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GEOMETRIC SYNTOMIC COHOMOLOGY AND VECTOR BUNDLES ON THE FARGUES-FONTAINE CURVE

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Abstract

We show that geometric syntomic cohomology lifts canonically to the category of Banach-Colmez spaces and study its relation to extensions of modifications of vector bundles on the Fargues-Fontaine curve. We include some computations of geometric syntomic cohomology spaces: they are finite rank $\mathbf{Q}_p$-vector spaces for ordinary varieties, but in the nonordinary case, these cohomology spaces carry much more information; in particular they can have a nontrivial C-rank. This dichotomy is reminiscent of the Hodge-Tate period map for p-divisible groups.

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1. Introduction

As is well known, syntomic cohomology is a p-adic analog of Deligne-Beilinson cohomology. Recall that the latter is an absolute Hodge cohomology [2]; i.e., it can be computed as Ext groups in the category of mixed Hodge structures, a feature that makes definition of regulators straightforward. In [9], it was shown that this is also the case for syntomic cohomology of varieties over a p-adic local field K: it is an absolute p-adic Hodge cohomology; i.e., it can be computed as Ext groups in a category of admissible filtered φ-modules of Fontaine. In this paper we prove an analog of this statement for geometric syntomic cohomology (i.e., for varieties over $\overline{K}$): it can be computed as Ext groups in the category of effective filtered φ-modules over $\overline{K}$ (what Fargues would call "φ-modules jaugés"). It follows that it has an extra rigid structure; namely, it comes from a complex of finite dimensional Banach-Colmez spaces. Hence the geometric syntomic cohomology groups are finite dimensional Banach-Colmez spaces. In the case of ordinary varieties, these groups are of Dimension $(0, h)$; i.e., they are finite dimensional $\mathbf{Q}_p$-vector spaces, but in the nonordinary case, these groups can have Dimension (d, h), with $d \geq 1$, and thus carry much more information, as we show in the example of the symmetric square of an elliptic curve. The interested reader will find examples of computations in Section 7.

We are now going to explain in more detail what we have said above. Recall that for a log-smooth variety $\mathscr{X}$ over $\mathscr{O}_K$ – a complete discrete valuation ring of mixed characteristic $(0, p)$ with field of fractions K and perfect residue field – the arithmetic syntomic cohomology of X is defined as a filtered Frobenius eigenspace of its crystalline cohomology

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, r) := [\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})^{\varphi=p^r} \to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})/F^r], \quad r \geq 0. \tag{1.1}$$

We write $[\cdots]$ for the mapping fiber. For a variety X over K, this sheafifies well in the h-topology[1] and yields syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)$, $r\geq 0$, of X [17]. This cohomology comes equipped with a period map to étale cohomology

$$\rho_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)\to \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X,\mathbf{Q}_p(r))$$

that is a quasi-isomorphism after taking the truncation $\tau_{\leq r}$. Syntomic cohomology approximates better p-adic motivic cohomology than étale cohomology does; in particular, étale p-adic regulators from K-theory factor through syntomic cohomology.

As was shown in [9], syntomic cohomology is an absolute p-adic Hodge cohomology. Namely, the data of the Hyodo-Kato cohomology $\mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K}})$ and the de Rham cohomology $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})$ together with the Hyodo-Kato quasi-isomorphism $\iota_{\mathrm{dR}}:\mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K}})\otimes_{F^{\mathrm{nr}}}\overline{K}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})$, where F is the maximal absolutely unramified subfield of K and F^{nr}–its maximal unramified extension, allows one to canonically associate to any variety X over K a complex $\mathrm{R}\Gamma_{\mathrm{DF}_K}(X,r)$ of Fontaine's admissible filtered (φ,N,G_K)-modules. One proves that

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)\simeq \mathrm{Hom}_{D^b(\mathrm{DF}_K)}(\mathbb{1},\mathrm{R}\Gamma_{\mathrm{DF}_K}(X,r)),\quad r\geq 0.$$

Syntomic cohomology from (1.1) has a geometric version. Geometric syntomic cohomology is defined as a filtered Frobenius eigenspace of geometric crystalline cohomology

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}},r):=[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})^{\varphi=p^r}\to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})/F^r],\quad r\geq 0, \tag{1.2}$$

where $\overline{K}$ is an algebraic closure of K and $\mathscr{O}_{\overline{K}}$ its ring of integers. For a variety X over $\overline{K}$, this also sheafifies well in the h-topology[2] and yields syntomic cohomology of X [17]

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)\simeq [\mathrm{R}\Gamma_{\mathrm{cr}}(X)^{\varphi=p^r}\to \mathrm{R}\Gamma_{\mathrm{cr}}(X)/F^r].$$

This cohomology comes equipped with a period map to étale cohomology

$$\rho_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)\to \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X,\mathbf{Q}_p(r))$$

that is a quasi-isomorphism after taking the truncation $\tau_{\leq r}$. In particular, the groups $H^i_{\mathrm{syn}}(X,r)$, $i\leq r$, are finite rank $\mathbf{Q}_p$-vector spaces.

[1] Contrary to crystalline cohomology itself, which does not sheafify well. The sheafification process uses the fact that h-topology has a basis consisting of smooth varieties with semistable compactifications [3].

[2] Here crystalline cohomology itself also sheafifies well and yields well-behaved crystalline cohomology $\mathrm{R}\Gamma_{\mathrm{cr}}(X)$ of X.

The first main result of this paper is that geometric syntomic is an absolute p-adic Hodge cohomology. To explain what this means, we note that we have the isomorphisms

$$H^i_{\mathrm{cr}}(X) \simeq H^i_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{cr}}, \quad H^i_{\mathrm{cr}}(X)/F^r \simeq (H^i_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r,$$

where $\mathbf{B}^*_*$ denotes rings of p-adic periods, and an isomorphism

$$\iota_{\mathrm{dR}} : H^i_{\mathrm{cr}}(X) \otimes_{\mathbf{B}^+_{\mathrm{cr}}} \mathbf{B}^+_{\mathrm{dR}} \xrightarrow{\sim} H^i_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}}.$$

Hence the data

$$(H^i_{\mathrm{cr}}(X) \otimes_{\mathbf{B}^+_{\mathrm{cr}}} \mathbf{B}^+, (H^i_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}}, F^r), \iota_{\mathrm{dR}})$$

gives an effective filtered φ-module over $\overline{K}$.[3] The category of such objects is equivalent to the category of effective modifications of vector bundles on the Fargues-Fontaine curve.[4] For the above data: the vector bundle is associated to the φ-module over $\mathbf{B}^+_{\mathrm{cr}}$ given by $H^i_{\mathrm{cr}}(X)$ and it is modified at infinity by the $\mathbf{B}^+_{\mathrm{dR}}$-lattice $F^r(H^i_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})$. As in the arithmetic case this data can be lifted to complexes and we obtain the first main result of this paper.

Theorem 1.1.

(1) *To every variety X over $\overline{K}$ one can associate a canonical complex of effective filtered φ-modules $\mathrm{R}\Gamma_{\mathrm{DF}_{\overline{K}}}(X, r)$, $r \geq 0$.*

(2) *There is a canonical quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, r) \simeq \mathrm{R}\,\mathrm{Hom}_+(\mathbb{1}, \mathrm{R}\Gamma_{\mathrm{DF}_{\overline{K}}}(X, r)), \quad r \geq 0,$$

where $\mathrm{R}\,\mathrm{Hom}_+$ *denotes the derived* Hom *in the category of effective filtered φ-modules over $\overline{K}$.*

For an effective φ-module M, the complex $\mathrm{R}\,\mathrm{Hom}_+(\mathbb{1}, M)$ has nontrivial cohomology only in degrees $0, 1$; we call them $H^0_+(\overline{K}, M)$ and $H^1_+(\overline{K}, M)$. Hence the above spectral sequence reduces to the short exact sequence

$$(1.3) \quad 0 \to H^1_+(\overline{K}, H^{i-1}_{\mathrm{DF}_{\overline{K}}}(X, r)) \to H^i_{\mathrm{syn}}(X, r) \to H^0_+(\overline{K}, H^i_{\mathrm{DF}_{\overline{K}}}(X, r)) \to 0.$$

This sequence can easily be seen to arise from *the fundamental* (*long*) *exact sequence*

$$(1.4) \quad \begin{aligned} &\to (H^{i-1}_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \to H^i_{\mathrm{syn}}(X, r) \\ &\quad \to (H^i_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+)^{\varphi = p^r} \xrightarrow{\iota_{\mathrm{dR}}} (H^i_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \to . \end{aligned}$$

The terms

$$(H^i_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+)^{\varphi = p^r}, \quad (H^i_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r$$

[3] φ-module jaugé in the original terminology of Fargues.

[4] However, the equivalence does not preserve relevant exact structures.

are the key examples of p-adic Banach spaces that are C-points, $C = \widehat{\overline{K}}$, of finite dimensional Banach-Colmez spaces [6]. The latter are defined, roughly, as finite rank C-vector spaces modulo finite rank $\mathbf{Q}_p$-vector spaces, rigidified as functors on a subcategory of perfectoid spaces. The second main result of this paper states that this is also the case for geometric syntomic cohomology.

Theorem 1.2. *There exists a canonical complex of Banach-Colmez spaces* $\mathbb{R}\Gamma_{\rm syn}(X,r)$ *such that*

(1) $\mathbb{R}\Gamma_{\rm syn}(X,r)(C) \simeq \mathrm{R}\Gamma_{\rm syn}(X,r)$*; in particular,* $H^i\mathbb{R}\Gamma_{\rm syn}(X,r)(C) \simeq H^i_{\rm syn}(X,r)$.

(2) *The fundamental exact sequence* (1.4) *lifts canonically to the category of Banach-Colmez spaces.*

(3) *The syntomic period map* [17]

$$\rho_{\rm syn} : \mathrm{R}\Gamma_{\rm syn}(X,r) \to \mathrm{R}\Gamma_{\rm ét}(X,\mathbf{Q}_p(r))$$

can be lifted canonically to the category of Banach-Colmez spaces. The H^0*-term in the exact sequence* (1.3) *is equal to the image of this period map.*

(4) *The exact sequence* (1.3) *can be lifted canonically to the category of Banach-Colmez spaces, the* H^1*-term is the identity component of* $H^i\mathbb{R}\Gamma_{\rm syn}(X,r)$*, and the* H^0*-term is the space of its connected components.*

We also show that Theorems 1.1 and 1.2 have analogs for semistable formal schemes.

1.0.1. Notation and conventions. For a field L, let $\mathscr{V}ar_L$ be the category of varieties over L.

We will use a shorthand for certain homotopy limits. Namely, if $f : C \to C'$ is a map in the dg derived category of an abelian category, we set

$$[\, C \xrightarrow{f} C' \,] := \mathrm{holim}(C \to C' \leftarrow 0).$$

We also set

$$\begin{bmatrix} C_1 & \xrightarrow{f} & C_2 \\ \downarrow & & \downarrow \\ C_3 & \xrightarrow{g} & C_4 \end{bmatrix} := [[C_1 \xrightarrow{f} C_2] \to [C_3 \xrightarrow{g} C_4]],$$

where the diagram in the brackets is a commutative diagram in the dg derived category.

2. Preliminaries

Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field K of characteristic zero and with perfect residue field k of characteristic p. Let $\overline{K}$ be an algebraic closure of K and let $\mathscr{O}_{\overline{K}}$ denote the integral closure of $\mathscr{O}_K$ in $\overline{K}$; set $C := \overline{K}^{\wedge}$ and let $C^\flat$ be its tilt with valuation v. Let $W(k) = \mathscr{O}_F$ be the ring of Witt vectors of k with fraction field F. Set $G_K = \mathrm{Gal}(\overline{K}/K)$, and let $\varphi = \varphi_{W(\overline{k})}$ be the absolute Frobenius on $W(\overline{k})$.

In this section we will briefly recall some facts from p-adic Hodge Theory that may not yet be classical.

2.1. Finite dimensional Banach-Colmez spaces. Recall [6] that a finite dimensional Banach-Colmez space W is, morally, a finite dimensional C-vector space up to a finite dimensional $\mathbf{Q}_p$-vector space. It has a *Dimension*[5] $\mathrm{Dim}\, W = (a, b)$, where $a = \dim W \in \mathbf{N}$, the *dimension of* W, is the dimension of the C-vector space and $b = \mathrm{ht}\, W \in \mathbf{Z}$, the *height of* W, is the dimension of the $\mathbf{Q}_p$-vector space. More precisely, a *Banach-Colmez space* $\mathbb{W}$ is a functor $\Lambda \mapsto \mathbb{W}(\Lambda)$, from the category of sympathetic algebras (spectral Banach algebras, such that $x \mapsto x^p$ is surjective on $\{x,\ |x-1| < 1\}$; such an algebra is, in particular, perfectoid) to the category of $\mathbf{Q}_p$-Banach spaces. Trivial examples of such objects are:

- finite dimensional $\mathbf{Q}_p$-vector spaces V, with associated functors $\Lambda \mapsto V$ for all Λ,
- $\mathbb{V}^d$, for $d \in \mathbf{N}$, with $\mathbb{V}^d(\Lambda) = \Lambda^d$ for all Λ.

