbb{N}$ and $Gr\chi = id$. So χ is an isomorphism. This finishes the proof of the lemma. As a consequence, we get the following corollary :

Corollary 3.2.3 *The complex $P^\bullet \underset{\mathcal{O}_X}{\otimes} \mathcal{N}$ is a resolution of $\mathcal{N}$ by locally free $\mathcal{D}(\mathcal{L}_X)$-modules.*

Hence, the complex $R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{N}, \mathcal{D}(\mathcal{L}_X))$ is quasi isomorphic to the complex $\left(\mathcal{H}om_{\mathcal{O}_X} \left(\bigwedge^\bullet \mathcal{L}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{N}, \mathcal{D}(\mathcal{L}_X) \right), \delta \right)$ where δ is given by

$$\forall \phi \in \mathcal{H}om_{\mathcal{O}_X} \left(\bigwedge^p \mathcal{L}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{N}, \mathcal{D}(\mathcal{L}_X) \right), \forall (\xi_1, \ldots, \xi_p) \in \mathcal{L}_X, \forall v \in \mathcal{N}$$

$$
\begin{aligned}
\delta\phi\left(\xi_1 \wedge \ldots \wedge \xi_p \otimes v\right) &= \sum_{i<j}(-1)^{i+j}\phi\left([\xi_i,\xi_j] \wedge \ldots \wedge \widehat{\xi_i} \wedge \ldots \wedge \widehat{\xi_j} \wedge \ldots \xi_p\right) \\
&+ \sum_i (-1)^{i-1}\xi_i\phi\left(\xi_1 \wedge \ldots \wedge \widehat{\xi_i} \wedge \ldots \wedge \xi_p \otimes v\right) \\
&- \sum_i (-1)^{i-1}\phi\left(\xi_1 \wedge \ldots \wedge \widehat{\xi_i} \wedge \ldots \wedge \xi_p \otimes \xi_i v\right)
\end{aligned}
$$

For the study of the complex $\left(\mathcal{H}om_{\mathcal{O}_X}\left(\bigwedge^{\bullet}\mathcal{L}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{N}, \mathcal{D}(\mathcal{L}_X)\right), \delta\right)$ and the end of the proof, we refer the reader to [C1] (theorem 5.4.1).

4 The Trace morphism

Let $\Phi = (f, F)$ be a Lie algebroid morphism from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$. We will factorize it as follows $\Phi = \Pi \circ \Upsilon_2 \circ \Upsilon_1$ where

- $\Upsilon_1 = (u_1, U_1)$ is the Lie algebroid morphism from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_{X\times Y}, \omega_{X\times Y})$ defined by

$$
\begin{aligned}
u_1 \,:\, X &\rightarrow X \times Y \\
x &\mapsto (x, f(x)) \\
U_1 \,:\, \mathcal{L}_X &\rightarrow \mathcal{O}_X \underset{p_1^{-1}\mathcal{O}_{X\times Y}}{\otimes} p_1^{-1}\mathcal{L}_{X\times Y} \simeq \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{L}_Y \oplus \mathcal{L}_X \\
D &\mapsto F(D) + D
\end{aligned}
$$

- $\Upsilon_2 = (id, U_2)$ is the Lie algebroid morphism from $(\mathcal{L}_{X\times Y}, \omega_{X\times Y})$ to $(\mathcal{P}_{X\times Y}, \overline{\omega}_{X\times Y})$ defined by

$$\forall D \in p_1^{-1}\mathcal{L}_X,\ \forall \Delta \in p_2^{-1}\mathcal{L}_Y,\ U_2(1 \otimes D) = \omega_{X\times Y}(1 \otimes D),\ \ U_2(1 \otimes \Delta) = 1 \otimes \Delta$$

- $\Pi = (p_2, \pi)$ is the Lie algebroid morphism from $(\mathcal{P}_{X\times Y}, \overline{\omega}_{X\times Y})$ to $(\mathcal{L}_Y, \omega_Y)$ defined by the natural projections.

