

# An inverse image functor for Lie algebroids

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## Abstract

Extending material from the theory of  $\mathcal{D}$ -modules to Lie algebroids (see also [C3]), we introduce an inverse image functor and show that it preserves coherence under appropriate circumstances. As in the case of  $\mathcal{D}$ -modules, a suitable non characteristicity notion enables us to give a sufficient condition in order for the duality functor and the inverse image functor to commute. This generalizes a result for  $\mathcal{D}$ -modules due to Kashiwara-Kawai-Sato but, even in the case of  $\mathcal{D}$ -modules, our proof is different from theirs. In particular, we obtain a new duality formula for complexes of modules over an ordinary Lie algebra and, as a special case, we get a new adjunction formula for modules over Lie algebras. Moreover, our result will shed some light on the behaviour of Poisson cohomology (a notion introduced by Lichnerowicz) under a Poisson map (in the analytic case).

## 1 Introduction

We will follow the notation of [K-S-1] for sheaf theory. Let  $X$  be a complex manifold,  $\mathcal{O}_X$  the sheaf of holomorphic functions on  $X$ ,  $\Theta_X$  the sheaf of holomorphic vector fields and  $\mathcal{D}_X$  the sheaf of rings of differential operators on  $X$ . A complex Lie algebroid over  $X$  is a pair  $(\mathcal{L}_X, \omega_{\mathcal{L}_X})$  where

- $\mathcal{L}_X$  is endowed with the structure of a sheaf of  $\mathbb{C}$ -Lie algebras and with that of a locally free  $\mathcal{O}_X$ -module of constant finite rank.
- $\omega$  is an  $\mathcal{O}_X$ -linear morphism of sheaves of  $\mathbb{C}$ -Lie algebras from  $\mathcal{L}_X$  to  $\Theta_X$  (called the anchor map) such that the following compatibility relation

holds

$$[\xi, f\zeta] = \omega(\xi)(f)\zeta + f[\xi, \zeta]$$

for any  $(\xi, \zeta) \in \mathcal{L}_X$  and any  $f \in \mathcal{O}_X$ .

The Lie algebroid  $\mathcal{L}_X$  gives rise to the sheaf of algebras of generalized differential operators generated by  $\mathcal{O}_X$  and  $\mathcal{L}_X$ ,  $\mathcal{D}(\mathcal{L}_X)$ . If  $\mathcal{L}_X = \Theta_X$  and  $\omega = \text{id}$  then  $\mathcal{D}(\mathcal{L}_X)$  is the sheaf of rings of differential operators over  $X$ . If  $X$  is a point,  $\mathcal{L}_X$  is a Lie algebra and  $\mathcal{D}(\mathcal{L}_X)$  is its enveloping algebra. But there are many more Lie algebroids. In particular, Poisson manifolds give rise to Lie algebroids (if  $X$  is a Poisson manifold, then  $(X, \Omega_X^1)$  is a Lie algebroid) and actions of groups over manifolds give rise to Lie algebroids. Denote by  $R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X, -)$  the right derived functor of  $\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X, -)$ . It is interesting to notice that  $R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X, \mathcal{O}_X)$  computes Lie algebra cohomology with values in  $\mathcal{O}_X = \mathbb{C}$  if  $X$  is a point, De Rham cohomology if  $\mathcal{L}_X = \Theta_X$  and Poisson cohomology if  $X$  is a Poisson manifold and  $\mathcal{L}_X = \Omega_X^1$  ([Li], [Hu]).

The construction of  $\mathcal{D}(\mathcal{L}_X)$  is analogous to the construction of  $\mathcal{D}_X$ . This remark gave us the idea (see [C2], [C3]) of extending material from the theory of  $\mathcal{D}$ -modules to Lie algebroids. Recall that basic concepts in the theory of  $\mathcal{D}$ -modules are due to Bernstein and Kashiwara. We refer the reader to [Bj], [Bo], [Ho] and [S2] for an introduction to  $\mathcal{D}$ -modules theory.

Let  $d_{\mathcal{L}_X}$  be the rank of  $\mathcal{L}_X$ . As in the  $\mathcal{D}$ -modules case,  $\mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \Lambda^{d_{\mathcal{L}_X}} \mathcal{L}_X$  is endowed with a  $\mathcal{D}(\mathcal{L}_X) \otimes \mathcal{D}(\mathcal{L}_X)$ -module structure and we have the following duality functor in the derived category of bounded complexes of left  $\mathcal{D}(\mathcal{L}_X)$ -modules with coherent cohomology  $D_{coh}^b(\mathcal{D}(\mathcal{L}_X))$  (see [C3]) :

$$\underline{D}_{\mathcal{L}_X}(\mathcal{N}^\bullet) = R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)} \left( \mathcal{N}^\bullet, \mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \Lambda^{d_{\mathcal{L}_X}} \mathcal{L}_X \right) [d_{\mathcal{L}_X}]$$

which is defined for any  $\mathcal{N}^\bullet \in D_{coh}^b(\mathcal{D}(\mathcal{L}_X))$ . A Lie algebroid morphism ([A-K], [C2], [C3])  $\Phi$  from  $(\mathcal{L}_X, \omega_{\mathcal{L}_X})$  to  $(\mathcal{L}_Y, \omega_{\mathcal{L}_Y})$  is a pair  $(f, F)$  where  $f$  is an analytic map from  $X$  to  $Y$  and  $F$  is an  $\mathcal{O}_X$ -module morphism from  $\mathcal{L}_X$  to  $f^*\mathcal{L}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{L}_Y$  with some requirements (see section 2.3 for details). One of the requirements gives the existence of a transfer  $(\mathcal{D}(\mathcal{L}_X) \otimes f^{-1}\mathcal{D}(\mathcal{L}_Y)^{op})$ -bimodule

$$\mathcal{D}_{\mathcal{L}_X \rightarrow \mathcal{L}_Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}(\mathcal{L}_Y)$$

(extending the transfer bimodule in the  $\mathcal{D}$ -modules case, see [C3]). Denote by  $\overset{L}{\otimes}$  the derived functor of the tensor product. We define ([C3]), as in the  $\mathcal{D}$ -modules case an inverse image functor  $\underline{\Phi}^{-1} : D^b(\mathcal{D}(\mathcal{L}_Y)) \rightarrow D^b(\mathcal{D}(\mathcal{L}_X))$  as follows.

$$\underline{\Phi}^{-1}(\mathcal{R}^\bullet) = \mathcal{D}_{\mathcal{L}_X \rightarrow \mathcal{L}_Y} \overset{L}{\otimes}_{f^{-1}\mathcal{D}(\mathcal{L}_Y)} f^{-1}\mathcal{R}^\bullet.$$

In this article, we will study this inverse image functor. Generalizing the  $\mathcal{D}$ -modules case (see [S2] for an exposition), we introduce the notion of non characteristicity and we prove the theorem (due to Kashiwara in the case of  $\mathcal{D}$ -modules):

**Theorem 3.4.4** *Let  $(X, \mathcal{L}_X)$  and  $(Y, \mathcal{L}_Y)$  be Lie algebroids over the complex manifolds  $X$  and  $Y$  respectively. Let  $\Phi = (f, F)$  be a Lie algebroid morphism from  $(X, \mathcal{L}_X)$  to  $(Y, \mathcal{L}_Y)$ . Let  $\mathcal{R}^\bullet$  be an object of  $D_{coh}^b(\mathcal{D}(\mathcal{L}_Y))$ . Assume that  $\mathcal{R}^\bullet$  is non characteristic with respect to  $\Phi$ . Then  $\underline{\Phi}^{-1}(\mathcal{R}^\bullet)$  is in  $D_{coh}^b(\mathcal{D}(\mathcal{L}_X))$ .*

Then we establish the following duality theorem which generalizes a result of  $\mathcal{D}$ -modules due to Sato, Kawai and Kashiwara ([SKK]). We do not know whether the proof of [SKK] can be generalized to Lie algebroids and, even in the  $\mathcal{D}$ -modules case, our proof is different from theirs.

**Theorem 4.1.1** *Let  $\Phi = (f, F)$  be a Lie algebroid morphism from  $(X, \mathcal{L}_X)$  to  $(Y, \mathcal{L}_Y)$ . Let  $\mathcal{R}^\bullet$  be an object of  $D_{coh}^b(\mathcal{D}(\mathcal{L}_Y))$  which is supposed to be non characteristic with respect to  $\Phi$ . Then there is a functorial isomorphism from  $\underline{D}_{\mathcal{L}_X} \circ \underline{\Phi}^{-1}(\mathcal{R}^\bullet)$  to  $\underline{\Phi}^{-1} \circ \underline{D}_{\mathcal{L}_Y}(\mathcal{R}^\bullet)$ .*

As a particular case, we get a duality formula for complexes of modules over Lie algebras.

Combining Theorem 4.1.1 with the results of [C3] concerning the direct image, we obtain adjunction formulas in the Lie algebroids setting (see [K-S 2] for the  $\mathcal{D}$ -modules case). In particular, we get a new adjunction formula for modules over Lie algebras. Moreover, our result will shed some light on the behaviour of Poisson cohomology under a Poisson map (in the analytic case).

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### Notation

If  $\mathcal{A}$  is a sheaf of rings on  $X$ , one denotes by  $D^b(\mathcal{A})$  (respectively  $D_{coh}^b(\mathcal{A})$ ) the derived category of bounded complexes of left  $\mathcal{A}$ -modules (respectively of left  $\mathcal{A}$ -modules with coherent cohomology). In the sequel, we will take  $\mathcal{A} = \mathcal{D}(\mathcal{L}_X)$  or  $\mathcal{A} = \mathcal{D}(\mathcal{L}_X) \otimes \mathcal{D}(\mathcal{L}_X)$ . If  $\mathcal{M}^\bullet$  and  $\mathcal{M}'^\bullet$  are objects of  $D^b(\mathcal{D}(\mathcal{L}_X))$ , one puts

$$\mathcal{E}xt_{\mathcal{D}(\mathcal{L}_X)}^i(\mathcal{M}^\bullet, \mathcal{M}'^\bullet) = H^i\left(R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{M}^\bullet, \mathcal{M}'^\bullet)\right).$$

For the theory of filtered sheaves of rings, we refer the reader to [S2]. Let  $\mathcal{F}\mathcal{A}$  be a filtered sheaf of rings. One writes  $Gr\mathcal{F}\mathcal{A}$  for the associated graded sheaf of rings. If  $\mathcal{F}\mathcal{M}$  is a filtered  $\mathcal{F}\mathcal{A}$ -module, one denotes by  $Gr\mathcal{F}\mathcal{M}$  the associated graded  $Gr\mathcal{F}\mathcal{A}$ -module. Let  $\mathcal{F}\mathcal{M}$  and  $\mathcal{F}\mathcal{N}$  be two  $\mathcal{F}\mathcal{A}$ -modules. A morphism of filtered  $\mathcal{F}\mathcal{A}$ -modules from  $\mathcal{F}\mathcal{M}$  to  $\mathcal{F}\mathcal{N}$ ,  $\mathcal{F}u$ , is a morphism  $\mathcal{F}u : \mathcal{M} \rightarrow \mathcal{N}$  of the underlying  $\mathcal{A}$ -modules such that

$$\mathcal{F}u(\mathcal{F}_k\mathcal{M}) \subset \mathcal{F}_k\mathcal{N}.$$

The group of morphisms of  $\mathcal{F}\mathcal{A}$ -modules from  $\mathcal{F}\mathcal{M}$  to  $\mathcal{F}\mathcal{N}$  will be denoted by  $\mathcal{H}om_{\mathcal{F}\mathcal{A}}(\mathcal{F}\mathcal{M}, \mathcal{F}\mathcal{N})$ . With this notion of morphism, the category of  $\mathcal{F}\mathcal{A}$ -modules is an additive category. Let  $k \in \mathbb{Z}$ . One denotes by  $\mathcal{F}\mathcal{M}(k)$  the sheaf of  $\mathcal{F}\mathcal{A}$ -modules endowed with the filtration  $\mathcal{F}_{k+n}\mathcal{M}$ . One defines also the sheaf of filtered groups  $\mathcal{F}\mathcal{H}om_{\mathcal{F}\mathcal{A}}(\mathcal{F}\mathcal{M}, \mathcal{F}\mathcal{N})$  by setting

$$\mathcal{F}_k\mathcal{H}om_{\mathcal{F}\mathcal{A}}(\mathcal{F}\mathcal{M}, \mathcal{F}\mathcal{N}) = \mathcal{H}om_{\mathcal{F}\mathcal{A}}(\mathcal{F}\mathcal{M}, \mathcal{F}\mathcal{N}(k)).$$

To an element  $\mathcal{F}u$  of  $\mathcal{H}om_{\mathcal{F}\mathcal{A}}(\mathcal{F}\mathcal{M}, \mathcal{F}\mathcal{N})$ , one associates a morphism  $Gr\mathcal{F}u$  from  $Gr\mathcal{F}\mathcal{M}$  to  $Gr\mathcal{F}\mathcal{N}$ . One defines also the sheaf of groups  $\mathcal{H}om_{Gr\mathcal{F}\mathcal{A}}(Gr\mathcal{F}\mathcal{M}, Gr\mathcal{F}\mathcal{N})$  and the sheaf of graded groups  $G\mathcal{H}om_{Gr\mathcal{F}\mathcal{A}}(Gr\mathcal{F}\mathcal{M}, Gr\mathcal{F}\mathcal{N})$  (see [S2]). In the category of  $\mathcal{F}\mathcal{A}$ -modules, the notion of strict morphism is defined. Let  $\mathcal{F}u$  be an element of

$\mathcal{H}om_{\mathcal{F}\mathcal{A}}(\mathcal{F}\mathcal{M}, \mathcal{F}\mathcal{N})$ . Denote by  $\text{Ker}\mathcal{F}u$  the sheaf kernel of  $\mathcal{F}u$  filtered with the family  $\text{Ker}\mathcal{F}u \cap \mathcal{F}_k\mathcal{M}$ . Similarly, one defines the sheaf of graded  $\text{Gr}\mathcal{F}\mathcal{A}$ -modules  $\text{KerGr}\mathcal{F}u$ . Recall that  $\mathcal{F}u$  is strict if and only if  $\text{GrKer}\mathcal{F}u = \text{KerGr}\mathcal{F}u$ .