A Banach-Colmez space $\mathbb{W}$ is *finite dimensional* if it "is equal to $\mathbb{V}^d$, for some $d \in \mathbf{N}$, up to finite dimensional $\mathbf{Q}_p$-vector spaces." More precisely, we ask that there exists finite dimensional $\mathbf{Q}_p$-vector spaces V_1, V_2 and exact sequences

$$0 \to V_1 \to \mathbb{Y} \to \mathbb{V}^d \to 0, \quad 0 \to V_2 \to \mathbb{Y} \to \mathbb{W} \to 0,$$

so that $\mathbb{W}$ is obtained from $\mathbb{V}^d$ by "adding V_1 and moding out by V_2". Then $\dim \mathbb{W} = d$ and $\mathrm{ht}\, \mathbb{W} = \dim_{\mathbf{Q}_p} V_1 - \dim_{\mathbf{Q}_p} V_2$. (We are, in general, only interested in $\mathbb{W}(C)$ but, without the extra structure, it would be impossible to speak of its Dimension.)

Proposition 2.1.

(i) *The Dimension of a finite dimensional Banach-Colmez space is independent of the choices made in its definition.*

[5] In [6], the dimension is called the "dimension principale", noted $\dim_{\mathrm{pr}}$, and the height is called the "dimension résiduelle", noted $\dim_{\mathrm{res}}$, and the Dimension is called simply the "dimension".

(ii) *If* $f : \mathbb{W}_1 \to \mathbb{W}_2$ *is a morphism of finite dimensional Banach-Colmez spaces, then* $\operatorname{Ker} f$*,* $\operatorname{Coker} f$*, and* $\operatorname{Im} f$ *are finite dimensional Banach-Colmez spaces, and we have*

$$\operatorname{Dim}\mathbb{W}_1 = \operatorname{Dim}\operatorname{Ker} f + \operatorname{Dim}\operatorname{Im} f \quad \textit{and} \quad \operatorname{Dim}\mathbb{W}_2 = \operatorname{Dim}\operatorname{Coker} f + \operatorname{Dim}\operatorname{Im} f.$$

(iii) *If* $\dim \mathbb{W} = 0$*, then* $\operatorname{ht}\mathbb{W} \geq 0$*.*

(iv) *If* $\mathbb{W}$ *has an increasing filtration such that the successive quotients are* $\mathbb{V}^1$*, and if* $\mathbb{W}'$ *is a sub-Banach-Colmez space of* $\mathbb{W}$*, then* $\operatorname{ht}\mathbb{W}' \geq 0$*.*

Proof. The first two points are the core of the theory [6, Theorem 0.4]. The third point is obvious, and the fourth is [7, Lemme 2.6]. □

Remark 2.2. A more conceptual definition of the category of finite dimensional Banach-Colmez spaces as well as its relationship to the category of coherent sheaves on the Fargues-Fontaine curve was described recently by Le Bras in [16].

2.2. Period rings.

2.2.1. Period rings. Main references for this section are [10], [11], [12]. Let $\mathbf{B}^+_{\mathrm{cr}}$, $\mathbf{B}^+_{\mathrm{st}}$, $\mathbf{B}^+_{\mathrm{dR}}$, $\mathbf{B}_{\mathrm{cr}}$, $\mathbf{B}_{\mathrm{st}}$, $\mathbf{B}_{\mathrm{dR}}$ be the Fontaine's rings of crystalline, semistable, and de Rham periods, respectively. Let $\iota = \iota_p : \mathbf{B}_{\mathrm{st}} \hookrightarrow \mathbf{B}_{\mathrm{dR}}$ be the canonical embedding.

Let $\mathscr{E} = W(C^\flat)[1/p]$. Define

$$\mathbf{B}^b = \{ \sum_{n \gg -\infty} [x_n]p^n \in \mathscr{E} \,|\, \exists\, K \quad \forall n \, |x_n| \leq K \},$$

$$\mathbf{B}^{b,+} = \{ \sum_{n \gg -\infty} [x_n]p^n \in \mathscr{E} | x_n \in \mathscr{O}_{C^\flat} \} = W(\mathscr{O}_{C^\flat})[1/p].$$

For $x = \sum_n [x_n]p^n \in \mathbf{B}^b$ and $\rho \in]0,1[$, $r \geq 0$, set

$$|x|_\rho = \sup_n |x_n|\rho^n, \qquad v_r(x) = \inf_{n\in\mathbf{Z}} \{v(x_n) + nr\}.$$

If $\rho = p^{-r} \in]0,1[$, we have $|x|_\rho = p^{-v_r(x)}$. For $r \geq 0$, v_r is a valuation on $\mathbf{B}^b$; thus, for $\rho \in]0,1]$, $|\cdot|_\rho$ is a multiplicative norm. One defines the rings $\mathbf{B}$ and $\mathbf{B}^+$ as the completions of $\mathbf{B}^b$ and $\mathbf{B}^{b,+}$, respectively, with respect to $(|\cdot|_\rho)_{\rho\in]0,1[}$. For a compact interval $I \subset]0,1[$, the ring $\mathbf{B}_I$ is the completion of $\mathbf{B}^b$ with respect to $(|\cdot|_\rho)_{\rho\in I}$. It is a PID. The rings $\mathbf{B}$, $\mathbf{B}^+$ are $\mathbf{Q}_p$-Fréchet algebras, and $\mathbf{B}^+$ is the closure of $\mathbf{B}^{b,+}$ in $\mathbf{B}$. The ring $\mathbf{B}_I$ is a $\mathbf{Q}_p$-Banach algebra, and we have

$$\mathbf{B} = \varprojlim_{I\in]0,1[} \mathbf{B}_I.$$

We have

(1) $\mathbf{B}^+ = \bigcap_{n\geq 0} \varphi^n(\mathbf{B}^+_{\mathrm{cr}})$.[6]

[6]Hence $\mathbf{B}^+$ is the ring $\mathbf{B}^+_{\mathrm{rig}}$ from classical p-adic Hodge Theory.

(2) For an F-isocrystal D, $(D\otimes_F \mathbf{B}^+_{\mathrm{cr}})^{\varphi=1}=(D\otimes_F \mathbf{B}^+)^{\varphi=1}=(D\otimes_F \mathbf{B})^{\varphi=1}$.

(3) For $d<0$, $\mathbf{B}^{\varphi=p^d}=0$; $\mathbf{B}^{\varphi=1}=\mathbf{Q}_p$; and for $d\geq 0$, $\mathbf{B}^{\varphi=p^d}=(\mathbf{B}^+)^{\varphi=p^d}$.

(4) There is a natural map $\iota:\mathbf{B}\to\mathbf{B}^+_{\mathrm{dR}}$ compatible with the embedding $\mathbf{B}^+_{\mathrm{cr}}\to\mathbf{B}^+_{\mathrm{dR}}$.

Let $\mathbf{B}^+_{\log}$ be the period ring defined in the same way as $\mathbf{B}^+_{\mathrm{st}}$ but starting from $\mathbf{B}^+$ instead of from $\mathbf{B}^+_{\mathrm{cr}}$. We will denote by $\iota:\mathbf{B}^+_{\log}\to\mathbf{B}^+_{\mathrm{dR}}$ the canonical imbedding. We have a canonical map $\mathbf{B}^+_{\log}\to\mathbf{B}^+_{\mathrm{st}}$ compatible with all the structures. For an F-isocrystal D, we have

$$(D\otimes_F \mathbf{B}^+_{\mathrm{st}})^{\varphi=1,N=0}=(D\otimes_F \mathbf{B}^+_{\log})^{\varphi=1,N=0}. \tag{2.1}$$

The Robba ring $\mathscr{R}$ is defined as

$$\mathscr{R}=\varinjlim_{\rho\to 0}\mathbf{B}_{]0,\rho]},$$

where

$$\mathbf{B}_{]0,\rho]}:=\varprojlim_{0<\rho'\leq\rho}\mathbf{B}_{[\rho',\rho]}$$

is the completion of $\mathbf{B}^b$ with respect to $(|\cdot|_{\rho'})_{0<\rho'\leq\rho}$. Since $\varphi:\mathbf{B}_{]0,\rho]}\xrightarrow{\sim}\mathbf{B}_{]0,\rho^p]}$ the ring $\mathscr{R}$ is equipped with a bijective Frobenius. By [15, Theorem 2.9.6], the ring $\mathbf{B}_{]0,\rho]}$ is Bézout. Any closed ideal of $\mathbf{B}_{]0,\rho]}$ is principal. Hence the ring $\mathscr{R}$ is Bézout as well.

2.2.2. Period rings and some Banach-Colmez spaces. Recall [6] that the above rings of periods can be also defined starting from any sympathetic algebra instead of C. One obtains Rings (of periods) $\mathbb{B}^+,\mathbb{B}^+_{\mathrm{st}},\mathbb{B}^+_{\mathrm{dR}}$ and natural morphisms $\iota:\mathbb{B}^+_{\mathrm{st}}\hookrightarrow\mathbb{B}^+_{\mathrm{dR}}$, $\iota:\mathbb{B}^+\hookrightarrow\mathbb{B}^+_{\mathrm{dR}}$. We have $\mathbf{B}^+=\mathbb{B}^+(C)$, $\mathbf{B}^+_{\mathrm{st}}=\mathbb{B}^+_{\mathrm{st}}(C)$, $\mathbf{B}^+_{\mathrm{dR}}=\mathbb{B}^+_{\mathrm{dR}}(C)$.

The category $\mathscr{BC}$ of finite dimensional Banach-Colmez spaces is abelian. The functor of C-points

$$\mathscr{BC}\to\text{Banach},\quad X\mapsto X(C)$$

is exact and faithful. In particular, if $C^\bullet\in C^b(\mathscr{BC})$ is a complex of finite dimensional Banach-Colmez spaces, then we have that its cohomology $H^i(C^\bullet)$ is a Banach-Colmez space as well, and for its C-points, we have $H^i(C^\bullet)(C)=H^i(C^\bullet(C))$.

Recall that a φ-module (over F) is a finite rank vector space over F equipped with a bijective semilinear Frobenius $\varphi:D\to D$. A (φ,N)-module (over F) is a finite φ-module (over F) equipped with a monodromy operator $N:D\to D$ such that $N\varphi=p\varphi N$. A filtered (φ,N)-module (over K) is a finite (φ, N)-module over F such that $D_K := D \otimes_F K$ is a filtered K-vector space.

To D, a finite filtered (φ, N)-module over K, and to $r \geq 0$, one can associate Banach-Colmez spaces

$$D \mapsto \mathbb{X}^r_{\mathrm{st}}(D) = (D \otimes_F t^{-r}\mathbb{B}^+_{\mathrm{st}})^{\varphi=1,N=0} = (D \otimes_F \mathbb{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r},$$
$$D \mapsto \mathbb{X}^r_{\mathrm{dR}}(D) = (D_K \otimes_K t^{-r}\mathbb{B}^+_{\mathrm{dR}})/F^0 = (D_K \otimes_K \mathbb{B}^+_{\mathrm{dR}})/F^r.$$

These are exact functors. We also have a natural transformation $\iota : \mathbb{X}^r_{\mathrm{st}}(D) \to \mathbb{X}^r_{\mathrm{dR}}(D)$ induced by the morphism $\iota : \mathbb{B}^+_{\mathrm{st}} \hookrightarrow \mathbb{B}^+_{\mathrm{dR}}$.

A filtered (φ, N, G_K)-module is a tuple $(D, \varphi, N, \rho, F^\bullet)$, where

(1) D is a finite dimensional F^{nr}-vector space;
(2) $\varphi : D \to D$ is a bijective semilinear Frobenius map;
(3) $N : D \to D$ is an F^{nr}-linear monodromy map such that $N\varphi = p\varphi N$;
(4) ρ is an F^{nr}-semilinear G_K-action on D (hence $\rho|I_K$ is linear) that is smooth, i.e., all vectors have open stabilizers, and that commutes with φ and N;
(5) $F^\bullet$ is a decreasing finite filtration of $D_K := (D \otimes_{F^{\mathrm{nr}}} \overline{K})^{G_K}$ by K-vector spaces.

The above functors extend to exact functors

$$D \mapsto \mathbb{X}^r_{\mathrm{st}}(D) = (D \otimes_{F^{\mathrm{nr}}} t^{-r}\mathbb{B}^+_{\mathrm{st}})^{\varphi=1,N=0} = (D \otimes_{F^{\mathrm{nr}}} \mathbb{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r},$$
$$D \mapsto \mathbb{X}^r_{\mathrm{dR}}(D) = (D_K \otimes_K t^{-r}\mathbb{B}^+_{\mathrm{dR}})/F^0 = (D_K \otimes_K \mathbb{B}^+_{\mathrm{dR}})/F^r,$$

from filtered (φ, N, G_K)-modules to finite dimensional Banach-Colmez spaces. We also have a natural transformation $\iota : \mathbb{X}^r_{\mathrm{st}}(D) \to \mathbb{X}^r_{\mathrm{dR}}(D)$.