Let X and Y be two complex analytic manifolds of dimension x and y respectively and $f : X \rightarrow Y$ an analytic map. Schneiders constructed a canonical arrow in $D^b(\mathcal{D}_Y^{op})$

$$\underline{f_!}(\Omega_X[x]) \rightarrow \Omega_Y[y].$$

We refer the reader to [S1], [S3] or [S-S] for details.

In the next two paragraphs we will define a trace morphism for the Lie algebroid morphisms Υ_1 and Π.

4.1 A trace morphism for Υ_1

Assume that we are in the case of $\Upsilon_1 = (u_1, U_1)$. Then we will construct an arrow from $\underline{\Upsilon}_{1!}(\Omega_X[x])$ to $\Omega_{X\times Y}[x+y]$ using the two following lemmas.

Lemma 4.1.1 *Let $\Phi = (f, F)$ be a Lie algebroid map from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$. There exists a natural right $f^{-1}(\mathcal{D}(\mathcal{L}_Y))$ -linear morphism from $\Omega_X[x] \underset{\mathcal{D}(\mathcal{L}_X)}{\otimes} \mathcal{D}_{\mathcal{L}_X\to\mathcal{L}_Y}$ to $\Omega_X[x] \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y}$.*

Proof of the lemma 4.1.1

It is not hard to see, using the definition of a Lie algebroid morphism, that the map

$$\begin{array}{rcl} \Omega_X[x] \underset{\mathcal{D}(\mathcal{L}_X)}{\otimes} \mathcal{D}_{\mathcal{L}_X\to\mathcal{L}_Y} & \to & \Omega_X[x] \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y} \\ \sigma \otimes (1 \otimes v) & \mapsto & \sigma \otimes (1 \otimes \omega_Y(v)) \end{array}$$

is well defined. Moreover, it is clearly a right $f^{-1}\mathcal{D}(\mathcal{L}_Y)$ -module morphism.

Lemma 4.1.2 *$\mathcal{D}_{\mathcal{L}_X\to\mathcal{L}_{X\times Y}}$ is a locally free left $\mathcal{D}(\mathcal{L}_X)$-module.*

Proof of the lemma 4.1.2 :

Let us first remark the following isomorphism

$$\mathcal{D}_{\mathcal{L}_X\to\mathcal{L}_{X\times Y}} \simeq \mathcal{D}(\mathcal{L}_X) \underset{\mathcal{O}_X}{\otimes} \left(\mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{D}(\mathcal{L}_Y) \right)$$

where the tensor product concerns the left $\mathcal{O}_X$-module structure of $\mathcal{D}(\mathcal{L}_X)$. It is a left $\mathcal{D}(\mathcal{L}_X)$-module isomorphism if an element ξ of $\mathcal{L}_X$ such that $F(\xi) = \sum_i a_i \otimes \xi_i$ acts on the right hand side as follows.

$$\begin{array}{l} \forall u \in \mathcal{D}(\mathcal{L}_X), \forall v \in f^{-1}\mathcal{D}(\mathcal{L}_Y), \\ \xi \cdot (u \otimes 1 \otimes v) = \xi u \otimes 1 \otimes v + \sum_i a_i u \otimes 1 \otimes \xi_i v. \end{array}$$

Let now V be an open subset of Y such that $\mathcal{L}_V$ is an $\mathcal{O}_V$ free module with basis $(\partial_1, \ldots, \partial_n)$. Let W be an open subset of X such that $f(W) \subset V$.

Endow $\underset{\alpha_1,\dots,\alpha_n}{\oplus} \mathcal{D}(\mathcal{L}_W) \otimes \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ with left multiplication. Then let us show that the following left $\mathcal{D}(\mathcal{L}_W)$-module morphism χ

$$\begin{array}{ccc} \underset{\alpha_1,\dots,\alpha_n}{\oplus} \mathcal{D}(\mathcal{L}_W) \otimes \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} & \to & \mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_{X \times Y|W}} \\ \sum\limits_{\alpha_1,\dots,\alpha_n} v_{\alpha_1,\dots,\alpha_n} \otimes \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} & \to & \sum\limits_{\alpha_1,\dots,\alpha_n} v_{\alpha_1,\dots,\alpha_n} \cdot (1 \otimes \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}). \end{array}$$