An exact sequence of  $\mathcal{F}\mathcal{A}$ -modules is a sequence

$$\mathcal{F}\mathcal{M} \xrightarrow{\mathcal{F}u} \mathcal{F}\mathcal{N} \xrightarrow{\mathcal{F}v} \mathcal{F}\mathcal{P}$$

such that  $\text{Ker}\mathcal{F}_kv = \text{Im}\mathcal{F}_ku$ .

A finite free  $\mathcal{F}\mathcal{A}$ -module is a  $\mathcal{F}\mathcal{A}$ -module of the form  $\bigoplus_{i=1}^n \mathcal{F}\mathcal{A}(r_i)$ . If  $\mathcal{F}\mathcal{M}$  is a finite free  $\mathcal{F}\mathcal{A}$ -module, then the underlying sheaf of modules of  $\mathcal{F}\mathcal{H}om_{\mathcal{F}\mathcal{A}}(\mathcal{F}\mathcal{M}, \mathcal{F}\mathcal{N})$  is  $\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ . A finite free resolution of the  $\mathcal{F}\mathcal{A}$ -module  $\mathcal{F}\mathcal{M}$  is an exact sequence (of  $\mathcal{F}\mathcal{A}$ -modules) of the form

$$\dots \longrightarrow \bigoplus_{i=1}^{n_1} \mathcal{F}\mathcal{A}(r_{i,1}) \longrightarrow \bigoplus_{i=1}^{n_0} \mathcal{F}\mathcal{A}(r_{i,0}) \longrightarrow \mathcal{F}\mathcal{M} \longrightarrow 0.$$

## 2 Lie algebroids

### 2.1 Definitions

Let  $X$  be a complex analytic manifold and let  $\mathcal{O}_X$  be the sheaf of holomorphic functions on  $X$ . Let  $\Theta_X$  be the  $\mathcal{O}_X$ -module of holomorphic vector fields on  $X$ .

**Definition 2.1.1** *A sheaf in Lie algebras  $\mathcal{L}_X$  is a sheaf of  $\mathbb{C}$ -vector spaces such that for any open subset  $U$ ,  $\mathcal{L}_X(U)$  is equipped with the structure of a Lie algebra and the restriction morphisms are Lie algebra homomorphisms.*

A morphism between two sheaves of Lie algebras  $\mathcal{L}_X$  and  $\mathcal{M}_X$  is a  $\mathbb{C}_X$ -module morphism which is a Lie algebra morphism on each open subset.

**Definition 2.1.2** *A complex Lie algebroid over  $X$  is a pair  $(\mathcal{L}_X, \omega)$  where*

- $\mathcal{L}_X$  is a locally free  $\mathcal{O}_X$ -module of finite constant rank,
- $\mathcal{L}_X$  is a sheaf of  $\mathbb{C}$ -Lie algebras,

- $\omega : \mathcal{L}_X \rightarrow \Theta_X$  is an  $\mathcal{O}_X$ -linear morphism of sheaves of  $\mathbb{C}$ -Lie algebras such that the following compatibility relation holds :

$$\forall (\xi, \zeta) \in \mathcal{L}_X^2, \quad \forall f \in \mathcal{O}_X, \quad [\xi, f\zeta] = \omega(\xi)(f)\zeta + f[\xi, \zeta]$$

for any  $(\xi, \zeta)$  in  $\mathcal{L}_X$  and for any  $f$  in  $\mathcal{O}_X$ .

One calls  $\omega$  the anchor map. When there is no ambiguity, we will drop the anchor map in the notation of the Lie algebroid. If  $(\mathcal{L}_X, \omega)$  is a Lie algebroid over  $X$ , then  $(\mathcal{L}_{X|U}, \omega|_U)$  is a Lie algebroid over  $U$  which will be denoted by  $\mathcal{L}_U$ .

A Lie algebroid  $(\mathcal{L}_X, \omega)$  gives rise to the sheaf of generalized differential operators generated by  $\mathcal{O}_X$  and  $\mathcal{L}_X$  which is denoted by  $\mathcal{D}(\mathcal{L}_X)$ .

**Definition 2.1.3**  $\mathcal{D}(\mathcal{L}_X)$  is the sheaf associated with the presheaf:

$$U \mapsto T_{\mathbb{C}}^+(\mathcal{O}_X(U) \oplus \mathcal{L}_X(U)) / J_U$$

where  $J_U$  is the two sided ideal generated by the relations

$$\forall (f, g) \in \mathcal{O}_X(U), \quad \forall (\xi, \zeta) \in \mathcal{L}_X(U)^2$$

- 1)  $f \otimes g = fg$
- 2)  $f \otimes \xi = f\xi$
- 3)  $\xi \otimes \zeta - \zeta \otimes \xi = [\xi, \zeta]$
- 4)  $\xi \otimes f - f \otimes \xi = \omega(\xi)(f)$

$\mathcal{D}(\mathcal{L}_X)$  is endowed with the filtration  $(\mathcal{F}_n \mathcal{D}(\mathcal{L}_X))_{n \in \mathbb{N}} = (\mathcal{D}(\mathcal{L}_X)_n)_{n \in \mathbb{N}}$  defined as follows:

$$\begin{aligned} \mathcal{D}(\mathcal{L}_X)_0 &= \mathcal{O}_X \\ \mathcal{D}(\mathcal{L}_X)_n &= \mathcal{D}(\mathcal{L}_X)_{n-1} \cdot \mathcal{L}_X + \mathcal{D}(\mathcal{L}_X)_{n-1} \end{aligned}$$

The anchor map  $\omega_{\mathcal{L}_X} : \mathcal{L}_X \rightarrow \Theta_X$  induces a sheaf of rings morphism from  $\mathcal{D}(\mathcal{L}_X)$  to  $\mathcal{D}_X$ .

As  $\mathcal{L}_X$  is a locally free  $\mathcal{O}_X$ -module of finite rank, we have the following theorem due to Rinehart [R].

**Theorem 2.1.4** *The sheaves of  $\mathcal{O}_X$ -algebras  $S_{\mathcal{O}_X}(\mathcal{L}_X)$  and  $Gr\mathcal{FD}(\mathcal{L}_X)$  are isomorphic.*

*Proof of theorem 2.1.4:*

The proof of this theorem is done in the affine case (that is to say the case of a Lie-Rinehart algebra) in [R]. The case of Lie algebroids follows.

**Proposition 2.1.5** *The sheaf  $\mathcal{D}(\mathcal{L}_X)$  is coherent, noetherian and has finite global homological dimension. Moreover, there exists an integer  $p$  such that, locally, every coherent  $\mathcal{D}(\mathcal{L}_X)$ -module  $\mathcal{M}$  has a free resolution of less than  $p$  or equal to  $p$ . In other words, any point  $x$  has an open neighborhood  $V$  such that there exists a resolution of the type*

$$0 \rightarrow \mathcal{D}(\mathcal{L}_V)^{l_p} \rightarrow \dots \rightarrow \mathcal{D}(\mathcal{L}_V)^{l_0} \rightarrow \mathcal{M}|_V \rightarrow 0.$$

The proof of proposition 2.1.5 in the  $\mathcal{D}$ -modules case extends to the setting of Lie algebroids ([S2] p.14).

A sheaf of rings (respectively graded rings) is syzygic if it is coherent, has finite global homological dimension and syzygic fibers (i.e any finite type  $S(\mathcal{L}_{X,x})$ -module has a finitely free resolution of finite length).

**Proposition 2.1.6** *The sheaf  $S(\mathcal{L}_X)$  is syzygic. Hence, locally, any coherent  $\mathcal{FD}(\mathcal{L}_X)$ -module  $\mathcal{FM}$  has a finite free resolution. In other words, any point  $x$  has an open neighborhood  $V$  such that there exists a resolution of the type*

$$\dots \rightarrow \bigoplus_{i=1}^{n_1} \mathcal{FD}(\mathcal{L}_V)(r_{i,1}) \rightarrow \bigoplus_{i=1}^{n_0} \mathcal{FD}(\mathcal{L}_V)(r_{i,0}) \rightarrow \mathcal{FM}|_V \rightarrow 0.$$

See [S2] for details.

**Proposition 2.1.7** *Let  $\mathcal{M}^\bullet$  be a complex of  $\mathcal{FD}(\mathcal{L}_X)$ -modules. If  $H^j(\text{Gr}\mathcal{M}^\bullet)$  is a coherent  $S(\mathcal{L}_X)$ -module, then  $H^j(\mathcal{M}^\bullet)$  is a coherent  $\mathcal{D}(\mathcal{L}_X)$ -module.*

*Proof of proposition 2.1.7:*

We put

$$\mathcal{M}^\bullet : 0 \rightarrow \mathcal{M}_0 \xrightarrow{d_0} \mathcal{M}_1 \rightarrow \dots$$

For each  $i$ , endow  $\text{Ker}d_i$  and  $\text{Im}d_{i-1}$  with the induced filtration. We have the following strict short exact sequence (i.e the arrows are strict morphisms)

$$0 \rightarrow \text{Im}d_{i-1} \rightarrow \text{Ker}d_i \rightarrow \mathcal{H}^i(\mathcal{M}^\bullet) \rightarrow 0.$$

Consequently, we get the short exact sequence

$$0 \rightarrow \text{Gr}\mathcal{I}md_{i-1} \rightarrow \text{Gr}\mathcal{K}erd_i \rightarrow \text{Gr}\mathcal{H}^i(\mathcal{M}^\bullet) \rightarrow 0.$$

And, as  $\text{Gr}\mathcal{K}erd_i \subset \text{Ker}\text{Gr}d_i$  and  $\mathcal{I}m\text{Gr}d_i \subset \text{Gr}\mathcal{I}md_i$ , we have the following inclusion

$$\text{Gr}\mathcal{H}^i(\mathcal{M}^\bullet) \subset \mathcal{H}^i(\text{Gr}\mathcal{M}^\bullet)$$

from which the proposition follows.

## 2.2 Resolution of $\mathcal{O}_X$ as a left $\mathcal{D}(\mathcal{L}_X)$ -module.

Let  $(\mathcal{L}_X, \omega_X)$  be a Lie algebroid over  $X$ . Set  $\mathcal{L}_X^* = \text{Hom}_{\mathcal{O}_X}(\mathcal{L}_X, \mathcal{O}_X)$ . Consider the graded left  $\mathcal{D}(\mathcal{L}_X)$ -module  $\mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \Lambda^\bullet(\mathcal{L}_X) = \bigoplus_n \mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \Lambda^n \mathcal{L}_X$  where  $\mathcal{D}(\mathcal{L}_X)$  acts by left multiplication. It is known ([R] p 200) that the endomorphism of degree -1,  $d$ , defined below is a differential on  $\mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \Lambda^\bullet(\mathcal{L}_X)$ .

$$\begin{aligned} \forall v \in \mathcal{D}(\mathcal{L}_X), \quad \forall \xi_i \in \mathcal{L}_X, \\ d(v \otimes \xi_1 \wedge \dots \wedge \xi_n) &= \sum_{i=1}^n (-1)^{i-1} v \xi_i \otimes \xi_1 \wedge \dots \wedge \widehat{\xi}_i \wedge \dots \wedge \xi_n \\ &+ \sum_{k < i} (-1)^{i+k} v \otimes [\xi_k, \xi_i] \wedge \xi_1 \wedge \dots \wedge \widehat{\xi}_k \wedge \dots \wedge \widehat{\xi}_i \wedge \dots \wedge \xi_n \\ d(v \otimes 1) &= \omega_X(v)(1) \end{aligned}$$

where the notation  $\widehat{x}$  means that  $x$  is omitted.

**Theorem 2.2.1** *Let  $(\mathcal{L}_X, \omega_X)$  be a Lie algebroid. The complex  $K_{\mathcal{L}_X}^\bullet$  defined by*

$$\forall n \in \mathbb{Z}, \quad K_{\mathcal{L}_X}^{-n} = \mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \Lambda^n(\mathcal{L}_X)$$

*and the differential above is a resolution of  $\mathcal{O}_X$  by locally free left  $\mathcal{D}(\mathcal{L}_X)$ -modules. Filter  $K_{\mathcal{L}_X}^i$  by  $\mathcal{F}_k K_{\mathcal{L}_X}^i = \mathcal{F}_k \mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \Lambda^i \mathcal{L}_X$ . Then*

$$0 \rightarrow K_{\mathcal{L}_X}^{d_{\mathcal{L}_X}} \rightarrow K_{\mathcal{L}_X}^{d_{\mathcal{L}_X}-1}(1) \rightarrow \dots \rightarrow K_{\mathcal{L}_X}^1(d_{\mathcal{L}_X} - 1) \rightarrow K_{\mathcal{L}_X}^0(d_{\mathcal{L}_X}) \rightarrow \mathcal{O}_X \rightarrow 0$$

*is a resolution of  $\mathcal{O}_X$  by locally free left  $\mathcal{F}\mathcal{D}(\mathcal{L}_X)$ -modules.*

*Proof of theorem 2.2.1 :* See [R] p 202.

