Duality properties for quantum groups

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Abstract : Some duality properties for induced representations of enveloping algebras involve the character $Trad_{\mathfrak{g}}$. We extend them to deformation Hopf algebras A_h of a noetherian Hopf k-algebra A_0 satistying $Ext^i_{A_0}(k, A_0) = \{0\}$ except for $i = d$ where it is isomorphic to k. These duality properties involve the character of A_h defined by right multiplication on the one dimensional free $k[[h]]$-module $Ext^d_{A_h}(k[[h]], A_h)$. In the case of quantized enveloping algebras, this character lifts the character $Trad_{\mathfrak{g}}$. We also prove Poincaré duality for such deformation Hopf algebras in the case where A_0 is of finite homological dimension. We explain the relation of our construction with quantum duality.

1. Introduction

In this article k will be a field of characteristic 0 and we set $K = k[[h]]$.

Let A_0 be a noetherian algebra. We assume moreover that k has a left A_0-module structure such that there exists an integer d satisfying

$$\begin{cases} Ext^i_{A_0}(k, A_0) = \{0\} \text{ if } i \neq d \\ Ext^d_{A_0}(k, A_0) \simeq k. \end{cases}$$

It follows from Poincaré duality that any finite dimensional Lie algebra $\mathfrak{g}$ verifies these assumptions. In this case $d = dim\mathfrak{g}$ and the character defined by the right representation of $U(\mathfrak{g})$ on $Ext^{dim\mathfrak{g}}_{U(\mathfrak{g})}(k, U(\mathfrak{g}))$ is $Trad_{\mathfrak{g}}$ ([C1]). The algebra of regular fonctions on an affine algebraic Poisson group and algebra of formal power series also satisfy these hypothesis. Let A_h be a deformation algebra of A_0. Assume that there exists an A_h-module structure on K that reduces modulo h to the A_0-module structure we started with. The following theorem constructs a new character of A_h, which will be denoted by θ_{A_h}.

Theorem 5.0.7
With the assumptions made above, one has :
a) $Ext^i_{A_h}(K, A_h) = \{0\}$ *is zero if* $i \neq d$
b) $Ext^d_{A_h}(K, A_h)$ *is a free* K*-module of dimension one. The right* A_h*-module structure given by right multiplication lifts that of* A_0 *on* $Ext^d_{A_0}(k, A_0)$.

The right A_h-module $Ext^d_{A_h}(K, A_h)$ will be denoted by Ω_{A_h}. If there is an ambiguity, the integer d will be written d_{A_h}.

Theorem 5.0.7 applies to universal quantum enveloping algebras, quantization of affine algebraic Poisson groups and to quantum formal series Hopf algebras.

Let $\mathfrak{g}$ be a Lie bialgebra. Denote by $F[\mathfrak{g}]$ the formal series Poisson algebra $U(\mathfrak{g})^*$. If $U_h(\mathfrak{g}^*)$ is a quantum enveloping algebra such that $U_h(\mathfrak{g}^*)/hU_h(\mathfrak{g}^*)$ is isomorphic to $U(\mathfrak{g}^*)$ as a coPoisson Hopf algebra, we show that one may construct a resolution of the trivial $U_h(\mathfrak{g}^*)$-module $k[[h]]$ that lifts the Koszul resolution of the trivial $U(\mathfrak{g}^*)$-module k. If $F_h[\mathfrak{g}]$ is a quantum formal series algebras such that $F_h[\mathfrak{g}]/hF_h[\mathfrak{g}]$ is isomorphic to $F[\mathfrak{g}]$ as a Poisson Hopf algebra, we construct a resolution of the trivial $F_h[\mathfrak{g}]$-module that lifts the Koszul resolution of the trivial $F[\mathfrak{g}]$-module k and that respects quantum duality ([Dr], [Ga]). This construction is not explicit but it allows to show that, if $F_h[\mathfrak{g}]$ and $U_h(\mathfrak{g}^*)$ are linked by quantum duality, the following equality holds $\theta_{F_h[\mathfrak{g}]} = h\theta_{U_h(\mathfrak{g}^*)}$.

As an application of theorem 5.0.7, we show Poincaré duality :

Theorem 8.1.1

We make the same assumtions as above. Let M be an A_h-module. Assume that K is an A_h-module of finite projective dimension. One has an isomorphism of K-modules for all integer i :

$$Ext^i_{A_h}(K, M) \simeq Tor^{A_h}_{d_{A_h}-i}(\Omega_{A_h}, M).$$

From now on, we assume that A_h is a deformation Hopf algebra.

Brown and Levasseur ([B-L]) and Kempf ([Ke]) had shown that, in the semisimple context, the Ext-dual of a Verma module is a Verma module. In [C1], we have extended this result to the Ext-dual of an induced representation of any Lie superalgebra. In this article, we show that this result can be generalized to quantum groups provided that the quantization is functorial. Such a functorial quantization has been constructed by Etingof and Kazdhan ([E-K1], [E-K2], [E-K3], [E-S]). As the result holds for quantized universal enveloping algebras, for quantized functions algebras and for quantum formal series Hopf algebras, we state it in the more general setting of Hopf algebras.

Corollary 8.2.2

Let A_h (respectively B_h) be a topological Hopf deformation of A_0 (respectively B_0). We assume that there exists a morphism of Hopf algebras from B_h to A_h such that A_h is a flat B_h^{op}-module. We also assume that B_h satisfies the condition of the theorem 5.0.7. Let V be a B_h-module which is a free finite dimensional K-module. Then

a) $Ext^i_{A_h}\left(A_h \underset{B_h}{\otimes} V, A_h\right)$ is $\{0\}$ if i is different from d_{B_h}.

b) The right A_h-module $Ext^{d_{B_h}}_{A_h}\left(A_h \underset{B_h}{\otimes} V, A_h\right)$ *is isomorphic to* $(\Omega_{B_h} \otimes V^*) \underset{B_h}{\otimes} A_h$ where $\Omega_{B_h} \otimes V^*$ is endowed with the following right B_h-module structure :

$$\begin{aligned}
&\forall u \in B_h\, \forall f \in V^*,\ \forall \omega \in \Omega_{B_h},\\
&(\omega \otimes f) \cdot u = \lim_{n\to+\infty} \sum_j \theta_{B_h}(u'_{j,n})\omega \otimes f \cdot S_h^2(u''_{j,n})\\
&\Delta(u) = \lim_{n\to+\infty} \sum_j u'_{j,n} \otimes u''_{j,n}.
\end{aligned}$$

S_h being the antipode of B_h.

Proposition 8.2.3 *Let A_h be a Hopf deformation of A_0, B_h be a Hopf deformation of B_0 and C_h be a Hopf deformation of C_0. We assume that there exists a morphism of Hopf algebras from B_h to A_h and a morphism of Hopf algebras from C_h to A_h such that A_h is a flat B_h^{op}-module and a flat C_h^{op}-module. We also assume that B_h and C_h satisfies the hypothesis of theorem 5.0.7. Let V (respectively W) be a B_h-module (respectively C_h-module) which is a free finite dimensional K-module. Then, for all integer n, one has an isomorphism*

$$\begin{aligned}
&Ext^{n+d_{B_h}}_{A_h}\left(A_h \underset{B_h}{\otimes} V, A_h \underset{C_h}{\otimes} W\right)\\
&\simeq Ext^{n+d_{C_h}}_{A_h}\left((\Omega_{C_h} \otimes W^*) \underset{C_h}{\otimes} A_h, (\Omega_{B_h} \otimes V^*) \underset{B_h}{\otimes} A_h\right).
\end{aligned}$$

The right B_h (respectively C_h)-module structure on $\Omega_{B_h} \otimes V^*$ (respectively $\Omega_{C_h} \otimes W^*$) are as in Corollary 8.2.2.

Remark :

Proposition 8.2.3 is already known in the case where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h}$ and $\mathfrak{k}$ are Lie subalgebras of $\mathfrak{g}$, A_h, B_h and C_h are their corresponding enveloping algebras. In this case one has $d_{B_h} = dim\mathfrak{h}$ and $d_{C_h} = dim\mathfrak{k}$. More precisely :

Generalizing a result of G. Zuckerman ([B-C]), A. Gyoja ([G]) proved a part of this theorem (namely the case where $\mathfrak{h} = \mathfrak{g}$ and $n = dim\mathfrak{h} = dim\mathfrak{k}$) under the assumptions that $\mathfrak{g}$ is split semi-simple and $\mathfrak{h}$ is a parabolic subalgebra of $\mathfrak{g}$. D.H Collingwood and B. Shelton ([C-S]) also proved a duality of this type (still under the semi-simple hypothesis) but in a slighly different context.

M. Duflo [Du2] proved proposition 8.2.3 for a $\mathfrak{g}$ general Lie algebra, $\mathfrak{h} = \mathfrak{k}$, $V = W^*$ being one dimensional representations.

Proposition 8.2.3 is proved in full generality in the context of Lie superalgebras in [C1].

Wet set $A_h^e = A_h \otimes A_h^{op}$. Using the properties as a Hopf algebra (as in [C2]), we show that all the $Ext^i_{\widehat{A_h^e}}(A_h, A_h \underset{k[[h]]}{\widehat{\otimes}} A_h)$'s are zero except one. More precisely :

Proposition 8.3.1 *Assume that A_h satisfies the conditions of the theorem 5.0.7. Assume moreover that $A_0 \otimes A_0^{op}$ is noetherian. Consider $A_h \underset{k[[h]]}{\widehat{\otimes}} A_h$ with the following $\widehat{A_h^e}$-module structure :*

$$\forall(\alpha, \beta, x, y) \in A_h, \quad \alpha \cdot (x \otimes y) \cdot \beta = \alpha x \otimes y\beta.$$

a) $HH^i_{A_h}(A_h \widehat{\underset{k[[h]]}{\otimes}} A_h)$ *is zero if* $i \neq d_{A_h}$.

b) The $\widehat{A^e_h}$*-module* $HH^{d_{A_h}}_{A_h}(A_h \widehat{\underset{k[[h]]}{\otimes}} A_h)$ *is isomorphic to* $\Omega_{A_h} \otimes A_h$ *with the following* $\widehat{A^e_h}$*-module structure :*

$$\forall (\alpha, \beta, x) \in A_h, \quad \alpha \cdot (\omega \otimes x) \cdot \beta = \omega \theta_{A_h}(\beta'_i) \otimes S(\beta''_i) x S^{-1}(\alpha)$$

where $\alpha = \sum_i \alpha'_i \otimes \alpha''_i$ *(to be taken in the topological sense)*

This result has already been obtained in [D-E] for a deformation of the algebra of regular functions on a smooth algebraic affine variety. From this, as in [VdB], we deduce a relation between Hochschild homology and Hochschild cohomology for the ring A_h.

We start the article by a study of algebras endowed with a decreasing filtration and filtered modules over such algebras. Our study relies on the use of the associated graded algebra and graded module and on the use of the topology defined by a decreasing filtration. We apply this study to deformation algebras endowed with the h-adic filtration and filtered modules over such algebras. In [K-S], a study of the derived category of A_h-modules is carried out using the right derived functor of the functor $M \mapsto \dfrac{M}{hM}$.

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2. Graded linear algebra

In this section, we fix notation about graded linear algebra. A graded k-algebra GA is the data of a k-algebra with unit and a family of k-vector spaces $(G_tA)_{t\in\mathbb{Z}}$ of A satisfying :

$$\begin{aligned} &a. \quad A = \oplus_{t\in\mathbb{Z}} G_tA \\ &b. \quad 1 \in G_0A \\ &c. \quad G_tA \cdot G_lA \subset G_{t+l}A. \end{aligned}$$

We will also assume that $G_tA = 0$ for $t < 0$.

A graded GA-module GM is the data of a GA-module and a family of k-vector space $(G_tM)_{t\in\mathbb{Z}}$ of GM such that

$$\begin{aligned} &GM = \oplus_{t\in\mathbb{Z}} G_tM \\ &G_tA \cdot G_lM \subset G_{t+l}M \end{aligned}$$

We will always also assume that $G_tM = 0$ if $t << 0$.

Let GM and GN be two graded GA-modules. A morphism of graded GA-modules from GM to GN is a morphism of GA-modules $f : GM \to GN$ such that $f(G_tM) \subset G_tN$. The group of morphisms of graded GA-modules from GM to GN will be denoted $\mathrm{Hom_{GA}(GM, GN)}$. With this notion of morphisms, the category of graded GA-modules is abelian. Thus it is suitable for homological algebra.

For $r \in \mathbb{Z}$ and any graded GA-module GM, we define the shifted graded GA-module $GM(r)$ to be the GA-module GM endowed with the grading defined by

$$\forall t \in \mathbb{Z}, \quad G_tM(r) = G_{t+r}M.$$

Let us denote $\underline{\mathrm{Hom}}_{GA}(GM, GN)$ the graded group defined by setting

$$G_t\underline{\mathrm{Hom}}_{GA}(GM, GN) = Hom_{GA}(GM, GN(t)).$$

The ith right derived functor of the functor $\underline{\mathrm{Hom}}_{GA}(-, N)$ will be denoted $\underline{\mathrm{Ext}}^i_{GA}(-, N)$.

A graded GA-module GL is finite free if there are integers $d_1, d_2, \dots, d_n$ such that

$$GL \simeq \overset{n}{\underset{i=1}{\oplus}} GA(-d_i).$$

A graded GA-module GM is of finite type if there exists a finite free graded GA-module GL and an exact sequence in the category of graded GA-modules

$$GL \to GM \to 0.$$

This means that there are homogeneous elements $m_1 \in G_{d_1}M, \dots, m_n \in G_{d_n}M$ such that any $m \in G_dM$ may be written as

$$m = \sum_{i=1}^{n} a_{d-d_i} m_{d_i}$$

where $a_{d-d_i} \in G_{d-d_i}A$.

A graded ring GA is noetherian if any graded GA-submodule of a graded GA-module of finite type is of finite type.

In the sequel, all the GA-modules we will consider will be graded so that we will say "GA-module" for "graded GA-module".

3. Decreasing filtrations

In this section, we give results about decreasing filtrations. These results are proved in [Schn] in the framework of increasing filtrations. For the sake of completeness, we give detailled proofs of the results even if most of our proofs are obtained by adjusting those of Schneiders.

We will consider a k-algebra endowed with a decreasing filtration $...F_{t+1}A \subset F_tA \subset \cdots \subset F_1A \subset F_0A = A$. The order of an element a, $o(a)$, is the biggest t such that $a \in F_tA$. The principal symbol of a is the image of a in $F_{o(a)}/F_{o(a)+1}$. It will be denoted by $[a]$.

A filtered module over FA is the data of an A-module M and a family $(F_tM)_{t\in\mathbb{Z}}$ of k-subspaces such that

- $\bigcup_{t\in\mathbb{Z}} F_tM = M$
- $F_{t+1}M \subset F_tM$
- $F_tA \cdot F_lM \subset F_{t+l}M$

We will assume that $F_tM = M$ for $t << 0$. We have the notion of principal symbol. We endow such a module with the topology for which a basis a neighborhoods is $(F_tM)_{t\in\mathbb{Z}}$. The topological space M is Hausdorff if and only if $\cap_{t\in\mathbb{Z}} F_tM = \{0\}$. If M is Hausdorff, the topology defined by the filtration is defined by the following metric

$$\forall (x, y) \in FM,\ d(x, y) = \| x - y \| \quad \text{with}$$
$$\| x - y \| = 2^{-t} \quad \text{where } t = Sup\{j \in \mathbb{Z} \mid x - y \in F_jM\}$$

Note that M is Hausdorff if and only if the natural map from M to $\varprojlim_{t\in\mathbb{Z}} \frac{M}{F_tM}$ is injective. The metric space (M,d) is complete if and only if the natural map from M to $\varprojlim_{t\in\mathbb{Z}} \frac{M}{F_tM}$ is an isomorphism.

Example :

Let k be a field and set $K = k[[h]]$. If V is a K-module, it is endowed with the following decreasing filtration $\cdots \subset h^nV \subset h^{n-1}V \subset \cdots \subset hV \subset V$.
The topology induced by this filtration is the h-adic topology.

Recall the following result :

Lemma 3.0.1. *Let N be a Hausdorff filtered module. Let P be a submodule of N which is closed in N. Let p the canonical projection from N to N/P.*

a) The topology defined by the filtration $p(F_tN)$ on N/P is the quotient topology. N/P is Hausdorff and its topology is defined by the distance $d(\bar{x}, \bar{y}) = \| \bar{x} - \bar{y} \|$ where

$$\| \bar{x} \| = Inf\{\| a \|, a \in \bar{x}\}$$

b) If N is complete, then N/P is complete for the quotient topology.

Proof of the lemma :

a) As P is closed in N, then $\bar{0}$ is closed in N/P. Thus, its complement in N/P, U, is open. Let $\bar{x}$ an element of N/P different from $\bar{0}$. As U is open, there exists $n \in \mathbb{N}$ such that $\bar{x} \in p(F_nN) + \bar{x} \subset U$. Hence $\bar{x} \notin p(F_nN)$ and we have proved that $\bigcap_{n\in\mathbb{N}} p(F_nN) = \{\bar{0}\}$. Hence N/P is Hausdorff. It is easy to check that the open ball of center 0 and radius 2^{-t} in N/P for the distance defined is $p(F_{t+1}N)$.

b) we refer to [Schw] p 245. □.

Let FM and FN be two filtered FA-modules. A filtered morphism $Fu : FM \to FN$ is a morphism $u : M \to N$ of the underlying A-modules such that $u(F_tM) \subset F_tN$. It is continuous if we endow M and N with the topology defined by the filtrations. Denote by F_tu the morphism $u_{|F_tM} : F_tM \to F_tN$. Denote by $Hom_{FA}(FM, FN)$ the group of filtered morphisms from FM to FN. The kernel of Fu is the kernel of u filtered by the family $KerFu \cap F_tM$. If M is complete and N is Hausdorff, then $KerFu$, endowed with the induced topology is complete.

To a filtered ring FA is associated a graded ring GA defined by

$$GA = \underset{t\in\mathbb{N}}{\oplus} G_tA \text{ with } G_tA = F_tA/F_{t+1}A$$

the multiplication being induced by that of FA. To a filtered FA-module FM is associated a graded GA-module GM defined by setting

$$GM = \underset{t\in\mathbb{Z}}{\oplus} G_tM \text{ with } G_tM = F_tM/F_{t+1}M$$

the action of GA on GM being induced by that of FA. If x is in F_tM, we will write $\sigma_t(x)$ for the class of x in $F_tM/F_{t+1}M$. A filtered morphism of FA-modules $Fu : FM \to FN$ induces a morphism of abelian groups $G_tu : G_tM \to G_tN$ and a morphism of GA-modules $Gu : GM \to GN$.