is an isomorphism. Filter these two sheaves of algebras as follows:

$$\mathcal{F}_n \mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_{X \times Y|W}} = \mathcal{D}(\mathcal{L}_W)_n \underset{\mathcal{O}_W}{\otimes} \left(\mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{D}(\mathcal{L}_Y) \right)_{|W}$$

$$\mathcal{F}_n \left(\sum_{\alpha_1,\dots,\alpha_n} \mathcal{D}(\mathcal{L}_W) \otimes \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} \right) = \sum_{\alpha_1,\dots,\alpha_n} \mathcal{D}(\mathcal{L}_W)_n \otimes \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}.$$

χ is a filtered morphism over $\mathbb{N}$. Its graded associated morphism is identity. So χ is an isomorphism.

Proposition 4.1.3 *There is a natural morphism in $\mathcal{D}^b_{coh}(\mathcal{D}(\mathcal{L}_{X \times Y})^{op})$ from $\underline{\Upsilon_1}_!(\Omega_X[x])$ to $\Omega_{X \times Y}[x+y]$*

Proof of the proposition 4.1.3:

Consider the chain of morphisms

$$\begin{array}{rcl} \underline{\Upsilon_1}_!(\Omega_X[x]) & = & Ru_! \left(\Omega_X[x] \overset{L}{\underset{\mathcal{D}(\mathcal{L}_X)}{\otimes}} \mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_{X \times Y}} \right) \\ & \simeq & Ru_! \left(\Omega_X[x] \underset{\mathcal{D}(\mathcal{L}_X)}{\otimes} \mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_{X \times Y}} \right) \\ & \to & Ru_! \left(\Omega_X[x] \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X \to X \times Y} \right) \\ & \simeq & Ru_! \left(\Omega_X[x] \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{D}_{X \to X \times Y} \right) \\ & \to & \Omega_{X \times Y}[x+y] \end{array}$$

where the second isomorphism is due to the fact that $\mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_{X \times Y}}$ is a flat $\mathcal{D}(\mathcal{L}_X)$-left module and the first arrow is due to the previous lemma. In the last isomorphism, we used the fact that u is a closed immersion and the last arrow is the classical trace morphism.

Corollary 4.1.4 *There is a natural morphism in* $D^b\left(\mathcal{D}(\mathcal{L}_{X\times Y})^{op} - \mathcal{D}(\mathcal{L}_{X\times Y})^{op}\right)$*-bimodules from* $\underline{\underline{\Phi}}_!\left(\mathcal{K}_{\mathcal{L}_X}\right)$ *to* $\mathcal{K}_{\mathcal{L}_{X\times Y}}$

Proof of the corollary 4.1.4

The following sequence of morphisms provides the morphism we are looking for:

$$\begin{array}{rcl} \underline{\underline{\Phi}}_!\left(\mathcal{K}_{\mathcal{L}_X}\right) & = & \underline{\underline{\Phi}}_!\left(\Omega_X[x] \underset{\mathcal{O}_X}{\otimes} \mathcal{D}(\mathcal{L}_X)\right) \\ & \xrightarrow{\sim} & \underline{\underline{\Phi}}_!\left(\Omega_X[x]\right) \underset{\mathcal{O}_{X\times Y}}{\otimes} \mathcal{D}(\mathcal{L}_{X\times Y}) \\ & \rightarrow & \Omega_{X\times Y}[x+y] \underset{\mathcal{O}_{X\times Y}}{\otimes} \mathcal{D}(\mathcal{L}_{X\times Y}). \end{array}$$

Note that the first arrow is due to 3.1.4 and the last one is the previous proposition.

4.2 The case of the projection Π

Proposition 4.2.1 *There is a natural morphism in* $D^b(\mathcal{D}(\mathcal{L}_Y)^{op})$ *from* $\underline{\Pi}_!\left(\Omega_{X\times Y}[x+y]\right)$ *to* $\Omega_Y[y]$.