We call  $K_{\mathcal{L}_X}^\bullet$  the Koszul resolution of the left  $\mathcal{D}(\mathcal{L}_X)$ -module  $\mathcal{O}_X$ .



## 2.3 Lie algebroid morphisms

**Definition 2.3.1** Let  $(\mathcal{L}_X, \omega_X)$  and  $(\mathcal{L}_Y, \omega_Y)$  be Lie algebroids over  $X$  and  $Y$  respectively. A morphism  $\Phi$  from  $(\mathcal{L}_X, \omega_X)$  to  $(\mathcal{L}_Y, \omega_Y)$  is a pair  $(f, F)$  such that

- $f : X \rightarrow Y$  is a holomorphic map
- $F : \mathcal{L}_X \rightarrow f^*\mathcal{L}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{L}_Y$  is an  $\mathcal{O}_X$ -module morphism such that the two following conditions are satisfied:

1) The diagram

$$\begin{array}{ccc} \mathcal{L}_X & \xrightarrow{F} & f^*\mathcal{L}_Y \\ \omega_X \downarrow & & \downarrow f^*\omega_Y \\ \Theta_X & \xrightarrow{Tf} & f^*\Theta_Y \end{array}$$

commutes (where  $Tf$  is the differential of  $f$ ).

2) Let  $\xi$  and  $\eta$  be two sections of  $\mathcal{L}_X$ . Put  $F(\xi) = \sum_{i=1}^m a_i \otimes \xi_i$  and  $F(\eta) = \sum_{j=1}^m b_j \otimes \eta_j$  with  $a_i, b_j \in \mathcal{O}_X$  and  $\xi_i, \eta_j \in f^{-1}\mathcal{L}_Y$ . Then

$$F([\xi, \eta]) = \sum_{j=1}^n \omega_X(\xi)(b_j) \otimes \eta_j - \sum_{i=1}^n \omega_X(\eta)(a_i) \otimes \xi_i + \sum_{i,j} a_i b_j \otimes [\xi_i, \eta_j].$$

The condition 2) is equivalent to the following property :  $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}(\mathcal{L}_Y)$  endowed with the two operations below is a left  $\mathcal{D}(\mathcal{L}_X)$ -module.

$$\forall (a, b) \in \mathcal{O}_X^2, \forall \xi \in \mathcal{L}_X, \forall v \in f^{-1}\mathcal{D}_Y$$

$$\begin{aligned} a \cdot (b \otimes v) &= ab \otimes v \\ \xi \cdot (b \otimes v) &= \omega_X(\xi)(b) \otimes v + \sum_i b a_i \otimes \xi_i v \end{aligned}$$

(where  $F(\xi) = \sum_i a_i \otimes \xi_i$  with  $a_i \in \mathcal{O}_X$  and  $\xi_i \in f^{-1}\mathcal{L}_Y$ ).

Our definition ([C2], [C3]) coincides with that of Almeida and Kumpera ([A-K]).

**Notation :**

$\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}(\mathcal{L}_Y)$  considered as a  $\mathcal{D}(\mathcal{L}_X) \otimes f^{-1}\mathcal{D}(\mathcal{L}_Y)^{op}$ -module will be denoted  $\mathcal{D}_{\mathcal{L}_X \rightarrow \mathcal{L}_Y}$  (as in [C3]).

**Remark :**

Let  $(\mathcal{L}_X, \omega_{\mathcal{L}_X})$  be a Lie algebroid. Then  $\text{Ker}\omega_{\mathcal{L}_X}$  endowed with the operations below is a  $\mathcal{D}(\mathcal{L}_X)$ -module which is coherent as an  $\mathcal{O}_X$ -module: for any  $f \in \mathcal{O}_X$ , any  $D \in \mathcal{L}_X$  and any  $\Delta \in \text{Ker}\omega_{\mathcal{L}_X}$ ,

$$\begin{aligned} D \cdot \Delta &= [D, \Delta] \\ f \cdot \Delta &= f\Delta \end{aligned}$$

The composition of two Lie algebroid morphisms is a Lie algebroid morphism.

## 2.4 Examples

1) The Lie algebroid  $(X, \Theta_X, id)$  gives rise to the usual ring of differential operators. Moreover, if  $f : X \rightarrow Y$  is a morphism between complex analytic manifolds, then its differential  $Tf : \Theta_X \rightarrow \mathcal{O}_X \otimes_{f^{-1}\Theta_Y} f^{-1}\Theta_Y$  defines a Lie algebroid morphism  $(f, Tf)$  from  $(X, \Theta_X)$  to  $(Y, \Theta_Y)$ .

2) Let  $\mathfrak{g}$  be a finite dimensional Lie algebra. It is a Lie algebroid over a point with trivial anchor map. The ring of differential operators in this case is the universal enveloping algebra of  $\mathfrak{g}$ . Lie algebroid morphisms generalize Lie algebra morphisms.

3) Let  $\mathfrak{g}$  be a Lie algebra. Assume that there is a Lie algebra morphism  $\sigma : \mathfrak{g} \rightarrow \Theta_X$ . Then  $\mathcal{O}_X \otimes \mathfrak{g}$  has a natural Lie algebroid structure with anchor map  $\omega$  defined by

$$\forall f \in \mathcal{O}_X, \forall \xi \in \mathfrak{g}, \omega(f \otimes \xi) = f\sigma(\xi).$$

The Lie algebra bracket on  $\mathcal{O}_X \otimes \mathfrak{g}$  is given

$$[f \otimes \xi, g \otimes \eta] = f\sigma(\xi)(g) \otimes \eta - g\sigma(\eta)(f) \otimes \xi + fg \otimes [\xi, \eta].$$

Let  $G$  and  $G'$  be two complex Lie groups with Lie algebras  $\mathfrak{g}$  and  $\mathfrak{g}'$  and let  $\chi : G \rightarrow G'$  be a Lie group morphism. We will denote by  $d\chi$  the differential of  $\chi$  at the unity. Let  $X$  (respectively  $X'$ ) be an analytic manifold with action of  $G$  (respectively  $G'$ ). Let  $f : X \rightarrow X'$  be an equivariant map in the sense that

$$\forall g \in G, \forall x \in X, f(g \cdot x) = \chi(g) \cdot f(x).$$

Let us define  $F : \mathcal{O}_X \otimes \mathfrak{g} \rightarrow \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_{X'}} f^{-1}(\mathcal{O}_{X'} \otimes \mathfrak{g}')$  by

$$F(f \otimes \xi) = f \otimes 1 \otimes d\chi(\xi)$$

Then  $(f, F)$  is a Lie algebroid morphism from  $\mathcal{O}_X \otimes \mathfrak{g}$  to  $\mathcal{O}_{X'} \otimes \mathfrak{g}'$ .

4) Take the same notation as in the example 5. Assume that  $X = V$  is a finite dimensional vector space and that  $G$  is a connected algebraic group acting on  $V$ . Put  $LieG = \mathfrak{g}$  and  $\mathcal{L}_V = \omega(\mathcal{O}_V \otimes \mathfrak{g})$ . If  $\max_{v \in V} \dim G \cdot v = \dim G$ , then  $\mathcal{L}_V$  is a locally free  $\mathcal{O}_V$ -module ([P] p. 186) and  $(V, \mathcal{L}_V)$  (with the natural embedding as an anchor) is a Lie algebroid.

5) Let  $X$  be an analytic Poisson manifold. The Poisson bracket on  $\mathcal{O}_X$  is denoted by  $\{, \}$ . The  $\mathcal{O}_X$ -module of differential forms of degree 1,  $\Omega_X^1$ , is endowed with a natural Lie algebroid structure (see [Hu]) with anchor map

$$\begin{aligned} \Omega_X^1 &\rightarrow \Theta_X \\ fdg &\mapsto f\{g, \bullet\}. \end{aligned}$$

Recall that the Lie bracket on  $\Omega_X^1$  is given by

$$[f \otimes da, g \otimes db] = fg \otimes d\{a, b\} + f\{a, g\} \otimes db - g\{b, f\} \otimes da.$$

Let  $Y$  be another Poisson analytic manifold and let  $f : X \rightarrow Y$  be a Poisson map. We endow the  $\mathcal{O}_X$ -module  $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^1$  with a Lie algebroid structure as follows : We define the Lie algebra bracket by

$$\begin{aligned} \forall (a, a') \in \mathcal{O}_X, \quad \forall (b, b', v, v') \in \mathcal{O}_Y \\ [a \otimes dv, a' \otimes dv'] &= a\{v \circ f, a'\} \otimes dv' - a'\{v' \circ f, a\} \otimes dv \\ &+ aa' \otimes d\{v, v'\} \end{aligned}$$

and the anchor map by

$$\begin{aligned} \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^1 &\rightarrow \Theta_X \\ a \otimes dv &\mapsto a\{v \circ f, \cdot\}. \end{aligned}$$

The couple  $(f, id)$  is a Lie algebroid morphism from  $\left(X, \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^1\right)$  to  $(Y, \Omega_Y^1)$ .

We construct the following correspondence :

$$\begin{array}{ccc} & \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^1 & \\ (id, F) \swarrow & & \searrow (f, id) \\ (X, \Omega_X^1) & & (Y, \Omega_Y^1), \end{array}$$

where  $F$  is defined as follows : for all  $\alpha$  in  $\mathcal{O}_X$  and all  $\beta$  in  $\mathcal{O}_Y$ ,

$$\begin{aligned} \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^1 &\rightarrow \Omega_X^1 \\ \alpha \otimes \beta dq &\mapsto \alpha(\beta \circ f)d(q \circ f). \end{aligned}$$

6) Let  $(X, \mathcal{L}_X)$  be a Lie algebroid. We associate to it the abelian Lie algebroid  $(X, \mathcal{L}_X^{ab})$  defined by

- $\mathcal{L}_X^{ab} = \mathcal{L}_X$
- The Lie bracket on  $\mathcal{L}_X^{ab}$  is zero
- The anchor map on  $\mathcal{L}_X^{ab}$  is 0.

Let  $\Phi = (f, F)$  be a Lie algebroid morphism from  $(X, \mathcal{L}_X)$  to  $(Y, \mathcal{L}_Y)$ . It induces a Lie algebroid morphism  $\Phi^{ab} = (f, F^{ab})$  from  $(X, \mathcal{L}_X^{ab})$  to  $(Y, \mathcal{L}_Y^{ab})$ .

For other examples of Lie algebroids, see [M] and [C3].

## 2.5 The characteristic variety

Let  $(X, \mathcal{L}_X)$  be a Lie algebroid over  $X$ . Let  $L_X$  be the vector bundle associated with  $\mathcal{L}_X$  and let  $\pi$  be the projection from  $L_X^*$  to  $X$ . As in the  $\mathcal{D}$ -modules case ([S2]), one can define the notion of good filtration for a coherent  $\mathcal{D}(\mathcal{L}_X)$ -module (see [Bj] p. 24). Locally, each coherent  $\mathcal{D}(\mathcal{L}_X)$ -module admits a good filtration. Let  $\mathcal{N}$  be a coherent  $\mathcal{D}(\mathcal{L}_X)$ -module and

let  $U$  be an open subset on which  $\mathcal{N}|_U$  admits a good filtration. The subset  $\text{Supp} \left( \mathcal{O}_{L_U^*} \otimes_{\pi^{-1}\mathcal{S}(\mathcal{L}_U)} \pi^{-1} \text{Gr} \mathcal{N}|_U \right)$  does not depend on the good filtration. The characteristic variety,  $\text{char}(\mathcal{N})$ , is the closed conic subset of  $L_X^*$  defined by

$$\text{Supp} \left( \mathcal{O}_{L_U^*} \otimes_{\pi^{-1}\mathcal{S}(\mathcal{L}_U)} \pi^{-1} \text{Gr} \mathcal{N} \right) = \text{char}(\mathcal{N}) \cap L_U^*.$$

If  $\mathcal{N}^\bullet$  is an element of  $D^b(\mathcal{D}(\mathcal{L}_X))$ , set

$$\text{char}(\mathcal{N}^\bullet) = \bigcup_{j \in \mathbb{Z}} \text{char} H^j(\mathcal{N}^\bullet).$$

Then  $\text{char} \mathcal{N}^\bullet = \text{char} \mathcal{N}^\bullet[1]$ . As in the  $\mathcal{D}$ -modules case, one can show the following property: If  $\mathcal{N}'^\bullet \rightarrow \mathcal{N}^\bullet \rightarrow \mathcal{N}''^\bullet \xrightarrow{+1}$  is a distinguished triangle in  $D_{\text{coh}}^b(\mathcal{D}(\mathcal{L}_X))$ , then  $\text{char} \mathcal{N}^\bullet \subset \text{char} \mathcal{N}'^\bullet \cup \text{char} \mathcal{N}''^\bullet$ .

### 3 Some operations for modules on Lie algebroids

In this section, we generalize some basic notions of  $\mathcal{D}$ -modules theory due to Bernstein and Kashiwara. We refer the reader to [Bj], [Bo], [Ho] and [S2] for an exposition.

#### 3.1 Left and right-modules

The following proposition is classical for  $\mathcal{D}$ -modules and is easy to generalize to Lie algebroids ([C2], [C3]).