An arrow $Fu : FM \to FN$ is strict if it satisfies $u(F_tM) = u(M) \cap F_tN$.

An exact sequence of FA-modules is a sequence

$$FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$$

such that $KerF_tv = ImF_tu$. It follows from this definition that Fu is strict. if moreover Fv is strict, we say that it is a strict exact sequence.

Proposition 3.0.2. *a) Consider $Fu : FM \to FN$ and $Fv : FN \to FP$ two filtered FA-morphisms such that $Fv \circ Fu = 0$. If the sequence*

$$FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$$

is strict exact, then

$$GM \xrightarrow{Gu} GN \xrightarrow{Gv} GP$$

is exact.

b) Conversely, assume that FM is complete for the topology defined by the filtration and that FN is Hausdorff for the topology defined by the filtration. If the sequence

$$GM \xrightarrow{Gu} GN \xrightarrow{Gv} GP$$

is exact, then the sequence

$$FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$$

is strict exact.

Proof of the proposition :

a) Let $n_t \in G_tN$ be such that $G_tv(n_t) = 0$. There is $n'_t \in F_tN$ such that $n_t = \sigma_t(n'_t)$. Hence $v(n'_t) \in F_{t+1}P$. Since Fv is strict, we find $n''_{t+1} \in F_{t+1}N$ such that $v(n''_{t+1}) = v(n'_t)$. Then $v(n'_t - n''_{t+1}) = 0$ and there is $m_t \in F_tM$ such that $u(m_t) = n'_t - n''_{t+1}$. This shows that

$$G_tu\left(\sigma_t(m_t)\right) = \sigma_t(n'_t) = n_t.$$

b) Let us prove that Fv is strict. Assume that $p_t \in F_tP \cap Imv$. Let l be the biggest integer such that $p_t = v(n_l)$ with $n_l \in F_lN$. We need to show that $l \geq t$. Assume that $l < t$. One has

$$G_lv\left(\sigma_l(n_l)\right) = \sigma_l\left(v(n_l)\right) = \sigma_l(p_t) = 0.$$

Hence $\exists m_l \in F_lM$ such that $G_lu\left(\sigma_l(m_l)\right) = \sigma_l(n_l)$. Thus we have

$$n_l - u(m_l) \in F_{l+1}M \text{ and } v\left(n_l - u(m_l)\right) = p_t$$

which contredicts the definition of l.

Let us prove that $KerF_tv = ImF_tu$. Let $n_t \in KerF_tv$. One has : $G_t(v)\left(\sigma_t(n_t)\right) = 0$. Hence there exists m_t in F_tM such that

$$\sigma_t(n_t) = G_t(u)\left(\sigma_t(m_t)\right).$$

Hence $n_t - u(m_t) \in KerFv \cap F_{t+1}N$. We can reproduce the previous reasoning to $n_t - u(m_t)$ and produce an element m_{t+1} in $F_{t+1}M$ such that $n_t - u(m_t + m_{t+1}) \in$

$KerFv \cap F_{t+2}N$. The sequence $\mathcal{U}_p = \sum_{l=0}^{p} m_{t+l}$ is a Cauchy sequence, hence it converges and $n_t = u\left(\sum_{l=0}^{\infty} m_{t+l}\right)$. □

Corollary 3.0.3. *Let FA be a filtered k-algebra and let FM and FN two FA-modules. Let $Fu : FM \to FN$ be a morphism of FA-modules. Then* $G\mathrm{KerFu} \subset \mathrm{KerGFu}$ *and* $\mathrm{ImGFu} \subset \mathrm{GImFu}$*. Assume moreover that FM is complete and FN is Hausdorff, then the following conditions are equivalent :*
(a) Fu is strict
(b) $G\mathrm{KerFu} = \mathrm{KerGFu}$
(c) $\mathrm{ImGFu} = \mathrm{GImFu}$.

Proof :
One has :

$$\begin{aligned}
&F_t\mathrm{Keru} = \mathrm{Keru} \cap \mathrm{F_tM}\\
&F_t\mathrm{Imu} = \mathrm{Imu} \cap \mathrm{F_tN}\\
&G_t\mathrm{Keru} = \frac{\mathrm{F_tM} \cap \mathrm{Keru}}{\mathrm{F_{t+1}M} \cap \mathrm{Keru}}\\
&\mathrm{KerG_tu} = \frac{\mathrm{F_tM} \cap \mathrm{u^{-1}(F_{t+1}N)}}{\mathrm{F_{t+1}M} \cap \mathrm{u^{-1}(F_{t+1}N)}}\\
&\mathrm{ImG_tu} = \frac{\mathrm{u(F_tM)}}{\mathrm{F_{t+1}N} \cap \mathrm{u(F_tM)}}\\
&G_t\mathrm{Imu} = \frac{\mathrm{Imu} \cap \mathrm{F_tN}}{\mathrm{Imu} \cap \mathrm{F_{t+1}M}}
\end{aligned}$$

The second part of the corollary follows from applying the previous proposition to the strict exact sequence $FM \to \mathrm{Imu} \to 0$.
Indeed Fu is strict if and only the following sequence

$$FM \xrightarrow{Fu} \mathrm{Imu} \to 0$$

is a strict exact sequence of FA-modules when Imu is endowed with the induced topology. Then we apply 3.0.2 .

Let us recall this well known result about complexes of filtered modules.

Proposition 3.0.4. *Let $(M^\bullet, d^\bullet)$ be a complex of complete FA-modules. $H^i(M^\bullet)$ is filtered as follows $F_tH^i(M^\bullet) = \dfrac{Kerd_i \cap F_tM^i + Imd_{i-1}}{Imd_{i-1}} \simeq \dfrac{Kerd_i \cap F_tM^i}{Imd_{i-1} \cap F_tM^{i-1}}$. If d_i and d_{i-1} are strict, then $GH^i(M^\bullet)$ is isomorphic to $H^i(GM^\bullet)$*

Proof of the proposition 3.0.4:
We consider the following exact sequence

$$0 \to Imd_{i-1} \to Kerd_i \xrightarrow{p} H^iM^\bullet \to 0.$$

we endow $Kerd_i$ and Imd_{i-1} with the induced filtration. One has

$$F_tKerd_i = Kerd_i \cap F_tM^i$$
$$F_tImd_{i-1} = Imd_{i-1} \cap F_tM^i$$
$$p(F_tKerd^i) = \frac{Kerd_i \cap F_tM^i + Imd_i}{Imd_i} = F_tH^i(M^\bullet).$$

The exact sequence above is strict exact. It stays exact if one takes the graded modules. Thus, we have the following exact sequence of GA-modules

$$0 \to GImd_{i-1} \to GKerd_i \xrightarrow{p} GH^iM^\bullet \to 0.$$

Then $GH^i(M^\bullet) \simeq \dfrac{GKerd_i}{GImd_{i-1}} \simeq \dfrac{KerGd_i}{ImGd_{i-1}} \simeq H^i(GM^\bullet)$. This finishes the proof of the proposition. □

Remark :
The isomorphism from $G_tH^i(M^\bullet)$ to $H^i(G_tM^\bullet)$ is given by

$$\begin{array}{rcl} G_tH^i(M^\bullet) & \to & H^i(G_tM^\bullet) \\ \sigma_t cl(x) & \mapsto & cl(\sigma_t(x)). \end{array}$$

For any $r \in \mathbb{Z}$ and for any FA-module FM, we define the shifted module $FM(r)$ as the module M endowed with the filtration $(F_{t+r}M)_{t\in\mathbb{Z}}$.

An FA-module module is finite free if it is isomorphic to an FA-module of the type $\oplus_{i=1}^{p} FA(-d_i)$ where $d_1, \ldots, d_p$ are integers. An FA-module FM is of finite type if there exists a strict epimorphism $FL \to FM$ where FL is a finite free FA-module. This means that we can find $m_1 \in F_{d_1}M, \ldots, m_p \in F_{d_p}M$ such that any $m \in F_dM$ may be written as

$$m = \sum_{i=1}^{p} a_{d-d_i} m_i$$

where $a_{d-d_i} \in F_{d-d_i}A$.

Proposition 3.0.5. *Let FA be a filtered k-algebra and FM be an FA-module.*

a) If FM is an FA-module of finite type generated by $(s_1, \ldots, s_r)$ then GM is a GA-module of finite type generated by $([s_1], \ldots, [s_r])$. Conversely, assume that FA is complete for the topology given by the filtration and that FM is a FA-module which is Hausdorff for the topology defined by the filtration. If GM is a GA-module of finite type generated by $([s_1], \ldots, [s_r])$, then FM is an FA-module of finite type generated by $(s_1, \ldots, s_r)$

b) If FM is a finite free FA-module, then GM is a finite free GA-module. Conversely, assume that FA is complete for the topology given by the filtration and FM is a FA-module Hausdorff for the topology defined by the filtration. If GM is a finite free GA-module, then FM is a finite free FA-module.

Proof of the proposition :

a) If FM is an FA-module of finite type, then there is a strict exact sequence $\oplus_{i=1}^{N} FA(-d_i) \to FM \to 0$. If we apply proposition 3.0.2, we see that GM is a GA-module of finite type. Conversely, assume that GM is a GA-module of finite type generated by $\sigma_1 = [s_1], \ldots, \sigma_r = [s_r]$. Assume that $s_i \in F_{d_i}M - F_{d_i-1}M$. Let x in F_nM. There exists $a_{i,0} \in G_{n-d_i}A$ such that

$$\sigma_n(x) = \sum_{i=1}^{r} a_{i,0}\sigma_i.$$

Let $\alpha_{i,0} \in F_{n-d_i}A$ such that $\sigma_{n-d_i}(\alpha_{i,0}) = a_{i,0}$. We have

$$x - \sum_{i=1}^{r} \alpha_{i,0}s_i \in F_{n+1}M.$$

Reasoning in the same way, one can construct $\alpha_{i,1} \in F_{n-d_i+1}A$ such that

$$x - \sum_{i=1}^{r} (\alpha_{i,0} + \alpha_{i,1})s_i \in F_{n+2}M$$

Going on that way, we construct an element $\sum_{j=1}^{\infty} \alpha_{i,j}$ in $F_{n-d_i}A$ such that

$$x = \sum_{i=1}^{r} \left(\sum_{j=1}^{\infty} \alpha_{i,j} \right) s_i.$$

Hence FM is a finite type FA-module.

b) apply proposition 3.0.2.□

Definition 3.0.6. *A filtered k-algebra is said to be (filtered) noetherian if it satisfies one of the following equivalent conditions :*

- *Any filtered submodule (not necessarily a strict submodule) of a finite type FA-module is of finite type*
- *Any filtered ideal (not necessarily a strict ideal) of FA is of finite type.*

Proposition 3.0.7. *Let FA be a filtered complete k-algebra and denote by GA its associated graded algebra. If GA is graded noetherian, then FA is filtered noetherian.*

Proof of the proposition :

We assume that GA is a noetherian algebra. We need to prove that a filtered submodule FM' of a finitely generated FA-module FM is finitely generated.

First we assume that FM is Hausdorff. For this case, we reproduce the proof of [Sch].

If FM' is strict, then the associated GA-module GM' is a submodule of the GA-module GM associated to FM. Since GA is noetherian and GM is finitely generated so is GM' and the conclusion follows.

To prove the general case, we may assume that the image of the inclusion $FM' \to FM$ is equal to FM. In this case, using a finite systeme of generators of FM, it is easy to find an integer l such that

$$F_tM' \subset F_tM \subset F_{t-l}M'.$$

We will prove the result by induction on l.

For $l = 1$, let us introduce the auxiliary GA-modules

$$GK_0 = \oplus_{t\in\mathbb{Z}} F_t M'/F_{t+1}M$$
$$GK_1 = \oplus_{t\in\mathbb{Z}} F_t M/F_t M'$$

These modules satisfy the exact sequences

$$0 \to GK_0 \to GM \to GK_1 \to 0$$

$$0 \to GK_1(1) \to GM' \to GK_0 \to 0$$

Since GM is a finite type GA-module, so are GK_0 and GK_1. Hence GM' is also finitely generated and the conclusion follows.

For $l > 1$, we define the auxiliary FA-module FM'' by setting

$$F_t M'' = F_{t+1}M + F_t M'.$$

Since we have

$$F_t M'' \subset F_t M \subset F_{t-1} M''$$

the preceeding discussion shows that FM'' is finitely generated. Moreover

$$F_t M' \subset F_t M'' \subset F_{t-(l-1)} M'$$

and the conclusion follows from the induction hypothesis.

We no longer assume that FM is Hausdorff

As FM is a finite type FA-module, there exists a strict exact sequence

$$FL = \overset{n}{\underset{i=1}{\oplus}} FA(-d_i) \xrightarrow{p} FM \to 0.$$

We will denote by p_t the map from $F_t L$ to $F_t M$ induced by p. As p is strict, the map p_t is surjective. Let FM' be a submodule (not necessarily strict) of FM. Then $p^{-1}(FM')$ is an FA-submodule of FL if we endow it with the filtration

$$F_t\left[p^{-1}(M')\right] = p_t^{-1}(F_t M') = p^{-1}(F_t M') \cap F_t L.$$

As FL is Hausdorff, we know from the first part of the proof that the FA-module $p^{-1}M'$ is finite type. Hence there exist $\alpha_1 \in F_{\delta_1}\left[p^{-1}M'\right], \ldots, \alpha_p \in F_{\delta_p}\left[p^{-1}M'\right]$ such that any x of $F_d\left[p^{-1}M'\right]$ can be written

$$x = \sum_{i=1}^{p} a_{d-\delta_i}\alpha_i \text{ with } a_{d-\delta_i} \in F_{d-\delta_i}A.$$

Let y in $F_d M'$. As p is strict, there exist $x \in F_d\left[p^{-1}M'\right]$ such that $y = p(x)$. Then y can be written

$$y = \sum_{i=1}^{p} a_{d-\delta_i} p(\alpha_i) \text{ with } a_{d-\delta_i} \in F_{d-\delta_i}A.$$

We have proved that FM' is a finite type FA-module $\square$.

Proposition 3.0.8. *Assume that FA is noetherian for the topology given by the filtration. Any FA-module of finite type has an infinite resolution by finite free FA-modules i.e there is an exact sequence*

$$\cdots \to FL_s \to FL_{s-1} \to \cdots \to FL_0 \to FM \to 0$$

where each FL_s is a finite free FA-module.

Remark :
For such a resolution of FM, the sequence

$$\cdots \to GL_s \to GL_{s-1} \to \cdots \to GL_0 \to GM \to 0$$

is a resolution of the GA-module GM.

Proposition 3.0.9. *Assume FA is noetherian and complete. If GA is of finite (left) global homological dimension, so is A.*

Proof : we adjust the proof of [Schn] proposition 10.3.5. to decreasing filtrations. Let us start by a lemma.

Lemma 3.0.10. *If FN is a finite type FA-module, then it is complete.*

First we assume that FN is Hausdorff. Let FN be a finite type Hausdorff FA-module. We have a strict exact sequence

$$FL = \oplus_{i=1}^{n} FA(-d_i) \xrightarrow{p} FN \to 0.$$

The filtration on FN is given by $p(F_t L)$. Let us endow the kernel K of p with the induced topology. We have a strict exact sequence

$$0 \to FK \to FL \to FN \to 0.$$

As N is Hausdorff, $K = p^{-1}(\{0\})$ is closed in FL. The filtered FA-module FN is isomorphic to FL/K, endowed with the quotient topology. Hence, FN is complete (see lemma 3.0.1).

We no longer assume that FN is Hausdorff. From the first case, FK, endowed with the induced topology is complete and hence closed in FL. As $FN \simeq FL/K$, the FA-module FN is Hausdorff.

Lemma 3.0.11. *Assume that FA is noetherian and complete. Then, for any FA-module of finite type FM and any complete FA-module FN,*

$$\underline{\mathrm{Ext}}^j_{GA}(GM, GN) = 0 \Longrightarrow \mathrm{Ext}^j_A(M, N) = 0.$$

Let

$$\cdots \to FL_n \to FL_{n-1} \to \cdots \to FL_0 \to FM \to 0$$

be a filtered resolution of FN by finite free FA-modules. Applying the graduation functor, we get a resolution

$$\cdots \to GL_n \to GL_{n-1} \to \cdots \to GL_0 \to GM \to 0.$$

Assuming $\underline{\mathrm{Ext}}^j_{GA}(GM, GN) = \{0\}$ means that the sequence

$$\underline{Hom}_{GA}(GL_{j-1}, GN) \to \underline{Hom}_{GA}(GL_j, GN) \to \underline{Hom}_{GA}(GL_{j+1}, GN)$$

is an exact sequence of GA-modules. When $FL = \overset{n}{\underset{i=1}{\oplus}} FA(-d_i)$ is finite free, the FA-module $FHom(FL, FN) = \oplus_{i=1}^{n} FN(d_i)$ is complete and the natural map

$$\mathrm{GFHom}_{FA}(FL, FN) \to \underline{\mathrm{Hom}}_{GA}(GL, GN)$$

is an isomorphism. Hence the sequence

$$\mathrm{FHom_{FA}(FL_{j-1}, FN) \to FHom_{FA}(FL_j, FN) \to FHom_{FA}(FL_{j+1}, FN)}$$

is a strict exact sequence of FA-modules (proposition 3.0.2). When FL is finite free, the underlying module of $\mathrm{FHom}_{FA}(FL, FN)$ is $\mathrm{Hom_A(L,N)}$. This finishes the proof of the lemma.