Proof of the proposition 4.2.1:

Let p_1 be the projection of $X \times Y$ onto the first factor.

$$\begin{array}{l} \underline{\Pi}_!\left(\Omega_{X\times Y}[x+y]\right) \\ = Rp_{2!}\left(\Omega_{X\times Y}[x+y] \overset{L}{\underset{\mathcal{D}(\mathcal{P}_{X\times Y})}{\otimes}} \left(\mathcal{O}_{X\times Y} \underset{p_2^{-1}\mathcal{O}_Y}{\otimes} p_2^{-1}\mathcal{D}(\mathcal{L}_Y)\right)\right)[x+y] \\ = Rp_{2!}\left(\mathcal{O}_{X\times Y} \underset{p^{-1}\mathcal{O}_X \otimes p_2^{-1}\mathcal{O}_Y}{\otimes} \left(p_1^{-1}\left(\Omega_X[x] \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X\right) \otimes p_2^{-1}\Omega_Y[y]\right)\right) \\ = Rp_{2!}\left(\Omega_{X\times Y}[x+y] \overset{L}{\underset{\mathcal{D}_{X\times Y}}{\otimes}} \left(\mathcal{O}_{X\times Y} \underset{p_2^{-1}\mathcal{O}_Y}{\otimes} p_2^{-1}\mathcal{D}_Y\right)\right)[x+y] \\ \rightarrow \Omega_Y[y]. \end{array}$$

where the arrow is the classical trace morphism.

Corollary 4.2.2 *There is a natural morphism in* $D^b\left(\mathcal{D}\left(\mathcal{L}_Y\right)^{op} - \mathcal{D}\left(\mathcal{L}_Y\right)^{op}\right)$ *from* $\underline{\underline{\Pi}}_!\left(\mathcal{K}_{\mathcal{P}_{X\times Y}}\right)$ *to* $\mathcal{K}_{\mathcal{L}_Y}$

The proof of the corollary 4.2.2 is the same as the proof of the corollary 4.1.4.

Remark : In the case of a relative analytic map, we recover the trace morphism constructed in [S1] (see also in [S-S]).

4.3 A duality morphism

Theorem 4.3.1 *Let X and Y be two complex manifolds of dimension x and y respectively. Let $(\mathcal{L}_X, \omega_X)$ (respectively $(\mathcal{L}_Y, \omega_Y)$) be a Lie algebroid over X (respectively Y). Let $\Phi = (f, F)$ be a Lie algebroid morphism from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$. Let $\mathcal{M}^\bullet$ be an element of $D^b_{good}(\mathcal{D}(\mathcal{L}_X)^{op})$ such that f is proper on the support of $\mathcal{M}^\bullet$. There is a functorial isomorphism from $\underline{\Phi}_! \Delta_{\mathcal{L}_X}(\mathcal{M}^\bullet)$ to $\Delta_{\mathcal{L}_Y} \underline{\Phi}_!(\mathcal{M}^\bullet)$ in $D^b_{good}(\mathcal{D}(\mathcal{L}_Y)^{op})$*

Proof of the theorem 4.3.1:

We will make use of the three following lemmas. In the lemmas 4.3.2 and 4.3.3, the notations are the same as in the previous section.

Lemma 4.3.2 *Let $\mathcal{M}^\bullet$ be an object of $D^b_{coh}(\mathcal{D}(\mathcal{L}_X)^{op})$. There is a canonical isomorphism from $\underline{\Upsilon}_{1!} \Delta_{\mathcal{L}_X}(\mathcal{M}^\bullet)$ to $\Delta_{\mathcal{L}_{X\times Y}} \underline{\Upsilon}_{1!}(\mathcal{M}^\bullet)$ in $D^b_{coh}(\mathcal{D}(\mathcal{L}_{X\times Y})^{op})$*

Proof of the lemma 4.3.2

We start by constructing a morphism from $\underline{\Upsilon}_{1!} \Delta_{\mathcal{L}_X}(\mathcal{M}^\bullet)$ to $\Delta_{\mathcal{L}_{X\times Y}} \underline{\Upsilon}_{1!}(\mathcal{M}^\bullet)$. Our construction will be analog to [S-S].