**Proposition 3.1.1** *a) If  $\mathcal{N}$  and  $\mathcal{N}'$  are left  $\mathcal{D}(\mathcal{L}_X)$ -modules, then  $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{N}'$  endowed with the two following operations :*

$$\begin{aligned} \forall a \in \mathcal{O}_X, \forall n \in \mathcal{N}, \forall n' \in \mathcal{N}', \forall D \in \mathcal{L}_X \\ a \cdot (n \otimes n') &= a \cdot n \otimes n' = n \otimes a \cdot n' \\ D \cdot (n \otimes n') &= D \cdot n \otimes n' + n \otimes D \cdot n' \end{aligned}$$

*is a left  $\mathcal{D}(\mathcal{L}_X)$ -module.*

b) If  $\mathcal{M}$  (respectively  $\mathcal{N}$ ) is a right (respectively a left)  $\mathcal{D}(\mathcal{L}_X)$ -module, then  $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$  endowed with the two following operations :

$$\begin{aligned} \forall a \in \mathcal{O}_X, \forall m \in \mathcal{M}, \forall n \in \mathcal{N}, \forall D \in \mathcal{L}_X \\ (m \otimes n) \cdot a = m \otimes a \cdot n = m \cdot a \otimes n \\ (m \otimes n) \cdot D = m \cdot D \otimes n - m \otimes D \cdot n \end{aligned}$$

is a right  $\mathcal{D}(\mathcal{L}_X)$ -module.

c) If  $\mathcal{M}$  and  $\mathcal{M}'$  are two right  $\mathcal{D}(\mathcal{L}_X)$ -modules, then  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}')$  endowed with the two following operations

$$\begin{aligned} \forall \phi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}'), \forall m \in \mathcal{M}, \forall a \in \mathcal{O}_X, \forall D \in \mathcal{L}_X \\ (a \cdot \phi)(m) = \phi(m) \cdot a \\ (D \cdot \phi)(m) = -\phi(m) \cdot D + \phi(m \cdot D) \end{aligned}$$

is a left  $\mathcal{D}(\mathcal{L}_X)$ -module.

d) If  $\mathcal{N}$  and  $\mathcal{N}'$  are two left  $\mathcal{D}(\mathcal{L}_X)$ -modules, then  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{N}')$  endowed with the two following operations

$$\begin{aligned} \forall \phi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{N}'), \forall m \in \mathcal{M}, \forall a \in \mathcal{O}_X, \forall D \in \mathcal{L}_X \\ (a \cdot \phi)(m) = a \cdot \phi(m) \\ (D \cdot \phi)(m) = D \cdot \phi(m) - \phi(D \cdot m) \end{aligned}$$

is a left  $\mathcal{D}(\mathcal{L}_X)$ -module.

The following theorem is now a consequence of the previous proposition.

**Theorem 3.1.2** *Let  $\mathcal{E}$  be a right  $\mathcal{D}(\mathcal{L}_X)$ -module which is a locally free  $\mathcal{O}_X$ -module of rank one. The functor  $\mathcal{N}^\bullet \mapsto \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet$  establishes an equivalence of categories between complexes of left and complexes of right  $\mathcal{D}(\mathcal{L}_X)$ -modules. Its inverse functor is given by  $\mathcal{M}^\bullet \mapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M}^\bullet)$ .*

It is a well known fact that  $\Omega_X^{\dim X}$  (the sheaf of differential forms of maximal degree) is endowed with a right  $\mathcal{D}_X$ -module structure (see [S2] p.9, [Bo] p. 226 ). By the morphism  $\mathcal{D}(\mathcal{L}_X) \rightarrow \mathcal{D}_X$ ,  $\Omega_X^{\dim X}$  has a structure of right  $\mathcal{D}(\mathcal{L}_X)$ -module. Hence theorem 3.1.2 applies in particular if  $\mathcal{E} = \Omega_X^{\dim X}$ . Put

$$\mathcal{L}_X^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_X, \mathcal{O}_X)$$

and let  $d_{\mathcal{L}_X}$  be the rank of  $\mathcal{L}_X$ . Then one may take  $\mathcal{E} = \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*)$ . Indeed,  $\mathcal{L}_X$  acts on  $\mathcal{L}_X^*$  as follows :

$$\langle D, \Delta \cdot \lambda \rangle = \langle [D, \Delta], \lambda \rangle + \Delta(\langle D, \lambda \rangle)$$

for any  $(D, \Delta) \in \mathcal{L}_X^2$  and any  $\lambda \in \mathcal{L}_X^*$ . Hence  $\mathcal{L}_X$  acts on  $\Lambda(\mathcal{L}_X^*)$ . The action of an element  $D$  of  $\mathcal{L}_X$  on  $\Lambda(\mathcal{L}_X^*)$  is called the Lie derivative of  $D$  and is denoted  $L_D$ . Put  $\det(\mathcal{L}_X^*) = \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*)$ . Then  $\det(\mathcal{L}_X^*)$ , endowed with the following two operations,

$$\begin{aligned} \forall \sigma \in \det(\mathcal{L}_X^*), \forall D \in \mathcal{L}_X, \forall a \in \mathcal{O}_X \\ \sigma \cdot a &= a \cdot \sigma \\ \sigma \cdot D &= -L_D(\sigma) \end{aligned}$$

is a right  $\mathcal{D}(\mathcal{L}_X)$ -module (see [C1]).

Consider

$$\begin{aligned} \mathcal{H}_{\mathcal{L}_X} &= \mathcal{H}om_{\mathcal{O}_X}(\Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*), \mathcal{D}(\mathcal{L}_X)) \\ &= \mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*) \end{aligned}$$

and

$$\begin{aligned} \overline{\mathcal{H}}_{\mathcal{L}_X} &= \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{D}(\mathcal{L}_X)) \\ &= \mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \Omega_X^{-1} \end{aligned}$$

They are endowed with a natural  $\mathcal{D}(\mathcal{L}_X) \otimes \mathcal{D}(\mathcal{L}_X)$ -left module structure (the first left  $\mathcal{D}(\mathcal{L}_X)$ -module structure is given by left multiplication, the second one is obtained by proposition 3.1.1 c) .

### 3.2 Duality functor

If  $\mathcal{N}^\bullet$  is an element of  $D_{coh}^b(\mathcal{D}(\mathcal{L}_X))$ , we set

$$\underline{D}_{\mathcal{L}_X}(\mathcal{N}^\bullet) = R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{N}^\bullet, \mathcal{H}_{\mathcal{L}_X})[d_{\mathcal{L}_X}].$$

Since the natural arrow  $\mathcal{N}^\bullet \mapsto \underline{D}_{\mathcal{L}_X}(\underline{D}_{\mathcal{L}_X}(\mathcal{N}^\bullet))$  is an isomorphism (see [C3]), one calls  $\underline{D}_{\mathcal{L}_X}$  a duality functor. Similarly, one sets

$$\overline{\Delta}_{\mathcal{L}_X}(\mathcal{N}^\bullet) = R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{N}^\bullet, \overline{\mathcal{H}}_{\mathcal{L}_X})[dim X].$$

$\underline{\Delta}_{\mathcal{L}_X}$  is a duality functor.

Let  $\mathcal{N}$  and  $\mathcal{N}'$  be two  $\mathcal{D}(\mathcal{L}_X)$ -modules. We endow  $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{N}')$  with a left  $\mathcal{D}(\mathcal{L}_X)$ -module structure as follows. Let  $\mathcal{I}^\bullet$  be an injective resolution of  $\mathcal{N}'$  in the category of left  $\mathcal{D}(\mathcal{L}_X)$ -modules. Then, as  $\mathcal{D}(\mathcal{L}_X)$  is a flat  $\mathcal{O}_X$ -module,  $\mathcal{I}^\bullet$  is an injective resolution of  $\mathcal{N}'$  in the category of  $\mathcal{O}_X$ -modules. Hence, one has

$$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{N}') \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{I}^\bullet).$$

As, by proposition 3.1.1, the right-hand side is endowed with a left  $\mathcal{D}(\mathcal{L}_X)$ -module structure, so is the left-hand side. In other words, deriving the functor  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \bullet)$  in the category of left  $\mathcal{D}(\mathcal{L}_X)$ -modules is the same as deriving it in the category of  $\mathcal{O}_X$ -modules.

**Lemma 3.2.1** *Let  $\mathcal{N}$  and  $\mathcal{N}'$  be two left  $\mathcal{D}(\mathcal{L}_X)$ -modules. The objects  $R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{N}, \mathcal{N}')$  and  $R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X, R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{N}'))$  are isomorphic in  $D^b(\mathbb{C}_X)$*

*Proof of lemma 3.2.1 :*

Let  $K_{\mathcal{L}_X}^\bullet$  be the Koszul resolution of  $\mathcal{O}_X$  and let  $\mathcal{N}' \rightarrow \mathcal{I}^\bullet$  be an injective resolution of the  $\mathcal{D}(\mathcal{L}_X)$ -module  $\mathcal{N}'$ . One has the following sequence of isomorphisms :

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X, R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{N}')) &\simeq \mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(K_{\mathcal{L}_X}^\bullet, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{I}^\bullet)) \\ &\simeq \mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}\left(K_{\mathcal{L}_X}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{I}^\bullet\right) \\ &\simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{N}, \mathcal{N}') \end{aligned}$$

**Proposition 3.2.2** *If  $\mathcal{N}$  is a left  $\mathcal{D}(\mathcal{L}_X)$ -module which is coherent as an  $\mathcal{O}_X$ -module, then  $\underline{D}_{\mathcal{L}_X}(\mathcal{N})$  and  $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{O}_X)$  are isomorphic in  $D^b(\mathcal{O}_X)$ .*

This proposition is well known for  $\mathcal{D}$ -modules (see [Ho] p. 93). Note that in the  $\mathcal{D}$ -module case, the  $\mathcal{O}_X$ -module  $\mathcal{N}$  is necessarily locally free of finite rank.

*Proof of proposition 3.2.2 :*

If  $\mathcal{N}$  is a locally free  $\mathcal{O}_X$ -module of finite rank, the proposition was already proved in [C3].



Assume now that  $\mathcal{N}$  is only a coherent  $\mathcal{O}_X$ -module, then one has the following sequence of left  $\mathcal{D}(\mathcal{L}_X)$ -modules.

$$\begin{aligned} \underline{D}_{\mathcal{L}_X}(\mathcal{N}) &\simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{O}_X, R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{H}_{\mathcal{L}_X})) [d_{\mathcal{L}_X}] \\ &\simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}\left(\mathcal{O}_X, R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{O}_X) \otimes_{\mathcal{O}_X}^L \mathcal{H}_{\mathcal{L}_X}\right) [d_{\mathcal{L}_X}] \\ &\simeq \underline{D}_{\mathcal{L}_X}(\mathcal{O}_X) \otimes_{\mathcal{O}_X}^L R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{O}_X) \\ &\simeq R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{O}_X). \end{aligned}$$

The first isomorphism follows from lemma 3.2.1, the second one follows from the fact that  $\mathcal{N}$  is a coherent  $\mathcal{O}_X$ -module, the third one follows from the fact that  $\mathcal{O}_X$  is a coherent  $\mathcal{D}(\mathcal{L}_X)$ -module and the last isomorphism follows from  $\underline{D}_{\mathcal{L}_X}(\mathcal{O}_X) \simeq \mathcal{O}_X$ .

**Corollary 3.2.3** *Let  $\mathcal{M}^\bullet$  and  $\mathcal{N}^\bullet$  be two elements of  $D_{\text{coh}}^b(\mathcal{D}(\mathcal{L}_X))$ . There is an isomorphism from  $R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$  to  $R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}\left(\mathcal{O}_X, \underline{D}_{\mathcal{L}_X}(\mathcal{M}^\bullet) \otimes_{\mathcal{O}_X}^L \mathcal{N}^\bullet\right)$*

*Proof of corollary 3.2.3 :*

Let  $K_{\mathcal{L}_X}^\bullet$  the Koszul resolution of  $\mathcal{O}_X$  and let  $\mathcal{P}^\bullet \rightarrow \mathcal{M}_{|V}^\bullet$  be a bounded locally free resolution of  $\mathcal{M}_{|V}^\bullet$ . The morphism we are looking for,  $\chi_{\mathcal{L}_X}(\mathcal{P}^\bullet, \mathcal{N}^\bullet)$  or  $\chi_{\mathcal{L}_X}$  for short, can be made explicite as follows : It is a morphism from  $\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}^\bullet(\mathcal{P}^\bullet, \mathcal{N}^\bullet)$  to  $\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}^\bullet\left(K_{\mathcal{L}_X}^\bullet, \mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}^\bullet(\mathcal{P}^\bullet, \mathcal{H}_{\mathcal{L}_X}) \otimes_{\mathcal{O}_X}^L \mathcal{N}^\bullet\right) [d_{\mathcal{L}_X}]$ , and if  $\phi_p$  is in  $\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}^p(\mathcal{P}^\bullet, \mathcal{N}^\bullet)$  then  $\chi(\phi_p)$  is defined by

$$\begin{aligned} \forall \alpha \in \mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X}^L \Lambda_{\mathcal{O}_X}^q(\mathcal{L}_X), \quad \chi(\phi_p)(\alpha) &= 0 \text{ if } q \neq d_{\mathcal{L}_X} \\ \forall \omega \in \Lambda_{\mathcal{O}_X}^{d_{\mathcal{L}_X}}(\mathcal{L}_X), \quad \chi(\phi_p)(1 \otimes \omega)(e_{i,r}) &= (1 \otimes \omega) \otimes \phi_p(e_{i,r}) \end{aligned}$$

where  $(e_{i,r})_{i \in [1, n_r]}$  is a basis of the free module  $\mathcal{P}^r$ . It is easy to check that  $\chi$  is a morphism of complexes

**Remark :**

Note that if  $\mathcal{M}^\bullet$  and  $\mathcal{N}^\bullet$  are bounded complexes of filtered  $\mathcal{FD}(\mathcal{L}_X)$ -modules, we could take a resolution  $\mathcal{P}^\bullet$  of  $\mathcal{M}_{|V}^\bullet$  by finite free  $\mathcal{FD}(\mathcal{L}_V)$ -modules.