Recall that we have (cf. [6, Proposition 10.6])

$$\operatorname{Dim} \mathbb{X}^r_K(D) = (r \dim_K D_K - \sum_{i=1}^{r} \dim F^i D_K, 0),$$
$$\operatorname{Dim} \mathbb{X}^r_{\mathrm{st}}(D) = \sum_{r_i \leq r} (r - r_i, 1),$$

where the r_i's are the slopes of φ, repeated with multiplicity. In particular, if $F^{r+1}D_K = 0$ and if all r_i's are $\leq r$ (we let $r(D)$ be the smallest r with these properties), then

$$\operatorname{Dim} \mathbb{X}^r_{\mathrm{st}}(D) = (r \dim_{F^{\mathrm{nr}}} D - t_N(D), \dim_{F^{\mathrm{nr}}} D)$$

and

$$\operatorname{Dim} \mathbb{X}^r_{\mathrm{dR}}(D) = (r \dim_K D_K - t_H(D_K), 0).$$

Here $t_N(D) = v_p(\det \varphi)$, and $t_H(D) = \sum_{i \geq 0} i \dim_K(F^i D_K / F^{i+1} D_K)$.

The kernel of the map $\iota : \mathbb{X}^r_{\rm st}(D) \to X^r_{\rm dR}(D)$ is $V_{pst}(D)$ if $r \geq r(D)$ ([6, Proposition 10.14]), where $V_{pst}(D) := (D \otimes_{F^{\rm nr}} \mathbf{B}_{\rm st})^{\varphi=1,N=0} \cap F^0(D_K \otimes_K \mathbf{B}_{\rm dR})$.

2.3. Vector bundles on the Fargues-Fontaine curve. Main references for this section are [10], [11], [12].

2.3.1. Definitions. Let $X_{\rm FF}$ be the (algebraic) Fargues-Fontaine curve associated to the tilt $C^\flat$ and to $\mathbf{Q}_p$. We have

$$X_{\rm FF} = \mathrm{Proj}(P) := \mathrm{Proj}(\oplus_{d\geq 0} P_d), \quad P_d := (\mathbf{B}^+_{\rm cr})^{\varphi=p^d}.$$

In the above definition we can replace $\mathbf{B}^+_{\rm cr}$ by $\mathbf{B}$ or $\mathbf{B}^+$. The map $\theta : \mathbf{B}^+_{\rm cr} \to C$ determines a distinguished point $\infty \in X_{\rm FF}(C)$. We have the canonical identification $\mathbf{B}^+_{\rm dR} \xrightarrow{\sim} \widehat{\mathscr{O}}_{X,\infty}$; let t be the uniformizing element of $\mathbf{B}^+_{\rm dR}$ ($t \in P_1 - \{0\}$) so that $\mathbf{B}_{\rm dR} = \mathbf{B}^+_{\rm dR}[1/t]$. $X_{\rm FF}$ is a regular noetherian scheme of Krull dimension one which is locally a spectrum of a Dedekind domain.

Let $\mathrm{Bun}_{X_{\rm FF}}$ denote the category of vector bundles on $X_{\rm FF}$. Since $X_{\rm FF}$ is locally a spectrum of a Dedekind domain $\mathrm{Bun}_{X_{\rm FF}}$ is a quasi-abelian category[7] [1, 1.2.16]. It is thus equipped with the induced kernel-cokernel exact structure: a *short exact sequence*

$$0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$$

is a pair of morphisms (f, g) such that $M = \ker(g), P = \mathrm{coker}(f)$. This is the same as the natural exact structure: locally, use the embedding of the category of torsion-free modules of finite rank into its left abelian envelope, the category of finitely generated modules.[8]

For every $d \in \mathbf{Z}$, there exists a line bundle $\mathscr{O}(d) := \widetilde{P[d]}$. This is a line bundle of degree d and every line bundle on the curve $X_{\rm FF}$ is isomorphic to some $\mathscr{O}(d)$. One defines degree of a vector bundle as the degree of its determinant bundle, slope as degree divided by rank. For every slope $\lambda \in \mathbf{Q}$, there exists a stable bundle $\mathscr{O}(\lambda)$ with slope λ. These vector bundles are constructed in the following way. For $\lambda = d/h$, $h \geq 1$, $(d, h) = 1$, one takes the base change $X_{{\rm FF},h} := X_{{\rm FF},\mathbf{Q}_{p^h}}$, where $\mathbf{Q}_{p^h}$ is the degree h unramified extension of $\mathbf{Q}_p$, and defines $\mathscr{O}(\lambda) = \mathscr{O}(d, h) := \pi_*\mathscr{O}(d)$, $\pi : X_{{\rm FF},h} \to X_{\rm FF}$. We have

$$\mathscr{O}(d, h) = \widetilde{M(d, h)}, \qquad M(d, h) = \bigoplus_{i\in\mathbf{N}} (\mathbf{B}^+)^{\varphi^h = p^{ih+d}}.$$

[7] An additive category with kernels and cokernels is called *quasi-abelian* if every pullback of a strict epimorphism is a strict epimorphism and every pushout of a strict monomorphism is a strict monomorphism. Equivalently, an additive category with kernels and cokernels is called *quasi-abelian* if $\mathrm{Ext}(\cdot, \cdot)$ is bifunctorial.

[8] Recall that a short sequence in a quasi-abelian category is exact if and only if it is exact in its left abelian envelope.

This is a vector bundle on X_{FF} of rank h and degree d, hence of slope λ. The global sections functor and the functor $V \mapsto V \otimes_{\mathbf{Q}_p} \mathscr{O}$ induce an equivalence of categories between semistable vector bundles of slope zero and finite dimensional $\mathbf{Q}_p$-vector spaces (see Section 2.3.3). Since semistable vector bundles of slope zero have vanishing H^1 this is an equivalence of exact categories.

2.3.2. Cohomology of vector bundles. We have

$$\mathscr{O}(\lambda) \otimes \mathscr{O}(\mu) \simeq \bigoplus_{\text{finite}} \mathscr{O}(\lambda+\mu); \mathscr{O}(\lambda)^{\vee} = \mathscr{O}(-\lambda);$$
$$\mathrm{Hom}(\mathscr{O}(\lambda), \mathscr{O}(\mu)) = 0, \quad \lambda > \mu;$$
$$\mathrm{Ext}(\mathscr{O}(\lambda), \mathscr{O}(\mu)) = \bigoplus_{\text{finite}} H^1(X_{\mathrm{FF}}, \mathscr{O}(\mu-\lambda)) = 0, \quad \lambda \leq \mu.$$

Example 2.3 ([10, 12.1,12.3])**.** We have

$$(2.2) \qquad \begin{aligned} H^0(X_{\mathrm{FF}}, \mathscr{O}(d)) &= \begin{cases} P_d & \text{if } d \geq 0, \\ 0 & \text{if } d < 0, \end{cases} \\ H^1(X_{\mathrm{FF}}, \mathscr{O}(d)) &= \begin{cases} 0 & \text{if } d \geq 0, \\ \mathbf{B}^+_{\mathrm{dR}}/(t^{-d}\mathbf{B}^+_{\mathrm{dR}} + \mathbf{Q}_p) & \text{if } d < 0. \end{cases} \end{aligned}$$

To obtain this we write $X_{\mathrm{FF}} \setminus \{\infty\} = \mathrm{Spec}\, \mathbf{B}_e$ for $\mathbf{B}_e = \mathbf{B}_{\mathrm{cr}}^{\varphi=1} = (\mathbf{B}^+[1/t])^{\varphi=1}$, a principal ideal domain. There is an equivalence of exact categories

$$\mathrm{Bun}_{X_{\mathrm{FF}}} \xrightarrow{\sim} \mathscr{C}, \quad \mathscr{E} \mapsto (\Gamma(X_{\mathrm{FF}} \setminus \{\infty\}, \mathscr{E}), \widehat{\mathscr{E}}_{\infty}).$$

Here $\mathscr{C}$ is the category of $\mathbf{B}$-pairs [5], i.e., the category of pairs (M, W), where W is a free $\mathbf{B}^+_{\mathrm{dR}}$-module of finite type and $M \subset W[1/t]$ is a sub-free $\mathbf{B}_e$-module of finite type such that $M \otimes_{\mathbf{B}_e} \mathbf{B}_{\mathrm{dR}} \xrightarrow{\sim} W[1/t]$. If $\mathscr{E}$ corresponds to the pair (M, W), then its cohomology can be computed by the following Čech complex:

$$\mathrm{R}\Gamma(X_{\mathrm{FF}}, \mathscr{E}) = (M \oplus W \xrightarrow{\partial} W[1/t]), \quad \partial(x,y) = x - y.$$

The line bundle $\mathscr{O}(d)$ corresponds to the pair $(\mathbf{B}_e, t^{-d}\mathbf{B}^+_{\mathrm{dR}})$. Hence

$$\mathrm{R}\Gamma(X_{\mathrm{FF}}, \mathscr{O}(d)) = (\mathbf{B}_e \oplus t^{-d}\mathbf{B}^+_{\mathrm{dR}} \to \mathbf{B}_{\mathrm{dR}}).$$

From this, since $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_e + \mathbf{B}^+_{\mathrm{dR}}$ and $\mathbf{B}_e \cap \mathbf{B}^+_{\mathrm{dR}} = \mathbf{Q}_p$, we get (2.2). Both $H^0(X_{\mathrm{FF}}, \mathscr{O}(d))$ and $H^1(X_{\mathrm{FF}}, \mathscr{O}(d))$ are C-points of finite dimensional Banach-Colmez spaces. These spaces have Dimensions $(d,1)$, resp., $(0,0)$, for $d \geq 0$, and $(0,0)$, resp. $(-d,-1)$, for $d < 0$. Hence the Euler characteristic $\chi(X_{\mathrm{FF}}, \mathscr{O}(d)) = (d,1)$.

More generally, for $\lambda = d/h$, $h \geq 1$, $(d,h) = 1$, we have

$$H^0(X_{\rm FF}, \mathscr{O}(d,h)) = \begin{cases} (\mathbf{B}^+)^{\varphi^h = p^d} & \text{if } d \geq 0, \\ 0 & \text{if } d < 0, \end{cases}$$

(2.3)

$$H^1(X_{\rm FF}, \mathscr{O}(d,h)) = \begin{cases} 0 & \text{if } d \geq 0, \\ \mathbf{B}^+_{\rm dR}/(t^{-d}\mathbf{B}^+_{\rm dR} + \mathbf{Q}_{p^h}) & \text{if } d < 0. \end{cases}$$

The vector bundle $\mathscr{O}(d,h)$ corresponds to the pair $((\mathbf{B}^+[1/t])^{\varphi^h = p^d}, (\mathbf{B}^+_{\rm dR})^h)$, with the gluing map $u : (\mathbf{B}^+[1/t])^{\varphi^h = p^d} \to (\mathbf{B}_{\rm dR})^h$ defined as $x \mapsto (x, \varphi(x), \ldots, \varphi^{h-1}(x))$. Hence

$$\mathrm{R}\Gamma(X_{\rm FF}, \mathscr{O}(d,h)) = ((\mathbf{B}^+[1/t])^{\varphi^h = p^d} \oplus (\mathbf{B}^+_{\rm dR})^h \xrightarrow{u - \mathrm{can}} (\mathbf{B}_{\rm dR})^h),$$

and computation (2.3) follows since $\mathbf{B}_{\rm dR} = \mathbf{B}_e + \mathbf{B}^+_{\rm dR}$ and $(\mathbf{B}^+[1/t])^{\varphi^h = 1} \cap \mathbf{B}^+_{\rm dR} = \mathbf{Q}_{p^h}$.

Again, both $H^0(X_{\rm FF}, \mathscr{O}(d,h))$ and $H^1(X_{\rm FF}, \mathscr{O}(d,h))$ are C-points of finite dimensional Banach-Colmez spaces. These spaces have Dimensions (d,h), resp., $(0,0)$, for $d \geq 0$, and $(0,0)$, resp., $(-d,-h)$, for $d < 0$. Hence the Euler characteristic $\chi(X_{\rm FF}, \mathscr{O}(d,h)) = (d,h)$.

2.3.3. Classification of vector bundles. We have the following classification theorem for vector bundles on $X_{\rm FF}$.

Theorem 2.4 (Fargues-Fontaine [11, Theorem 6.9])**.**

(1) *The semistable vector bundles of slope λ on $X_{\rm FF}$ are the direct sums of $\mathscr{O}(\lambda)$.*

(2) *The Harder-Narasimhan filtration of a vector bundle on $X_{\rm FF}$ is split.*

(3) *There is a bijection*

$$\{\lambda_1 \geq \cdots \geq \lambda_n | n \in \mathbf{N}, \lambda_i \in \mathbf{Q}\} \xrightarrow{\sim} \mathrm{Bun}_{X_{\rm FF}} / \sim (\lambda_1, \ldots, \lambda_n) \mapsto [\bigoplus_{i=1}^n \mathscr{O}(\lambda_i)].$$

Let $\mathrm{Mod}_{\mathscr{R}}(\varphi)$ be the category of φ-modules over the Robba ring $\mathscr{R}$, i.e., finite type projective $\mathscr{R}$-modules D[9] equipped with a φ-linear isomorphism $\varphi : D \xrightarrow{\sim} D$. Since $\mathscr{R}$ is Bézout, this implies [1, 1.2.7] that $\mathrm{Mod}_{\mathscr{R}}(\varphi)$ is quasi-abelian. The induced kernel-cokernel exact structure is the same as the natural exact structure: use the embedding of the category $\mathrm{Mod}_{\mathscr{R}}$ into its left abelian envelope, the category of finitely generated $\mathscr{R}$-modules.