$$\begin{aligned}
&\underline{\Upsilon}_{1!} \Delta_{\mathcal{L}_X}(\mathcal{M}^\bullet) \\
&= Ru_! \left(R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{M}^\bullet, \mathcal{K}_{\mathcal{L}_X}) \overset{L}{\underset{\mathcal{D}(\mathcal{L}_X)}{\otimes}} \mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_{X\times Y}} \right) \\
&\xrightarrow{\sim} Ru_! \left(R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)} \left(\mathcal{M}^\bullet, \mathcal{K}_{\mathcal{L}_X} \overset{L}{\underset{\mathcal{D}(\mathcal{L}_X)}{\otimes}} \mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_{X\times Y}} \right) \right) \\
&\to Ru_! \left(R\mathcal{H}om_{u^{-1}\mathcal{D}(\mathcal{L}_{X\times Y})} \left(\mathcal{M}^\bullet \overset{L}{\underset{\mathcal{D}(\mathcal{L}_X)}{\otimes}} \mathcal{D}_{\mathcal{L}_X \to \mathcal{L}_{X\times Y}}, \mathcal{K}_{\mathcal{L}_X} \overset{L}{\underset{\mathcal{D}(\mathcal{L}_X) \otimes \mathcal{D}(\mathcal{L}_X)}{\otimes}} \mathcal{D}^2_{\mathcal{L}_X \to \mathcal{L}_{X\times Y}} \right) \right) \\
&\to R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_{X\times Y})} \left(\underline{\Upsilon}_{1!}(\mathcal{M}^\bullet), \underline{\underline{\Upsilon}}_{1!}(\mathcal{K}_{\mathcal{L}_X}) \right) \\
&\to R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_{X\times Y})} \left(\underline{\Upsilon}_{1!}(\mathcal{M}^\bullet), \mathcal{K}_{\mathcal{L}_{X\times Y}} \right)
\end{aligned}$$

where the first arrow is due to the coherence of $\mathcal{M}^\bullet$ and the last arrow comes from 4.1.4. To show that this morphism is an isomorphism, we proceed as in [S-S] (p. 36).

Lemma 4.3.3 *Let Z be an analytic complex manifold. Let $\mathcal{L}_Z$ and $\mathcal{P}_Z$ be two Lie algebroids over Z. Let $\Upsilon_2 = (id, U_2)$ be a Lie algebroid morphism from $\mathcal{L}_Z$ to $\mathcal{P}_Z$. Let $\mathcal{M}^\bullet$ be an element of $D^b_{coh}(\mathcal{D}(\mathcal{L}_Z)^{op})$. The complexes $\underline{\Upsilon}_{2!}\Delta_{\mathcal{L}_Z}(\mathcal{M}^\bullet)$ and $\Delta_{\mathcal{P}_Z}\underline{\Upsilon}_{2!}(\mathcal{M}^\bullet)$ are functorially isomorphic in $D^b_{coh}(\mathcal{D}(\mathcal{P}_Z)^{op})$.*

Proof of the lemma 4.3.3:
On one hand, we have the following chain of isomorphisms:

$$\begin{aligned}
\underline{\Upsilon}_{2!}\Delta_{\mathcal{L}_Z}(\mathcal{M}^\bullet) &= R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_Z)}(\mathcal{M}^\bullet, \mathcal{K}_{\mathcal{L}_Z}) \overset{L}{\underset{\mathcal{D}(\mathcal{L}_Z)}{\otimes}} \mathcal{D}(\mathcal{P}_Z) \\
&\simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_Z)}\left(\mathcal{M}^\bullet, \mathcal{K}_{\mathcal{L}_Z} \overset{L}{\underset{\mathcal{D}(\mathcal{L}_Z)}{\otimes}} \mathcal{D}(\mathcal{P}_Z)\right) \\
&\simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_Z)}\left(\mathcal{M}^\bullet, \Omega_Z \underset{\mathcal{O}_Z}{\otimes} \mathcal{D}(\mathcal{P}_Z)\right)
\end{aligned}$$

where the first isomorphism is due to the fact that $\mathcal{M}^\bullet$ is with coherent cohomology. On another hand, it is easy to see that

$$\Delta_{\mathcal{P}_Z}\Upsilon_2(\mathcal{M}^\bullet) \simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{P}_Z)}\left(\mathcal{M}^\bullet, \Omega_Z \underset{\mathcal{O}_Z}{\otimes} \mathcal{D}(\mathcal{P}_Z)\right).$$

This finishes the proof of the lemma 4.3.3.