Then  $\mathcal{F}\mathcal{H}om_{\mathcal{F}\mathcal{D}(\mathcal{L}_X)}^\bullet \left( K_{\mathcal{L}_X}^\bullet, \mathcal{F}\mathcal{H}om_{\mathcal{F}\mathcal{D}(\mathcal{L}_X)}^\bullet (\mathcal{P}^\bullet, \mathcal{H}_{\mathcal{L}_X}) \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet \right) [d_{\mathcal{L}_X}]$  and  $\mathcal{F}\mathcal{H}om_{\mathcal{F}\mathcal{D}(\mathcal{L}_X)}^\bullet (\mathcal{P}^\bullet, \mathcal{N}^\bullet)$  are complexes of filtered  $\mathcal{F}\mathcal{D}(\mathcal{L}_V)$ -modules. The morphism  $\chi_{\mathcal{L}_X} (\mathcal{P}^\bullet, \mathcal{N}^\bullet)$  is a morphism of filtered complexes and

$$(\#) \quad Gr\chi_{\mathcal{L}_X} (\mathcal{P}^\bullet, \mathcal{N}^\bullet) = \chi_{\mathcal{L}_X^{ab}} (Gr\mathcal{P}^\bullet, Gr\mathcal{N}^\bullet).$$

**Corollary 3.2.4** *Let  $\mathcal{M}^\bullet$  and  $\mathcal{N}^\bullet$  be two elements of  $D_{coh}^b(\mathcal{D}(\mathcal{L}_X))$ . The groups  $\text{Hom}_{D^b(\mathcal{D}(\mathcal{L}_X))}(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$  and  $\text{Hom}_{D^b(\mathcal{D}(\mathcal{L}_X))} \left( \mathcal{O}_X, \underline{\mathcal{D}}_{\mathcal{L}_X}(\mathcal{M}^\bullet) \otimes_{\mathcal{O}_X}^L \mathcal{N}^\bullet \right)$  are isomorphic.*

### 3.3 Direct images

In this paragraph we recall results of [C3].

Let  $\Phi = (f, F)$  be a Lie algebroid morphism from  $(\mathcal{L}_X, \omega_X)$  to  $(\mathcal{L}_Y, \omega_Y)$ . Let  $\mathcal{M}^\bullet$  be an object of  $D^b(\mathcal{D}(\mathcal{L}_X)^{op})$ . In [C3], the direct image functor is defined by

$$\underline{\Phi}_! (\mathcal{M}^\bullet) = Rf_! \left( \mathcal{M}^\bullet \otimes_{\mathcal{D}(\mathcal{L}_X)}^L \mathcal{D}_{\mathcal{L}_X \rightarrow \mathcal{L}_Y} \right).$$

Then  $\underline{\Phi}_! (\mathcal{M}^\bullet)$  is in  $D^b(\mathcal{D}(\mathcal{L}_Y)^{op})$ . If  $\Phi = (f, Tf)$ , we recover the  $\mathcal{D}$ -module construction (see [S2] for example). Then  $\mathcal{D}_{\Theta_X \rightarrow \Theta_Y}$  is denoted by  $\mathcal{D}_{X \rightarrow Y}$  and  $\underline{\Phi}_!$  is denoted by  $\underline{f}_!$ .

**Proposition 3.3.1** *Let  $\Phi$  and  $\Psi$  be Lie algebroids morphism from  $(\mathcal{L}_X, \omega_X)$  to  $(\mathcal{L}_Y, \omega_Y)$  and from  $(\mathcal{L}_Y, \omega_Y)$  to  $(\mathcal{L}_Z, \omega_Z)$  respectively, then*

$$\underline{\Psi}_! \circ \underline{\Phi}_! = (\underline{\Psi} \circ \underline{\Phi})_!.$$

The proof of proposition 3.3.1 is similar to the  $\mathcal{D}$ -modules case (see [Bo] p. 251).

We recall here a definition due to Kashiwara (see [S-S]).

**Definition 3.3.2** *A right coherent  $\mathcal{D}(\mathcal{L}_X)$ -module is good if, for any compact subset  $K$  of  $X$ , there exists an open neighborhood  $U$  of  $K$  such that  $\mathcal{M}|_U$  has a filtration  $(\mathcal{M}_k)_{k \in [1, n]}$  by coherent right  $\mathcal{D}(\mathcal{L}_U)$ -submodules such that each quotient  $\mathcal{M}_k / \mathcal{M}_{k-1}$  is generated by a coherent  $\mathcal{O}_U$ -module.*

Note that if  $X$  is a smooth algebraic variety, all the coherent  $\mathcal{D}(\mathcal{L}_X)$ -modules are good. Good  $\mathcal{D}(\mathcal{L}_X)$ -modules form a thick subcategory of the category of coherent  $\mathcal{D}(\mathcal{L}_X)$ -modules. The associated full subcategory of  $D^b(\mathcal{D}(\mathcal{L}_X)^{op})$  consisting of objects with good cohomology is denoted by  $D_{good}^b(\mathcal{D}(\mathcal{L}_X)^{op})$ .

**Theorem 3.3.3** *Assume that  $\mathcal{M}^\bullet$  is in  $D_{good}^b(\mathcal{D}(\mathcal{L}_X)^{op})$  and that  $f$  is proper on  $\text{Supp}(\mathcal{M})$ , then  $\underline{\Phi}_!(\mathcal{M})$  is in  $D_{good}^b(\mathcal{D}(\mathcal{L}_Y)^{op})$ .*

The proof of Schneiders ([S2] p. 38) in the case of  $\mathcal{D}$ -modules extends without any change to our situation. The particular case where  $f$  is projective and  $\mathcal{M}$  has a global good filtration was treated in [Ka].

**Theorem 3.3.4** *Let  $X$  and  $Y$  be two complex manifolds. Let  $(\mathcal{L}_X, \omega_X)$  and  $(\mathcal{L}_Y, \omega_Y)$  be Lie algebroids over  $X$  and  $Y$  respectively. Let  $\Phi = (f, F)$  be a Lie algebroid morphism from  $(\mathcal{L}_X, \omega_X)$  to  $(\mathcal{L}_Y, \omega_Y)$ . Let  $\mathcal{M}^\bullet$  be an element of  $D_{good}^b(\mathcal{D}(\mathcal{L}_X)^{op})$  such that  $f$  is proper on the support of  $\mathcal{M}^\bullet$ . Then there is a functorial isomorphism from  $\underline{\Phi}_! \Delta_{\mathcal{L}_X}(\mathcal{M}^\bullet)$  to  $\Delta_{\mathcal{L}_Y} \underline{\Phi}_!(\mathcal{M}^\bullet)$  in  $D_{good}^b(\mathcal{D}(\mathcal{L}_Y)^{op})$ .*

Theorem 3.3.4 generalizes a result in Schneiders' thesis [S1] (see also the work of Schapira-Schneiders [S-S]) where the case of relative differential operators is treated. The algebraic smooth case had been previously treated by Bernstein ([Be], [Bo], [Ho]) (in the  $\mathcal{D}$ -modules context) for a proper morphism. Moreover Mebkhout had treated the absolute case (i.e  $Y$  consists of a single point, see corollary 4.3.6 in [Me1], [Me2]).

### 3.4 Inverse image

Let  $\Phi = (f, F)$  be a Lie algebroid morphism from  $(\mathcal{L}_X, \omega_X)$  to  $(\mathcal{L}_Y, \omega_Y)$ . Let  $\mathcal{R}^\bullet$  be an object of  $D^b(\mathcal{D}(\mathcal{L}_Y))$ . Set

$$\underline{\Phi}^{-1}(\mathcal{R}^\bullet) = \mathcal{D}_{\mathcal{L}_X \rightarrow \mathcal{L}_Y} \underset{f^{-1}\mathcal{D}(\mathcal{L}_Y)}{\overset{L}{\otimes}} f^{-1}\mathcal{R}^\bullet.$$

Then  $\underline{\Phi}^{-1}(\mathcal{R}^\bullet)$  is in  $D^b(\mathcal{D}(\mathcal{L}_X))$ . We call it the inverse image of  $\mathcal{R}^\bullet$  by  $\Phi$ . If  $\Phi = (f, Tf)$ , we recover the  $\mathcal{D}$ -module construction (see [S2] for example). Then  $\mathcal{D}_{\Theta_X \rightarrow \Theta_Y}$  is denoted by  $\mathcal{D}_{X \rightarrow Y}$  and  $\underline{\Phi}^{-1}$  is denoted by  $\underline{f}^{-1}$ .

**Remark :**

If  $X = Y$  and  $f = id$ , then  $\underline{\Phi}^{-1}(\mathcal{R}^\bullet)$  is nothing but  $\mathcal{R}^\bullet$  considered as an element of  $D^b(\mathcal{D}(\mathcal{L}_X))$ . We will write  $\underline{\Phi}^{-1}(\mathcal{R}^\bullet) = \mathcal{R}^\bullet_{|\mathcal{L}_X}$ .

To define the inverse image of an object of  $D^b(\mathcal{D}(\mathcal{L}_Y)^{op})$ , as in the  $\mathcal{D}$ -modules case, one uses the  $(f^{-1}\mathcal{D}(\mathcal{L}_Y) \otimes \mathcal{D}(\mathcal{L}_X)^{op})$ -bimodule  $\mathcal{D}_{\mathcal{L}_Y \leftarrow \mathcal{L}_X}$  defined by

$$\mathcal{D}_{\mathcal{L}_Y \leftarrow \mathcal{L}_X} = \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*) \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathcal{L}_X \rightarrow \mathcal{L}_Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Lambda^{d_{\mathcal{L}_Y}}(\mathcal{L}_Y).$$

**Proposition 3.4.1** *Let  $\Phi$  and  $\Psi$  be Lie algebroids morphisms from  $(\mathcal{L}_X, \omega_X)$  to  $(\mathcal{L}_Y, \omega_Y)$  and from  $(\mathcal{L}_Y, \omega_Y)$  to  $(\mathcal{L}_Z, \omega_Z)$  respectively . Then*

$$\underline{\Phi}^{-1} \circ \underline{\Psi}^{-1} = (\underline{\Psi} \circ \underline{\Phi})^{-1}.$$

The proof of proposition 3.4.1 is analogous to the  $\mathcal{D}$ -modules case (see [Bo] p. 251).

**Proposition 3.4.2** *Let  $\Phi$  be a Lie algebroid morphism from  $(\mathcal{L}_X, \omega_X)$  to  $(\mathcal{L}_Y, \omega_Y)$ . There is an isomorphism between  $\underline{\Phi}^{-1}\left(\mathcal{M}^\bullet \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{N}^\bullet\right)$  and  $\underline{\Phi}^{-1}(\mathcal{M}^\bullet) \overset{L}{\otimes}_{\mathcal{O}_X} \underline{\Phi}^{-1}(\mathcal{N}^\bullet)$  for any  $\mathcal{M}^\bullet, \mathcal{N}^\bullet$  in  $D^b(\mathcal{D}(\mathcal{L}_Y))$*

The proof of proposition 3.4.2 is analogous to the  $\mathcal{D}$ -modules case. We refer the reader to [Bj].

**Proposition 3.4.3** *Let  $\Phi = (f, F)$  be a Lie algebroid morphism from  $(X, \mathcal{L}_X)$  to  $(Y, \mathcal{L}_Y)$  and let  $\Phi^{ab} = (f, F^{ab})$  be the Lie algebroid morphism it induces between the abelian Lie algebroids  $(X, \mathcal{L}_X^{ab})$  and  $(Y, \mathcal{L}_Y^{ab})$ . Let  $\mathcal{R}$  be a filtered  $\mathcal{FD}(\mathcal{L}_Y)$ -module and let*

$$\mathcal{L}^\bullet : \dots \rightarrow \bigoplus_{i=1}^{n_1} \mathcal{FD}(\mathcal{L}_Y)(r_{i,1}) \rightarrow \bigoplus_{i=1}^{n_0} \mathcal{FD}(\mathcal{L}_Y)(r_{i,0}) \rightarrow \mathcal{R} \rightarrow 0$$

*be a filtered resolution of  $\mathcal{R}$ . Then  $\underline{\Phi}^{-1}(\mathcal{R})$  is isomorphic to  $\mathcal{O}_{f^{-1}V} \otimes_{f^{-1}\mathcal{O}_V} f^{-1}\mathcal{L}^\bullet$  and we filter it by the image of the complex  $\mathcal{O}_{f^{-1}V} \otimes_{f^{-1}\mathcal{O}_V} f^{-1}\mathcal{F}_k\mathcal{L}^\bullet$ . One has an isomorphism*

$$Gr\underline{\Phi}^{-1}(\mathcal{R}) \simeq \underline{\Phi}^{ab^{-1}}(Gr\mathcal{R}).$$

*Proof of proposition 3.4.3 :*

The image of the complex  $\mathcal{O}_{f^{-1}V} \otimes_{f^{-1}\mathcal{O}_V} f^{-1}\mathcal{F}_k\mathcal{L}^\bullet$  in  $\mathcal{O}_{f^{-1}V} \otimes_{f^{-1}\mathcal{O}_V} f^{-1}\mathcal{F}\mathcal{L}^\bullet$  is isomorphic to the complex  $\mathcal{O}_{f^{-1}V} \otimes_{f^{-1}\mathcal{O}_V} f^{-1}\mathcal{F}_k\mathcal{L}^\bullet$  itself because the components of  $\mathcal{L}^\bullet$  are of the type  $\mathcal{F}\mathcal{D}(\mathcal{L}_Y)(r)$ . As  $Gr\mathcal{F}\mathcal{L}^i$  is a free  $\mathcal{O}_V$ -module, the complexes  $Gr\left(\mathcal{O}_{f^{-1}V} \otimes_{f^{-1}\mathcal{O}_V} f^{-1}\mathcal{F}\mathcal{L}^\bullet\right)$  and  $\left(\mathcal{O}_{f^{-1}V} \otimes_{f^{-1}\mathcal{O}_V} f^{-1}Gr\mathcal{F}\mathcal{L}^\bullet\right)$  are isomorphic. This finishes the proof of the proposition.