Let $\mathrm{Mod}_{\mathbf{B}}(\varphi)$ be the category of finite type projective $\mathbf{B}$-modules D equipped with a semi-linear isomorphism $\varphi : D \to D$. It is an exact category.

[9]Since $\mathscr{R}$ is Bézout, an $\mathscr{R}$-module D is projective of finite type if and only if it is torsion-free of finite type if and only if it is free of finite type.

We have the following diagram of equivalences of exact categories [11, Proposition 7.16, Theorem 7.18]:

$$\mathrm{Mod}_{\mathscr{R}}(\varphi) \xleftarrow{\sim} \mathrm{Mod}_{\mathbf{B}}(\varphi) \xrightarrow{\sim} \mathrm{Bun}_{X_{\mathrm{FF}}}. \tag{2.4}$$

The first map is induced by the inclusion $\mathbf{B} \subset \mathscr{R}$. Via this map, the classification theorem of Kedlaya for φ-modules over $\mathscr{R}$ [15] yields that there is a bijection

$$\{\lambda_1 \geq \cdots \geq \lambda_n | n \in \mathbf{N}, \lambda_i \in \mathbf{Q}\} \xrightarrow{\sim} \mathrm{Mod}_{\mathbf{B}}(\varphi)/ \sim (\lambda_1, \dots, \lambda_n) \mapsto [\bigoplus_{i=1}^{n} \mathbf{B}(-\lambda_i)]. \tag{2.5}$$

In particular, every φ-module over $\mathbf{B}$ is free of finite type.

The second map in (2.4) is defined by sending

$$D \mapsto \mathscr{E}(D), \tag{2.6}$$

where $\mathscr{E}(D)$ is the sheaf associated to the P-graded module $\bigoplus_{d \geq 0} D^{\varphi = p^d}$. Hence $\mathscr{E}(\mathbf{B}(i)) = \mathscr{O}(-i)$ and we have $D \otimes \mathbf{B}^+_{\mathrm{dR}} \xrightarrow{\sim} \widehat{\mathscr{E}(D)}_\infty$. This equivalence of exact categories implies that the category $\mathrm{Mod}_{\mathbf{B}}(\varphi)$ is also quasi-abelian.[10] Since the canonical exact structure on $\mathrm{Bun}_{X_{\mathrm{FF}}}$ agrees with the quasi-abelian kernel-cokernel exact structure it follows that this is also the case in $\mathrm{Mod}_{\mathbf{B}}(\varphi)$. Since we know that this is also the case in $\mathrm{Mod}_{\mathscr{R}}(\varphi)$, it follows that the first map in (2.4) is an equivalence of exact categories.

2.3.4. Modifications of vector bundles and filtered φ-modules. We will recall the definitions of these categories [12]. A *modification* of vector bundles on the curve X_{FF} is a triple $(\mathscr{E}_1, \mathscr{E}_2, u)$, where $\mathscr{E}_1$, $\mathscr{E}_2$ are vector bundles on X_{FF} and u is an isomorphism $\mathscr{E}_{1|X_{\mathrm{FF}} \setminus \{\infty\}} \simeq \mathscr{E}_{2|X_{\mathrm{FF}} \setminus \{\infty\}}$. Modification $(\mathscr{E}_1, \mathscr{E}_2, u)$ is called *effective* if $u(\mathscr{E}_1) \subset \mathscr{E}_2$; it is called *admissible* if $\mathscr{E}_1$ is a semistable vector bundle of slope zero. Let $\mathscr{M}$, $\mathscr{M}^+$, $\mathscr{M}^{\mathrm{ad}}$ denote the corresponding exact categories.

Modifications can be also described as pairs $(\mathscr{E}, \Lambda)$, where $\mathscr{E}$ is a vector bundle and Λ is a $\mathbf{B}^+_{\mathrm{dR}}$-lattice $\Lambda \subset \widehat{\mathscr{E}}_\infty[1/t]$. To a modification $(\mathscr{E}_1, \mathscr{E}_2, u)$ one associates the pair $(\mathscr{E}_2, u(\widehat{\mathscr{E}}_{1,\infty}))$. Effective modifications correspond to pairs $(\mathscr{E}, \Lambda)$ such that $\Lambda \subset \widehat{\mathscr{E}}_\infty$. This correspondence preserves exact structures.

A filtered φ-module (over $\overline{K}$)[11] (D, Λ) consists of a φ-module D over $\mathbf{B}^+$ and a $\mathbf{B}^+_{\mathrm{dR}}$-lattice $\Lambda \subset D \otimes \mathbf{B}_{\mathrm{dR}}$. It is *effective* if $\Lambda \subset D \otimes \mathbf{B}^+_{\mathrm{dR}}$. A filtered φ-module is called *admissible* if $D^{\varphi=1} \cap \Lambda$ is a $\mathbf{Q}_p$-vector space of rank equal to the rank of D over $\mathbf{B}^+$. Denote by $\mathrm{DF}_{\overline{K}}, \mathrm{DF}^+_{\overline{K}}, \mathrm{DF}^{\mathrm{ad}}_{\overline{K}}$ the corresponding exact

[10] We note here that the ring $\mathbf{B}$ is not Bézout.

[11] Fargues' [12, 4.2.2] original name was "φ-module jaugé".

categories. We have an equivalence of categories $\mathrm{DF}_{\overline{K}} \xrightarrow{\sim} \mathscr{M}$ that induces equivalences between the effective and admissible subcategories. We note that this is not an equivalence of exact categories.

3. Extensions of modifications

In this section we study extensions in the categories of modifications and of filtered φ-modules over $\overline{K}$.

3.1. Extensions of φ-modules. We start with extensions of φ-modules. Let $\mathrm{Mod}_{\mathbf{B}^+}$ be the category of free $\mathbf{B}^+$-modules of finite rank. It is an exact category (with a split exact structure), and we will denote by $\mathscr{D}_{\mathbf{B}^+}$, $D_{\mathbf{B}^+}$ its derived dg category and derived category, respectively. We note that, since the exact structure is split, all quasi-isomorphisms are actually homotopy equivalences. In particular, $D_{\mathbf{B}^+} = H_{\mathbf{B}^+}$ (the homotopy category). For $D_1, D_2 \in \mathscr{D}^b_{\mathbf{B}^+}$, we have a quasi-isomorphism $\mathrm{Hom}_{\mathscr{D}^b_{\mathbf{B}^+}}(D_1, D_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{B}^+}(D_1, D_2)$. We have similar statements for analogous categories $\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}}$ and $\mathrm{Mod}_{\mathbf{B}_{\mathrm{dR}}}$. We note that, since $\mathbf{B}^+_{\mathrm{dR}}$ is a PID and $\mathbf{B}_{\mathrm{dR}}$ is a field, these two categories are quasi-abelian (with the kernel-cokernel exact structure equal to the natural one) [1, 1.2.7]. In $\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}}$, a morphism is strict if and only if its cokernel taken in the category of $\mathbf{B}^+_{\mathrm{dR}}$-modules is torsion-free or, equivalently, its kernel is t-saturated in the ambient module.[12]

Let $\mathrm{Mod}_{\mathbf{B}^+}(\varphi)$ be the category of free $\mathbf{B}^+$-modules D of finite rank equipped with an isomorphism $\varphi^* D \xrightarrow{\sim} D$. It is an exact category. The exact structure is split; i.e., Ext^1 in $\mathrm{Mod}_{\mathbf{B}^+}(\varphi)$ is trivial. To see this, one first proves a classification of φ-modules over $\mathbf{B}^+$ analogous to the one for $\mathbf{B}$ in (2.5), then computes Ext^1 for the simple modules [12, Theorem 7.23, Proposition 7.25].

Let $\mathrm{Mod}_{\widehat{F^{\mathrm{nr}}}}(\varphi)$ be the category of finite rank modules over $\widehat{F^{\mathrm{nr}}}$ with a semilinear isomorphism $\varphi : D \to D$. It is an exact category with split exact structure. There is an (exact) functor $\mathrm{Mod}_{\widehat{F^{\mathrm{nr}}}}(\varphi) \to \mathrm{Mod}_{\mathbf{B}^+}(\varphi)$, $D \mapsto D \otimes_{\widehat{F^{\mathrm{nr}}}} \mathbf{B}^+$. Using the Dieudonné-Manin decomposition in $\mathrm{Mod}_{\widehat{F^{\mathrm{nr}}}}(\varphi)$ we see that this functor induces bijection on objects of the two categories. Clearly though we have many more morphisms in $\mathbf{B}^+$-modules.

We will denote by $\mathscr{D}_{\mathbf{B}^+}(\varphi)$, $D_{\mathbf{B}^+}(\varphi)$ the derived categories of $\mathrm{Mod}_{\mathbf{B}^+}(\varphi)$. For $D_1, D_2 \in \mathrm{Mod}_{\mathbf{B}^+}(\varphi)$, let $\mathrm{Hom}_{\mathbf{B}^+,\varphi}(D_1, D_2)$ denote the group of Frobenius

[12] A $\mathbf{B}^+_{\mathrm{dR}}$-module N is called t-saturated in a $\mathbf{B}^+_{\mathrm{dR}}$-module M, for $N \hookrightarrow M$, if every $x \in M$ such that $tx \in N$ is actually in N.

morphisms. We have the exact sequence

(3.1)
$$0 \to \mathrm{Hom}_{\mathbf{B}^+,\varphi}(D_1, D_2) \to \mathrm{Hom}_{\mathbf{B}^+}(D_1, D_2) \xrightarrow{\delta} \mathrm{Hom}_{\mathbf{B}^+}(D_1, \varphi_* D_2) \to 0,$$

where $\delta : x \mapsto \varphi_{D_2} x - \varphi_*(x)\varphi_{D_1}$. Hence, $\mathrm{Hom}_{\mathbf{B}^+,\varphi}(D_1, D_2) = \mathrm{Cone}(\delta)[-1]$. It follows[13] that, for $D_1, D_2 \in \mathscr{D}^b_{\mathbf{B}^+}(\varphi)$, we have quasi-isomorphisms

$$\mathrm{Hom}_{\mathscr{D}^b_{\mathbf{B}^+}(\varphi)}(D_1, D_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{B}^+,\varphi}(D_1, D_2)$$

and

$$\mathrm{Hom}_{\mathscr{D}^b_{\mathbf{B}^+}(\varphi)}(D_1, D_2) \simeq [\mathrm{Hom}_{\mathbf{B}^+}(D_1, D_2) \xrightarrow{\delta} \mathrm{Hom}_{\mathbf{B}^+}(D_1, \varphi_* D_2)],$$

where $\mathrm{Hom}_{\mathbf{B}^+}(-,-)$ is the Hom-complex.

We compute similarly that (with the obvious notation), for $D_1, D_2 \in \mathscr{D}^b_{\mathbf{B}}(\varphi)$, we have quasi-isomorphisms

$$\mathrm{Hom}_{\mathscr{D}^b_{\mathbf{B}}(\varphi)}(D_1, D_2) \simeq \mathrm{Hom}_{\mathbf{B},\varphi}(D_1, D_2) \simeq [\mathrm{Hom}_{\mathbf{B}}(D_1, D_2) \xrightarrow{\delta} \mathrm{Hom}_{\mathbf{B}}(D_1, \varphi_* D_2)]$$

The inclusion $\mathbf{B}^+ \subset \mathbf{B}$ defines an equivalence of categories [11, 7.7]

(3.2)
$$\mathrm{Mod}_{\mathbf{B}^+}(\varphi) \xrightarrow{\sim} \mathrm{Mod}_{\mathbf{B}}(\varphi).$$

This is shown in [11] by proving that the above functor is fully faithful [11, Proposition 7.20], what implies, by the classification of φ-modules over $\mathbf{B}$ (2.5), an analogous classification of φ-modules over $\mathbf{B}^+$. The wanted equivalence of categories follows. On the other hand, if we equip $\mathrm{Mod}_{\mathbf{B}^+}(\varphi)$ with the natural exact structure, the functor (3.2) is not an equivalence of exact categories. It is because for the natural exact structure $\mathrm{Ext}(\mathbf{B}^+, \mathbf{B}^+(1)) = 0$ (by (3.1)), but for the quasi-abelian kernel-cokernel exact structure

$$\begin{aligned}\mathrm{Ext}(\mathbf{B}^+, \mathbf{B}^+(1)) = \mathrm{Ext}(\mathbf{B}, \mathbf{B}(1)) = \mathrm{Ext}(\mathscr{O}, \mathscr{O}(-1)) = H^1(X_{\mathrm{FF}}, \mathscr{O}(-1)) \\ = C/\mathbf{Q}_p.\end{aligned}$$

The corresponding exact sequences have cokernel maps which are not surjective for the natural exact structure. In what follows we will work exclusively with the natural exact structure.