Lemma 4.3.4 *Let $\mathcal{M}^\bullet$ be an element of $D^b_{good}(\mathcal{D}(\mathcal{P}_{X\times Y}))$ such that p_2 is proper on the support of $\mathcal{M}^\bullet$. There is a canonical isomorphism from $\underline{\Pi}_!\Delta_{\mathcal{P}_{X\times Y}}(\mathcal{M}^\bullet)$ to $\Delta_{\mathcal{L}_Y}\underline{\Pi}_!(\mathcal{M}^\bullet)$ in $D^b_{good}(\mathcal{D}(\mathcal{L}_Y))$.*

Proof of the lemma 4.3.4

To construct our morphism, we proceed as in the previous lemma. To show that the morphism constructed is an isomorphism, we adopt the method of [S-S]. By a "devissage" argument, it is enough to treat the case where $\mathcal{M}$ is a $\mathcal{D}(\mathcal{P}_{X\times Y})$ which admits a good filtration in a neighborhood of any compact. It is easy to show that, in a neighborhood of a compact K, such a module $\mathcal{M}$ has a resolution by modules of the type $\mathcal{N}_0 \underset{\mathcal{D}_{X\times Y|Y}}{\otimes} \mathcal{D}(\mathcal{P}_{X\times Y})$ where $\mathcal{N}_0$ is a $\mathcal{D}_{X\times Y|Y}$-module admitting a good filtration and such that $Supp(\mathcal{N}_0) \subset Supp(\mathcal{M})$ (see [S-S] lemma 2.10). For a module of the type $\mathcal{N}_0 \underset{\mathcal{D}_{X\times Y|Y}}{\otimes} \mathcal{D}(\mathcal{P}_{X\times Y})$, the morphism is an isomorphism because we know from

[S1] (see also [S-S]) that the duality functor and the direct image commute for the $\mathcal{D}_{X\times Y|Y}$-module $\mathcal{N}_0$ (see [S-S] p 40 for details).

The theorem is now a consequence of the previous lemmas.

Remarks :

1) The theorem 4.3.1 generalizes Schneiders' thesis ([S1], [S2]) where the case of relative differential operators is treated. The algebraic smooth case had been previously treated by Bernstein ([Be], [Bo], [Ho]) (in the $\mathcal{D}$-modules context) for a proper morphism. Moreover Mebkhout had treated the absolute case (i.e Y consists in one point, see corollary 4.3.6) in [Me1], [Me2].

2) In [C2], the theorem 4.3.1 was proved in the case of a closed embedding between Lie Rinehart algebras (that is to say for affine algebraic varieties).

3) If $\mathcal{L}_X = \mathcal{L}_Y = \{0\}$, we recover the Ramis-Ruget-Verdier duality in the special case of analytic manifolds ([R-R], [R-R-V]): Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module and let $f : X \to Y$ be an analytic map proper on $Supp(\mathcal{G})$, we have

$$Rf_! R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \Omega_X[x]) \simeq R\mathcal{H}om_{\mathcal{O}_Y}(Rf_!\mathcal{G}, \Omega_Y[y]).$$

4) If $\mathfrak{h}$ and $\mathfrak{g}$ are two finite dimensional Lie algebras, F is a Lie algebra morphism from $\mathfrak{h}$ to $\mathfrak{g}$ and V a finite dimensional $\mathfrak{h}$-module, then we have the following isomorphism in $D^b(U(\mathfrak{g})^{op})$

$$RHom_{U(\mathfrak{g})}\left(V \overset{L}{\underset{U(\mathfrak{h})}{\otimes}} U(\mathfrak{g}), U(\mathfrak{g})\right)[-dim\mathfrak{g}] \simeq \left(V^* \otimes \Lambda^{dim\mathfrak{h}}\mathfrak{h}^*\right) \overset{L}{\underset{U(\mathfrak{h})}{\otimes}} U(\mathfrak{g}).$$