The following question arises naturally: Let  $\mathcal{R}^\bullet$  be an element of  $D_{coh}^b(\mathcal{D}(\mathcal{L}_Y))$ . Give a sufficient condition so that  $\underline{\Phi}^{-1}(\mathcal{R}^\bullet)$  is in  $D_{coh}^b(\mathcal{D}(\mathcal{L}_X))$ .

To do so, we introduce, as in the  $\mathcal{D}$ -modules case, the notion of non characteristicity.

Let  $L_X$  (respectively  $L_Y$ ) be the vector bundle associated to  $\mathcal{L}_X$  (respectively  $\mathcal{L}_Y$ ). We have the following diagram

$$L_X^* \xleftarrow{^tF} X \times_Y L_Y^* \xrightarrow{F_\pi} L_Y^*$$

where, for  $x \in X$  and  $\lambda \in L_{f(x)}^*$ , one has

$$\begin{aligned} ^tF(x, f(x), \lambda) &= (x, ^tF(\lambda)) \\ F_\pi(x, f(x), \lambda) &= (f(x), \lambda). \end{aligned}$$

Denote by  $NS(X \times_Y L_Y^*)$  the null section of  $X \times_Y L_Y^*$ . Let  $\mathcal{R}^\bullet$  be in  $D_{coh}^b(\mathcal{D}(\mathcal{L}_Y))$ . We will say that  $\mathcal{R}^\bullet$  is non characteristic with respect to  $\Phi$  if the following inclusion holds.

$$F_\pi^{-1}(\text{char}(\mathcal{R}^\bullet)) \cap \{(x, f(x), \lambda) \in X \times_Y L_Y^* \mid \lambda \circ F_x = 0\} \subset NS(X \times_Y L_Y^*).$$

Kashiwara has shown that, in the  $\mathcal{D}$ -modules case, the non characteristicity condition ensures that  $\underline{\Phi}^{-1}(\mathcal{R}^\bullet)$  is in  $D_{coh}^b(\mathcal{D}(\mathcal{L}_X))$  (see [S2]). We will now generalize this result to Lie algebroids.

**Theorem 3.4.4** *Let  $(X, \mathcal{L}_X)$  and  $(Y, \mathcal{L}_Y)$  be Lie algebroids over the complex manifolds  $X$  and  $Y$  respectively. Let  $\Phi = (f, F)$  be a Lie algebroid morphism from  $(X, \mathcal{L}_X)$  to  $(Y, \mathcal{L}_Y)$ . Let  $\mathcal{R}^\bullet$  be an element of  $D_{coh}^b(\mathcal{D}(\mathcal{L}_Y))$  which is supposed to be non characteristic with respect to  $\Phi$ . Then  $\underline{\Phi}^{-1}(\mathcal{R}^\bullet)$  is in  $D_{coh}^b(\mathcal{D}(\mathcal{L}_X))$  and*

$$\text{Char}(\underline{\Phi}^{-1}(\mathcal{R}^\bullet)) \subset {}^tFF_\pi(\text{Char}\mathcal{R}^\bullet).$$

*Proof of theorem 3.4.4*

Let  $\mathcal{R}^\bullet$  be an element of  $D_{coh}^b(\mathcal{D}(\mathcal{L}_Y))$  which is supposed to be non characteristic for  $\Phi$ . By a standard induction argument on the number of non zero cohomology groups of  $\mathcal{R}^\bullet$ , one sees that it is enough to prove the theorem for a module which we will denote by  $\mathcal{R}$ . The question being local, we may assume

$$\begin{aligned}\mathcal{L}_X &= \mathcal{O}_X e_1 \oplus \dots \oplus \mathcal{O}_X e_r \\ \mathcal{L}_Y &= \mathcal{O}_Y e_{r+1} \oplus \dots \oplus \mathcal{O}_Y e_m.\end{aligned}$$

One may shrink  $Y$  so that  $\mathcal{R}$  is endowed with a good filtration. The module  $Gr\mathcal{R}$  is then a graded coherent  $S(\mathcal{L}_Y)$ -module non characteristic for the morphism  $\Phi^{ab}$  (the morphism induced by  $\Phi$  on the abelian Lie algebroids  $\mathcal{L}_X^{ab}$  and  $\mathcal{L}_Y^{ab}$ ).

Hence propositions 2.1.7 and 3.4.3 show that it is enough to prove the theorem in the case of  $\Phi^{ab}$  (that is to say the abelian case) for a graded coherent  $S(\mathcal{L}_Y)$ -module non characteristic for  $\Phi^{ab}$ . Introduce the following maps :

$$\begin{aligned}t : X &\rightarrow X \times Y \\ x &\mapsto (x, f(x)) \\ q : X \times Y &\rightarrow Y \\ (x, y) &\mapsto f(x) \\ T : \mathcal{L}_X &\rightarrow \mathcal{L}_X \oplus \left( \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{L}_Y \right) \\ D &\mapsto D + F(D)\end{aligned}$$

and the following abelian Lie algebroids :

$$\begin{aligned}&\left( X \times Y, \mathcal{P}_{X \times Y}^{ab} = \mathcal{O}_{X \times Y} \otimes_{p_1^{-1}\mathcal{O}_X} p_1^{-1}T(\mathcal{L}_X^{ab}) \right) \\ &\left( X \times Y, \mathcal{Q}_{X \times Y}^{ab} = \mathcal{O}_{X \times Y} \otimes_{q^{-1}\mathcal{O}_Y} q^{-1}\mathcal{L}_Y^{ab} \right).\end{aligned}$$

$(id, T)$  is a Lie algebroid isomorphism from  $(X, \mathcal{L}_X^{ab})$  to  $(X, T(\mathcal{L}_X^{ab}))$ . The pair  $(t, id)$  defines a Lie algebroid morphism from  $(X, T(\mathcal{L}_X^{ab}))$  to

$(X \times Y, \mathcal{P}_{X \times Y}^{ab})$ . Let  $\psi$  be the map

$$\begin{aligned} \mathcal{P}_{X \times Y}^{ab} &\rightarrow \mathcal{Q}_{X \times Y}^{ab} \\ b \otimes (D + \sum_i a_i \otimes \Delta_i) &\mapsto b \left( \sum_i a_i \otimes \Delta_i \right). \end{aligned}$$

The pair  $\Psi^{ab} = (id, \psi)$  is a Lie algebroid morphism from  $\mathcal{P}_{X \times Y}^{ab}$  to  $\mathcal{Q}_{X \times Y}^{ab}$ . Lastly the pair  $(q, id)$  is a Lie algebroid morphism from  $(X \times Y, \mathcal{Q}_{X \times Y}^{ab})$  to  $(Y, \mathcal{L}_Y^{ab})$ . Then we have  $(f, F^{ab}) = (q, id) \circ (id, \psi) \circ (t, id) \circ (id, T)$ . This factorization shows that it is enough to prove our proposition in the cases where  $(f, F^{ab}) = (id, F^{ab})$  and  $(f, F^{ab}) = (f, id)$ .

a) Case where  $\Phi^{ab} = (f, id)$ .

Necessarily  $\mathcal{L}_X^{ab} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{L}_Y^{ab}$ . In this case any module is non characteristic with respect to  $\Phi^{ab}$  and  $\underline{\Phi}^{ab^{-1}}(S_{\mathcal{O}_Y}(\mathcal{L}_Y^{ab})) = S_{\mathcal{O}_X}(\mathcal{L}_X^{ab})$ . Thus the coherence is established.

b) Case where  $\Phi^{ab} = (id, F^{ab})$ .

We introduce the abelian Lie algebroid over  $Z = X \times Y$ ,  $(Z, \mathfrak{a}_Z = \mathcal{P}_Z^{ab} \oplus \mathcal{Q}_Z^{ab})$  and we factorize  $(id, F^{ab}) = (id, \Pi^{ab}) \circ (id, I^{ab})$  where

$$\begin{aligned} I^{ab} : \mathcal{P}_Z^{ab} &\rightarrow \mathfrak{a}_Z \\ D &\mapsto D + F^{ab}(D) \end{aligned}$$

$$\begin{aligned} \Pi^{ab} : \mathfrak{a}_Z &\rightarrow \mathcal{Q}_Z^{ab} \\ D + \Delta &\mapsto \Delta \end{aligned}$$

The case of  $(id, \Pi^{ab})$  is easy. Let us now treat the case of  $(id, I^{ab})$ . Put  $\mathcal{P}'_Z^{ab} = I^{ab}(\mathcal{P}_Z^{ab})$ . Introduce the following abelian Lie algebroids with zero anchor (over  $Z$ ):

$$V_j = \mathcal{P}'_Z^{ab} \oplus \mathcal{O}_Z e_{r+1} \dots \oplus \mathcal{O}_Z e_{r+j}.$$

We have the following sequence of Lie algebroids embeddings :

$$V_0 = \mathcal{P}'_Z^{ab} \hookrightarrow V_1 \hookrightarrow \dots \hookrightarrow V_{m-r} = \mathfrak{a}_Z.$$

So we may assume that  $m - r = 1$  and we are in the following situation

$$I^{ab} : \mathcal{P}'_Z{}^{ab} \hookrightarrow \mathcal{P}'_Z{}^{ab} \oplus \mathcal{O}_Z e_{r+1}.$$

Let  $\eta$  be the element of  $\mathfrak{a}_Z^*$  such that  $\eta|_{\mathcal{P}'_Z} = 0$  and  $\eta(e_{r+1}) = 1$ .

Restricting  $Z$ , we can take  $(\sigma_1, \dots, \sigma_q)$  to be a generating system of the  $S(\mathfrak{a}_Z)$ -module  $\mathcal{R}$ . For each  $j$ , denote by  $\mathcal{I}_j$  the annihilating (graded) ideal of  $\sigma_j$  in  $S(\mathfrak{a}_Z)$ . The module  $\frac{S(\mathfrak{a}_Z)}{\mathcal{I}_j}$  is non characteristic for  $\Phi^{ab}$ .

Let  $z$  be in  $Z$ . If  $t \neq 0$ , the pair  $(z, te_{r+1}^*)$  is not in  $\text{char}(\frac{S(\mathfrak{a}_Z)}{\mathcal{I}_j})$ . So there exists an homogeneous element  $s_j$  in  $\mathcal{I}_j$  such that  $s_j(z, te_{r+1}^*) \neq 0$ . Put

$$s_j = \sum_{l=0}^{p_j} \alpha_{l,j} e_{r+1}^l$$

with  $\alpha_{l,j} \in S^{p_j-l}(\mathcal{P}'_Z)$ . Shrinking  $Z$ , we may assume that  $\alpha_{p_j,j}$  is an invertible element of  $\mathcal{O}_Z$ . We have a natural epimorphism

$$\bigoplus_{j=1}^q \frac{S_{\mathcal{O}_Z}(\mathfrak{a}_Z)}{\langle s_j \rangle} \rightarrow \text{Gr} \mathcal{R} \rightarrow 0.$$

So it is enough to prove that  $\frac{S_{\mathcal{O}_Z}(\mathfrak{a}_Z)}{\langle s_j \rangle}$  is a coherent  $S_{\mathcal{O}_Z}(\mathcal{P}'_Z{}^{ab})$ -module. And this is obvious because Euclidean division in  $S_{\mathcal{O}_Z}(\mathfrak{a}_Z)$  provides an isomorphism of  $S_{\mathcal{O}_Z}(\mathcal{P}'_Z)$ -modules  $\chi : \frac{S_{\mathcal{O}_Z}(\mathfrak{a}_Z)}{\langle s_j \rangle} \rightarrow S_{\mathcal{O}_Z}(\mathcal{P}'_Z)^{p_j}$ . This finishes the proof of the proposition.

## 4 A duality theorem

This section is devoted to the generalization of a duality theorem due to Kashiwara, Kawai and Sato in the  $\mathcal{D}$ -modules case (see [SKK]).