3.2. Extensions of filtered φ-modules. We consider the category $\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}}$ of pairs (Λ, M), $\Lambda \subset M$, where $M \in \mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}}$, and Λ is a $\mathbf{B}^+_{\mathrm{dR}}$-lattice in $M[1/t]$. If this does not cause confusion we will write simply M for the pair (M, Λ). It is easy to check that this category is quasi-abelian. A morphism $(f_\Lambda, f_M) : (\Lambda, M) \to (\Lambda', M')$ is strict [19, 1.1.3] if and only if so are the morphisms f_Λ and f_M. The same observation holds for kernels and

[13] By an argument analogous to the one used in [4, 1.11].

cokernels. This is, perhaps, slightly surprising but follows from the fact that strict subobjects in the category $\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}}$ are t-saturated.

Let $M_1, M_2 \in C^b(\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}})$. Define the complex

$$\operatorname{Hom}^F(M_1, M_2) := [\operatorname{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(M_1, M_2) \xrightarrow{\mathrm{can}} \operatorname{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(\Lambda_1, M_2/\Lambda_2)], \tag{3.3}$$

where $\operatorname{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(-,-)$ denotes the Hom-complex in the category of $\mathbf{B}^+_{\mathrm{dR}}$-modules.

Lemma 3.1. *Let $M_1, M_2 \in \mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}}$. We have*

$$H^i \operatorname{Hom}^F(M_1, M_2) \simeq \begin{cases} \operatorname{Hom}_{\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}}}(M_1, M_2) & \text{if } i = 0, \\ \operatorname{Ext}_{\mathbf{B}^+_{\mathrm{dR}}}(M_1/\Lambda_1, M_2/\Lambda_2) & \text{if } i = 1, \\ 0 & \text{if } i \geq 2, \end{cases}$$

where $\operatorname{Ext}_{\mathbf{B}^+_{\mathrm{dR}}}(-,-)$ denotes Ext *in the category of $\mathbf{B}^+_{\mathrm{dR}}$-modules.*

Proof. The cases of $i = 0$ and $i \geq 2$ are clear. To show that

$$H^1 \operatorname{Hom}^F(M_1, M_2) \simeq \operatorname{Ext}_{\mathbf{B}^+_{\mathrm{dR}}}(M_1/\Lambda_1, M_2/\Lambda_2),$$

we first write the map in the definition (3.3) as a composition

$$\operatorname{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(M_1, M_2) \to \operatorname{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(M_1, M_2/\Lambda_2) \to \operatorname{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(\Lambda_1, M_2/\Lambda_2). \tag{3.4}$$

We consider the commutative diagram (we set $\operatorname{Hom}(-,-) := \operatorname{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(-,-)$ and $\operatorname{Ext}(-,-) := \operatorname{Ext}_{\mathbf{B}^+_{\mathrm{dR}}}(-,-)$):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Hom}(M_1/\Lambda_1, \Lambda_2) & \to & \operatorname{Hom}(M_1, \Lambda_2) & \to & \operatorname{Hom}(\Lambda_1, \Lambda_2) & \to & \operatorname{Ext}(M_1/\Lambda_1, \Lambda_2) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Hom}(M_1/\Lambda_1, M_2) & \to & \operatorname{Hom}(M_1, M_2) & \to & \operatorname{Hom}(\Lambda_1, M_2) & \to & \operatorname{Ext}(M_1/\Lambda_1, M_2) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Hom}(M_1/\Lambda_1, M_2/\Lambda_2) & \to & \operatorname{Hom}(M_1, M_2/\Lambda_2) & \to & \operatorname{Hom}(\Lambda_1, M_2/\Lambda_2) & \to & \operatorname{Ext}(M_1/\Lambda_1, M_2/\Lambda_2) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}$$

The rows of the diagram are exact because M_1 is a projective module in the category of $\mathbf{B}^+_{\mathrm{dR}}$-modules. The second and the third columns are exact because M_1 and Λ_1 are projective, respectively. Hence the first map in the composition (3.4) is surjective, and the second one has cokernel isomorphic to $\operatorname{Ext}(M_1/\Lambda_1, M_2/\Lambda_2)$, as wanted. □

Lemma 3.2. *Let M_1, M_2 be two complexes in $C^b(\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}})$. We have a natural quasi-isomorphism*

$$\operatorname{Hom}_{\mathscr{D}^b(\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}})}(M_1, M_2) \simeq \operatorname{Hom}^F(M_1, M_2). \tag{3.5}$$

Proof. Assume first that $M_1, M_2 \in \mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}}$. By Lemma 3.1, H^0's of both sides of (3.5) are naturally isoomorphic to $\mathrm{Hom}_{\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}}}(M_1, M_2)$.

For $i \geq 2$, we claim that $H^i \mathrm{Hom}_{\mathscr{D}^b(\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}})}(M_1, M_2) = 0$. To see that note that we have the natural isomorphism

$$H^i \mathrm{Hom}_{\mathscr{D}^b(\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}})}(M_1, M_2) \simeq \mathrm{Ext}^i_{\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}}}(M_1, M_2)$$

with the group of Yoneda extensions. Moreover, the natural map

$$\mathrm{Ext}^i_{\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}}}(M_1, M_2) \to \mathrm{Ext}^i_{\mathbf{B}^+_{\mathrm{dR}}}(M_1/\Lambda_1, M_2/\Lambda_2), \quad i \geq 0,$$

is injective. Indeed, first pass from Yoneda extensions in the category of $\mathbf{B}^+_{\mathrm{dR}}$-modules to Yoneda extensions in the category of torsion $\mathbf{B}^+_{\mathrm{dR}}$-modules, then lift Yoneda equivalence relations from the category of torsion $\mathbf{B}^+_{\mathrm{dR}}$-modules to the category $\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}}$. But $\mathrm{Ext}^i_{\mathbf{B}^+_{\mathrm{dR}}}(M_1/\Lambda_1, M_2/\Lambda_2) = 0$, $i \geq 2$, as wanted.

For $i = 1$, we know from the above that the natural map

$$\mathrm{Ext}^1_{\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}}}(M_1, M_2) \to \mathrm{Ext}^1_{\mathbf{B}^+_{\mathrm{dR}}}(M_1/\Lambda_1, M_2/\Lambda_2)$$

is injective. It is easy to check that it is also surjective. Since, by Lemma 3.1, $H^1 \mathrm{Hom}^F(M_1, M_2) \simeq \mathrm{Ext}^1_{\mathbf{B}^+_{\mathrm{dR}}}(M_1/\Lambda_1, M_2/\Lambda_2)$, we are done.

Now, we go back to $M_1, M_2 \in C^b(\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}})$. Consider the functor $\mathrm{Hom}^F(M_1, -)$ on $C^b(\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}})$. We have

$$Z^0 \mathrm{Hom}^F(M_1, -) \simeq \mathrm{Hom}_{C^b(\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}})}(M_1, -).$$

Also, if M_2 is strictly acyclic, $\mathrm{Hom}^F(M_1, M_2)$ is acyclic. Arguing similarly with respect to M_1, we conclude that we have a natural (in both variables) map

$$\mathrm{Hom}_{\mathscr{D}^b(\mathrm{FMod}_{\mathbf{B}^+_{\mathrm{dR}}})}(M_1, M_2) \to \mathrm{Hom}^F(M_1, M_2).$$

To see that it is a quasi-isomorphism we reduce, by an easy devissage, to the case when both M_1, M_2 are concentrated in one degre, and we treated that case above. □

Similarly, we define the quasi-abelian category $\mathrm{FMod}_{\mathbf{B}_{\mathrm{dR}}}$ of pairs (Λ, M), $\Lambda \subset M$, where $M \in \mathrm{Mod}_{\mathbf{B}_{\mathrm{dR}}}$, and Λ is a $\mathbf{B}^+_{\mathrm{dR}}$-lattice in M. This time, for $M_1, M_2 \in \mathscr{D}^b(\mathrm{FMod}_{\mathbf{B}_{\mathrm{dR}}})$, we have

$$\mathrm{Hom}_{\mathscr{D}^b(\mathrm{FMod}_{\mathbf{B}_{\mathrm{dR}}})}(M_1, M_2) \simeq \mathrm{Hom}_{\mathrm{FMod}_{\mathbf{B}_{\mathrm{dR}}}}(M_1, M_2).$$

Indeed, since $\mathbf{B}^+_{\mathrm{dR}}$ is a DVR, elementary divisors theory gives us that every exact sequence

$$0 \to (\Lambda_1, M_1) \to (\Lambda_2, M_2) \to (\Lambda_3, M_3) \to 0$$

splits.

Let $M = (D, \Lambda)$, $T = (D', \Lambda')$ be two complexes in $C^b(\mathrm{DF}^+_{\overline{K}})$, $C^b(\mathrm{DF}_{\overline{K}})$, respectively. Define the complexes $\mathrm{Hom}^+(M,T)$, $\mathrm{Hom}(M,T)$ as the following homotopy fibers:

$$\begin{aligned}\mathrm{Hom}^+(M,T) &:= [\mathrm{Hom}_{\mathbf{B}^+,\varphi}(D,D') \oplus \mathrm{Hom}^F((\Lambda, D_{\mathbf{B}^+_{\mathrm{dR}}}), (\Lambda', D'_{\mathbf{B}^+_{\mathrm{dR}}})) \\ &\qquad \xrightarrow{\mathrm{can}-\mathrm{can}} \mathrm{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(D_{\mathbf{B}^+_{\mathrm{dR}}}, D'_{\mathbf{B}^+_{\mathrm{dR}}})], \\ \mathrm{Hom}(M,T) &:= [\mathrm{Hom}_{\mathbf{B}^+,\varphi}(D,D') \oplus \mathrm{Hom}_{F\,\mathrm{Mod}_{\mathbf{B}_{\mathrm{dR}}}}((\Lambda, D_{\mathbf{B}_{\mathrm{dR}}}), (\Lambda', D'_{\mathbf{B}_{\mathrm{dR}}})) \\ &\qquad \xrightarrow{\mathrm{can}-\mathrm{can}} \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}}}(D_{\mathbf{B}_{\mathrm{dR}}}, D'_{\mathbf{B}_{\mathrm{dR}}})].\end{aligned}$$

Proposition 3.3. *We have* $(* = \llcorner\!\lrcorner, \mathrm{ad})$

$$\mathrm{Hom}_{\mathscr{D}^b(\mathrm{DF}^{+,*}_{\overline{K}})}(M,T) \simeq \mathrm{Hom}^+(M,T), \quad \mathrm{Hom}_{\mathscr{D}^b(\mathrm{DF}^{*}_{\overline{K}})}(M,T) \simeq \mathrm{Hom}(M,T).$$

Proof. Using the computations done above, especially Lemma 3.2, the proof is analogous to the one of [9, Proposition 2.7]: note that $\mathrm{Cone}(M \xrightarrow{\mathrm{Id}} M)$, for $M \in \mathscr{D}^b(\mathrm{DF}^+_{\overline{K}})$, $M \in \mathscr{D}^b(\mathrm{DF}_{\overline{K}})$, is acyclic and that the category of semistable vector bundles of slope zero is closed under extensions (in the category of vector bundles). □

Let $\mathbb{1} := (\mathbf{B}^+, \mathbf{B}^+_{\mathrm{dR}})$ be the unit filtered φ-module. For $M = (D, \Lambda)$, $M \in \mathrm{DF}^+_{\overline{K}}$, and $M \in \mathrm{DF}_{\overline{K}}$ we set $H^*_+(\overline{K}, M) := H^* \mathrm{Hom}^+(\mathbb{1}, M)$ and $H^*(\overline{K}, M) := H^* \mathrm{Hom}(\mathbb{1}, M)$, respectively. We have

$$(3.6)\quad \begin{aligned} H^i_+(\overline{K}, M) &= \begin{cases} D^{\varphi=1} \cap \Lambda, & i = 0, \\ (D \otimes_{\mathbf{B}^+} \mathbf{B}^+_{\mathrm{dR}})/(\Lambda + D^{\varphi=1}), & i = 1, \\ 0, & i \geq 2, \end{cases} \\ H^i(\overline{K}, M) &= \begin{cases} D^{\varphi=1} \cap \Lambda_M, & i = 0, \\ (D \otimes_{\mathbf{B}^+} \mathbf{B}_{\mathrm{dR}})/(\Lambda + D^{\varphi=1}), & i = 1, \\ 0, & i \geq 2. \end{cases} \end{aligned}$$

This is because

$$\mathrm{Hom}^F(\mathbb{1}, M) = [\mathrm{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(\mathbf{B}^+_{\mathrm{dR}}, M) \to \mathrm{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(\mathbf{B}^+_{\mathrm{dR}}, M/\Lambda)] \simeq [M \to M/\Lambda] \xrightarrow{\sim} \Lambda.$$

Moreover, the complex

$$\mathrm{Hom}^+(\mathbb{1}, M) = (D^{\varphi=1} \to (D \otimes_{\mathbf{B}^+} \mathbf{B}^+_{\mathrm{dR}})/\Lambda)$$

can be lifted canonically to a complex of finite dimensional Banach-Colmez spaces. To see that, set, for a sympathetic algebra A,

$$D(A) := D \otimes_{\mathbf{B}^+} \mathbb{B}^+(A), \quad \Lambda(A) := \Lambda \otimes_{\mathbf{B}^+_{\mathrm{dR}}} \mathbb{B}^+_{\mathrm{dR}}(A), \tag{3.7}$$
$$\mathrm{Hom}^+(\mathbb{1}, M)(A) := (D(A)^{\varphi=1} \to (D \otimes_{\mathbf{B}^+} \mathbb{B}^+_{\mathrm{dR}}(A))/\Lambda(A)).$$

The bottom complex is clearly a complex of Banach-Colmez spaces such that $\mathrm{Hom}^+(\mathbb{1}, M)(C) = \mathrm{Hom}^+(\mathbb{1}, M)$. The fact that this is a complex of finite dimensional Banach-Colmez spaces follows from Section 2.2.2.