In the particular case where $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, we get

$$\mathrm{Ext}^i_{U(\mathfrak{g})}\left(V \underset{U(\mathfrak{h})}{\otimes} U(\mathfrak{g}), U(\mathfrak{g})\right) = \begin{cases} (V^* \otimes \bigwedge^{dim\mathfrak{h}} \mathfrak{h}^*) \underset{U(\mathfrak{h})}{\otimes} U(\mathfrak{g}) & \text{if } i = dim\mathfrak{h} \\ 0 & \text{if } i \neq dim\mathfrak{h}. \end{cases}$$

It is easy to remove the hypothesis that $\mathfrak{g}$ is finite dimensional. We recover a duality result proved independently by Brown and Levasseur ([B-L]) and by Kempf ([K]) when $\mathfrak{g}$ is semi-simple and V is a Verma module. This duality result was extended to general Lie algebras in [C1].

Proposition 4.3.5 *Let X and Y be two complex manifolds of complex dimension respectively x and y. Assume that $\mathcal{L}_X$ is a rank $d_{\mathcal{L}_X}$ Lie algebroid. Let $\Phi = (f, F)$ be a Lie algebroid morphism from $(\mathcal{L}_X, \omega_X)$ to $(\mathcal{L}_Y, \omega_Y)$*

such that f is proper. Let $\mathcal{E}$ be a right $\mathcal{D}(\mathcal{L}_X)$-module which is a locally free $\mathcal{O}_X$-module of finite rank. Then we have the following isomorphic in $\mathcal{D}^b_{good}(\mathcal{D}(\mathcal{L}_Y)^{op})$

$$\underline{\Phi}_!\left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\Omega_X[x])\underset{\mathcal{O}_X}{\otimes}\Lambda^{d_{\mathcal{L}_X}}\mathcal{L}_X^*[-d_{\mathcal{L}_X}]\right)\simeq\Delta_{\mathcal{L}_Y}\underline{\Phi}_!(\mathcal{E}).$$

the proposition follows from the theorem 4.3.1 by applying proposition 3.2.1.

Corollary 4.3.6 *Let X be a compact analytic manifold of complex dimension x. Let $(\mathcal{L}_X,\omega_X)$ be a Lie algebroid of rank $d_{\mathcal{L}_X}$. Let $\mathcal{N}$ be a left $\mathcal{D}(\mathcal{L}_X)$-module which is a locally free $\mathcal{O}_X$-module of finite rank. Then, for any i in $\mathbb{Z}$, $\mathrm{Ext}^i_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X,\mathcal{N})$ is finite dimensional and*

$$\mathrm{Ext}^{d_{\mathcal{L}_X}+x-i}_{\mathcal{D}(\mathcal{L}_X)}\left(\mathcal{O}_X,\mathcal{N}^*\underset{\mathcal{O}_X}{\otimes}\mathcal{H}om_{\mathcal{O}_X}\left(\Lambda^{d_X}\mathcal{L}_X^*,\Omega_X\right)\right)\simeq\mathrm{Ext}^i_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X,\mathcal{N})^*.$$

Proof of the corollary 4.3.6

Apply the previous proposition to the following Lie algebroid morphism

$$\mathcal{L}_X\xrightarrow{\omega_X}\Theta_X\to\Theta_{\{pt\}}$$

where $\{pt\}$ denotes the variety consisting in one point. The corollary follows then by using propositions 2.3.2 and 3.2.1.

Remarks :

1) If X is a point, we recover Poincaré duality for Lie algebras.
2) If $\mathcal{L}_X=0$, we recover Serre duality (see remark after proposition 3.2.1).
3) Our duality morphism is different than the one obtained for $\mathcal{C}^\infty$ Lie algebroids in [E-L-W] (see also [Hu]).