### 4.1 Statement

We will construct a functorial arrow  $\underline{D}_{\mathcal{L}_X} \Phi^{-1}(\mathcal{R}^\bullet) \rightarrow \Phi^{-1} \underline{D}_{\mathcal{L}_Y}(\mathcal{R}^\bullet)$  for any element  $\mathcal{R}^\bullet$  in  $D^b(\mathcal{D}(\mathcal{L}_Y))$  such that  $\Phi^{-1}(\mathcal{R}^\bullet)$  is in  $D_{\text{coh}}^b(\mathcal{D}(\mathcal{L}_X))$ . Our construction will be in particular valid if  $\mathcal{R}^\bullet$  is non characteristic for  $\Phi$ .



Using corollary 3.2.4 , the element 1 of  $\text{Hom}_{D^b(\mathcal{D}(\mathcal{L}_Y))}(\mathcal{R}^\bullet, \mathcal{R}^\bullet)$  provides an arrow

$$\mathcal{O}_Y \rightarrow \underline{D}_{\mathcal{L}_Y}(\mathcal{R}^\bullet) \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{R}^\bullet$$

in  $D^b(\mathcal{D}(\mathcal{L}_Y))$ . Hence an arrow

$$\begin{aligned} \underline{\Phi}^{-1}(\mathcal{O}_Y) = \mathcal{O}_X &\rightarrow \underline{\Phi}^{-1} \left( \underline{D}_{\mathcal{L}_Y}(\mathcal{R}^\bullet) \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{R}^\bullet \right) \\ &\simeq \underline{\Phi}^{-1} \left( \underline{D}_{\mathcal{L}_Y}(\mathcal{R}^\bullet) \right) \overset{L}{\otimes}_{\mathcal{O}_X} \underline{\Phi}^{-1}(\mathcal{R}^\bullet) \end{aligned}$$

Using the fact that  $\underline{D}_{\mathcal{L}_X}^2(\underline{\Phi}^{-1}(\mathcal{R}^\bullet)) \simeq \underline{\Phi}^{-1}(\mathcal{R}^\bullet)$  and corollary 3.2.4 again, we get an arrow

$$\underline{D}_{\mathcal{L}_X} \underline{\Phi}^{-1}(\mathcal{R}^\bullet) \rightarrow \underline{\Phi}^{-1} \underline{D}_{\mathcal{L}_Y}(\mathcal{R}^\bullet).$$

**Theorem 4.1.1** *Let  $\Phi$  be a Lie algebroid morphism from  $(X, \mathcal{L}_X)$  to  $(Y, \mathcal{L}_Y)$ . Let  $\mathcal{R}^\bullet$  be in  $D_{\text{coh}}^b(\mathcal{D}(\mathcal{L}_Y))$  which is supposed to be non characteristic with respect to  $\Phi$ . The functorial morphism from  $\underline{D}_{\mathcal{L}_X} \underline{\Phi}^{-1}(\mathcal{R}^\bullet)$  to  $\underline{\Phi}^{-1} \underline{D}_{\mathcal{L}_Y}(\mathcal{R}^\bullet)$  constructed above is an isomorphism.*

## 4.2 Proof of theorem 4.1.1

As the proof of theorem 3.4.4, the proof of theorem 4.1.1 is into two steps. First, we will reduce to the abelian case, then we will prove the theorem in the abelian case.

a) Reduction to the abelian case.

### Preliminary remark :

Let  $\mathcal{Q}^\bullet$  be a bounded complex of locally free  $\mathcal{D}(\mathcal{L}_X)$ -modules and  $\mathcal{N}^\bullet$  be a bounded complex of  $\mathcal{D}(\mathcal{L}_X)$ -modules. Let  $\mathbb{C}_X \xrightarrow{\alpha} \mathcal{S}^\bullet$  be a resolution of  $\mathbb{C}_X$  by c-soft sheaves. One has an isomorphism (see [K-S 3] exercise 4.7)

$$\text{Hom}_{D^b(\mathcal{D}(\mathcal{L}_X))}(\mathcal{Q}^\bullet, \mathcal{N}^\bullet) \simeq H^0 \left( X, \mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}^\bullet(\mathcal{Q}^\bullet, \mathcal{N}^\bullet) \overset{L}{\otimes}_{\mathbb{C}_X} \mathcal{S}^\bullet \right).$$

There is a morphism

$$\begin{aligned} H^0 \left( \mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}^\bullet(\mathcal{Q}^\bullet, \mathcal{N}^\bullet) \right) &\rightarrow \text{Hom}_{D^b(\mathcal{D}(\mathcal{L}_X))}(\mathcal{Q}^\bullet, \mathcal{N}^\bullet) \\ [\phi] &\mapsto [\phi \otimes \alpha(1)]. \end{aligned}$$

If  $[\phi]$  is an isomorphism, so is  $[\phi \otimes \alpha(1)]$ . We will say that the morphism  $[\phi \otimes \alpha(1)]$  is induced by  $\phi$ .

To show that the morphism we have constructed is an isomorphism, we may assume that  $\mathcal{R}^\bullet$  is a module (denoted  $\mathcal{R}$ ) which is non characteristic with respect to the morphism  $\Phi$  and endowed with a good filtration. Then  $Gr\mathcal{R}$  is a coherent  $S(\mathcal{L}_Y)$ -module non characteristic for  $\Phi^{ab}$ . As we may reason locally, we may take  $\mathcal{P}^\bullet$  to be a resolution of  $\mathcal{R}$  by finite free  $\mathcal{D}(\mathcal{L}_Y)$ -modules. With the notation of corollary 3.2.3, we see that the arrow  $\mathcal{O}_Y \rightarrow \underline{D}_{\mathcal{L}_Y}(\mathcal{R}) \underset{\mathcal{O}_Y}{\overset{L}{\otimes}} \mathcal{R}$  in  $D^b(\mathcal{D}(\mathcal{L}_Y))$  is induced by the arrow  $\Gamma(X, \chi_{\mathcal{L}_Y}(\mathcal{P}^\bullet, \mathcal{P}^\bullet))(id)$  from  $K_{\mathcal{L}_Y}^\bullet$  to  $\mathcal{H}om_{\mathcal{D}(\mathcal{L}_Y)}(\mathcal{P}^\bullet, \mathcal{H}_{\mathcal{L}_Y}) \underset{\mathcal{O}_Y}{\otimes} \mathcal{P}^\bullet[d_{\mathcal{L}_Y}]$ .

The arrow  $\mathcal{O}_X \rightarrow \underline{\Phi}^{-1}(\underline{D}_{\mathcal{L}_Y}(\mathcal{R})) \underset{\mathcal{O}_X}{\overset{L}{\otimes}} \underline{\Phi}^{-1}(\mathcal{R})$  in  $D^b(\mathcal{D}(\mathcal{L}_X))$  is induced by the morphism  $id_{\mathcal{O}_X} \otimes \Gamma(X, \chi_{\mathcal{L}_Y}(\mathcal{P}^\bullet, \mathcal{P}^\bullet))(id)$  from  $\mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}K_{\mathcal{L}_Y}^\bullet$  to  $\mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\left(\mathcal{H}om_{\mathcal{D}(\mathcal{L}_Y)}(\mathcal{P}^\bullet, \mathcal{H}_{\mathcal{L}_Y}) \underset{\mathcal{O}_Y}{\otimes} \mathcal{P}^\bullet\right)[d_{\mathcal{L}_Y}]$ . One sees that the arrow  $\underline{D}_{\mathcal{L}_X} \circ \underline{\Phi}^{-1}(\mathcal{R}) \rightarrow \underline{\Phi}^{-1} \circ \underline{D}_{\mathcal{L}_Y}(\mathcal{R})$  is also induced by a morphism and we want to see that this morphism is an isomorphism.

We have

$$Gr\Gamma(X, \chi_{\mathcal{L}_Y}(\mathcal{P}^\bullet, \mathcal{P}^\bullet))(id) = \Gamma(X, \chi_{\mathcal{L}_Y^{ab}}(Gr\mathcal{P}^\bullet, Gr\mathcal{P}^\bullet))(id)$$

$$Gr(id_{\mathcal{O}_X} \otimes f^{-1}\Gamma(X, \chi_{\mathcal{L}_Y}(\mathcal{P}^\bullet, \mathcal{P}^\bullet))(id)) = id_{\mathcal{O}_X} \otimes f^{-1}\Gamma(X, \chi_{\mathcal{L}_Y^{ab}}(Gr\mathcal{P}^\bullet, Gr\mathcal{P}^\bullet))(id).$$

Using the last remark of section 3.2 (equality #), one sees that it is enough to show the theorem 4.1.1 for the case of  $\Phi^{ab}$ .

b) Proof of theorem 4.1.1 in the abelian case.

From now on, we assume that  $\Phi = \Phi^{ab}$  and that  $\mathcal{R}$  is a graded  $S(\mathcal{L}_Y)$ -module non characteristic for  $\Phi^{ab}$ . As in the proof of theorem 3.4.4, we introduce the maps  $t, q, T$  as well as the abelian Lie algebroids  $\mathcal{P}_{X \times Y}^{ab}$  and  $\mathcal{Q}_{X \times Y}^{ab}$  and we factorize  $(f, F^{ab})$  into

$$(f, F^{ab}) = (q, id) \circ (id, \psi) \circ (t, id) \circ (id, T).$$

This factorization shows that it is enough to prove the theorem in the case where  $(f, F^{ab}) = (id, F^{ab})$  and  $(f, F^{ab}) = (f, id)$ .

Let us first treat the case where  $\Phi^{ab} = (f, id)$ . Necessarily  $\mathcal{L}_X = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{L}_Y$ . In this case any module is non characteristic with respect to  $\Phi^{ab}$  so that we can assume that  $\mathcal{R} = S(\mathcal{L}_Y)$ . Then  $\underline{\Phi}^{ab^{-1}}(S_{\mathcal{O}_Y}(\mathcal{L}_Y^{ab})) = S_{\mathcal{O}_X}(\mathcal{L}_X^{ab})$  and the morphism of functor  $\underline{D}_{\mathcal{L}_X} \circ (\underline{\Phi}^{ab})^{-1} \rightarrow (\underline{\Phi}^{ab})^{-1} \circ \underline{D}_{\mathcal{L}_Y}$  is the identity as easily checked for  $\mathcal{R} = S(\mathcal{L}_Y)$ . So the theorem is proved in the case where  $\Phi^{ab} = (f, Id)$ .

Let us now treat the case where  $\Phi^{ab} = (id, F)$ . As in the proof of theorem 3.4.4, we introduce the abelian Lie algebroid over  $Z = X \times Y$ ,  $(Z, \mathfrak{a}_Z = \mathcal{P}_Z^{ab} \oplus \mathcal{Q}_Z^{ab})$  and we factorize  $(id, F^{ab})$  into

$$(id, F^{ab}) = (id, \Pi^{ab}) \circ (id, I^{ab}).$$

We can treat the cases where  $(id, F^{ab}) = (id, \Pi^{ab})$  and  $(id, F^{ab}) = (id, I^{ab})$  independently.

Assume that  $(id, F^{ab}) = (id, \Pi^{ab})$ . Recall that  $\underline{D}_{\mathfrak{a}_Z} \left( \frac{S(\mathfrak{a}_Z)}{S(\mathfrak{a}_Z)\mathcal{P}_Z^{ab}} \right)$  is isomorphic to  $S(\mathcal{Q}_Z^{ab}) \otimes_{\mathcal{O}_Z} \Lambda^{d_{\mathcal{Q}_Z^{ab}}}(\mathcal{Q}_Z^{ab})[d_{\mathcal{Q}_Z^{ab}}]$  in  $D^b(S(\mathcal{Q}_Z^{ab}) \otimes S(\mathcal{Q}_Z^{ab})^{op})$  ([C2]). These remarks allow us to write the following sequence of isomorphisms in  $D^b(S(\mathcal{Q}_Z^{ab}))$ :

$$\begin{aligned} \underline{D}_{\mathfrak{a}_Z}(\mathcal{R}^\bullet) &\simeq \underline{D}_{\mathfrak{a}_Z} \left( \frac{S(\mathfrak{a}_Z)}{S(\mathfrak{a}_Z)\mathcal{P}_Z^{ab}} \otimes_{S(\mathcal{Q}_Z^{ab})} \mathcal{R}^\bullet \right) \\ &\simeq R\mathcal{H}om_{S(\mathcal{Q}_Z)} \left( \mathcal{R}^\bullet, \underline{D}_{\mathfrak{a}_Z} \left( \frac{S(\mathfrak{a}_Z)}{S(\mathfrak{a}_Z)\mathcal{P}_Z^{ab}} \right) \right) \\ &\simeq \underline{D}_{\mathcal{Q}_Z}(\mathcal{R}^\bullet). \end{aligned}$$

Let us now treat the case where  $\Phi^{ab} = (id, I^{ab})$ . Put  $\mathcal{P}'_Z = I^{ab}(\mathcal{P}_Z^{ab})$ . Reasoning as in the proof of theorem 3.4.4 and keeping the notation, we may assume that  $\mathcal{Q}_Y$  is a one dimensional free  $\mathcal{O}_Y$ -module and put  $\mathcal{Q}_Y = \mathcal{O}_Y e_{r+1}$ . We have then the following Lie algebroids embedding

$$\mathcal{P}'_Z \hookrightarrow \mathfrak{a}_Z = \mathcal{P}'_Z \oplus \mathcal{O}_Z e_{r+1}.$$

$\mathcal{R}$  is a graded coherent  $S(\mathbf{a}_Z)$ -module non characteristic for the morphism  $\Phi^{ab} = (id, I^{ab})$ .