Denote by d_{GA} the (left) global homological dimension of GA. Let M be a finite type A-module. One has an epimorphism

$$A^n \xrightarrow{p} M \to 0.$$

We endow M with the filtration $p(FA^n)$. Similarly, we endow N with a filtration FN such that FN is a finite FA-module. From the two previous lemmas, we deduce that for any finite type A modules M and N,

$$\mathrm{Ext^j_A(M,N) = 0 \ \ if \ \ j \geq d_{GA} + 1.}$$

Let now N be any A-module. We have $N = \varinjlim N'$ where N' runs over all finitely generated submodules of N. Let $L^\bullet$ be a resolution of M by finitely generated free A-modules. We have for all $j \geq d_{GA} + 1$

$$\begin{aligned}
\mathrm{Ext^j(M,N)} &= \mathrm{Ext^j(M, \varinjlim N')} \\
&= H^j\left(\mathrm{Hom_A(L^\bullet, \varinjlim N')}\right) \\
&= H^j\left(\varinjlim \mathrm{Hom_A(L^\bullet, N')}\right) \\
&= \varinjlim H^j\left(\mathrm{Hom_A(L^\bullet, N')}\right) \\
&= \varinjlim \mathrm{Ext^j_A(M, N')}
\end{aligned}$$

where, in the equality before the last equality, we used the fact that the functor $\varinjlim$ is exact because the set of finitely generated submodules of M is a directed set ([Ro] proposition 5.33). Thus we have proved : if M is a finitely generated A-module and N is any A-module, then

$$\mathrm{Ext^j(M,N) = \{0\} \ \ if \ \ j \geq d_{GA} + 1.}$$

From this, we deduce ([Ro] theorem 8.16), that the global (left) dimension of A is finite and inferior or equal to d_{GA}. □

Remark : The lemma 3.0.10 is proved in [K-S] in the case an A_h-module (A_h being a deformation algebra) endowed with the h-adic filtration.

4. Deformation algebras

4.1. **Definition and properties.** In this section k will be a field of characteristic 0 and we will set $K = k[[h]]$.

Definition 4.1.1. *A topologically free K-algebra A_h is a topologically free K-module together with a K-bilinear (multiplication) map $A_h \times A_h \to A_h$ making A_h into an associative algebra.*

Let A_0 be an associative k-algebra. A deformation of A_0 is topologically free K-algebra A_h such that $A_0 \simeq A_h/hA_h$ as algebras.

Remark :

If A_h is a deformation algebra of A_0, we may endow it with the h-adic filtration. We then have $GA_h = \underset{i\in\mathbb{N}}{\oplus} \dfrac{h^iA_h}{h^{i+1}A_h} \simeq A_0[h]$ as $k[h]$-algebra.

From proposition 3.0.6, we deduce that a deformation algebra of a noetherian algebra is noetherian.

Examples ([C-P]):

Before giving a list of examples, let us recall the following definition :

Definition 4.1.2. *A deformation of a Hopf algebra* $(A, \iota, \mu, \epsilon, \Delta, S)$ *over a field* k *is a topological Hopf algebra* $(A_h, \iota_h, \mu_h, \epsilon_h, \Delta_h, S_h)$ *over the ring* $k[[h]]$ *such that*
i) A_h *is isomorphic to* $A_0[[h]]$ *as a* $k[[h]]$*-module*
ii) A_h/hA_h *is isomorphic to* A_0 *as Hopf algebra.*

Example 1 : Quantized universal enveloping algebras (QUEA)

Definition 4.1.3. *Let* $\mathfrak{g}$ *be a Lie bialgebra. A Hopf algebra deformation of* $U(\mathfrak{g})$ *,* $U_h(\mathfrak{g})$*, such that* $\dfrac{U_h(\mathfrak{g})}{hU_h(\mathfrak{g})}$ *is isomorphic to* $U(\mathfrak{g})$ *as a coPoisson Hopf algebra is called a quantization of* $U(\mathfrak{g})$*.*

Quantizations of Lie bialgebras have been constructed in $[E - K1]$.

Example 2 : Quantization of affine algebraic Poisson groups

Definition 4.1.4. *A quantization of an affine algebraic Poisson group* $(G, \{,\})$ *is a Hopf algebra deformation* $\mathcal{F}_h(G)$ *of the Hopf algebra* $\mathcal{F}(G)$ *of regular functions on* G*, such that* $\dfrac{\mathcal{F}_h(G)}{h\mathcal{F}_h(G)}$ *is isomorphic to* $(\mathcal{F}(G), \{,\})$ *as Poisson Hopf algebra.*

Quantization of affine algebraic Poisson groups have been constructed by Etingof and Kazhdan ([E-S], see also [C-P] for the case where G is simple).

Examples 3: Quantum formal series Hopf algebras (QFSHA)
The vector space dual $U(\mathfrak{g})^*$ of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra can be identified with an algebra of formal power series and it has a natural Hopf algebra structure, provided we interpret the tensor product $U(\mathfrak{g})^* \otimes U(\mathfrak{g})^*$ in a suitable completed sense. If $\mathfrak{g}$ is a Lie bialgebra, $U(\mathfrak{g})^*$ is a Hopf Poisson algebra.

Definition 4.1.5. *A quantum formal series Hopf algebra is a topological Hopf algebra* B_h *over* $k[[h]]$ *such that* $\dfrac{B_h}{hB_h}$ *is isomorphic to* $U(\mathfrak{g})^*$ *as a topological Poisson Hopf algebra for some finite dimensional Lie bialgebra.*

The following proposition is proved in [K-S] (theorem 2.6)

Proposition 4.1.6. *Let* A_h *be a deformation algebra of* A_0 *and let* M *be an* A_h*-module. Assume that*
(i) M *has no* h*-torsion*
(ii)M/hM *is a flat* A_0*-module*
(iii) $M = \varprojlim_n M/h^n M$
then M *is a flat* A_h*-module.*

5. A quantization of the character *trad*

Theorem 5.0.7. *Let A_0 be a noetherian k-algebra and let A_h be a deformation of A_0. Assume that k has a left A_0-module structure such that there exists an integer d such that*

$$\begin{cases} Ext^i_{A_0}(k, A_0) = \{0\} \text{ if } i \neq d \\ Ext^d_{A_0}(k, A_0) \simeq k \end{cases}$$

Assume that K is endowed with a A_h-module structure which reduces modulo h to the A_0-module structure on k we started with. Then

a) $Ext^i_{A_h}(K, A_h)$ is zero if $i \neq d$.

b) $Ext^d_{A_h}(K, A_h)$ is a free K-module of dimension 1. By right multiplication, it is a right A_h-module. It is a lift of the right A_0-module structure (given by right multiplication) on $Ext^d_{A_0}(k, A_0)$.

Notation : The right A_h-module $Ext^d_{A_h}(k, A_h)$ will be denoted Ω_{A_h} and the character defined by this action θ_{A_h}.

Remark : In [K-S] (paragraph 6), Kashiwara and Schapira make a similar construction in the set up of DQ-algebroids. In [C2], it is shown that a result similar to theorem 5.0.7 holds for $U_q(\mathfrak{g})$ ($\mathfrak{g}$ semi-simple).

Example 1 : Quantized universal enveloping algebras

Poincaré duality gives us the following result for any finite dimensional Lie algebra.

$$\begin{cases} Ext^i_{U(\mathfrak{g})}(k, U(\mathfrak{g})) = \{0\} \text{ if} i \neq 0 \\ Ext^{dim\mathfrak{g}}_{U(\mathfrak{g})}(k, U(\mathfrak{g})) \simeq \Lambda^{dim\mathfrak{g}}(\mathfrak{g}^*) \end{cases} .$$

The character defined by the right action of $U(\mathfrak{g})$ on $Ext^{dim\mathfrak{g}}_{U(\mathfrak{g})}(k, U(\mathfrak{g}))$ is $trad_{\mathfrak{g}}$ ([C1]). Thus, the character defined by the theorem 5.0.7 is a quantization of the character $trad_{\mathfrak{g}}$.

• If $\mathfrak{g}$ is a complex semi-simple algebra, as $H^1(\mathfrak{g}, k) = \{0\}$ ([H-S] p 247), there exists a unique lift of the trivial representation of $U_h(\mathfrak{g})$, hence the representation $\Omega_{U_h(\mathfrak{g})}$ is the trivial representation.

• Let $\mathfrak{a}$ be a k-Lie algebra. Denote by $\mathfrak{a}_h$ the Lie algebra obtained from $\mathfrak{a}$ by multiplying the bracket of $\mathfrak{a}$ by h. Thus, for any elements X and Y of $\mathfrak{a}_h \simeq \mathfrak{a}$,

$$[X, Y]_{\mathfrak{a}_h} = h[X, Y]_{\mathfrak{a}}.$$

Denote by $\widehat{U(\mathfrak{a}_h)}$ the h-adic completion of $U(\mathfrak{a}_h)$. Then $\widehat{U(\mathfrak{a}_h)}$ is a Hopf deformation of $(\mathfrak{a}^{ab}, \delta = 0)$. The character $\theta_{\widehat{U(\mathfrak{a}_h)}}$ defined by the theorem in this case is given by

$$\forall X \in \mathfrak{a}, \;\; \theta_{\widehat{U(\mathfrak{a}_h)}}(X) = htrad_{\mathfrak{a}}(X).$$

Thus, even if $\mathfrak{g}$ is unimodular, the character defined by the right action of $U_h(\mathfrak{g})$ on $\Omega_{U_h(\mathfrak{g})} \simeq \wedge^{dim\mathfrak{g}}(\mathfrak{g}^*)[[h]]$ might not be trivial.

• We consider the following Lie algebra : $\mathfrak{a} = \overset{5}{\underset{i=1}{\oplus}} ke_i$ with non zero bracket $[e_2, e_4] = e_1$. Consider $k[[h]]$-Lie algebra structure on $\mathfrak{a}[[h]]$ defined by the following non zero brackets

$$[e_3, e_5] = he_3$$
$$[e_2, e_4] = 2e_1$$

$\widehat{U(\mathfrak{a}[[h]])}$ is a quantization of $U(\mathfrak{a})$. It is easy to see that

$$\theta_{\widehat{U(\mathfrak{a}[[h]])}}(e_i) = 0 \text{ if } i \neq 5$$
$$\theta_{\widehat{U(\mathfrak{a}[[h]])}}(e_5) = -h.$$

Example 2

The theorem 5.0.7 also applies to quantization of affine algebraic Poisson groups. If G is an affine algebraic Poisson group with neutral element e, we take k to be given by the counit of the Hopf algebra $\mathcal{F}(G)$. One has [A-K]

$$Ext^i_{\mathcal{F}(G)}(k, \mathcal{F}(G)) = \{0\} \text{ if } i \neq dimG$$
$$Ext^{dimG}_{\mathcal{F}(G)}(k, \mathcal{F}(G)) \simeq \wedge^{dimG}\left(\left(\mathcal{M}_e/\mathcal{M}_e^2\right)^*\right)$$

where

$$\mathcal{M}_e = \{f \in \mathcal{F}(G) \mid f(e) = 0\}.$$

Let $\mathfrak{g}$ be a real Lie algebra. The algebra of regular functions on $\mathfrak{g}^*$, $\mathcal{F}(\mathfrak{g}^*)$, is isomorphic to $S(\mathfrak{g})$ and is naturally equipped with a Poisson structure given by :

$$\forall X, Y \in \mathfrak{g}, \{X, Y\} = [X, Y].$$

In the example above, $\widehat{U(\mathfrak{g}_h)}$ is a quantization of the Poisson algebra $\mathcal{F}(\mathfrak{g}^*)$. $\mathcal{F}(\mathfrak{g}^*)$ acts trivially on $Ext^{dim\mathfrak{g}}_{\mathcal{F}(\mathfrak{g}^*)}(k, \mathcal{F}(\mathfrak{g}^*))$ whereas the action of $\mathcal{F}_h(\mathfrak{g}^*) \simeq \widehat{U(\mathfrak{g}_h)}$ on $Ext^{dim\mathfrak{g}}_{\mathcal{F}_h(\mathfrak{g}^*)}(k, \mathcal{F}_h(\mathfrak{g}^*))$ is not trivial.

Example 3 : The theorem 5.0.7 also applies to quantum formal series Hopf algebras.

Proof of the theorem 5.0.7:

Let us consider a resolution of the A_h-module K by filtered finite free A_h-modules

$$\cdots \to FL^{i+1} \xrightarrow{\partial_{i+1}} FL^i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} FL^1 \xrightarrow{\partial_1} FL^0 \to K \to \{0\}$$
$$FL^i = \oplus_{k=1}^{d_i} FA_h(-m_{j,i})$$

so that the graded complex

$$\ldots GL^{i+1} \xrightarrow{G\partial_{i+1}} GL^i \xrightarrow{G\partial_i} \cdots \to GL^1 \xrightarrow{G\partial_1} GL^0 \to k[h] \to \{0\}$$

is a resolution of the $A_0[h]$-module $k[h]$. Consider the complex $M^\bullet = (Hom_{A_h}(L^\bullet, A_h),^t \partial^\bullet)$. Recall that there is a natural filtration on $Hom_{A_h}(L^i, A_h)$ defined by

$$F_t Hom_{A_h}(L^i, A_h) = \{\lambda \in Hom_{A_h}(L^i, A_h) \mid \lambda\left(F_p L^i\right) \subset F_{t+p} A_h\}.$$

One has an isomorphism of right FA-modules

$$\mathrm{FHom}_{A_h}(L^i, A_h) = \overset{r_i}{\underset{j=1}{\oplus}} FA(m_{j,i})$$

Hence

$$\mathrm{GFHom}_{A_h}(L^i, A_h) \simeq \underline{Hom}_{GA_h}(GL^i, GA_h)$$

and the complex $\underline{\mathrm{Hom}}_{GA_h}(GL^i, GA_h)$ computes $\underline{Ext}^i_{GA_h}(k[h], GA_h)$. We have the following isomorphisms of right $A_0[h]$-modules.

$$\underline{\mathrm{Ext}}^i_{GA_h}(k[h], GA_h) \simeq \underline{\mathrm{Ext}}^i_{A_0[h]}(k[h], A_0[h]) \simeq \mathrm{Ext}^i_{A_0}(k, A_0)[h].$$

If $i \neq d$, then $\underline{\mathrm{Ext}}^i_{GA_h}(k[h], GA_h) = \{0\}$. This means that the sequence

$$\underline{Hom}_{GA}(GL_{i-1}, GA_h) \overset{{}^tG\partial_i}{\longrightarrow} \underline{Hom}_{GA}(GL_i, GA_h) \overset{{}^tG\partial_{i+1}}{\longrightarrow} \underline{Hom}_{GA}(GL_{j+1}, GA_h)$$

is an exact sequence of GA_h-modules. Hence, applying 3.0.2 the sequence

$$\mathrm{FHom_{FA}(FL_{i-1}, FN)} \overset{{}^t\partial_i}{\longrightarrow} \mathrm{FHom_{FA}(FL_i, FN)} \overset{{}^t\partial_{i+1}}{\longrightarrow} \mathrm{FHom_{FA}(FL_{i+1}, FN)}$$

is strict exact. As FL_i is finite free, the underlying module of $\mathrm{FHom}_{FA}(FL_i, FN)$ is $\mathrm{Hom_A(L_i, N)}$. Hence we have proved that $Ext^i_{A_h}(K, A_h) = \{0\}$ if $i \neq d$.

We have also proved that all the maps ${}^t\partial_i$ are strict. Hence, by proposition 3.0.4, we have for all integer i

$$GExt^i_{A_h}(k[[h]], A_h) \simeq \underline{Ext}^i_{GA_h}(k[h], A_0[h]) \simeq Ext^i_{A_0}(k, A_0)[h]$$

The FA_h-modules $\mathrm{Ext^i_{A_h}(K, A_h)}$ are finite type FA-modules, hence they are Hausdorff and even complete (see lemma 3.0.10).

As $\mathrm{Ext^d_{A_h}(K, A_h)}$ is Hausdorff and $GExt^d_{A_h}(k[[h]], A_h) \simeq Ext^d_{A_0}(k, A_0)[h]$, the $k[[h]]$-module $\mathrm{Ext^d_{A_h}(K, A_h)}$ is a one dimensional.

This finishes the proof of the theorem 5.0.7. □

From now on, we assume that A_h is a topological Hopf algebra and that its action on K is given by the counit. The antipode of A_h will be denoted S_h.

If V is a left A_h-module, we set V^r (respectively V^ρ) the right A_h-module defined by

$$\forall a \in A_h,\ \forall v \in V,\ v \cdot_{S_h} a = S_h(a) \cdot v \ \text{ (respectively } \ v \cdot_{S_h^{-1}} a = S_h^{-1}(a) \cdot v).$$

Similarly, if W is a right A_h-module, we set W^l (respectively W^λ) the left A_h-module defined by

$$\forall a \in A_h,\ \forall w \in W,\ a \cdot_{S_h} w = w \cdot S_h(a) \ \text{ (respectively } \ a \cdot_{S_h^{-1}} w = w \cdot S_h^{-1}(a)).$$

One has $(V^r)^\lambda = V$, $(V^\rho)^l = V$, $\left(W^l\right)^\rho = W$ and $\left(W^\lambda\right)^r = W$. Thus, we have defined two (in the case where $S_h^2 \neq id$) equivalences of categories between the category of left A_h-modules and the category of right A_h-modules, that is to say left A_h^{op}-modules.

Let $Mod\,(A_h)$ be the abelian category of left A_h-modules and $D\,(Mod(A_h))$ be the derived category of the abelian category $Mod(A_h)$. We may consider A_h as an $A_h \otimes A_h^{op}$-module. Introduce the functor D_{A_h} from $D\,(Mod\,(A_h))$ to $D\,(Mod\,(A_h^{op}))$

$$\forall M^\bullet \in D\,(A_h)\,,\ \ D_{A_h}(M^\bullet) = RHom_{A_h}\,(M^\bullet, A_h)\,.$$

If M is a finitely generated module, the canonical arrow $M \to D_{A_h^{op}} \circ D_{A_h}(M)$ is an isomorphism.

Let V be a left A_h-module, then, by transposition, $V^* = Hom_K(V, K)$ is naturally endowed with a right A_h-module structure. Using the antipode, we can also see it as a left module structure. Thus, one has :

$$\forall u \in A_h\, \forall f \in V^*,\ \ u \cdot f = f \cdot S_h(u).$$

We endow $\Omega_{A_h} \otimes V^*$ with the following right A_h-module structure :

$$\forall u \in A_h \, \forall f \in V^*, \ \forall \omega \in \Omega_{A_h},$$
$$(\omega \otimes w) \cdot u = \lim_{n \to +\infty} \sum_j \theta_{A_h}(u'_{j,n}) \omega \otimes f \cdot S_h^2(u''_{j,n})$$

where $\Delta(u) = \lim_{n \to +\infty} \sum_j u'_{j,n} \otimes u''_{j,n}$.

Theorem 5.0.8. *Let V be an A_h-module free of finite type as a $k[[h]]$-module. Then $D_{A_h}(V)$ and $V^* \otimes \Omega_{A_h}$ are isomorphic in $D(A_h^{op})$.*

Proof of the theorem :

In the proof of this theorem, we will make use of the following lemma (see [Du1], [C1]).