Applying theorem 4.3.1, we get

Corollary 4.3.7 *Let X, Y, Z be three analytic manifolds of dimension respectively x, y and z. Let $\mathcal{L}_X$ (respectively $\mathcal{L}_Y$, $\mathcal{L}_Z$) be a Lie algebroid over X (respectively Y, Z) of rank $d_{\mathcal{L}_X}$ (respectively $d_{\mathcal{L}_Y}$, $d_{\mathcal{L}_Z}$). Let $\Phi=(f,F)$ (respectively $\Psi=(g,G)$) be a Lie algebroid morphism from $\mathcal{L}_X$ (respectively $\mathcal{L}_Z$) to $\mathcal{L}_Y$. We assume that f and g are proper. Let $\mathcal{M}$ (respectively $\mathcal{N}$) be* *a $\mathcal{D}(\mathcal{L}_X)$ (respectively $\mathcal{D}(\mathcal{L}_Z)$) -right module which is a locally free $\mathcal{O}_X$ (respectively $\mathcal{O}_Z$)-module of finite rank. Introduce the right $\mathcal{D}(\mathcal{L}_X)$ (respectively $\mathcal{D}(\mathcal{L}_Z)$)-modules $\widetilde{\mathcal{M}}$ (respectively $\widetilde{\mathcal{N}}$)*

$$\begin{aligned}\widetilde{\mathcal{M}} &= \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \Omega_X) \underset{\mathcal{O}_X}{\otimes} \bigwedge^{d_{\mathcal{L}_X}} \mathcal{L}_X^* \\ \widetilde{\mathcal{N}} &= \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{M}, \Omega_Z) \underset{\mathcal{O}_Z}{\otimes} \bigwedge^{d_{\mathcal{L}_Z}} \mathcal{L}_Z^*.\end{aligned}$$

For all n in $\mathbb{Z}$, we have the following isomorphism

$$\mathcal{E}xt^{n-d_{\mathcal{L}_Z}+z}_{\mathcal{D}(\mathcal{L}_Y)}\left(\underline{\Phi}_!(\widetilde{\mathcal{M}}), \underline{\Psi}_!(\widetilde{\mathcal{N}})\right) \simeq \mathcal{E}xt^{n-d_{\mathcal{L}_X}+x}_{\mathcal{D}(\mathcal{L}_Y)}\left(\underline{\Psi}_!(\mathcal{N}), \underline{\Phi}_!(\mathcal{M})\right).$$

Proof of the corollary 4.3.7

$$\begin{aligned}&R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_Y)}\left(\underline{\Phi}_!(\widetilde{\mathcal{M}}), \underline{\Psi}_!(\widetilde{\mathcal{N}})\right)\\ &R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_Y)}\left(\underline{\Phi}_!\Delta_{\mathcal{L}_X}(\mathcal{M})[d_{\mathcal{L}_X} - x], \underline{\Psi}_!\Delta_{\mathcal{L}_Z}(\mathcal{N})[d_{\mathcal{L}_Z} - z]\right)\\ &\simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_Y)}\left(\Delta_{\mathcal{L}_Y}\underline{\Phi}_!(\mathcal{M}), \Delta_{\mathcal{L}_Y}\underline{\Psi}_!(\mathcal{N})\right)[d_{\mathcal{L}_Z} - d_{\mathcal{L}_X} - z + x]\\ &\simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_Y)}\left(\underline{\Psi}_!(\mathcal{N}), \underline{\Phi}_!(\mathcal{M})\right)[d_{\mathcal{L}_Z} - d_{\mathcal{L}_X} - z + x]\end{aligned}$$

Corollary 4.3.7 was proved in [C1] in the case where $\mathcal{L}_Y = \mathfrak{g}$ is a Lie algebra and $\mathcal{L}_X = \mathfrak{h}$, $\mathcal{L}_Z = \mathfrak{t}$ are Lie subalgebras of $\mathfrak{g}$. The proof of [C1] is different from the one in this paper. It is inspired by a method used by M. Duflo ([D]). See also [B-C], [G], [C-S] and [D] for particular cases of this duality in the Lie algebra context.

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