Remark 3.4. Let $(\mathscr{E}_1, \mathscr{E}_2, u)$ be an effective modification and let M be the associated effective filtered φ-module. We have an exact sequence of sheaves on X_{FF}

$$0 \to \mathscr{E}_1 \xrightarrow{u} \mathscr{E}_2 \to i_{\infty *}(\widehat{\mathscr{E}}_{2,\infty}/u(\widehat{\mathscr{E}}_{1,\infty})) \to 0.$$

We get from it the long exact sequence of cohomology groups

$$\begin{aligned} 0 \to & H^0(X_{\mathrm{FF}}, \mathscr{E}_1) \to H^0(X_{\mathrm{FF}}, \mathscr{E}_2) \to \widehat{\mathscr{E}}_{2,\infty}/u(\widehat{\mathscr{E}}_{1,\infty}) \\ & \to H^1(X_{\mathrm{FF}}, \mathscr{E}_1) \to H^1(X_{\mathrm{FF}}, \mathscr{E}_2) \to 0. \end{aligned}$$

Since $H^0(X_{\mathrm{FF}}, \mathscr{E}(D)) = D^{\varphi=1}$, by (3.6), we have that

$$\begin{aligned} & H^0_+(\overline{K}, M) = H^0(X_{\mathrm{FF}}, \mathscr{E}_1), \\ & H^1_+(\overline{K}, M) = \ker(H^1(X_{\mathrm{FF}}, \mathscr{E}_1) \to H^1(X_{\mathrm{FF}}, \mathscr{E}_2)). \end{aligned}$$

In particular, if $(\mathscr{E}_1, \mathscr{E}_2, u)$ is effective and admissible, then $H^1_+(\overline{K}, M) = 0$ because $H^1(X_{\mathrm{FF}}, \mathscr{E}_1) = 0$.

Remark 3.5. We note that if $M = \mathbb{1}$, then $H^1_+(\overline{K}, M) = 0$ and $H^1(\overline{K}, M) = \mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}$; hence effective filtered φ-modules are not closed under extensions in the category of filtered φ-modules. On the other hand, admissible filtered φ-modules are closed under extensions (because semistable vector bundles of slope zero are closed under extensions in the category of vector bundles).

3.3. Extensions of modifications. Extensions of modifications of vector bundles can be computed in an analogous way; we will just list the results. Let $M = (D, \Lambda)$, $T = (D', \Lambda')$ be two complexes in $C^b(\mathscr{M}^+)$, $C^b(\mathscr{M})$, respectively, with D, D', complexes of φ-modules over $\mathbf{B}$. Define the respective complexes $\mathrm{Hom}^+(M, T)$, $\mathrm{Hom}(M, T)$ as the following homotopy fibers:

$$\begin{aligned} \mathrm{Hom}^+(M, T) := [&\mathrm{Hom}_{\mathbf{B},\varphi}(D, D') \oplus \mathrm{Hom}^F((\Lambda, D_{\mathbf{B}^+_{\mathrm{dR}}}), (\Lambda', D'_{\mathbf{B}^+_{\mathrm{dR}}})) \\ & \xrightarrow{\mathrm{can}-\mathrm{can}} \mathrm{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(D_{\mathbf{B}^+_{\mathrm{dR}}}, D'_{\mathbf{B}^+_{\mathrm{dR}}})], \\ \mathrm{Hom}(M, T) := [&\mathrm{Hom}_{\mathbf{B},\varphi}(D, D') \oplus \mathrm{Hom}_{F\,\mathrm{Mod}_{\mathbf{B}_{\mathrm{dR}}}}((\Lambda, D_{\mathbf{B}_{\mathrm{dR}}}), (\Lambda', D'_{\mathbf{B}_{\mathrm{dR}}})) \\ & \xrightarrow{\mathrm{can}-\mathrm{can}} \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}}}(D_{\mathbf{B}_{\mathrm{dR}}}, D'_{\mathbf{B}_{\mathrm{dR}}})]. \end{aligned}$$

Proposition 3.6. *We have* $(* = \sqcup, \mathrm{ad})$

$$\mathrm{Hom}_{\mathscr{D}^b(\mathscr{M}^{+,*})}(M,T) \simeq \mathrm{Hom}^+(M,T), \quad \mathrm{Hom}_{\mathscr{D}^b(\mathscr{M}^*)}(M,T) \simeq \mathrm{Hom}(M,T).$$

Let $\mathbb{1} := (\mathbf{B}, \mathbf{B}_{\mathrm{dR}}^+)$ be the unit modification. For $M = (D, \Lambda)$, $M \in \mathscr{M}^+$ and $M \in \mathscr{M}$, we set $H_+^*(\mathscr{M}, M) := H^* \mathrm{Hom}^+(\mathbb{1}, M)$ and $H^*(\mathscr{M}, M) := H^* \mathrm{Hom}(\mathbb{1}, M)$, respectively. We have the following long exact sequence $(* = +, \sqcup)$:

$$0 \to H_*^0(\mathscr{M}, M) \to D^{\varphi=1} \to (D \otimes \mathbf{B}_{\mathrm{dR}}^*)/\Lambda \to H_*^1(\mathscr{M}, M) \to D/(1-\varphi)D \to 0.$$

Moreover, for $i \geq 2$, $H_*^i(\mathscr{M}, M) = 0$.

We conclude that, for an effective filtered φ-module $M = (D, \Lambda)$ over $\overline{K}$, we have $H_+^i(\overline{K}, M) \hookrightarrow H_+^i(\mathscr{M}, M_{\mathbf{B}})$, where $M_{\mathbf{B}} := (D \otimes_{\mathbf{B}^+} \mathbf{B}, \Lambda)$. More specifically, we have

(1) $H_+^0(\overline{K}, M) = H_+^0(\mathscr{M}, M_{\mathbf{B}})$,
(2) $H_+^1(\overline{K}, M) \hookrightarrow H_+^1(\mathscr{M}, M_{\mathbf{B}})$ with cokernel

$$H^1(X_{\mathrm{FF}}, \mathscr{E}(D)) = D/(1-\varphi)D,$$

(3) $H_+^i(\overline{K}, M) = H_+^i(\mathscr{M}, M_{\mathbf{B}}) = 0 \quad$ for $i \geq 2$.

The following proposition shows that cohomology of effective modifications recovers only the cohomology of the "smaller" modified vector bundle on the Fargues-Fontaine curve.

Proposition 3.7. *Let* $T = (\mathscr{E}_1, \mathscr{E}_2, u)$ *be an effective modification. We have a canonical quasi-isomorphism*

$$\mathrm{R}\,\mathrm{Hom}_{\mathscr{M}^+}(\mathbb{1}, T) \simeq \mathrm{R}\Gamma(X_{\mathrm{FF}}, \mathscr{E}_1).$$

Proof. Let $T = (\mathscr{E}_1, \mathscr{E}_2, u)$ and $T' = (\mathscr{E}_1', \mathscr{E}_2', u')$ be two complexes of effective modifications. We have

$$\mathrm{R}\,\mathrm{Hom}_{\mathscr{M}^+}(T, T') = [\mathrm{R}\,\mathrm{Hom}_{\mathrm{Bun}_{X_{\mathrm{FF}}}}(\mathscr{E}_1, \mathscr{E}_1') \oplus \mathrm{R}\,\mathrm{Hom}_{\mathrm{Bun}_{X_{\mathrm{FF}}}}(\mathscr{E}_2, \mathscr{E}_2') \xrightarrow{u'_* - u^*} \mathrm{R}\,\mathrm{Hom}_{\mathrm{Bun}_{X_{\mathrm{FF}}}}(\mathscr{E}_1, \mathscr{E}_2')].$$

If we apply this to the unit modification $\mathbb{1}$ and $T = (\mathscr{E}_1, \mathscr{E}_2, u)$ we find that

$$\begin{aligned}\mathrm{R}\,\mathrm{Hom}_{\mathscr{M}^+}(\mathbb{1}, T) &= [\mathrm{R}\,\mathrm{Hom}_{\mathrm{Bun}_{X_{\mathrm{FF}}}}(\mathscr{O}, \mathscr{E}_1) \oplus \mathrm{R}\,\mathrm{Hom}_{\mathrm{Bun}_{X_{\mathrm{FF}}}}(\mathscr{O}, \mathscr{E}_2) \xrightarrow{u_* - \mathrm{Id}} \mathrm{R}\,\mathrm{Hom}_{\mathrm{Bun}_{X_{\mathrm{FF}}}}(\mathscr{O}, \mathscr{E}_2)] \\ &= \mathrm{R}\,\mathrm{Hom}_{\mathrm{Bun}_{X_{\mathrm{FF}}}}(\mathscr{O}, \mathscr{E}_1) = \mathrm{R}\Gamma(X_{\mathrm{FF}}, \mathscr{E}_1),\end{aligned}$$

as wanted. ∎

4. Geometric syntomic cohomology

We will recall the definition of geometric syntomic cohomology defined in [17] and list its basic properties.

4.1. Definitions. For $X \in \mathscr{V}ar_{\overline{K}}$, we have the rational crystalline cohomology $\mathrm{R}\Gamma_{\mathrm{cr}}(X)$ defined in [4] using h-topology. It is a filtered dg perfect $\mathbf{B}^+_{\mathrm{cr}}$-algebra equipped with the Frobenius action φ. The Galois group G_K acts on $\mathscr{V}ar_{\overline{K}}$ and it acts on $X \mapsto \mathrm{R}\Gamma_{\mathrm{cr}}(X)$ by transport of structure. If X is defined over K, then G_K acts naturally on $\mathrm{R}\Gamma_{\mathrm{cr}}(X)$.

For $r \geq 0$, one defines [17] geometric syntomic cohomology of X as the rth filtered Frobenius eigenspace of crystalline cohomology

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,r) := [F^r\mathrm{R}\Gamma_{\mathrm{cr}}(X) \xrightarrow{1-\varphi_r} \mathrm{R}\Gamma_{\mathrm{cr}}(X)].$$

In the case when X is the generic fiber of a proper semistable scheme $\mathscr{X}$ over $\mathscr{O}_K$, this agrees with the (continuous) logarithmic syntomic cohomology of Fontaine-Messing-Kato.

The above definition is convenient to study period maps, but for computations a different definition is more convenient. We will explain it now. There is a natural map $\gamma : \mathrm{R}\Gamma_{\mathrm{cr}}(X) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}}$. Here $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$ is the Deligne's de Rham cohomology. It is a filtered perfect complex of $\overline{K}$-vector spaces. We equip it with the Hodge-Deligne filtration. It follows from the degeneration of the Hodge-de Rham spectral sequence that the differentials in $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$ are strict for the filtration (cf. [14, Proposition 8.3.1]).

The complex $\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}}$ is a perfect complex of free $\mathbf{B}^+_{\mathrm{dR}}$-modules. It is filtered by a perfect complex of $\mathbf{B}^+_{\mathrm{dR}}$-lattices. The cohomology of

$$F^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}}), i \geq 0,$$

is torsion-free: this follows from the degeneration of the Hodge-de Rham spectral sequence. Moreover, this implies that this complex is strict: kernels of differentials are t-saturated because the complex is perfect, and images are t-saturated in the kernels because cohomology is torsion-free; this implies that the images are t-saturated in the ambient modules, as wanted.

For $r \geq 0$, the geometric syntomic cohomology of X can be defined in the following way:

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,r) := \left[\mathrm{R}\Gamma_{\mathrm{cr}}(X) \xrightarrow{(1-\varphi_r,\gamma)} \mathrm{R}\Gamma_{\mathrm{cr}}(X) \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r\right]$$

$$\xleftarrow{\sim} \left[\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}} & \xrightarrow{(1-\varphi_r,\iota_{\mathrm{dR}}\otimes\iota)} & \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}} \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \\ \downarrow N & & \downarrow (N,0) \\ \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}} \end{array}\right].$$

We set $\varphi_i := \varphi/p^i$. Here $\mathrm{R}\Gamma_{\mathrm{HK}}(X)$ is the Beilinson's Hyodo-Kato cohomology of X [4]. It is a complex of finite rank (φ, N)-modules over F^{nr}. It comes equipped with the Hyodo-Kato quasi-isomorphism

$$\iota_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \overline{K} \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

The second quasi-isomorphism in the diagram uses the quasi-isomorphism

$$\iota_{\mathrm{cr}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X) \otimes_{\mathbf{B}^+_{\mathrm{cr}}} \mathbf{B}^+_{\mathrm{st}}$$

that is compatible with the action of φ and N. We will write

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X)^{\tau}_{\mathbf{B}^+_{\mathrm{cr}}} := (\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{N=0}.$$

We have a trivialization

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{cr}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(X)^{\tau}_{\mathbf{B}^+_{\mathrm{cr}}} = (\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{N=0},$$
$$x \mapsto \exp(N(x)\log([\tilde{p}])),$$

where $\tilde{p}$ is a sequence of p^nth roots of p. This yields a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{HK}}(X)^{\tau}_{\mathbf{B}^+_{\mathrm{cr}}} \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X)$. Both maps are compatible with Frobenius and monodromy.