Lemma 5.0.9. *Let W be a left A_h-module. $A_h \widehat{\otimes} W$ is endowed with two different structures of $A_h \otimes A_h^{op}$-modules. The first one denoted $\left(A_h \widehat{\otimes} W\right)_1$ is described as follows : Let w be an element of W and let u, a be two elements of A_h. We set $\Delta(a) = \lim_{n \to +\infty} \sum_i a'_{i,n} \otimes a''_{i,n}$. Then*

$$(u \otimes w) \cdot a = ua \otimes w$$
$$a \cdot (u \otimes w) = \lim_{n \to +\infty} \sum_i a'_{i,n} u \otimes a''_{i,n} \cdot w$$

The second one denoted $\left(A_h \widehat{\otimes} W\right)_2$ is described as follows : Then

$$a \cdot (u \otimes w) = au \otimes w$$
$$(u \otimes w) \cdot a = \lim_{n \to +\infty} \sum_i u a'_{i,n} \otimes S_h(v''_{i,n}) \cdot w$$

The $A_h \otimes A_h^{op}$-modules $\left(A_h \widehat{\otimes} W\right)_1$ and $\left(A_h \widehat{\otimes} W\right)_2$ are isomorphic.

Proof of the lemma :

The map

$$\begin{array}{rcl} \Psi : \left(A_h \widehat{\otimes} W\right)_2 & \to & \left(A_h \widehat{\otimes} W\right)_1 \\ u \otimes w & \mapsto & \lim_{n \to +\infty} \sum_i u'_{i,n} \otimes u''_{i,n} \cdot w \end{array}$$

where $\Delta(u) = \lim_{n \to +\infty} \sum_i u'_{i,n} \otimes u''_{i,n}$ is an isomorphism of $A_h \otimes A_h^{op}$-modules from $\left(A_h \widehat{\otimes} W\right)_2$ to $\left(A_h \widehat{\otimes} W\right)_1$. Moreover

$$\Psi^{-1}(u \otimes w) = \sum u'_{i,n} \otimes S_h(u''_{i,n}) \cdot w.$$

This finishes the proof of the lemma.

Let $L^\bullet$ be a resolution of K by free A_h-modules. We endow $L^i \otimes V$ with the following left A_h-module structure :

$$a \cdot (l \otimes v) = \lim_{n \to +\infty} \sum_i a'_{i,n} \cdot l \otimes a''_{i,n} \cdot v.$$

Then $L^\bullet \otimes V$ is a resolution of V by free A_h-modules. Using the relation

$$a \cdot l \otimes v = \lim_{n \to +\infty} \sum_i a'_{i,n} \left[l \otimes S_h(a''_{i,n}) \cdot v \right]$$

one shows the following sequence of A_h-isomorphisms

$$\begin{array}{rcl} D_{A_h}(V) & \simeq & Hom_{A_h}(L\otimes V, A_h) \\ & \simeq & Hom_{A_h}(L, (A_h\otimes V^*)_1) \\ & \simeq & Hom_{A_h}(L, (A_h\otimes V^*)_2) \\ & \simeq & RHom_{A_h}(K, A_h)\otimes V^*.\square \end{array}$$

6. Link with quantum duality

6.1. **Recollection on the quantum dual principle.** The quantum dual principle ([Dr], see [G] for a detailed treatment) states that there exist two functors, namely $()' : QUEA \to QFSA$ and $()^\vee : QFSA \to QUEA$ which are inverse of each other. If $U_h(\mathfrak{g})$ is a quantization of $U(\mathfrak{g})$ and $F_h[[\mathfrak{g}]]$ is a quantization of $F[[\mathfrak{g}]] = U(\mathfrak{g})^*$, then $U_h(\mathfrak{g})'$ is a quantization of $F[[\mathfrak{g}^*]]$ and $F_h[[\mathfrak{g}]]^\vee$ is a quantization of $U(\mathfrak{g}^*)$.

Let's recall the construction of the functor $()^\vee : QFSA \to QUEA$ which is the one we will need. Let $\mathfrak{g}$ be a Lie bialgebra and $F_h[[\mathfrak{g}]]$ a quantization of $F[[\mathfrak{g}]] = U(\mathfrak{g})^*$. For simplicity we will write F_h instead of $F_h[[\mathfrak{g}]]$. If ϵ_h denotes the counit of F_h, set $I := \epsilon_h^{-1}(hk[[h]])$ and $J = Ker\epsilon_h$. Let

$$F_h^\times := \sum_{n\geq 0} h^{-n}I^n = \sum_{n\geq 0}(h^{-1}I)^n = \bigcup_{n\geq 0}(h^{-1}I)^n$$

be the $k[[h]]$-subalgebra of $k((h)) \underset{k[[h]]}{\otimes} F_h$ generated by $h^{-1}I$. As $I = J + hF_h$, one has $F_h^\times = \sum_{n\geq 0} h^{-n}J^n$. Define $F_h^\vee$ to be the h-adic completion of the $k[[h]]$-module $F_h^\times$. The coproduct (respectively counit, antipode) on F_h provides a coproduct (respectively counit, antipode) on $F_h^\vee$ and $F_h^\vee$ is endowed with a Hopf algebra structure. A precise description of $F_h^\vee$ is given in [G]. Let us recall it as we will need it for our computations. The algebras F_h/hF_h and $k[[\bar{x}_1,\dots,\bar{x}_n]]$ are isomorphic. We denote $\pi : F_h \to F_h/hF_h$ be the natural projection. We may choose $x_j \in \pi^{-1}(\bar{x}_j)$ for any j such that $\epsilon_h(x_j) = 0$, then F_h and $k[[x_1,\dots,x_n,h]]$ are isomorphic as $k[[h]]$- topological module and J is the set of formal series f whose degree in the x_j, $\partial_X(f)$ (that is the degree of the lowest degree monomials occuring in the series with non zero coefficients) is strictly positive. As F_h/hF_h is commutative, one has

$$x_ix_j - x_jx_i = h\chi_{i,j}$$

with $\chi_{i,j} \in F_h$. As $\chi_{i,j}$ is in J, it can be written as follows :

$$\chi_{i,j} = \sum_{a=1}^{n} c_a(h)x_a + f_{i,j}(x_1,\dots,x_n,h)$$

with $\partial_X(f_{i,j}) > 1$. If $\check{x}_i = h^{-1}x_j$, then

$$F_h^\vee = \{f = \sum_{r\in\mathbb{N}} P_r(\check{x}_1,\dots,\check{x}_n)h^r \mid P_r(X_1,\dots,X_n) \in k[X_1,\dots,X_n]\}.$$

Thus $F_h^\vee$ and $k[\check{x}_1,\dots,\check{x}_n][[h]]$ are isomorphic as a topological $k[[h]]$-modules. One has

$$\check{x}_i\check{x}_j - \check{x}_j\check{x}_i = \sum_{a=1}^{n} c_a(h)\check{x}_a + h^{-1}\check{f}_{i,j}(\check{x}_1,\dots,\check{x}_n,h)$$

where $\check{f}_{i,j}(\check{x}_1, \ldots, \check{x}_n, h)$ is obtained from $f_{i,j}(x_1, \ldots, x_n)$ by writing $x_j = h\check{x}_j$. The element $h^{-1}\check{f}_{i,j}(\check{x}_1, \ldots, \check{x}_n, h)$ is in $hk[\check{x}_1, \ldots, \check{x}_n][[h]]$. The k-span of the set of cosets $\{e_i = \check{x}_i \bmod hF_h^\vee\}$ is a Lie algebra isomorphic to $\mathfrak{g}^*$. The map $\Psi : F_h^\vee \to U(\mathfrak{g}^*)[[h]]$ defined by

$$\Psi\left(\sum_{r\in\mathbb{N}} P_r(\check{x}_1, \ldots, \check{x}_n)h^r\right) = \sum_{r\in\mathbb{N}} P_r(e_1, \ldots, e_n)h^r$$

is an isomorphism of topological $k[[h]]$-modules. The algebra $\dfrac{F_h^\vee}{hF_h^\vee}$ is isomorphic to $U(\mathfrak{g}^*)$ and $F_h(\mathfrak{g})^\vee$ is a quantization of the coPoisson Hopf algebra $U(\mathfrak{g}^*)$. Denote by $\cdot_h$ multiplication on F_h and its transposition to $U(\mathfrak{g}^*)[[h]]$ by Ψ. To compute $e_1^{a_1} \ldots e_n^{a_n} \cdot_h e_1^{b_1} \ldots e_n^{b_n}$ we proceed as follows : we compute $\check{x}_1^{a_1} \ldots \check{x}_n^{a_n} \cdot_h \check{x}_1^{b_1} \ldots \check{x}_n^{b_n}$ in $F_h^\vee$ and write it under the form $\sum_{r\in\mathbb{N}} P_r(\check{x}_1, \ldots, \check{x}_n)h^r$. Then

$$e_1^{a_1} \ldots e_n^{a_n} \cdot_h e_1^{b_1} \ldots e_n^{b_n} = \sum_{r\in\mathbb{N}} P_r(e_1, \ldots, e_n)h^r.$$

If u and v are in $U(\mathfrak{g}^*)[[h]]$, one writes $u * v = \sum_{r\in\mathbb{N}} h^r \mu_r(u, v)$. One knows that the first non zero μ_r is a 1-cocycle of the Hochschild cohomology.

If P in $k[X_1, \ldots, X_n]$ can be written $P = \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} X_1^{i_1} \ldots X_n^{i_n}$, one sets

$$P^\otimes(e_1, \ldots, e_n) = \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} e_1^{\otimes i_1} \ldots e_n^{\otimes i_n}$$

and if $g \in k[X_1, \ldots, X_n][[h]]$ can be written $g = \sum_{i=1}^{r} P_r(X_1, \ldots, X_r)h^r$, then one sets :

$$g^\otimes(e_1, \ldots, e_n) = \sum_{i=1}^{r} P_r^\otimes(e_1, \ldots, e_r)h^r.$$

Fact :

$(F_h)^\vee$ is isomorphic as an algebra to

$$U_h(\mathfrak{g}^*) \simeq \frac{T_{k[[h]]}\left(\overset{n}{\underset{i=1}{\oplus}} k[[h]]e_i\right)}{I}$$

where I is the closure (in the h-adic topology) of the two sided ideal generated by the relations

$$e_i \otimes e_j - e_j \otimes e_i = \sum_{k=1}^{n} c_k(h)e_k + h^{-1}\check{f}_{i,j}^\otimes(e_1, \ldots, e_n, h).$$

Let us prove this fact. Let $\Omega : T_{k[[h]]}\left(\overset{n}{\underset{i=1}{\oplus}} k[[h]]e_i\right) \to F_h^\vee$ that sends e_i to $\check{x}_i$. One has $I \subset Ker\Omega$ and we need to prove that $Ker\Omega \subset I$. Let $\mathcal{R}$ be in $T_{k[[h]]}\left(\overset{n}{\underset{i=1}{\oplus}} k[[h]]e_i\right)$ be such that $\Omega(\mathcal{R}) = 0$. Then, modulo h, we get $\overline{\Omega}(\overline{\mathcal{R}}) = 0$.

Hence there exist $(u^0_{i,j})$ and $(v^0_{i,j})$ in $T_k(\overset{n}{\underset{i=1}{\oplus}} ke_i)$ such that

$$\overline{\mathcal{R}} = \sum_{i,j} u^0_{i,j} \otimes (e_i \otimes e_j - e_j \otimes e_i - [e_i, e_j]) \otimes v^0_{i,j}$$

and $\mathcal{R} - \sum_{i,j} u^0_{i,j} \otimes \left(e_i \otimes e_j - e_j \otimes e_i - \sum_{a=1}^{n} c_a(h) e_a - h^{-1} \check{f}^{\otimes}_{i,j}(e_1, \ldots, e_n, h) \right) \otimes v^0_{i,j} \in hKer\Omega$. Hence there exist $\mathcal{R}_1 \in Ker\Omega$ be such that

$$\mathcal{R} - \sum_{i,j} u^0_{i,j} \otimes \left(e_i \otimes e_j - e_j \otimes e_i - \sum_{a=1}^{n} c_a(h) e_a - h^{-1} \check{f}^{\otimes}_{i,j}(e_1, \ldots, e_n, h) \right) \otimes v^0_{i,j} = h\mathcal{R}_1.$$

Reproducing the same reasoning, we find $\left(u^1_{i,j}\right)$ and $\left(v^1_{i,j}\right)$ in $T_k\left(\overset{n}{\underset{i=1}{\oplus}} ke_i\right)$ such that

$$\mathcal{R}_1 - \sum_{i,j} u^1_{i,j} \otimes \left(e_i \otimes e_j - e_j \otimes e_i - \sum_{a=1}^{n} c_a(h) e_a - h^{-1} \check{f}_{i,j}(e_1, \ldots, e_n, h) \right) \otimes v^1_{i,j} = h\mathcal{R}_2$$

and going on like this, we show that $\mathcal{R}$ is in I.

6.2. **Deformation of the Koszul complex.** Let $\mathfrak{a}$ be a k-Lie algebra. There is a well known resolution of the trivial $U(\mathfrak{a})$-module, namely the Koszul resolution $K = (U(\mathfrak{a}) \otimes \wedge^{\bullet}\mathfrak{a}, \partial)$ where

$$\partial\left(u \otimes X_1 \wedge \cdots \wedge X_n\right) = \sum_{i=1}^{n} (-1)^{i-1} u X_i \otimes X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_n$$
$$\sum_{i<j} (-1)^{i+j} u \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge \widehat{X_j} \wedge \cdots \wedge X_n.$$

We will now show that the Koszul resolution can be deformed.

Theorem 6.2.1. *Let $\mathfrak{a}$ be a Lie algebra and let $(e_1, \ldots, e_n)$ be a basis of $\mathfrak{a}$. Denote by $C^a_{i,j}$ the structure constants of $\mathfrak{a}$ with respect to the basis $(e_1, \ldots, e_n)$ so that we have $[e_i, e_j] = \sum_{k=1}^{n} C^a_{i,j} e_a$. Consider $U_h(\mathfrak{a})$ a deformation of $U(\mathfrak{a})$ given under the form*

$$U_h(\mathfrak{a}) \simeq \frac{T_{k[[h]]}\left(\overset{n}{\underset{i=1}{\oplus}} k[[h]] e_i\right)}{I}$$

where I is the closure (in the h-adic topology) of the two sided ideal generated by the relations

$$e_i \otimes e_j - e_j \otimes e_i - g^{\otimes}_{i,j}(e_1, \ldots, e_n, h)$$

where $g_{i,j}$ satisfies the following :

$$g_{i,j} \in k[X_1, \ldots, X_n][[h]] \text{ and } \partial_X(g_{i,j}) \geq 1$$
$$g^{\otimes}_{i,j}(e_1, \ldots, e_n) = \sum_{a=1}^{n} C^a_{i,j} e_a \bmod hT_{k[[h]]}\left(\overset{n}{\underset{i=1}{\oplus}} k[[h]] e_i\right).$$

$k[[h]]$ is an $U_h(\mathfrak{a})$-module (called the trivial $U_h(\mathfrak{a})$-module) if we let the e_i's act trivially. There exists a resolution of the trivial $U_h(\mathfrak{a})$-module $k[[h]]$, $K_h = (U_h(\mathfrak{a}) \otimes_k \wedge^\bullet \mathfrak{a}, \partial_h^\bullet)$, such that GrK_h is the resolution of the trivial $U(\mathfrak{a})[h]$-module $k[h]$.

Remarks :

1) Any quantized universal enveloping algebra, $U_h(\mathfrak{a})$, has a presentation as in the theorem because we might write it $U_h(\mathfrak{a}) = (U_h(\mathfrak{a})')^\vee$.

2) The proof of the theorem gives an algorithm to construct the resolution K_h.

3) By theorem 6.2.1, we even get a filtered resolution of the $FU_h(\mathfrak{g})$-module $k[[h]]$.

Proof of the theorem 6.2.1:

We will prove by induction that on q that one can construct $\partial_0^h, \dots, \partial_q^h$ morphisms of $U_h(\mathfrak{a})$-modules such that :

$\bullet \forall r \in [1, q], \partial_{r-1}^h \partial_r^h = 0.$

$$\bullet \partial_r^h(1 \otimes e_{p_1} \wedge \dots \wedge e_{p_r}) = \sum_{i=1}^{r} (-1)^{i-1} e_{p_i} \otimes e_{p_1} \wedge \dots \wedge \widehat{e_{p_i}} \wedge \dots \wedge e_{p_r}$$
$$+ \sum_{k<l} \sum_a (-1)^{k+l} C_{p_k,p_l}^a 1 \otimes e_a \wedge e_{p_1} \wedge \dots \wedge \widehat{e_{p_k}} \wedge \dots \wedge \widehat{e_{p_l}} \wedge \dots \wedge e_{p_r} + \alpha_{p_1,\dots,p_r}$$

with $\alpha_{p_1,\dots,p_r} \in hU_h(\mathfrak{a}) \otimes \wedge^{r-1}(\mathfrak{a})$ so that $G\partial_r^h$ is the qth differential of the Koszul complex of the trivial $U(\mathfrak{a})[h]$-module $k[h]$. From proposition 3.0.2, this implies that $\mathrm{Ker}\partial^h_{r-1} = \mathrm{Im}\partial^h_r$.

We take $\partial_0^h : U_h(\mathfrak{a}) \to k[[h]]$ to be the algebra morphism determined by $\partial_0^h(e_i) = 0$.

We take $\partial_1^h : U_h(\mathfrak{a}) \otimes_k \mathfrak{a} \to U_h(\mathfrak{a})$ to be the morphism of $U_h(\mathfrak{a})$-modules determined by $\partial_1^h(u \otimes e_i) = ue_i$.

One writes

$$g_{i,j}(e_1, \dots, e_n, h) = \sum_{a=1}^n P_{i,j}^a e_a + \sum_{a=1}^n C_{i,j}^a e_a$$

where the $P_{i,j}^a$'s are in $hU_h(\mathfrak{a})$.

We look for a morphism of $U_h(\mathfrak{a})$-modules, $\partial_2^h : U_h(\mathfrak{a}) \otimes_k \wedge^2 \mathfrak{a} \to U_h(\mathfrak{a}) \otimes_k \mathfrak{a}$ under the form

$$\partial_2^h(1 \otimes e_i \wedge e_j) = e_i \otimes e_j - e_j \otimes e_i - \sum_a C_{i,j}^a 1 \otimes e_a - \alpha_{i,j}$$

where $\alpha_{i,j}$ is in $hU_h(\mathfrak{a}) \otimes \mathfrak{a}$.