We know from the proof of theorem 3.4.4 that there exists an epimorphism

$$\bigoplus_{j=1}^q \frac{S_{\mathcal{O}_Z}(\mathbf{a}_Z)}{\langle u_j \rangle} \rightarrow \mathcal{R} \rightarrow 0$$

with  $u_j$  of the form

$$u_j = \sum_{l=0}^{p_j} \alpha_{l,j} e_{r+1}^l$$

with  $\alpha_{l,j} \in S^{p_j-l}(\mathcal{P}'_Z)$  and  $\alpha_{p_j,j}$  invertible in  $\mathcal{O}_Z$ .

Consequently, it is enough to show the theorem for  $\mathcal{R} = \frac{S_{\mathcal{O}_Z}(\mathbf{a}_Z)}{\langle u_j \rangle}$  and in this case it is easy to make the arrow

$$\underline{D}_{\mathcal{P}'_Z} \circ (\Phi^{ab})^{-1}(\mathcal{R}) \rightarrow (\Phi^{ab})^{-1} \circ \underline{D}_{\mathbf{a}_Z}(\mathcal{R})$$

explicit, and to see that it is an isomorphism. This finishes the proof of theorem 4.1.1.

**Corollary 4.2.1** *Let  $\Psi$  be a Lie algebroid morphism from  $(X, \mathcal{L}_X)$  to  $(Y, \mathcal{L}_Y)$ . Let  $\mathcal{R}$  be a left  $\mathcal{D}(\mathcal{L}_Y)$ -module which is coherent as an  $\mathcal{O}_Y$ -module. For short, put  $\mathcal{N}^* = R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}, \mathcal{O}_Y)$  with its  $\mathcal{D}(\mathcal{L}_Y)$ -module structure defined in section 3.2. There is a functorial isomorphism of right  $\mathcal{D}(\mathcal{L}_X)$ -modules from  $\mathcal{E}xt_{\mathcal{D}(\mathcal{L}_X)}^{d_{\mathcal{L}_X}+i} \left( \mathcal{D}_{\mathcal{L}_X \rightarrow \mathcal{L}_Y} \underset{f^{-1}\mathcal{D}(\mathcal{L}_Y)}{\overset{L}{\otimes}} f^{-1}\mathcal{N}, \mathcal{D}(\mathcal{L}_X) \right)$  to  $Tor_i^{f^{-1}\mathcal{D}(\mathcal{L}_Y)} \left( f^{-1}(\mathcal{N}^* \otimes \Lambda^{d_{\mathcal{L}_Y}}(\mathcal{L}_Y^*)) , \mathcal{D}_{\mathcal{L}_Y \leftarrow \mathcal{L}_X} \right)$ .*

## 5 Adjunction formulas and applications

### 5.1 Adjunction formulas

Generalizing the results of [K-S 2] ( chapter 7), we obtain the following adjunction formulas.

**Theorem 5.1.1** *Let  $(X, \mathcal{L}_X)$  and  $(Y, \mathcal{L}_Y)$  be two Lie algebroids over  $X$  and  $Y$  respectively. Set  $d_{\mathcal{L}_X} = \text{rank}(\mathcal{L}_X)$  and  $d_{\mathcal{L}_Y} = \text{rank}(\mathcal{L}_Y)$ . Let  $\Phi = (f, F)$  be*

a) Lie algebroid morphism from  $(X, \mathcal{L}_X)$  to  $(Y, \mathcal{L}_Y)$ .

a) Let  $\mathcal{M} \in D^b(\mathcal{D}(\mathcal{L}_Y))$  and  $\mathcal{N} \in D^b(\mathcal{D}(\mathcal{L}_X)^{op})$ . Then  $\underline{\Phi}_!(\mathcal{N}) \overset{L}{\otimes}_{\mathcal{D}(\mathcal{L}_Y)} \mathcal{M}$  and  $Rf_! \left( \mathcal{N} \overset{L}{\otimes}_{\mathcal{D}(\mathcal{L}_X)} \underline{\Phi}^{-1}(\mathcal{M}) \right)$  are isomorphic in  $D^b(\mathbb{C}_Y)$ .

b) Assume that  $\mathcal{M}$  is in  $D_{coh}^b(\mathcal{D}(\mathcal{L}_Y))$  and that  $\Phi$  is non characteristic for  $\mathcal{M}$ . The objects  $Rf_! R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\underline{\Phi}^{-1}(\mathcal{M}), \mathcal{N}) [d_{\mathcal{L}_X} - d_{\mathcal{L}_Y}]$  and  $R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_Y)}(\mathcal{M}, \underline{\Phi}_!(\mathcal{N}))$  are isomorphic in  $D^b(\mathbb{C}_Y)$ .

c) Assume that  $\mathcal{N}$  is in  $D_{good}^b(\mathcal{D}(\mathcal{L}_X))$  and that  $f$  is proper on  $\text{Supp}\mathcal{N}$ . Then  $Rf_* R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)}(\mathcal{N}, \underline{\Phi}^{-1}\mathcal{M}) [dimX - dimY]$  and  $R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_Y)} \left( \underline{\Phi}_!(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{N}), \Omega_Y \otimes_{\mathcal{O}_Y} \mathcal{M} \right)$  are isomorphic in  $D^b(\mathbb{C}_Y)$ .

*Proof of theorem 5.1.1 :* a) is straightforward , b) follows from theorem 4.1.1 and c) follows from theorem 3.3.4.

## 5.2 Applications

### Application 1

Note that the proof of theorem 4.1.1 holds in the case of Lie algebras over any field  $k$ . As a corollary of Theorem 5.1.1, we get the result below where we adopt the following notation : Let  $V$  be an  $n$  dimensional vector space, and let  $\det(V) = \Lambda^n(V)$ .

**Corollary 5.2.1** *Let  $\mathfrak{h}$  and  $\mathfrak{g}$  be two finite dimensional Lie algebras over a field and let  $\phi$  be a Lie algebra morphism from  $\mathfrak{h}$  to  $\mathfrak{g}$ . Let  $M$  be a finitely generated  $U(\mathfrak{g})$ -module. Assume that*

$$\text{char}(M) \cap \{\lambda \in \mathfrak{g}^* \mid \lambda|_{\phi(\mathfrak{h})} = 0\} = \{0\}.$$

*Let  $N$  be a  $\mathfrak{h}$ -module. Then, for all  $i$  in  $\mathbb{Z}$ , we have an isomorphism*

$$\text{Ext}_{U(\mathfrak{h})}^{i+dim\mathfrak{h}}(M|_{\mathfrak{h}}, N) \simeq \text{Ext}_{U(\mathfrak{g})}^{i+dim\mathfrak{g}} \left( M \otimes \det(\mathfrak{g}^*), (N \otimes \det(\mathfrak{h}^*)) \overset{L}{\otimes}_{U(\mathfrak{h})} U(\mathfrak{g}) \right).$$

## Application 2

Let  $X$  be an analytic Poisson manifold. We consider the Lie algebroid  $(X, \Omega_X^1)$  (see section 2.4). Let  $\mathcal{R}^\bullet$  be an element of  $D^b(\mathcal{D}(\Omega_X^1))$ . The cohomology of the complex  $R\mathcal{H}om_{\mathcal{D}(\Omega_X^1)}(\mathcal{O}_X, \mathcal{R}^\bullet)$  is called the canonical cohomology of  $X$  with values in  $\mathcal{R}^\bullet$  ([Li],[Hu]). We set

$$\mathcal{C}_{can}^\bullet(\mathcal{R}^\bullet) = R\mathcal{H}om_{\mathcal{D}(\Omega_X^1)}(\mathcal{O}_X, \mathcal{R}^\bullet).$$

Let  $(Y, \Omega_Y^1)$  be another holomorphic manifold endowed with a complex structure and let  $f : X \rightarrow Y$  be a Poisson map. Consider the correspondence

$$\begin{array}{ccc} & \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^1 & \\ \Phi=(id,F) \swarrow & & \searrow \Psi=(f,id) \\ (X, \Omega_X^1) & & (Y, \Omega_Y^1), \end{array}$$

where  $F$  is defined as follows : for all  $\alpha$  in  $\mathcal{O}_X$  and all  $\beta$  in  $\mathcal{O}_Y$ ,

$$\begin{aligned} \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^1 &\rightarrow \Omega_X^1 \\ \alpha \otimes \beta dq &\mapsto \alpha(\beta \circ f)d(q \circ f). \end{aligned}$$

Let  $\mathcal{N}$  be a left  $\mathcal{D}\left(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^1\right)$ -module. Taking  $\mathcal{M} = \mathcal{O}_Y$ , the adjunction formula tells us that we have the following isomorphisms

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}\left(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^1\right)}(\mathcal{O}_X, \mathcal{N})[dimY - dimX] &\simeq R\mathcal{H}om_{\mathcal{D}(\Omega_X^1)}(\mathcal{O}_X, \underline{\Phi}_! \mathcal{N}) \\ Rf_! R\mathcal{H}om_{\mathcal{D}\left(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^1\right)}(\mathcal{O}_X, \mathcal{N}) &\simeq R\mathcal{H}om_{\mathcal{D}(\Omega_Y^1)}(\mathcal{O}_Y, \underline{\Psi}_! \mathcal{N}) \end{aligned}$$

Noticing that  $\underline{\Psi}_!(\mathcal{N})$  is nothing but  $\mathcal{N}$  considered as a left  $\mathcal{D}(\Omega_Y^1)$ -module, we deduce the isomorphism

$$\mathcal{C}_{can,Y}^\bullet(\mathcal{N}) \simeq Rf_! \mathcal{C}_{can,X}^\bullet(\underline{\Phi}_! \mathcal{N})[dimX - dimY].$$

## Application 3

Let  $(X, \mathcal{L}_X)$  be a Lie algebroid over  $X$ . Let  $x_0$  be a point of  $X$ . Define the sheaf  $\mathcal{M}_{x_0}$  of ideals of  $\mathcal{O}_X$  by :

$$\mathcal{M}_{x_0}(U) = \{f \in \mathcal{O}_X(U) \mid f(x_0) = 0\}$$

for any open subset  $U$  of  $X$ . Define the (not necessarily locally free) Lie algebroid  $\mathcal{L}(\mathcal{M}_{x_0})$  by

$$\mathcal{L}(\mathcal{M}_{x_0}) = \{D \in \mathcal{L}_X \mid D(\mathcal{M}_{x_0}) \subset \mathcal{M}_{x_0}\}.$$

Let  $\mathcal{L}(x_0)$  be the Lie algebra

$$\frac{\mathcal{O}_X}{\mathcal{M}_{x_0}} \otimes_{\mathcal{O}_X} \mathcal{L}_X(\mathcal{M}_{x_0}) = \mathcal{L}_X(x_0).$$

Let us consider the Lie algebroid morphism  $\mathcal{I} = (\epsilon, I) : (\{pt\}, \mathcal{L}_X(x_0)) \rightarrow (X, \mathcal{L}_X)$  where  $\epsilon(\{pt\}) = x_0$  and  $I$  is defined by tensoring the canonical embedding  $\mathcal{L}_X(\mathcal{M}_{x_0}) \hookrightarrow \mathcal{L}_X$ . For any  $\mathcal{D}(\mathcal{L}_X)$ -module  $\mathcal{R}$  supposed to be non characteristic with respect to  $\mathcal{I}$ , the adjunction formula gives

$$Ext_{U(\mathcal{L}_X(x_0))}^{i+dim\mathcal{L}_X(x_0)} \left( \frac{\mathcal{R}}{\mathcal{M}_{x_0}\mathcal{R}}, W \right) \simeq Ext_{\mathcal{D}(\mathcal{L}_X)}^{i+rk\mathcal{L}_X} (\mathcal{R}, \underline{\mathcal{I}}_!(W)).$$

In particular,

$$H^{i+dim\mathcal{L}_X(x_0)}(\mathcal{L}_X(x_0), W) \simeq Ext_{\mathcal{D}(\mathcal{L}_X)}^{i+rk\mathcal{L}_X} (\mathcal{O}_X, \underline{\mathcal{I}}_!(W)).$$

**Example :**

Assume that  $X = V$  is a finite dimensional vector space on which a connected algebraic group  $G$  acts locally freely (that is  $max_{v \in V} dim G \cdot v = dim G$ ). Put  $Lie G = \mathfrak{g}$ . If  $\mathcal{L}_V = \mathcal{O}_V \otimes \mathfrak{g}$  (as in example 5) and  $x_0 = 0_V$ , then  $\mathcal{L}(0_V) = \mathfrak{g}$ . Consider the Lie algebroid  $\mathcal{M}_V = (V, \omega_V(\mathcal{O}_V \otimes \mathfrak{g}))$ . We have the isomorphism

$$H^{i+dim\mathfrak{g}}(\mathfrak{g}, W) \simeq Ext_{\mathcal{D}(\mathcal{M}_V)}^{i+rk\mathcal{M}_V} (\mathcal{O}_V, \underline{\omega_V \circ \mathcal{I}}_!(W)).$$

But  $dim\mathfrak{g} = rk\mathcal{M}_V$  ([P] p. 186). Assume that  $\mathfrak{g}$  is semi-simple. Then  $W$  can be written as a direct sum of irreducible representations. Let  $p$  be the multiplicity of  $k$  in  $W$ . Then

$$Ext_{\mathcal{D}(\mathcal{M}_V)}^i (\mathcal{O}_V, \underline{\omega_V \circ \mathcal{I}}_!(W)) = (\Lambda^i(\mathfrak{g})^{\mathfrak{g}})^p.$$

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