We can rewrite the above in the following form:

$$\begin{aligned}
&\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)\\
&\quad\xrightarrow{\sim} [\ \mathrm{R}\Gamma_{\mathrm{cr}}(X)^{\varphi=p^r} \xrightarrow{\quad\gamma\quad} (\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r\ ]\\
&\quad\xleftarrow{\sim} [\ (\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \xrightarrow{\iota_{\mathrm{dR}}\otimes\iota} (\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r\ ],
\end{aligned}$$

where we set

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X)^{\varphi=p^r} := [\mathrm{R}\Gamma_{\mathrm{cr}}(X) \xrightarrow{p^r-\varphi} \mathrm{R}\Gamma_{\mathrm{cr}}(X)]$$

and

$$\begin{aligned}
&(\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\\
&\quad := \left[\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}} & \xrightarrow{1-\varphi_r} & \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}}\\
\downarrow N & & \downarrow N\\
\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}}
\end{array}\right].
\end{aligned}$$

Alternatively, by formula (2.1), we can change the period ring $\mathbf{B}^+_{\mathrm{st}}$ to $\mathbf{B}^+_{\log}$ to obtain

$$\begin{aligned}
\mathrm{R}\Gamma_{\mathrm{syn}}(X,r) \xrightarrow{\sim} [(&\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\log})^{N=0,\varphi=p^r}\\
&\xrightarrow{\iota_{\mathrm{dR}}\otimes\iota} (\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r].
\end{aligned}$$

We will write $\mathrm{R}\Gamma_{\mathrm{HK}}(X)^{\tau}_{\mathbf{B}^+} := (\mathrm{R}\Gamma_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\log})^{N=0}$; we have a canonical trivialization $\mathrm{R}\Gamma_{\mathrm{HK}}(X)^{\tau}_{\mathbf{B}^+}\simeq \mathrm{R}\Gamma_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+$ (compatible with Frobenius and monodromy). With this notation, we have

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)\xrightarrow{\sim}[\,\mathrm{R}\Gamma_{\mathrm{HK}}(X)^{\tau,\varphi=p^r}_{\mathbf{B}^+}\longrightarrow(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^r\,]. \tag{4.1}$$

4.2. Basic properties. We will now list basic properties of geometric syntomic cohomology.

4.2.1. Syntomic period maps. Let $X\in\mathscr{V}ar_{\overline{K}}$. Recall that Beilinson [3], [4] defined comparison quasi-isomorphisms

$$\begin{aligned}\rho_{\mathrm{cr}}&:\mathrm{R}\Gamma_{\mathrm{cr}}(X)\otimes_{\mathbf{B}^+_{\mathrm{cr}}}\mathbf{B}_{\mathrm{cr}}\simeq\mathrm{R}\Gamma_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{cr}},\\ \rho_{\mathrm{HK}}&:\mathrm{R}\Gamma_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}\simeq\mathrm{R}\Gamma_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{st}},\\ \rho_{\mathrm{dR}}&:\mathrm{R}\Gamma_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}_{\mathrm{dR}}\simeq\mathrm{R}\Gamma_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}},\end{aligned}$$

which are compatible with the extra structures and with each other. For $r\geq 0$, they give us the syntomic period map [17]

$$\rho_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)\to\mathrm{R}\Gamma_{\text{ét}}(X,\mathbf{Q}_p(r)),$$

defined as follows:

$$\begin{aligned}&\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)\\ &\quad\simeq[(\mathrm{R}\Gamma_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{dR}}\otimes\iota}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^r]\\ &\quad\to[(\mathrm{R}\Gamma_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}})^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{dR}}\otimes\iota}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}_{\mathrm{dR}})/F^r]\\ &\quad\simeq[\mathrm{R}\Gamma_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}^{\varphi=p^r}_{\mathrm{cr}}\xrightarrow{\iota_{\mathrm{dR}}\otimes\iota}\mathrm{R}\Gamma_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}/F^r]\\ &\quad\xleftarrow{\sim}\mathrm{R}\Gamma_{\text{ét}}(X,\mathbf{Q}_p(r)).\end{aligned}\tag{4.2}$$

The last quasi-isomorphism follows from the fundamental exact sequence

$$0\to\mathbf{Q}_p(r)\to\mathbf{B}^{\varphi=p^r}_{\mathrm{cr}}\to\mathbf{B}_{\mathrm{dR}}/F^r\to 0.$$

In a stable range, the syntomic period map is a quasi-isomorphism.

Proposition 4.1 ([17, Proposition 4.6])**.** *The syntomic period morphism induces a quasi-isomorphism*

$$\rho_{\mathrm{syn}}:\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)\xrightarrow{\sim}\tau_{\leq r}\mathrm{R}\Gamma_{\text{ét}}(X,\mathbf{Q}_p(r)).$$

4.2.2. Homotopy property. Syntomic cohomology has homotopy invariance property.

Proposition 4.2. *Let $X\in\mathscr{V}ar_{\overline{K}}$ and let $f:\mathbb{A}^1_X\to X$ be the natural projection from the affine line over X to X. Then, for all $r\geq 0$, the pullback map*

$$f^*:\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{syn}}(\mathbb{A}^1_X,r)$$

is a quasi-isomorphism.

Proof. It suffices to show that the pullback maps

$$f^*: (H^i_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{cr}})^{\varphi=p^r}\to (H^i_{\mathrm{HK}}(\mathbb{A}^1_X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{cr}})^{\varphi=p^r},$$
$$f^*: (H^i_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^r\to (H^i_{\mathrm{dR}}(\mathbb{A}^1_X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^r$$

are isomorphisms. But this follows immediately from the fact that we have a filtered isomorphism

$$f^*: H^i_{\mathrm{dR}}(X)\xrightarrow{\sim} H^i_{\mathrm{dR}}(\mathbb{A}^1_X)$$

and hence, via the Hyodo-Kato isomorphism, also a Frobenius equivariant isomorphism

$$f^*: H^i_{\mathrm{HK}}(X)\xrightarrow{\sim} H^i_{\mathrm{HK}}(\mathbb{A}^1_X).$$

□

4.2.3. Projective space theorem. For $X\in\mathscr{V}ar_K$, we have the functorial syntomic Chern class map [17, 5.1]

$$c_1^{\mathrm{syn}}:\mathrm{Pic}(X)\to H^2_{\mathrm{syn}}(X,1).$$

For $X\in\mathscr{V}ar_{\overline{K}}$, it yields the syntomic Chern class map

$$c_1^{\mathrm{syn}}:\mathrm{Pic}(X)\to H^2_{\mathrm{syn}}(X,1).$$

We have the following projective space theorem for syntomic cohomology.

Proposition 4.3. *Let $\mathscr{E}$ be a locally free sheaf of rank $d+1$, $d\geq 0$, on a scheme $X\in\mathscr{V}ar_{\overline{K}}$. Consider the associated projective bundle $\pi:\mathbb{P}(\mathscr{E})\to X$. Then we have the following isomorphism:*

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{syn}}(\mathscr{O}(1))^i\cup\pi^*:\quad \bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{syn}}(X,r-i)\xrightarrow{\sim} H^a_{\mathrm{syn}}(\mathbb{P}(\mathscr{E}),r),\quad 0\leq d\leq r.$$

Here, the class $c_1^{\mathrm{syn}}(\mathscr{O}(1))\in H^2_{\mathrm{syn}}(\mathbb{P}(\mathscr{E}),1)$ refers to the class of the tautological bundle on $\mathbb{P}(\mathscr{E})$.

Proof. Just as in the proof of Proposition 5.2 from [17], the above projective space theorem can be reduced to the projective space theorems for the Hyodo-Kato and the Hodge cohomologies. We refer to [17] for details and notation.

To prove our proposition it suffices to show that for any ss-pair $(U,\overline{U})$ over K and the projective space $\pi:\mathbb{P}^d_{\overline{U}}\to\overline{U}$ of dimension d over $\overline{U}$ we have a projective space theorem for syntomic cohomology ($a\geq 0$):

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{syn}}(\mathscr{O}(1))^i\cup\pi^*:\quad \bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{syn}}((U,\overline{U})_{\overline{K}},r-i)$$
$$\xrightarrow{\sim} H^a_{\mathrm{syn}}((\mathbb{P}^d_U,\mathbb{P}^d_{\overline{U}})_{\overline{K}},r),\quad 0\leq d\leq r,$$

where the class $c_1^{\mathrm{syn}}(\mathscr{O}(1))\in H^2_{\mathrm{syn}}((\mathbb{P}^d_U,\mathbb{P}^d_{\overline{U}}),1)$ refers to the class of the tautological bundle on $\mathbb{P}^d_{\overline{U}}$.

By the distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{syn}}((U,\overline{U})_{\overline{K}},r) \to \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})_{\overline{K}})^{\varphi=p^r} \to (\mathrm{R}\Gamma_{\mathrm{dR}}((U,\overline{U})_{\overline{K}}) \otimes \mathbf{B}^+_{\mathrm{dR}})/F^r$$

and its compatibility with the action of c_1^{syn}, it suffices to prove the following two isomorphisms for the absolute log-crystalline complexes and for the filtered log de Rham complexes $(0 \leq d \leq r)$:

$$\begin{aligned}
(4.3) \qquad & \bigoplus_{i=0}^{d} c_1^{\mathrm{cr}}(\mathscr{O}(1))^i \cup \pi^* : \quad \bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{cr}}((U,\overline{U})_{\overline{K}}) \xrightarrow{\sim} H^a_{\mathrm{cr}}((\mathbb{P}^d_U,\mathbb{P}^d_{\overline{U}})_{\overline{K}}),\\
& \bigoplus_{i=0}^{d} c_1^{\mathrm{dR}}(\mathscr{O}(1))^i \cup \pi^* : \quad \bigoplus_{i=0}^{d} (H^{a-2i}_{\mathrm{dR}}(U_{\overline{K}}) \otimes \mathbf{B}^+_{\mathrm{dR}})/F^{r-i}\\
& \qquad \xrightarrow{\sim} (H^a_{\mathrm{dR}}(\mathbb{P}^d_{U,\overline{K}}) \otimes \mathbf{B}^+_{\mathrm{dR}})/F^r.
\end{aligned}$$

For the crystalline cohomology in (4.3), we can pass to the Hyodo-Kato cohomology (see [17, Section 5.1] for the necessary compatibility of Chern classes). There the projective space theorem

$$\begin{aligned}
\bigoplus_{i=0}^{d} c_1^{\mathrm{HK}}(\mathscr{O}(1))^i \cup \pi^* : \quad & \bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{HK}}((U,\overline{U})_{\overline{K}}) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{cr}}\\
& \xrightarrow{\sim} H^a_{\mathrm{HK}}((\mathbb{P}^d_U,\mathbb{P}^d_{\overline{U}})_{\overline{K}}) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{cr}}
\end{aligned}$$

follows immediately, via the Hyodo-Kato isomorphism, from the projective space theorem for the de Rham cohomology.

For the de Rham cohomology in (4.3), passing to the grading we obtain

$$\begin{aligned}
(4.4) \qquad \bigoplus_{i=0}^{d} c_1^{\mathrm{dR}}(\mathscr{O}(1))^i \cup \pi^* : \quad & \bigoplus_{i=0}^{d} \mathrm{gr}^{r-i}(H^{a-2i}_{\mathrm{dR}}(U_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})\\
& \xrightarrow{\sim} \mathrm{gr}^r(H^a_{\mathrm{dR}}(\mathbb{P}^d_{U,\overline{K}}) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}}).
\end{aligned}$$

Since, for a variety Y over $\overline{K}$,

$$\mathrm{gr}^r(H^a_{\mathrm{dR}}(Y) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}}) = \bigoplus_{i=0}^{r} H^{a-r+i}_{\mathrm{dR}}(Y,\Omega^{r-i}_Y) \otimes_{\overline{K}} C,$$

the isomorphism (4.4) follows from the projective space theorem for Hodge cohomology. We are done. □

4.2.4. Bloch-Ogus theory. The above implies that syntomic cohomology is representable by a motivic ring spectrum $\mathscr{S}$: the argument is the same as in Appendix B of [17]. We list the following consequences.

Proposition 4.4.

(1) *Syntomic cohomology is covariant with respect to projective morphisms of smooth varieties. More precisely, to a projective morphism of smooth K-varieties $f : Y \to X$ one can associate a Gysin morphism in syntomic cohomology*

$$f_* : H^i_{\mathrm{syn}}(Y, r) \to H^{i-2d}_{\mathrm{syn}}(X, r - d),$$

where d is the dimension of f.

(2) *We have the syntomic regulator*

$$r_{\mathrm{syn}} : H^{r,i}_M(X) \to H^i_{\mathrm{syn}}(X, r),$$

where $H^{r,i}_M(X)$ demotes the motivic cohomology. It is compatible with product, pullbacks, and pushforwards; via the period map it is compatible with the étale regulator.

(3) *The syntomic cohomology has a natural extension to h-motives:*

$$DM_h(K, \mathbf{Q}_p)^{op} \to D(\mathbf{Q}_p), \quad M \mapsto \mathrm{Hom}_{DM_h(K,\mathbf{Q}_p)}(M, \mathscr{S}),$$

and the syntomic regulator r_{syn} can be extended to motives.