One has

$$\partial_1^h \left(e_i \otimes e_j - e_j \otimes e_i - \sum_a C_{i,j}^a 1 \otimes e_a \right) = \sum_{a=1}^n P_{i,j}^a e_a = \partial_1^h \left(\sum_{a=1}^n P_{i,j}^a \otimes e_a \right).$$

We might take

$$\partial_2^h(1 \otimes e_i \wedge e_j) = e_i \otimes e_j - e_j \otimes e_i - \sum_{a=1}^n C_{i,j}^a 1 \otimes e_a - \sum_{a=1}^n P_{i,j}^a 1 \otimes e_a.$$

We have $\partial_1^h \circ \partial_2^h = 0$

Let $q \geq 2$. Assume that $\partial_0^h, \partial_1^h, \dots \partial_q^h$ are constructed and let us construct ∂_{q+1}^h as required.

We look for $\partial^h_{q+1}(1\otimes e_{p_1}\wedge\cdots\wedge e_{p_{q+1}})$ under the form

$$\partial^h_{q+1}(1\otimes e_{p_1}\wedge\cdots\wedge e_{p_{q+1}})=\sum_{i=1}^{q+1}(-1)^{i-1}e_{p_i}\otimes e_{p_1}\wedge\cdots\wedge\widehat{e_{p_i}}\wedge\cdots\wedge e_{p_{q+1}}$$
$$+\sum_{r<s}\sum_a(-1)^{r+s}C^a_{p_r,p_s}1\otimes e_a\wedge e_{p_1}\wedge\cdots\wedge\widehat{e_{p_r}}\wedge\cdots\wedge\widehat{e_{p_s}}\wedge\cdots\wedge e_{p_{q+1}}+\alpha_{p_1,\dots,p_{q+1}}$$

where $\alpha_{p_1,\dots,p_{q+1}}$ is in $hU_h(\mathfrak{a})\otimes\wedge^q(\mathfrak{a})$. The term

$$\partial^h_q\left(\sum_{i=1}^{q+1}(-1)^{i-1}e_{p_i}\otimes e_{p_1}\wedge\cdots\wedge\widehat{e_{p_i}}\wedge\cdots\wedge e_{p_{q+1}}\right)+$$
$$+\partial^h_q\left(\sum_{r<s}\sum_a C^a_{p_r,p_s}(-1)^{r+s}1\otimes e_a\wedge e_{p_1}\wedge\cdots\wedge\widehat{e_{p_r}}\wedge\cdots\wedge\widehat{e_{p_s}}\wedge\cdots\wedge e_{p_{q+1}}\right)$$

equals 0 modulo h. Hence it is in $hU_h(\mathfrak{a})\otimes\wedge^{q-1}(\mathfrak{a})$.

As $\partial^h_{q-1}\partial^h_q=0$, it is in $hKer\partial^h_{q-1}=hIm\partial^h_q$. The existence of $\alpha_{p_1,\dots,p_{q+1}}$ follows. Hence we have constructed ∂^h_{q+1} as required. The complex $K_h=\left(U_h(\mathfrak{a})\otimes\wedge^\bullet\mathfrak{a},\partial^h_\bullet\right)$ is a resolution of the trivial $U_h(\mathfrak{a})$-module $k[[h]]$.□

6.3. **Quantum duality and deformation of the Koszul complex.** We may construct resolutions of the trivial $F_h[\mathfrak{g}]$ and $F_h[\mathfrak{g}]^\vee$-modules that respects the quantum duality.

Theorem 6.3.1. *Let $\mathfrak{g}$ be a Lie bialgebra, $F_h[\mathfrak{g}]$ a QFSHA such that $\dfrac{F_h[\mathfrak{g}]}{hF_h[\mathfrak{g}]}$ is isomorphic to $F[\mathfrak{g}]$ as a topological Poisson Hopf algebra and $F_h[\mathfrak{g}]^\vee=U_h(\mathfrak{g}^*)$ the quantization of $U(\mathfrak{g}^*)$ constructed from $F_h[\mathfrak{g}]$ by the quantum duality principle. Let $\bar{x}_1,\dots,\bar{x}_n$ be elements of $F[\mathfrak{g}]$ such that $F[\mathfrak{g}]\simeq k[[\bar{x}_1,\dots,\bar{x}_n]]$. Choose $x_1,\dots,x_n$ elements of $F_h[\mathfrak{g}]$ such that $x_i=\bar{x}_i \mod h$ and $\epsilon_h(x_i)=0$. Then $U_h(\mathfrak{g}^*)\simeq k[\check{x}_1,\dots,\check{x}_n][[h]]$ with $\check{x}_i=h^{-1}x_i$. Let $(\epsilon_1,\dots,\epsilon_n)$ be a basis of $\mathfrak{g}^*$ and let $C^a_{i,j}$ the structural constants of $\mathfrak{g}^*$ with respect to this basis. We can construct a resolution of the trivial $F_h[\mathfrak{g}]$-module $K^h_\bullet=\left(F_h[\mathfrak{g}]\otimes\wedge\mathfrak{g}^*,\partial^h_q\right)$ of the form*

$$\partial^h_q\left(1\otimes\epsilon_{p_1}\wedge\cdots\wedge\epsilon_{p_q}\right)=\sum_{i=1}^{q}(-1)^{i-1}x_i\otimes\epsilon_{p_1}\wedge\cdots\wedge\widehat{\epsilon_{p_i}}\wedge\cdots\wedge\epsilon_{p_q}$$
$$+\sum_{r<s}\sum_a(-1)^{r+s}hC^a_{p_r,p_s}1\otimes\epsilon_a\wedge\epsilon_{p_1}\wedge\cdots\wedge\widehat{\epsilon_{p_r}}\wedge\cdots\wedge\widehat{\epsilon_{p_s}}\wedge\cdots\wedge\epsilon_{p_q}$$
$$+\sum_{t_1,\dots,t_{q-1}}h\alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}\otimes\epsilon_{t_1}\wedge\cdots\wedge\epsilon_{t_{q-1}}$$

such that $\alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}\in I=\epsilon_h^{-1}(hk[[h]])$. Set

$$\check{\alpha}^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}(\check{x}_1,\dots,\check{x}_n)=\alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}(x_1,\dots,x_n).$$

$\check{\alpha}_{p_1,\dots,p_q}^{t_1,\dots,t_{q-1}}$ *is in* $hk[\check{x}_1,\dots,\check{x}_n][[h]]$. *Define the morphism of* $U_h(\mathfrak{g}^*)$*-modules* $\check{\partial}_q^h$: $U_h(\mathfrak{g}^*)\otimes\wedge^q(\mathfrak{g}^*)\to U_h(\mathfrak{g}^*)\otimes\wedge^{q-1}(\mathfrak{g}^*)$ *by*

$$\check{\partial}_q^h\left(1\otimes\epsilon_{p_1}\wedge\cdots\wedge\epsilon_{p_q}\right)=\sum_{i=1}^n(-1)^{i-1}\check{x}_i\otimes\epsilon_{p_1}\wedge\cdots\wedge\widehat{\epsilon_{p_i}}\wedge\cdots\wedge\epsilon_{p_q}$$
$$+\sum_{r<s}\sum_a(-1)^{r+s}C^a_{p_r,p_s}1\otimes\epsilon_a\wedge\epsilon_{p_1}\wedge\cdots\wedge\widehat{\epsilon_{p_r}}\wedge\cdots\wedge\widehat{\epsilon_{p_s}}\wedge\cdots\wedge\epsilon_{p_q}$$
$$+\sum_{t_1,\dots,t_{q-1}}\check{\alpha}_{p_1,\dots,p_q}^{t_1,\dots,t_{q-1}}\otimes\epsilon_{t_1}\wedge\cdots\wedge\epsilon_{t_{q-1}}.$$

Then $\check{K}_h^\bullet=\left(U_h(\mathfrak{g}^*)\otimes\wedge^\bullet\mathfrak{g}^*,\check{\partial}_q^h\right)$ *is a resolution of the trivial* $U_h(\mathfrak{g}^*)$*-module.*

Proof of the theorem :

One sets $x_ix_j-x_jx_i=\sum_{a=1}^n hC^a_{i,j}x_a+hu^a_{i,j}x_a$. We know that $u^a_{i,j}$ is in I. We take

$$\partial_0^h=\epsilon_h$$
$$\partial_1^h(1\otimes\epsilon_i)=x_i$$

We set

$$\partial_2^h(1\otimes e_i\wedge e_j)=x_i\otimes\epsilon_j-x_j\otimes\epsilon_i-\sum_a hC^a_{i,j}\otimes\epsilon_a-h\sum_a u^a_{i,j}\otimes\epsilon_a.$$

We have $\partial_1^h\circ\partial_2^h=0$ and we may choose $\alpha^a_{i,j}=u^a_{i,j}$.

Assume that $\partial_0^h,\partial_1^h,\dots,\partial_q^h$ have been constructed such that

- $\forall r\in[1,q]\ \ \partial_{r-1}^h\partial_r^h=0$
- $\forall r\in[1,q]\ \ Im\partial_r^h=Ker\partial_{r-1}^h$ and satisfying the required relation.
- $\alpha_{p_1,p_2,\dots,p_r}^{q_1,\dots,q_{r-1}}\in I$.

and let us show that we can construct ∂_{q+1}^h satisfying these three conditions.

The computation below is in [Kn] p 173.

$$\partial_q^h\left(\sum_{i=1}^{q+1}(-1)^{i-1}x_{p_i}\otimes e_{p_1}\wedge\cdots\wedge\widehat{\epsilon_{p_i}}\wedge\cdots\wedge\epsilon_{p_{q+1}}\right)+$$
$$+\partial_q^h\left(\sum_{r<s}\sum_a hC^a_{p_r,p_s}(-1)^{r+s}1\otimes\epsilon_a\wedge\epsilon_{p_1}\wedge\cdots\wedge\widehat{\epsilon_{p_r}}\wedge\cdots\wedge\widehat{\epsilon_{p_s}}\wedge\cdots\wedge\epsilon_{p_{q+1}}\right)$$
$$=\sum_{j<i}(-1)^{i+j}(x_{p_i}x_{p_j}-x_{p_j}x_{p_i})\otimes\epsilon_1\wedge\cdots\wedge\widehat{\epsilon_{p_j}}\wedge\cdots\wedge\widehat{\epsilon_{p_i}}\wedge\cdots\wedge\epsilon_{p_{q+1}}$$
$$\sum_i\sum_{r<s,r,s\neq i}\sum_a(-1)^{r+s+\delta+i+1}hC^a_{p_r,p_s}x_{p_i}\otimes\epsilon_a\wedge\epsilon_{p_1}\wedge(\text{omit }p_r,p_s,p_i)\wedge\epsilon_{p_{q+1}}$$
$$\sum_{p_r<p_s}\sum_a(-1)^{r+s}hC^a_{p_r,p_s}x_a\otimes\epsilon_{p_1}\wedge(\text{ omit p}_\text{r},\text{p}_\text{s})\wedge\epsilon_{p_{n+1}}$$
$$+\sum_{r<s}\sum_a\sum_{p_i\neq p_r,p_s}(-1)^{r+s+i+\delta}hC^a_{p_r,p_s}x_{p_i}\otimes\epsilon_a\wedge\epsilon_{p_1}\wedge(\text{omit }p_r,p_s,p_i)\wedge\epsilon_{p_{q+1}}$$
$$\sum_{r<s}(-1)^{r+s}\sum_{k<l;k,l\neq r,s}(-1)^{k+l+\sigma}\sum_{a,b}h^2C^a_{p_r,p_s}C^b_{p_k,p_l}1\otimes\epsilon_a\wedge\epsilon_b\wedge\epsilon_1\wedge(\text{omit }p_k,p_l,p_r,p_s)\wedge\epsilon_{p_{q+1}}$$
$$+\sum_{r<s}(-1)^{r+s}\sum_{j\neq r,s}\sum_{a,b}(-1)^{j+\tau}h^2C^a_{p_r,p_s}C^b_{a,p_j}\otimes\epsilon_b\wedge\epsilon_{p_1}\wedge(\text{omit }p_j,p_r,p_s)\wedge\epsilon_{p_{q+1}}$$
$$+\sum_i(-1)^{i-1}x_{p_i}h\alpha_{p_1,\dots,\widehat{p_i},\dots,p_{q+1}}+\sum_{r<s}\sum_a(-1)^{r+s}hC^a_{p_r,p_s}h\alpha_{a,p_1,\dots,\widehat{p_r},\dots,\widehat{p_s},\dots,p_{q+1}}$$

where

$\delta = 1$ if $r < i < s$ and $\delta = 0$ otherwise
$\sigma = 1$ if exactly one of k and l is between r and s
$\tau = 1$ if $r < j < s$ and $\tau = 0$ otherwise

The fifth term and the sixth term cancel. The second term and fourth term cancel with each other so that we have

$$\partial_q^h\left(\sum_{i=1}^{q+1}\epsilon_{p_i}\otimes\epsilon_{p_1}\wedge\cdots\wedge\widehat{\epsilon_{p_i}}\wedge\cdots\wedge\epsilon_{p_{q+1}}\right)+$$
$$+\partial_q^h\left(\sum_{k<l}\sum_a hC^a_{p_k,p_l}1\otimes\epsilon_a\wedge\epsilon_{p_1}\wedge\cdots\wedge\widehat{\epsilon_{p_k}}\wedge\cdots\wedge\widehat{\epsilon_{p_l}}\wedge\cdots\wedge\epsilon_{p_{q+1}}\right)$$
$$=\sum_{j<i}(-1)^{i+j}\left(x_{p_i}x_{p_j}-x_{p_j}x_{p_i}-\sum_a hC^a_{p_i,p_j}x_a\right)\otimes\epsilon_1\wedge\cdots\wedge\widehat{\epsilon_{p_j}}\wedge\cdots\wedge\widehat{\epsilon_{p_i}}\wedge\cdots\wedge\epsilon_{p_{q+1}}$$
$$+\sum_i(-1)^{i-1}hx_{p_i}\alpha_{p_1,\dots,\widehat{p_i},\dots,p_{q+1}}+\sum_{r<s}(-1)^{r+s}h^2C^a_{p_r,p_s}\alpha_{a,p_1,\dots,\widehat{p_r},\dots,\widehat{p_l},\dots,p_{q+1}}.$$

As $\partial_{q-1}^h\partial_q^h=0$, the term

$$\partial_q^h\left(\sum_{i=1}^{q+1}(-1)^{i-1}e_{p_i}\otimes e_{p_1}\wedge\cdots\wedge\widehat{e_{p_i}}\wedge\cdots\wedge e_{p_{q+1}}\right)+$$
$$+\partial_q^h\left(\sum_{k<l}\sum_a(-1)^{k+l}C^a_{p_k,p_l}1\otimes e_a\wedge e_{p_1}\wedge\cdots\wedge\widehat{e_{p_k}}\wedge\cdots\wedge\widehat{e_{p_l}}\wedge\cdots\wedge e_{p_{q+1}}\right)$$

is in $hKer\partial_{q-1}^h=hIm\partial_q^h$. We can choose $\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}$ in $F_h[\mathfrak{g}]$ so that the expression above equals $-\partial_q^h\left(h\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}\right)$.

Let us now prove that $\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}$ is in I. It is easy to see that $-\partial_q^h\left(h\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}\otimes\epsilon_{t_1}\wedge\cdots\wedge\epsilon_{t_q}\right)$ is element of $I^3\otimes\wedge^q\mathfrak{g}^*$. Note that ∂_q^h sends $I^r\otimes\wedge^q\mathfrak{g}^*$ to $I^{r+1}\otimes\wedge^q\mathfrak{g}^*$. Let us write

$$\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}=\sum_{i_1,\dots,i_n}(\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})_{i_1,\dots,i_n}x_1^{i_1}\dots x_n^{i_n}$$

with $(\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})_{i_1,\dots,i_n}$ in $k[[h]]$. From the remarks we have just made, we see that $\partial_q^h\left(h\sum_{t_1,\dots,t_q}(\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})_{0,\dots,0}\epsilon_{t_1}\wedge\cdots\wedge\epsilon_{t_q}\right)$ is in $I^3\otimes\wedge^q\mathfrak{g}^*$. Hence $(\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})_{0,\dots,0}$ is in $hk[[h]]$.

As $\mathrm{Im}G\partial_{q+1}^h=\mathrm{Ker}G\partial_q^h$, one has $\mathrm{Im}\partial_{q+1}^h=\mathrm{Ker}\partial_q^h$.

Set

$$\check{\alpha}^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}(\check{x}_1,\dots,\check{x}_n)=\alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}(x_1,\dots,x_n).$$

$$\check\partial_0 = \epsilon$$
$$\check\partial_1(1\otimes\epsilon_i) = \check x_i$$
$$\check\partial_2(1\otimes\epsilon_i\wedge\epsilon_j) = \check x_i\otimes\epsilon_j - \check x_j\otimes\epsilon_j - \sum_a C^a_{i,j}\otimes\epsilon_a - \sum_a \check u^a_{i,j}\otimes\epsilon_a$$
$$\check\partial^h_{q+1}\left(1\otimes\epsilon_{p_1}\wedge\cdots\wedge\epsilon_{p_{q+1}}\right) = \sum_{i=1}^{q+1}(-1)^{i-1}\check x_i\otimes\epsilon_{p_1}\wedge\cdots\wedge\hat\epsilon_{p_i}\wedge\cdots\wedge\epsilon_{p_{q+1}}$$
$$+\sum_{r<s}\sum_a(-1)^{r+s}C^a_{p_r,p_s}1\otimes\epsilon_a\wedge\epsilon_{p_1}\wedge\cdots\wedge\hat\epsilon_{p_r}\wedge\cdots\wedge\hat\epsilon_{p_s}\wedge\cdots\wedge\epsilon_{p_{q+1}}$$
$$+\sum_{t_1,\ldots,t_{q-1}}\check\alpha^{t_1,\ldots,t_q}_{p_1,\ldots,p_{q+1}}\otimes\epsilon_{t_1}\wedge\cdots\wedge\epsilon_{t_q}.$$

If P is in F_h, one has

$$\partial_q(P\otimes\epsilon_{p_1}\wedge\cdots\wedge\epsilon_{p_q}) = h\check\partial(\check P\otimes\epsilon_{p_1}\wedge\cdots\wedge\epsilon_{p_q}).$$

The relation $\check\partial_q\check\partial_{q+1}=0$ is obtained by multiplying the relation $\partial^h_q\partial^h_{q+1}=0$ by h^{-2}. As $G\check\partial^h_q$ is the differential of the Koszul complex of the trivial $U(\mathfrak g^*)[h]$-module, the complex $\check K^\bullet_h = \left(U_h(\mathfrak g^*)\otimes\wedge^\bullet\mathfrak g^*, \check\partial^h_n\right)$ is a resolution of the trivial $U_h(\mathfrak g^*)$-module. □

6.4. **A link between θ_{F_h} and $\theta_{F_h^\vee}$.**

Theorem 6.4.1. *One has $\theta_{F_h} = h\theta_{F_h^\vee}$*

Proof of the theorem :

We keep the notation of the previous proposition and we will use the proof of the theorem 5.0.7.

The complex $\left(\wedge^\bullet\mathfrak g^*\otimes F_h, {}^t\partial^h_n\right)$ computes the $k[[h]]$-modules $\mathrm{Ext}^{\mathrm i}_{\mathrm{F_h}}(\mathrm{k[[h]],F_h})$. The cohomology class $cl\,(1\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is a basis of $\underline{\mathrm{Ext}}^n_{F[\mathfrak g][h]}(k[h],F[\mathfrak g][h]) \simeq G\mathrm{Ext}^{\mathrm n}_{\mathrm{F_h}}(\mathrm{k[[h]],F_h})$. Hence there exists $\sigma = 1+h\sigma_1+\cdots\in\mathrm{Ker}^{\mathrm t}\partial^{\mathrm h}_{\mathrm n}$ such that $[cl\,(\sigma\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)]$ is a basis of $G\mathrm{Ext}^{\mathrm n}_{\mathrm{F_h}}(\mathrm{k[[h]],F_h})$. As the filtration on $\mathrm{Ext}^{\mathrm n}_{\mathrm{F_h}}(\mathrm{k[[h]],F_h})$ is Hausdorff, the cohomology class $cl\,(\sigma\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is a basis of $\mathrm{Ext}^{\mathrm n}_{\mathrm{F_h}}(\mathrm{k[[h]],F_h})$.

Define $\check\sigma$ by

$$\check\sigma(\check x_1,\ldots,\check x_n) = \sigma(x_1,\ldots,x_n).$$

One has ${}^t\partial_n = h^t\check\partial_n$ and it is easy to check that $\check\sigma\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*$ is in $\mathrm{Ker}^{\mathrm t}\check\partial^{\mathrm h}_{\mathrm{n-1}}$. If we had

$$\check\sigma\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^* ={}^t\check\partial^h_{n-1}\left(\sum_{i=1}^n\check\sigma_i\otimes\epsilon_1^*\wedge\cdots\wedge\widehat{\epsilon_i^*}\wedge\cdots\wedge\epsilon_n^*\right),$$

then, reducing modulo h, we would get

$$\overline{\check\sigma}\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^* = \overline{{}^t\check\partial^h_{n-1}}\left(\sum_{i=1}^n\overline{\check\sigma_i}\otimes\epsilon_1^*\wedge\cdots\wedge\widehat{\epsilon_i^*}\wedge\cdots\wedge\epsilon_n^*\right).$$

This would implies that $cl\,(1\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is 0 in $\mathrm{Ext}^{\mathrm n}_{\mathrm{U(\mathfrak g^*)}}(\mathrm{k,U(\mathfrak g^*)})$, which is impossible because $cl\,(1\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is a basis of $\mathrm{Ext}^{\mathrm n}_{\mathrm{U(\mathfrak g^*)}}(\mathrm{k,U(\mathfrak g^*)})$. Thus $cl\,(\check\sigma\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is a non zero element of $Ext^{dim\mathfrak g^*}_{U_h(\mathfrak g^*)}(k[[h]],U_h(\mathfrak g^*))$. For all i in $[1,n]$, one has the relation

$$\sigma x_i\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^* = \theta_{F_h}(x_i)\sigma\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^* +{}^t\partial^h_n(\mu)$$

Let us write

$$\mu = \sum_i \mu_i \otimes \epsilon_1^* \wedge \cdots \wedge \widehat{\epsilon_i^*} \wedge \cdots \wedge \epsilon_n^*$$

with $\mu_i \in F_h[\mathfrak{g}]$. We set $\check{\mu}_i(\check{x}_1, \dots, \check{x}_n) = \mu_i(x_1, \dots, x_n)$ and

$$\check{\mu} = \sum_i \check{\mu}_i \otimes \epsilon_1^* \wedge \cdots \wedge \widehat{\epsilon_i^*} \wedge \cdots \wedge \epsilon_n^*.$$

Then we have

$$h\check{\sigma}\check{x}_i \otimes \epsilon_1^* \wedge \cdots \wedge \epsilon_n^* = \theta_{F_h}(x_i)\check{\sigma} \otimes \epsilon_1^* \wedge \cdots \wedge \epsilon_n^* + h^t \check{\partial}_n^h(\check{\mu}).$$

This finishes the proof of the theorem 6.4.1. □

7. Study of on example

We will now study explicitely an example suggested by B. Enriquez. Chloup ([Chl]) introduced the triangular Lie bialgebra ($\mathfrak{g} = kX_1 \oplus kX_2 \oplus kX_3 \oplus kX_4 \oplus kX_5$, $r = 4(X_2 \wedge X_3)$) where the non zero brackets are given by

$$[X_1, X_2] = X_3, \ \ [X_1, X_3] = X_4, \ \ [X_1, X_4] = X_5$$

and the cobracket $\delta_{\mathfrak{g}}$ is the following :

$$\forall X \in \mathfrak{g}, \ \ \delta(X) = X \cdot 4(X_2 \wedge X_3).$$

The dual Lie bialgebra of $\mathfrak{g}$ will be denoted ($\mathfrak{a} = ke_1 \oplus ke_2 \oplus ke_3 \oplus ke_4 \oplus ke_5, \delta$). The only non zero Lie bracket of $\mathfrak{a}$ is $[e_2, e_4] = 2e_1$ and its cobracket δ is non zero on the basis vectors e_3, e_4, e_5 :

$$\delta(e_3) = e_1 \otimes e_2 - e_2 \otimes e_1 = 2e_1 \wedge e_2, \ \ \delta(e_4) = 2e_1 \wedge e_3, \ \ \delta(e_5) = 2e_1 \wedge e_4.$$

The invertible element of $U(\mathfrak{g})[[h]]\widehat{\otimes}U(\mathfrak{g})[[h]]$, $R = exp\,(h(X_2 \otimes X_3 - X_3 \otimes X_2))$, satisfies the equations

$$R^{12}(\Delta \otimes 1)(R) = R^{23}(1 \otimes \Delta)(R)$$
$$(\epsilon \otimes id)(R) = 1 = (id \otimes \epsilon)(R).$$

Thus, we may twist the trivial deformation of $(U(\mathfrak{g})[[h]], \mu_0, \Delta_0, \iota_0, \epsilon_0, S_0)$ by R ([C-P] p. 130). The topological Hopf algebra obtained has the same multiplication, antipode , unit and counit but its coproduct is $\Delta^R = R^{-1}\Delta_0 R$. It is a quantization of $(\mathfrak{g}, r)$. We will denote it by $U_h(\mathfrak{g})$. The Hopf algebra $U_h(\mathfrak{g})^*$ is a QFSHA and $(U_h(\mathfrak{g})^*)^\vee$ is a quantization of $(\mathfrak{a}, \delta_{\mathfrak{a}})$. We will compute it explicitely.

Proposition 7.0.2. *a)* $(U(\mathfrak{g})^*)^\vee$ *is isomorphic as a topological Hopf algebra to the topological k[[h]]-algebra* $T_{k[[h]]}\left(k[[h]]e_1 \oplus k[[h]]e_2 \oplus k[[h]]e_3 \oplus k[[h]]e_4 \oplus k[[h]]e_5\right)/I$ *where* I *is the closure of the two-sided ideal generated by*

$$e_2 \otimes e_4 - e_4 \otimes e_2 - 2e_1$$
$$e_3 \otimes e_5 - e_5 \otimes e_3 - \frac{2}{3}h^2 e_1 \otimes e_1 \otimes e_1$$
$$e_4 \otimes e_5 - e_5 \otimes e_4 - \frac{1}{6}h^3 e_1 \otimes e_1 \otimes e_1 \otimes e_1$$
$$e_2 \otimes e_5 - e_5 \otimes e_2 + he_1 \otimes e_1$$
$$e_3 \otimes e_4 - e_4 \otimes e_3 + he_1 \otimes e_1$$
$$e_i \otimes e_j - e_j \otimes e_i \text{ if } \{i, j\} \neq \{2, 4\}, \ \{3, 5\}, \ \{4, 5\}, \ \{2, 5\}, \ \{3, 4\}$$

with the coproduct Δ_h, counit ϵ_h and antipode S defined as follows :

$$\begin{array}{l}\Delta_h(e_1) = e_1 \otimes 1 + 1 \otimes e_1 \\ \Delta_h(e_2) = e_2 \otimes 1 + 1 \otimes e_2 \\ \Delta_h(e_3) = e_3 \otimes 1 + 1 \otimes e_3 - he_2 \otimes e_1 \\ \Delta_h(e_4) = e_4 \otimes 1 + 1 \otimes e_4 - he_3 \otimes e_1 + \frac{h^2}{2} e_2 \otimes e_1^2 \\ \Delta_h(e_5) = e_5 \otimes 1 + 1 \otimes e_5 - he_4 \otimes e_1 + \frac{h^2}{2} e_3 \otimes e_1^2 - \frac{h^3}{6} e_2 \otimes e_1^3. \\ \forall i \in [1,5], \quad \epsilon_h(e_i) = 0 \\ \forall i \in [1,5], \quad S(e_i) = -e_i\end{array}$$

b) $(U(\mathfrak{g})^*)^\vee$ is not isomorphic to the trivial deformation of $U(\mathfrak{a})$, $U(\mathfrak{a})[[h]]$, as algebra.

Proof of the proposition

Let ξ_i be the element of $U(\mathfrak{g})^*$ defined by

$$< \xi_i, X_1^{a_1} X_2^{a_2} X_3^{a_3} X_4^{a_4} X_5^{a_5} >= \delta_{a_1,0} \dots \delta_{a_i,1} \dots \delta_{a_5,0}.$$

The algebras $U(\mathfrak{g})^*$ and $k[[\xi_1, \dots, \xi_n]]$ are isomorphic. The topological Hopf algebra $\left(U_h(\mathfrak{g})^*, {}^t\Delta_0^R = \cdot_h, {}^t\mu_0 = \Delta_h, {}^t\epsilon_0, {}^t\iota_0 = \epsilon_h, {}^tS_0\right)$ is a QFSHA. Remark that $U_h(\mathfrak{g})^*$ and $k[[\xi_1, \dots, \xi_n, h]]$ are isomorphic as $k[[h]]$-modules. The elements $\xi_1, \dots, \xi_n$ generate topologically the $k[[h]]$- algebra $U_h(\mathfrak{g})^*$ and satisfy $\epsilon_h(\xi_i) = 0$.

$< \xi_2 \otimes \xi_4 - \xi_4 \otimes \xi_2, \Delta^R(X_1^{a_1} \dots X_5^{a_5}) > \neq 0 \iff (a_1, a_2, a_3, a_4, a_5) = (1, 0, 0, 0, 0)$.

and $< \xi_2 \otimes \xi_4 - \xi_4 \otimes \xi_2, \Delta^R(X_1) >= 2h$. Hence $\xi_2 \cdot_h \xi_4 - \xi_4 \cdot_h \xi_2 = 2h\xi_1$.

$< \xi_3 \otimes \xi_5 - \xi_5 \otimes \xi_3, \Delta^R(X_1^{a_1} \dots X_5^{a_5}) > \neq 0 \iff (a_1, a_2, a_3, a_4, a_5) = (3, 0, 0, 0, 0)$

and $< \xi_3 \otimes \xi_5 - \xi_5 \otimes \xi_3, X_1^3 >= 4h$. Hence $\xi_3 \cdot_h \xi_5 - \xi_5 \cdot_h \xi_3 = \dfrac{2h^2}{3} \xi_1 \cdot_h \xi_1 \cdot_h \xi_1$.

$< \xi_4 \otimes \xi_5 - \xi_5 \otimes \xi_4, \Delta^R(X_1^{a_1} \dots X_5^{a_5}) > \neq 0 \iff (a_1, a_2, a_3, a_4, a_5) = (4, 0, 0, 0, 0)$.

and $< \xi_4 \otimes \xi_5 - \xi_5 \otimes \xi_4, \Delta(X_1^4) >= -4h$. Hence $\xi_4 \cdot_h \xi_5 - \xi_5 \cdot_h \xi_4 = \dfrac{-h^3}{6} \xi_1 \cdot_h \xi_1 \cdot_h \xi_1 \cdot_h \xi_1$.

$< \xi_2 \otimes \xi_5 - \xi_5 \otimes \xi_2, \Delta^R(X_1^{a_1} \dots X_5^{a_5}) > \neq 0 \iff (a_1, a_2, a_3, a_4, a_5) = (2, 0, 0, 0, 0)$.

and $< \xi_2 \otimes \xi_5 - \xi_5 \otimes \xi_2, \Delta(X_1^2) >= -2h$. Hence $\xi_2 \cdot_h \xi_5 - \xi_5 \cdot_h \xi_2 = -h\xi_1 \cdot_h \xi_1$.

$< \xi_3 \otimes \xi_4 - \xi_4 \otimes \xi_3, \Delta^R(X_1^{a_1} \dots X_5^{a_5}) > \neq 0 \iff (a_1, a_2, a_3, a_4, a_5) = (2, 0, 0, 0, 0)$.

and $< \xi_3 \otimes \xi_4 - \xi_4 \otimes \xi_3, \Delta^R(X_1^2) >= -2h$. Hence $\xi_3 \cdot_h \xi_4 - \xi_4 \cdot_h \xi_3 = -h\xi_1 \cdot_h \xi_1$. In the cases different from those mentionned above, $\xi_i \cdot_h \xi_j = \xi_j \cdot_h \xi_i$.

Let us now compute the coproduct Δ_h of $U_h(\mathfrak{g})^*$.

$< \Delta_h(\xi_3), X_1^{a_1} X_2^{a_2} X_3^{a_3} X_4^{a_4} X_5^{a_5} \otimes X_1^{b_1} X_2^{b_2} X_3^{b_3} X_4^{b_4} X_5^{b_5} > \neq 0 \iff$
$(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5) = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$ or $(0, 0, 0, 0, 0, 0, 0, 1, 0, 0)$ or $(0, 1, 0, 0, 0, 1, 0, 0, 0, 0)$

and $< \Delta_h(\xi_3), X_2X_1 >= -1$. Hence

$$\Delta_h(\xi_3) = \xi_3 \otimes 1 + 1 \otimes \xi_3 - \xi_2 \otimes \xi_1.$$

$< \Delta_h(\xi_4), X_1^{a_1} X_2^{a_2} X_3^{a_3} X_4^{a_4} X_5^{a_5} \otimes X_1^{b_1} X_2^{b_2} X_3^{b_3} X_4^{b_4} X_5^{b_5} > \neq 0 \iff$
$(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5) = (0, 0, 0, 1, 0, 0, 0, 0, 0, 0)$ or $(0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$ or $(0, 0, 1, 0, 0, 1, 0, 0, 0, 0)$ or $(0, 1, 0, 0, 0, 2, 0, 0, 0, 0)$.

Moreover

$$< \Delta_h(\xi_4), X_3 \otimes X_1 >= -1 \quad \text{and} \quad < \Delta_h(\xi_4), X_2 \otimes X_1^2 >= 1.$$

Hence

$$\Delta_h(\xi_4) = \xi_4 \otimes 1 + 1 \otimes \xi_4 - \xi_3 \otimes \xi_1 + \frac{1}{2}\xi_2 \otimes \xi_1 \cdot_h \xi_1.$$

$< \Delta_h(\xi_5), X_1^{a_1} X_2^{a_2} X_3^{a_3} X_4^{a_4} X_5^{a_5} \otimes X_1^{b_1} X_2^{b_2} X_3^{b_3} X_4^{b_4} X_5^{b_5} >\neq 0 \Longleftrightarrow$
$(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5) = (0,0,0,0,1,0,0,0,0,0)$ or $(0,0,0,0,0,0,0,0,0,1)$ or $(0,0,0,1,0,1,0,0,0,0)$ or $(0,0,1,0,0,2,0,0,0,0)$ or $(0,1,0,0,0,3,0,0,0,0)$.

Moreover

$$< \Delta_h(\xi_5), X_4 \otimes X_1 >= -1, \quad < \Delta_h(\xi_4), X_3 \otimes X_1^2 >= 1, \quad < \Delta_h(\xi_4), X_2 \otimes X_1^3 >= -1.$$

Hence

$$\Delta_h(\xi_5) = \xi_5 \otimes 1 + 1 \otimes \xi_5 - \xi_4 \otimes \xi_1 + \frac{1}{2}\xi_3 \otimes \xi_1 \cdot_h \xi_1 - \frac{1}{6}\xi_2 \otimes \xi_1 \cdot_h \xi_1 \cdot_h \xi_1.$$

We set $\check{\xi}_i = h^{-1}\xi_i$ and $e_i = \check{\xi}_i \bmod h\, (U(\mathfrak{g})^*)^\vee$. Let $\chi : (U(\mathfrak{g})^*)^\vee \to U(\mathfrak{a})[[h]]$ be the isomorphism of topologicall $k[[h]]$-modules defined by

$$\chi\left(\sum_{r\in\mathbb{N}} P_r(\check{\xi}_1, \dots, \check{\xi}_n) h^r\right) = \sum_{r\in\mathbb{N}} P_r(e_1, \dots, e_n) h^r.$$

From what we have reviewed in the first paragraph of this section, the first part of this theorem is proved.

If u and v are in $U(\mathfrak{a})$, one sets

$$u \cdot_h v = uv + \sum_{r=1}^{\infty} h^r \mu_r(u, v).$$

one has

$$\mu_1(e_3, e_4) = 0, \quad \mu_1(e_4, e_3) = e_1^2, \quad \mu_1(e_2, e_5) = 0, \quad \mu_1(e_5, e_2) = e_1^2.$$

Let us show now that μ_1 is a coboundary in the Hochschild cohomology. The Hochschild cohomology $HH^*(U(\mathfrak{a}), U(\mathfrak{a}))$ is computed by the complex $\left(Hom\,(U(\mathfrak{a})^{\otimes}, U(\mathfrak{a}))\,, b\right)$ where : if $f \in Hom\left(U(\mathfrak{a})^{\otimes n+1}, U(\mathfrak{a})\right)$, then

$$b(f)(a_0, \dots, a_n) = a_0 f(a_1, \dots, a_n) + \sum_{i=1}^{n} (-1)^i f(a_0, \dots, a_{i-1}a_i, \dots a_n) + f(a_0, \dots, a_{n-1}) a_n (-1)^n.$$

The Lie algebra cohomology of $\mathfrak{a}$ with coefficients in $U(\mathfrak{a})^{ad}$ (with the adjoint action), $H^*\left(\mathfrak{a}, U(\mathfrak{a})^{ad}\right)$, is computed by the Chevalley-Eilenberg complex $\left(Hom\,(\bigwedge \mathfrak{a}, U(\mathfrak{a}))\,, d\right)$ where : if $f \in Hom\left(\bigwedge^{n+1} \mathfrak{a}, U(\mathfrak{a})\right)$

$$\begin{aligned} d(f)(z_1, \dots, z_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i-1} z_i \cdot f(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{n+1}) \\ &+ \sum_{i<j} (-1)^{i+j} f([z_i, z_j], \dots, z_{i-1}, z_{i+1}, z_{j-1}, z_{j+1}, z_{n+1}). \end{aligned}$$

The map ([L] lemma 3.3.3) $\Psi^* : (Hom\,(U(\mathfrak{a})^{\otimes}, U(\mathfrak{a}))\,, b) \to \left(Hom\left(\bigwedge \mathfrak{a}, U(\mathfrak{a})^{ad}\right), d\right)$ defined by antisymmetrization

$$\Psi^*(f)(z_1,\dots,z_n) = f\left(\sum_{\sigma\in S_n} \epsilon(\sigma) z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(n)}\right)$$

is a morphism of complexes. One checks easily that

$$\Psi^*(\mu_1) = d\left(-\frac{1}{2}e_1e_2 \otimes e_3^* - \frac{1}{2}e_1e_4\otimes e_5^*\right).$$

There exists $\alpha \in Hom\,(U(\mathfrak{a}), U(\mathfrak{a}))$ such that $\mu_1 = b(\alpha)$. The map α is determined by

$$\begin{array}{l} \alpha_{|\mathfrak{a}} = -\frac{1}{2}e_1e_2 \otimes e_3^* - \frac{1}{2}e_1e_4\otimes e_5^* \\ \forall (u,v)\in U(\mathfrak{a}),\ \ \mu_1(u,v) = u\alpha(v) - \alpha(uv) + u\alpha(v) \end{array}$$

We set $\beta_h = id - h\alpha$. Then one has $\beta_h^{-1} = \sum_{i=0}^{\infty} h^i\alpha^i$. If u and v are elements of $U(\mathfrak{a})$, we put

$$u \cdot_h^{'} v = \beta_h^{-1}\left(\beta_h(u)\cdot_h \beta_h(v)\right).$$

Let's compute $e_i \cdot_h^{'} e_j - e_j \cdot_h^{'} e_i$. If i and j are different from 3 and 5, then $e_i \cdot_h^{'} e_j = e_i \cdot_h e_j$

$$\begin{aligned} e_1 \cdot_h^{'} e_3 - e_3\cdot_h^{'} e_1 &= f_h^{-1}\left[e_1 \cdot_h \left(e_3 + \frac{he_1e_2}{2}\right) - \left(e_3 + \frac{he_1e_2}{2}\right)\cdot_h e_1\right] \\ &= f_h^{-1}\left[e_1\cdot_h \frac{he_1e_2}{2} + \frac{he_1e_2}{2}\cdot_h e_1\right] \\ &= 0 \end{aligned}$$

Similarly, the following relations hold

$$e_1 \cdot_h^{'} e_5 = e_5\cdot_h^{'} e_1,\ \ e_2 \cdot_h^{'} e_3 = e_3\cdot_h^{'} e_2,\ \ e_2 \cdot_h^{'} e_5 = e_5\cdot_h^{'} e_2,\ \ e_3 \cdot_h^{'} e_4 = e_4\cdot_h^{'} e_3,$$

Let us now compute $e_3 \cdot_h^{'} e_5 - e_5\cdot_h^{'} e_3$. Easy computations lead to the following equalities : one has

$$\begin{array}{l} e_1e_2\cdot_h e_5 - e_5\cdot_h e_1e_2 = e_1^3 \\ e_3\cdot_h e_1e_4 - e_1e_4\cdot_h e_3 = -e_1^3 \\ e_1e_2\cdot_h e_1e_4 - e_1e_4\cdot_h e_1e_2 = 2e_1^3 \end{array}$$

One deduces easily from this that

$$e_3 \cdot_h^{'} e_5 - e_5\cdot_h^{'} e_3 = \frac{1}{6}h^2e_1^3.$$

Similarly, one has

$$e_4 \cdot_h^{'} e_5 - e_5\cdot_h^{'} e_4 = \frac{-h^2}{6}e_1^3.$$

The topological algebras $[U(\mathfrak{a})[[h]], \cdot_h]$ and $[U(\mathfrak{a})[[h]], \cdot_h^{'}]$ are isomorphic, hence their centers are isomorphic. Let us compute the center of $[U(\mathfrak{a})[[h]], \cdot_h^{'}]$. Let z be an element of the center $Z\,[U(\mathfrak{a})[[h]], \cdot_h^{'}]$. One writes z under the form $\sum_{n\geq 0} P_r(e_1,e_2,e_3,e_4,e_5)h^r$ (where the multiplications in $P_r(e_1,e_2,e_3,e_4,e_5)$ are $\cdot_h^{'}$). One has

$$e_2 \cdot_h^{'} z - z\cdot_h^{'} e_2 = \sum_{r\in\mathbb{N}} 2h^r\frac{\partial P_r}{\partial X_4}(e_1,e_2,e_3,e_4,e_5).$$

Hence the polynomials P_r don't depend on X_4 and z can be written $z = \sum_{n\geq 0} P_r(e_1, e_2, e_3, e_5)h^r$.

$$e_3 \cdot_h' z - z \cdot_h' e_3 = \sum_{r\in\mathbb{N}} \frac{1}{6} h^{r+2} \left(X_1^3 \frac{\partial P_r}{\partial X_5} \right) (e_1, e_2, e_3, e_5).$$

Hence the polynomials P_r don't depend on X_5 and z can be written $z = \sum_{n\geq 0} P_r(e_1, e_2, e_3)h^r$.

$$e_4 \cdot_h' z - z \cdot_h' e_4 = \sum_{r\in\mathbb{N}} -2h^r \frac{\partial P_r}{\partial X_2}(e_1, e_2, e_3).$$

Hence the polynomials P_r don't depend on X_2 and z can be written $z = \sum_{n\geq 0} P_r(e_1, e_3)h^r$.

$$e_5 \cdot_h' z - z \cdot_h' e_5 = \sum_{r\in\mathbb{N}} \frac{-1}{6} h^{r+2} \left(X_1^3 \frac{\partial P_r}{\partial X_3} \right) (e_1, e_3).$$

Hence the polynomials P_r don't depend on X_3 and z can be written $z = \sum_{n\geq 0} P_r(e_1)h^r$.

Hence

$$Z\left[U(\mathfrak{a})[[h]], \cdot_h'\right] = \{\sum_{n\geq 0} P_r(e_1)h^r \mid P_r \in k[X_1]\}.$$

But, the center of the trivial deformation of $U(\mathfrak{a})$ is

$$Z\left[U(\mathfrak{a})[[h]], \mu_0\right] = \{\sum_{n\geq 0} P_r(e_1, e_3, e_5)h^r \mid P_r \in k[X_1, X_3, X_5]\}.$$

The algebras $[U(\mathfrak{a})[[h]], \cdot_h']$ and $[U(\mathfrak{a})[[h]], \mu_0]$ are not isomorphic as their center are not isomorphic. □

Proposition 7.0.3. *We consider the quantized enveloping algebra of the proposition 7.0.2 We write the relations defining the ideal I as follows*

$$e_i \otimes e_j - e_j \otimes e_i - \sum_a C_{i,j}^a e_a - P_{i,j}.$$

As all the $P_{i,j}$'s are monomials in e_1's, the notation $\frac{P_{i,j}}{e_1}$ makes sense. The complex

$$0 \to U_h(\mathfrak{a}) \otimes \wedge^5 \mathfrak{a} \overset{\partial_5^h}{\to} U_h(\mathfrak{a}) \otimes \wedge^4 \mathfrak{a} \overset{\partial_4^h}{\to} \cdots \overset{\partial_2^h}{\to} U_h(\mathfrak{a}) \otimes \mathfrak{a} \overset{\partial_1^h}{\to} U_h(\mathfrak{a}) \overset{\partial_0^h}{\to} k[[h]] \to 0$$

where the morphisms of $U_h(\mathfrak{a})$, ∂_h^i, are described below is a resolution of the trivial $U_h(\mathfrak{a})$-module $k[[h]]$. We set

$$\partial_n(1 \otimes e_{p_1} \wedge \cdots \wedge e_{p_n}) = \sum_{i=1}^{n} (-1)^{i-1} e_{p_i} \otimes e_{p_1} \wedge \cdots \wedge \widehat{e_{p_i}} \wedge \cdots \wedge e_{p_n}$$
$$+ \sum_{k<l} (-1)^{k+l} \sum_a C_{p_k,p_l}^a 1 \otimes e_a \wedge e_{p_1} \wedge \cdots \wedge \widehat{e_{p_k}} \wedge \cdots \wedge \widehat{e_{p_l}} \wedge \cdots \wedge e_{p_n}.$$

Then
$\partial_0^h = \epsilon_h$
$\partial_1^h(1\otimes e_i) = e_i$
$\partial_2^h(1\otimes e_i\wedge e_j) = \partial_2(1\otimes e_i\wedge e_j) - \frac{P_{i,j}}{e_1}\otimes e_i$
$\partial_3^h(1\otimes e_i\wedge e_j\wedge e_k) = \partial_3(1\otimes e_i\wedge e_j\wedge e_k) - \frac{P_{i,j}}{e_1}\otimes e_1\wedge e_k + \frac{P_{i,k}}{e_1}\otimes e_1\wedge e_j - \frac{P_{j,k}}{e_1}\otimes e_1\wedge e_i$
$\partial_4^h(1\otimes e_1\wedge e_i\wedge e_j\wedge e_k) = \partial_4(1\otimes e_1\wedge e_i\wedge e_j\wedge e_k)$
$\partial_4^h(1\otimes e_2\wedge e_3\wedge e_4\wedge e_5) = \partial_4(1\otimes e_2\wedge e_3\wedge e_4\wedge e_5) + \frac{P_{3,5}}{e_1}\otimes e_1\wedge e_2\wedge e_4$
$\quad -\frac{P_{3,4}}{e_1}\otimes e_1\wedge e_2\wedge e_5 - \frac{P_{4,5}}{e_1}\otimes e_1\wedge e_2\wedge e_3 - \frac{P_{2,5}}{e_1}\otimes e_1\wedge e_3\wedge e_4$
$\partial_5^h(1\otimes e_1\wedge e_2\wedge e_3\wedge e_4\wedge e_5) = \partial_5(1\otimes e_1\wedge e_2\wedge e_3\wedge e_4\wedge e_5)$.

The character defined by the right multiplication of $U_h(\mathfrak{a})$ on $Ext^5_{U_h(\mathfrak{a})}(k[[h]], U_h(\mathfrak{a}))$ is zero.

Proof of the proposition : The resolution of $k[[h]]$ constructed in the proposition is obtained by applying the proof of theorem 6.2.1. Moreover, one has

$$\begin{array}{l} {}^t\partial_5(1\otimes e_1^*\wedge e_2^*\wedge e_3^*\wedge e_4^*) = e_5\otimes e_1^*\wedge e_2^*\wedge e_3^*\wedge e_4^*\wedge e_5^* \\ {}^t\partial_5(1\otimes e_1^*\wedge e_3^*\wedge e_4^*\wedge e_5^*) = -e_2\otimes e_1^*\wedge e_2^*\wedge e_3^*\wedge e_4^*\wedge e_5^* \\ {}^t\partial_5(1\otimes e_1^*\wedge e_2^*\wedge e_4^*\wedge e_5^*) = e_3\otimes e_1^*\wedge e_2^*\wedge e_3^*\wedge e_4^*\wedge e_5^* \\ {}^t\partial_5(1\otimes e_1^*\wedge e_2^*\wedge e_3^*\wedge e_5^*) = -e_4\otimes e_2^*\wedge e_3^*\wedge e_3^*\wedge e_4^*\wedge e_5^* \\ {}^t\partial_5(1\otimes e_2^*\wedge e_3^*\wedge e_4^*\wedge e_5^*) = e_1\otimes e_1^*\wedge e_2^*\wedge e_3^*\wedge e_4^*\wedge e_5^*. \end{array}$$

These equalities show that the character defined by the right multiplication of $U_h(\mathfrak{a})$ on $Ext^5_{U_h(\mathfrak{a})}(k[[h]], U_h(\mathfrak{a}))$ is zero.

8. Applications

8.1. **Poincaré duality.** Let M be an A_h^{op}-module and N an A_h-module. The right exact functor $M \underset{A_h}{\otimes} -$ has a left derived functor. We set $Tor^i_{A_h}(M,N) = L^i\left(M\underset{A_h}{\otimes} -\right)(N)$.

Theorem 8.1.1. *Let A_h be a deformation algebra of A_0 satisfying the hypothesis of theorem 5.0.7. Assume moreover that the A_h-module K is of finite projective dimension. Let M be an A_h-module. One has an isomorphism of K-modules*

$$Ext^i_{A_h}(K,M) \simeq Tor^{A_h}_{d_{A_h}-i}(\Omega_{A_h}, M).$$

Remark : Theorem 8.1.1 generalizes classical Poincaré duality ([Kn]).

Proof of the theorem
As the A_h-module K admits a finite length resolution by finitely generated projective A_h-modules, $P^\bullet \to K$, the canonical arrow

$$RHom_{A_h}(K, A_h) \overset{L}{\underset{A_h}{\otimes}} M \to RHom_{A_h}(K,M)$$

is an isomorphism in $D(ModA_h)$. Indeed the canonical arrow

$$Hom_{A_h}(P^\bullet, A_h) \underset{A_h}{\otimes} M \to Hom_{A_h}(P^\bullet, M)$$

is an isomorphism.

8.2. **Duality property for induced representations of quantum groups.**

From now on, we assume that A_h is a topological Hopf algebra.

In this section, we keep the notation of theorem 5.0.8. Let V be a left A_h-module, then, by transposition, $V^* = Hom_K(V, K)$ is naturally endowed with a right A_h-module structure. Using the antipode, we can also see V^* as a left module structure. Thus, one has :

$$\forall u \in A_h \, \forall f \in V^*, \;\; u \cdot f = f \cdot S(u).$$

We endow $\Omega_{A_h} \otimes V^*$ with the following right A_h-module structure :

$$\begin{aligned} &\forall u \in A_h \, \forall f \in V^*, \; \forall \omega \in \Omega_{A_h}, \\ &(\omega \otimes f) \cdot u = \lim_{n \to +\infty} \sum_j \theta_{A_h}(u'_{j,n}) \omega \otimes f \cdot S_h^2(u''_{j,n}) \end{aligned}$$

where $\Delta(u) = \lim_{n \to +\infty} \sum_j u'_{j,n} \otimes u''_{j,n}$.

Let A_h be a topological Hopf deformation of A_0 and B_h be a topological Hopf deformation of B_0. We assume moreover that there exists a morphism of Hopf algebras from B_h to A_h and that A_h is a flat B_h^{op}-module (by proposition 4.1.6 this is verified if the induced B_0-module structure on A_0 is flat). If V is an A_h-module, we can define the induced representation from V as follows :

$$Ind_{B_h}^{A_h}(V) = A_h \underset{B_h}{\otimes} V$$

on which A_h acts by left multiplication.

Proposition 8.2.1. *Let A_h be a topological Hopf deformation of A_0 and B_h be a topological deformation of B_0. We assume that there exists a morphism of Hopf algebras from B_h to A_h such that A_h is a flat B_h^{op}-module. We also assume that B_h satisfies the hypothesis of theorem 5.0.7. Let V be an B_h-module which is a free finite dimensional K-module. Then $D_{B_h}\left(Ind_{A_h}^{B_h}(V)\right)$ is isomorphic to $(\Omega_{B_h} \otimes V^*) \underset{B_h}{\otimes} A_h[-d_{B_h}]$ in $D\,(ModB_h^{op})$.*

Corollary 8.2.2. *Let A_h be a topological Hopf deformation of A_0 and B_h be a topological deformation of B_0. We assume that there exists a morphism of Hopf algebras from B_h to A_h such that A_h is a flat B_h^{op}-module. We also assume that B_h satisfies the condition of the theorem 5.0.7. Let V be a B_h-module which is a free finite dimensional K-module. Then*

a) $Ext^i_{A_h}\left(A_h \underset{B_h}{\otimes} V, A_h\right)$ is reduced to 0 if i is different from d_{B_h}.

b) The right A_h-module $Ext^{d_{B_h}}_{A_h}\left(A_h \underset{B_h}{\otimes} V, A_h\right)$ is isomorphic to $(\Omega_{B_h} \otimes V^*) \underset{B_h}{\otimes} A_h$.

Remarks :

Proposition 8.2.1 is already known in the case where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h}$ is a Lie subalgebras of $\mathfrak{g}$, A and B are the corresponding enveloping algebras.

In this case one has $d_{B_h} = dim\mathfrak{h}$ and $d_{C_h} = dim\mathfrak{k}$. More precisely : It was proved by Brown and Levasseur ([B-L] p. 410) and [Ke] in the case where $\mathfrak{g}$ is a finite dimensional semi-simple Lie algebra and $Ind_{U(\mathfrak{h})}^{U(\mathfrak{g})}(V)$ is a Verma-module. Proposition 8.2.3 is proved in full generality for Lie superalgebras in [C1].

Here are some examples of situations where we can apply the proposition 8.2.1:

Example 1 :
Let k be a field of characteristic 0. We set $K = k[[h]]$. Etingof and Kazhdan have constructed a functor Q from the category $LB(k)$ of Lie bialgebras over k to the category $HA(K)$ of topological Hopf algebras over K. If $(\mathfrak{g}, \delta)$ is a Lie bialgebra, its image by Q will be denoted $U_h(\mathfrak{g})$.
Let $\mathfrak{g}$ be a Lie bialgebra Let $\mathfrak{h}$ be a Lie sub-bialgebra of $\mathfrak{g}$. The functoriality of the quantization implies the existence of an embedding of Hopf algebras from $U_h(\mathfrak{h})$ to $U_h(\mathfrak{g})$ which satisfies all our hypothesis.

Example 2 : If $\mathfrak{g}$ is a Lie bialgebra, we will denote by $\mathcal{F}(\mathfrak{g})$ the formal group attached to it and $\mathcal{F}_h(\mathfrak{g})$ its Etingof Kazhdan quantization. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras and assume that there exists a surjective morphism of Lie bialgebras from $\mathfrak{g}$ to $\mathfrak{h}$. Then $\mathcal{F}_h(\mathfrak{g})$ is a flat $\mathcal{F}_h(\mathfrak{h})$-module and $A_h = \mathcal{F}_h(\mathfrak{g})$ and $B_h = \mathcal{F}_h(\mathfrak{h})$ satisfies the hypothesis of the theorem.

Example 3 :
If G is an affine algebraic Poisson group, we will denote by $\mathcal{F}(G)$ the algebra of regular functions on G and $\mathcal{F}_h(G)$ its Etingof Kazhdan quantization. Let G and H be affine algebraic Poisson groups. Assume that there is a Poisson group map $G \to H$ such that $\mathcal{F}(G)$ is a flat $\mathcal{F}(H)^{op}$-module. By functoriality of Etingof Kazhdan quantization, $A_h = \mathcal{F}_h(G)$ and $B_h = \mathcal{F}_h(H)$ satisfies the hypothesis of the theorem.

Proof of the proposition 8.2.1 :
We proceed as in [C1]. Let $L^\bullet \to V$ be a resolution of V by finite free B_h-modules. As A_h is a flat B_h^{op}-module, $A_h \underset{B_h}{\otimes} L^\bullet \to A_h \underset{B_h}{\otimes} V$ is a resolution of the A_h-module $A_h \underset{B_h}{\otimes} V$ by finite free A_h-modules.
We have the following sequence of isomorphisms in $D\,(ModA_h)$

$$\begin{aligned} RHom_{A_h}\left(A_h \underset{B_h}{\otimes} V, A_h\right) &\simeq Hom_{A_h}\left(A_h \underset{B_h}{\otimes} L^\bullet, A_h\right) \\ &\simeq Hom_{B_h}(L^\bullet, B_h) \underset{B_h}{\otimes} A_h \\ &\simeq (\Omega_{B_h} \otimes V^*) \underset{B_h}{\otimes} A_h[-d_{B_h}].\square \end{aligned}$$

We now extend to Hopf algebras another duality property for induced representations of Lie algebras ([C1]).

Proposition 8.2.3. *Let A_h be a Hopf deformation of A_0, B_h be a Hopf deformation of B_0 and C_h be a Hopf deformation of C_0. We assume that there exists a* *morphism of Hopf algebras from B_h to A_h and a morphism of Hopf algebras from C_h to A_h such that A_h is a flat B_h^{op}-module and a flat C_h^{op}-module. We also assume that B_h and C_h satisfies the hypothesis of theorem 5.0.7. Let V (respectively W) be an B_h-module (respectively C_h-module) which is a free finite dimensional K-module. Then, for all integer n, one has an isomorphism*

$$\begin{aligned} &Ext^{n+d_{B_h}}_{A_h}\left(A_h \underset{B_h}{\otimes} V, A_h \underset{C_h}{\otimes} W\right) \\ &\simeq Ext^{n+d_{C_h}}_{A_h^{op}}\left((\Omega_{C_h} \otimes W^*) \underset{C_h}{\otimes} A_h, (\Omega_{B_h} \otimes V^*) \underset{C_h}{\otimes} A_h\right) \end{aligned}$$

Remarks :

Proposition 8.2.3 is already known in the case where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h}$ and $\mathfrak{k}$ are Lie subalgebras of $\mathfrak{g}$, A, B and C are the corresponding enveloping algebras. In this case one has $d_{B_h} = dim\mathfrak{h}$ and $d_{C_h} = dim\mathfrak{k}$. More precisely :

Generalizing a result of G. Zuckerman ([B-C]), A. Gyoja ([G]) proved a part of this theorem (namely the case where $\mathfrak{h} = \mathfrak{g}$ and $n = dim\mathfrak{h} = dim\mathfrak{k}$) under the assumptions that $\mathfrak{g}$ is split semi-simple and $\mathfrak{h}$ is a parabolic subalgebra of $\mathfrak{g}$. D.H Collingwood and B. Shelton ([C-S]) also proved a duality of this type (still under the semi-simple hypothesis) but in a slighly different context.

M. Duflo [Du2] proved proposition 8.2.3 for a $\mathfrak{g}$ general Lie algebra, $\mathfrak{h} = \mathfrak{k}$, $V = W^*$ being one dimensional representations.

Proposition 8.2.3 is proved in full generality in the context of Lie superalgebras in [C1].

Proof of the proposition 8.2.3:

We will proceed as in [C2]. As $D_{A_h^{op}} \circ D_{A_h}\left(A_h \underset{B_h}{\otimes} V\right) = A_h \underset{B_h}{\otimes} V$, we have the following isomorphism

$$\begin{aligned} &Hom_{D(A_h)}\left(A_h \underset{B_h}{\otimes} V, A_h \underset{B_h}{\otimes} W\right) \\ &\simeq Hom_{D(A_h^{op})}\left[D_{A_h}\left(A_h \underset{C_h}{\otimes} W\right), D_{A_h}\left(A_h \underset{B_h}{\otimes} V\right)\right] \end{aligned}$$

the corollary follows now from proposition 8.2.1.

8.3. **Hochschild cohomology.** In this subsection, A_h is a topological Hopf algebra. We set $A_h^e = A_h \underset{k[[h]]}{\otimes} A_h^{op}$ and $\widehat{A_h^e} = A_h \underset{k[[h]]}{\widehat{\otimes}} A_h^{op}$. If M is an $\widehat{A_h^e}$-module, we set

$$\begin{aligned} HH^i_{A_h}(M) &= Ext^i_{\widehat{A_h^e}}(A_h, M) \\ HH_i^{A_h}(M) &= Tor_i^{\widehat{A_h^e}}(A_h, M) \end{aligned}$$

Proposition 8.3.1. *Assume that A_h satisfies the condition of the theorem 5.0.7. Assume moreover that $A_0 \otimes A_0^{op}$ is noetherian. Consider $A_h \underset{k[[h]]}{\widehat{\otimes}} A_h$ with the following $\widehat{A_h^e}$-module structure :*

$$\forall(\alpha, \beta, x, y) \in A_h, \quad \alpha \cdot (x \otimes y) \cdot \beta = \alpha x \otimes y\beta.$$

a) $HH^i_{A_h}(A_h \underset{k[[h]]}{\widehat{\otimes}} A_h)$ is zero if $i \neq d_{A_h}$.

b) The $\widehat{A_h^e}$-module $HH^{d_{A_h}}_{A_h}(A_h \widehat{\underset{k[[h]]}{\otimes}} A_h)$ is isomorphic to $\Omega_{A_h} \otimes A_h$ with the following $\widehat{A_h^e}$-module structure :

$$\forall(\alpha,\beta,x) \in A_h, \quad \alpha\cdot(\omega\otimes x)\cdot\beta = \omega\theta_{A_h}(\beta_i')\otimes S(\beta_i'')xS^{-1}(\alpha)$$

where $\alpha = \sum_i \alpha_i' \otimes \alpha_i''$ (to be taken in the topological sense)

This result was obtained in [D-E] for a deformation of the algebra of regular functions on a smooth algebraic affine variey.

Proof of the theorem :
The proof is analogous to that of [C2] (theorem 3.3.2).
Using the antipode S_h of A_h, we have the following isomorphism in $D\left(Mod\widehat{A_h^e}\right)$,

$$RHom_{\widehat{A_h^e}}\left(A_h, A_h\widehat{\otimes}A_h\right) \simeq RHom_{A_h\widehat{\otimes}A_h}\left((A_h)^{\#}, (A_h\widehat{\otimes}A_h)^{\#}\right).$$

where the structures on $(A_h)^{\#}$ and $(A_h\widehat{\otimes}A_h)^{\#}$ are given by :

$$\begin{array}{l}\forall(\alpha,\beta,u,v)\in A_h\\ (\alpha\otimes\beta)\cdot u = \alpha u S_h(\beta)\\ (\alpha\otimes\beta)\cdot(u\otimes v) = \alpha u\otimes vS_h(\beta)\\ (u\otimes v)\cdot\alpha\otimes\beta = u\alpha\otimes S_h(\beta)v.\end{array}$$

Using the version of lemma 5.0.9 for right modules (see [C2] lemma 1;1), one sees that $(A_h)^{\#}$ is isomophic to $(A_h\widehat{\otimes}A_h)\underset{A_h}{\otimes}K$ as an $A_h\widehat{\otimes}A_h$-module. we get

$$\begin{array}{rcl}RHom_{\widehat{A_h^e}}\left(A_h, A_h\widehat{\otimes}A_h\right) & \simeq & RHom_{A_h\widehat{\otimes}A_h}\left(A_h\widehat{\otimes}A_h\underset{A_h}{\otimes}K, (A_h\widehat{\otimes}A_h)^{\#}\right)\\ & \simeq & RHom_{A_h}\left(K, (A_h\widehat{\otimes}A_h)^{\#}\right)\\ & \simeq & RHom_{A_h}\left(K, A_h\right)\otimes_{A_h}(A_h\widehat{\otimes}A_h)^{\#}\\ & \simeq & \Omega_h\otimes_{A_h}(A_h\widehat{\otimes}A_h)^{\#}\end{array}$$

The isomorphism $id\otimes S_h^{-1}$ transforms $(A_h\widehat{\otimes}A_h)^{\#}$ into the natural $(A_h\widehat{\otimes}A_h)\otimes(A_h\widehat{\otimes}A_h)^{op}$-module $(A_h\widehat{\otimes}A_h)$-module $(A_h\widehat{\otimes}A_h)^{nat}$:

$$\begin{array}{l}\forall(\alpha,\beta,u,v)\in A_h\\ (\alpha\otimes\beta)\cdot(u\otimes v) = \alpha u\otimes \beta v\\ (u\otimes v)\cdot\alpha\otimes\beta = u\alpha\otimes v\beta.\end{array}$$

Then, using the lemma 5.0.9, one sees that $\Omega_h\otimes_{A_h}(A_h\widehat{\otimes}A_h)^{nat}$ is isomorphic to $\Omega_h\otimes A_h$ endowed with the following $(A_h\widehat{\otimes}A_h)^{op}$-module structure :

$$\forall(\alpha,\beta)\in A_h,\ (u\otimes v)\cdot\alpha\otimes\beta = \sum_i u\theta_{A_h}(\alpha_i')\otimes S(\alpha_i'')v\beta.$$

This finishes the proof of the proposition. □

We are in the case where $Ext^i_{\widehat{A_h^{op}}}(A_h, \widehat{A_h^e})$ is 0 unless when $i = d_{A_h}$, so we have a duality between Hochschild homology and Hochschild cohomology ([VdB]).

Corollary 8.3.2. *Let A_h be a k-algebra satisfying the hypothesis of theorem 5.0.7. Assume moreover that $A_0^e = A_0 \otimes A_0^{op}$ is noetherian and that the $\widehat{A_h^e}$-module A_h is finite projective dimension. Let M be an $\widehat{A_h^e}$-module. One has*

$$HH^i(M) \simeq HH_{d_{A_h}-i}(U \underset{A_h}{\otimes} M).$$

Proof of the corollary : The proof of the corollary is similar to that of [vdB].

First case : M is a finite type $\widehat{A_h^e}$-module. Let $P^\bullet \to A_h \to 0$ be a finite length finite type projective resolution of the $\widehat{A_h^e}$-module A_h and let $Q^\bullet \to M \to 0$ be a finite type projective resolution of the $\widehat{A_h^e}$-module M. As Q^i and $U \underset{A_h}{\otimes} Q^i$ are complete, one has the following sequence of isomorphisms :

$$\begin{aligned} HH^i_{\widehat{A_h^e}}(M) &\simeq H^i\left(Hom_{\widehat{A_h^e}}(P^\bullet, M)\right) \simeq H^i\left(Hom_{\widehat{A_h^e}}(P^\bullet, \widehat{A_h^e}) \underset{\widehat{A_h^e}}{\otimes} M\right) \\ &\simeq H^i\left(U[-d] \overset{L}{\underset{\widehat{A_h^e}}{\otimes}} M\right) \simeq H^{i-d_{A_h}}\left(U \underset{\widehat{A_h^e}}{\otimes} Q^\bullet\right) \simeq H^{i-d_{A_h}}\left((A_h \underset{A_h}{\otimes} U) \underset{\widehat{A_h^e}}{\otimes} Q^\bullet)\right) \\ &\simeq H^{i-d_{A_h}}\left(A_h \underset{\widehat{A_h^e}}{\otimes} (U \underset{A_h}{\otimes} Q^\bullet)\right) \simeq HH_{d_{A_h}-i}(U \underset{A_h}{\otimes} M). \end{aligned}$$

General case : We no longer assume that M is a finite type $\widehat{A_h^e}$-module. We have $M = \varinjlim M'$ where M' runs over all finitely generated submodules of M.

$$\mathrm{Ext}^i_{\widehat{A_h^e}}(\mathrm{A_h}, \mathrm{M}) = \mathrm{Ext}^i_{\widehat{A_h^e}}(\mathrm{A_h}, \varinjlim \mathrm{M'}) \simeq \varinjlim \mathrm{Ext}^i_{\widehat{A_h^e}}(\mathrm{A_h}, \mathrm{M'}) \simeq \varinjlim \mathrm{Tor}^{\widehat{A_h^e}}_{d_{A_h}-i}(\mathrm{A_h}, \mathrm{U} \underset{\widehat{A_h}}{\otimes} \mathrm{M'})$$
$$\simeq Tor^{\widehat{A_h^e}}_{d_{A_h}-i}(A_h, \varinjlim U \underset{\widehat{A_h}}{\otimes} M') \simeq Tor^{\widehat{A_h^e}}_{d_{A_h}-i}(A_h, U \underset{\widehat{A_h}}{\otimes} M)$$

where we used the fact that the functor $\varinjlim$ is exact because the set of finitely generated submodules of M is a directed set ([Ro] proposition 5.33)

9. Bibliography

[A-K] A.Altman and S. Kleiman : *Introduction to Grothendieck duality theory*, Lecture Notes in Mathematics **146**, Springer-Verlag.

[B-C] B.D Boe and D.H Collingwood : *A comparison theorem for the structures of induced representations* , J. algebra **94**, 1985, p. 511-545.

[B-L] K. A. Brown - T. Levasseur : *Cohomology of bimodules over enveloping algebras*, Mathematische Zeitschrift **189**, 1985, 511-545.

[C-P] V. Chari - A. Pressley : *A guide to quantum groups*, Cambridge University Press (1994).

[C1] S. Chemla : *Poincare duality for k-A Lie superalgebras*, Bulletin de la Société Mathématique de France **122**, 371-397.

[C2] S. Chemla : *Rigid dualizing complex for quantum algebras and algebras of generalized differential operators*, Journal of algebra **276** (2004) p.80-102.

[Chl] Chloup-Arnoult V. :*Linearization of some Poisson-Lie tensor*, Journal of geometry and physics **24** (1997) 46-52.

[C-S] D.H Collingwood- B. Shelton :*A duality theorem for extensions of induced highest weight modules*, Pacific J. Math **146** 2, 1990, 227-237.

[D-E] V. Dolgushev and P. Etingof : *Hochschild cohomology of quantized symplectic orbifolds and the Chen- Ruan cohomology*, Int. Math. Res. Not. 2005, **27**, 1657 -1688.

[Dr] V. G. Drindfeld : *Quantum groups*, Proc. Intern. Congress of Math. (Berkeley 1986)(1987), 798-820.

[Du1] M. Duflo : *Sur les idéaux induits dans les algèbres enveloppantes* Invent. Math. **67**, 385-393.

[Du2] M. Duflo : *Open problems in representation theory of Lie groups*, Proceedings of the eighteenth international symposium, division of mathematics, the Tanigushi foundation.

[E-K 1] P. Etingof - Kazhdan :*Quantization of Lie bialgebras I*, Selecta mathematica**2** (1996) p1-41.

[E-K 2] P. Etingof - Kazhdan :*Quantization of Lie bialgebras II*, Selecta mathematica.

[E-K 3] P. Etingof - Kazhdan :*Quantization of Poisson algebraic groups and Poisson homogeneous spaces*, ArXiv:q-alg/9510020v2.

[E-S] P. Etingof - O. Schiffman : *Lectures on quantum groups*, International Press of Boston Inc (2001).

[Ga] F. Gavarini : *The quantum duality principle*, Ann. Inst. Fourier, **52**, 3 (2002), 809-834.

[Gy] A. Gyoja : *A duality theorem for homomorphisms between generalized Verma modules* Preprint Kyoto University.

[H-S] P.J Hilton - U. Stammbach *A course in homological algebra*, Graduate text in mathematics, Springer Verlag.

[K-S] M. Kashiwara - P. Schapira : *Deformation quantization modules I* arXiv : math QA/0802.1245.

[Ke] G.R. Kempf : *The Ext-dual of a Verma module is a Verma module*, Journal of pure and applied algebra, **75**, 1991, p. 47-49.

[Kn] A. Knapp : *Lie groups, Lie algebras and cohomology*, Princeton university Press, 1988.

[L] J.L. Loday : *Cyclic homology*, Springer Verlag, Berlin (1998).

[Ro] J.J Rotman : *An introduction to homological algebra*, second edition, Springer.

[Schn] J-P Schneiders : *An introduction to D-modules*, Bulletin Soc. Royale Science Liege **63** (3-4)(1994) p. 223-295.

[Schw] L. Schwartz : *Topologie générale et analyse fonctionnelle*, Hermann.

[VdB] M. Van den Bergh : *A relation between Hochschild homology and cohomology for Gorenstein rings*, Proc. Amer. Math. Soc. **126** (1998), 1345-1348 and erratum in Proc. Amer. Math. Soc. **130** (2002), 2809-2810.