(4) *There exists a canonical syntomic Borel-Moore homology $H^{\mathrm{syn}}_*(-,*)$ such that the pair of functor $(H^*_{\mathrm{syn}}(-,*), H^{\mathrm{syn}}_*(-,*))$ defines a Bloch-Ogus theory.*

(5) *To the ring spectrum $\mathscr{S}$ there is associated a cohomology with compact support satisfying the usual properties.*

4.3. Fundamental (long) exact sequence. In this section, we will discuss certain exact sequences that involve syntomic cohomology. Recall that we have

(4.5)
$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{syn}}(X, r) =& [\mathrm{R}\Gamma_{\mathrm{cr}}(X)^{\varphi=p^r} \xrightarrow{\gamma} \mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r]\\ & [(\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_F \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \xrightarrow{\iota_{\mathrm{dR}}\otimes\iota} (\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r].\end{aligned}$$

This yields a long exact sequence of cohomology that simplifies quite a bit. Indeed, we have $H^j((\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_F \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}) = (H^j_{\mathrm{HK}}(X) \otimes_F \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}$ [17, Corollary 3.25]. It follows that $H^j(\mathrm{R}\Gamma_{\mathrm{cr}}(X)^{\varphi=p^r}) = H^j_{\mathrm{cr}}(X)^{\varphi=p^r} \simeq (\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{cr}})^{\varphi=p^r}$. By the degeneration of the Hodge-de Rham spectral sequence, we also have $H^j((\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r) = (H^j_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r$. Hence, from the mapping fiber (4.5), we get the following fundamental long exact sequence:

$$\begin{aligned}\to H^{i-1}_{\mathrm{cr}}(X)^{\varphi=p^r} &\xrightarrow{\gamma_{i-1}} (H^{i-1}_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \to H^i_{\mathrm{syn}}(X, r)\\ &\to H^i_{\mathrm{cr}}(X)^{\varphi=p^r} \xrightarrow{\gamma_i} (H^i_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \to,\end{aligned}$$

which we will write alternatively as

$$\begin{aligned}(4.6)\qquad &\to (H^{i-1}_{\mathrm{HK}}(X)\otimes_F \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\\ &\stackrel{\gamma_{i-1}}{\to} (H^{i-1}_{\mathrm{dR}}(X)\otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \to H^i_{\mathrm{syn}}(X,r)\\ &\to (H^{i}_{\mathrm{HK}}(X)\otimes_F \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \stackrel{\gamma_{i}}{\to} (H^{i}_{\mathrm{dR}}(X)\otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \to .\end{aligned}$$

It yields the exact sequence

$$(4.7)\qquad 0\to \operatorname{coker}\gamma_{i-1}\to H^i_{\mathrm{syn}}(X,r)\to \ker\gamma_i\to 0.$$

Example 4.5. Let $X=\operatorname{Spec}(\overline{K})$. We get the exact sequence

$$\begin{aligned}\to&(H^{i-1}_{\mathrm{dR}}(\overline{K})\otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \to H^i_{\mathrm{syn}}(\overline{K},r)\to H^i_{\mathrm{cr}}(\overline{K})^{\varphi=p^r}\\ &\to(H^{i}_{\mathrm{dR}}(\overline{K})\otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \to .\end{aligned}$$

Since $H^i_{\mathrm{cr}}(\overline{K})=H^i_{\mathrm{cr}}(F)\otimes_F\mathbf{B}^+_{\mathrm{cr}}$, we get that $H^0_{\mathrm{cr}}(\overline{K})=\mathbf{B}^+_{\mathrm{cr}}$ and $H^i_{\mathrm{cr}}(\overline{K})=0$ for $i>0$. Also, clearly, $H^0_{\mathrm{dR}}(\overline{K})=\overline{K}$ and $H^i_{\mathrm{dR}}(\overline{K})=0$ for $i>0$. Hence the above sequence becomes the fundamental exact sequence

$$0\to \mathbf{Q}_p(r)\to(\mathbf{B}^+_{\mathrm{cr}})^{\varphi=p^r}\to \mathbf{B}^+_{\mathrm{dR}}/F^r\to 0.$$

It implies that $H^0_{\mathrm{syn}}(\overline{K},r)\simeq \mathbf{Q}_p(r)$ and $H^i_{\mathrm{syn}}(\overline{K},r)=0$ for $i>0$.

4.3.1. Relation to extensions of φ-modules over $\overline{K}$. It turns out that the kernel and cokernel appearing in the exact sequence (4.7) are extension groups in the category of filtered φ-modules over $\overline{K}$. To see this, set $D^i(r):=H^i_{\mathrm{HK}}(X)^{\tau}_{\mathbf{B}^+}$ with Frobenius $\varphi_r=\varphi/p^r$ and set $\Lambda^i(r):=F^r(H^i_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})$. Let $H^i_{\mathrm{DF}_{\overline{K}}}(X,r):=(D^i(r),\Lambda^i(r))$. Since $H^i_{\mathrm{HK}}(X)\otimes_{\mathbf{B}^+}\mathbf{B}^+_{\mathrm{dR}}\simeq H^i_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}}$ (via the Hyodo-Kato isomorphism), we have $H^i_{\mathrm{DF}_{\overline{K}}}(X,r)\in \mathrm{DF}^+_{\overline{K}}$.

Lemma 4.6. *We have the following exact sequences:*

$$0\to H^1_+(\overline{K},H^{i-1}_{\mathrm{DF}_{\overline{K}}}(X,r))\to H^i_{\mathrm{syn}}(X,r)\to H^0_+(\overline{K},H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r))\to 0,$$

$$\begin{aligned}0\to \operatorname{Ker}(H^1(X_{\mathrm{FF}},\mathscr{E}(H^{i-1}_{\mathrm{DF}_{\overline{K}}}(X,r)))\to H^1(X_{\mathrm{FF}},\mathscr{E}(D^{i-1}(r))))\to H^i_{\mathrm{syn}}(X,r)\\ \to H^0(X_{\mathrm{FF}},\mathscr{E}(H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r)))\to 0.\end{aligned}$$

Moreover, for $i\leq r+1$ or $r\geq d$, there are natural isomorphisms

$$H^i_{\mathrm{syn}}(X,r)\stackrel{\sim}{\to} H^0_+(\overline{K},H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r))\simeq H^0(X_{\mathrm{FF}},\mathscr{E}(H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r))).$$

Proof. Since $\ker\gamma_i=D^i(r)^{\varphi_r=1}\cap\Lambda^i(r)$ and

$$\operatorname{coker}\gamma_i=(D^i(r)\otimes_{\mathbf{B}^+}\mathbf{B}^+_{\mathrm{dR}})/(\Lambda^i(r)+D^i(r)^{\varphi_r=1}),$$

the first exact sequence follows from (4.7) and (3.6). The second exact sequence follows from that and from the exact sequence of sheaves on X_{FF},

$$0\to\mathscr{E}(H^i_{\mathrm{DF}_{\overline{K}}}(X,r))\to\mathscr{E}(D^i(r))\to i_{\infty *}(D^i(r)\otimes_{\mathbf{B}^+}\mathbf{B}^+_{\mathrm{dR}}/\Lambda^i(r))\to 0.$$

The last statement of the lemma follows from the first exact sequence and the fact that, for $i \leq r$ or $r \geq d$, the filtered φ-module $H^i_{\mathrm{DF}_{\overline{K}}}(X,r)$ is admissible. To see the last claim, note that the variety X comes from a variety X_L defined over some finite extension L of K. Hence, by comparison theorems, the pair $(H^i_{\mathrm{HK}}(X_L), H^i_{\mathrm{dR}}(X_L))$ forms an admissible (φ, N, G_L)-module. Since $F^{r+1}H^i_{\mathrm{dR}}(X) = 0$, we can now simply quote [17, Corollary 2.10]. □

In the next section we will give a more conceptual reason for the existence of the sequences from the above lemma. We note that the above lemma implies that the fundamental (long) exact sequence (4.6) splits in a stable range: for $i \leq r$, we have the fundamental short exact sequences

$$0 \to H^i_{\mathrm{syn}}(X,r) \to (H^i_{\mathrm{HK}}(X) \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \xrightarrow{\iota_{\mathrm{dR}}\otimes\iota} (H^i_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \to 0.$$

5. The p-adic absolute Hodge cohomology

We will show in this section that the geometric syntomic cohomology is a p-adic absolute cohomology, that is, that to every variety over $\overline{K}$ one can associate a canonical complex of effective filtered φ-modules over $\overline{K}$ and syntomic cohomology is $\mathrm{R}\operatorname{Hom}$ from the trivial module to that complex. We will describe two methods of constructing such complexes.

5.1. Via geometric p-adic Hodge complexes. To every variety over $\overline{K}$ one can canonically associate Hyodo-Kato and de Rham cohomologies. Twisted by appropriate period rings these form what we will call geometric p-adic Hodge complexes. We will show that the derived ∞-category of such complexes is equivalent to the derived ∞-category of geometric effective filtered φ-modules over $\overline{K}$. The images in the latter category of the geometric p-adic Hodge complexes associated to varieties over $\overline{K}$ will give us the canonical complexes of effective filtered φ-modules over $\overline{K}$ (associated to varieties over $\overline{K}$) that we were looking for.

5.1.1. The category of geometric p-adic Hodge complexes. Let $\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}}$, $F\,\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}}$ be the categories $\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}}$, $F\,\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}}$ with the finiteness conditions dropped; i.e., we work with free $\mathbf{B}^+_{\mathrm{dR}}$-modules that are not necessarily of finite type. These are quasi-abelian categories. We have exact functors

$$\beta : \mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}} \to \mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}}, \quad \beta : F\,\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}} \to F\,\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}}.$$

We claim that, for $M_1, M_2 \in \mathscr{D}^b(\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}})$, resp., $M_1, M_2 \in \mathscr{D}^b(F\,\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}})$, we have

$$\mathrm{Hom}_{\mathscr{D}^b(\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}})}(M_1, M_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}^b(\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}})}(\beta(M_1), \beta(M_2)),$$
$$\mathrm{Hom}_{\mathscr{D}^d(F\,\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}})}(M_1, M_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}^d(F\,\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}})}(\beta(M_1), \beta(M_2)).$$

Indeed, by a simple devissage we may assume that $M_1, M_2 \in \mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}}$ or $M_1, M_2 \in F\,\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}}$. Then, in the first case, our claim follows from the fact that in both categories $\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}}$ and $\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}}$ short exact sequences split. In the second case, the result follows from the fact that

$$H^i\,\mathrm{Hom}_{\mathscr{D}^d(F\,\mathrm{Mod}_{\mathbf{B}^+_{\mathrm{dR}}})}(M_1, M_2) = 0$$

and $H^i\,\mathrm{Hom}_{\mathscr{D}^b(F\,\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}})}(\beta(M_1), \beta(M_2)) = 0$ for $i \geq 2$: for the first group this is shown in the proof of Lemma 3.2, and for the second one this can be proved in an analogous way.

Consider the exact monoidal functors

$$F_0 : \mathrm{Mod}_{\mathbf{B}^+}(\varphi) \to \mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}}, \quad D \mapsto D \otimes_{\mathbf{B}^+} \mathbf{B}^+_{\mathrm{dR}};$$
$$F_{\mathrm{dR}} : F\,\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}} \to \mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}}, \quad (\Lambda, M) \mapsto M.$$

We define the dg category $\mathscr{D}_{\mathrm{pH}}$ of *p-adic Hodge complexes* as the homotopy limit

$$\mathscr{D}_{\mathrm{pH}} := \mathrm{holim}(\mathscr{D}^b(\mathrm{Mod}_{\mathbf{B}^+}(\varphi)) \xrightarrow{F_0} \mathscr{D}^b(\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}}) \xleftarrow{F_{\mathrm{dR}}} \mathscr{D}^b(F\,\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}})).$$

We denote by D_{pH} the homotopy category of $\mathscr{D}_{\mathrm{pH}}$. An object of $\mathscr{D}_{\mathrm{pH}}$ consists of objects $M_0 \in \mathscr{D}^b(\mathrm{Mod}_{\mathbf{B}^+}(\varphi))$, $M_K \in \mathscr{D}^b(F\,\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}})$, and a quasi-isomorphism

$$F_0(M_0) \xrightarrow{a_M} F_{\mathrm{dR}}(M_K)$$

in $\mathscr{D}^b(\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}})$. We will denote the object above by $M = (M_0, M_K, a_M)$. The morphisms are given by the complex $\mathrm{Hom}_{\mathscr{D}_{\mathrm{pH}}}((M_0, M_K, a_M), (N_0, N_K, a_N))$:

$$\begin{aligned}
&\mathrm{Hom}^i_{\mathscr{D}^g_{\mathrm{pH}}}((M_0, M_K, a_M), (N_0, N_K, a_N)) \\
&\quad = \mathrm{Hom}^i_{\mathscr{D}^b(\mathrm{Mod}_{\mathbf{B}^+}(\varphi))}(M_0, N_0) \\
&\quad \oplus \mathrm{Hom}^i_{\mathscr{D}^b(F\,\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}})}(M_K, N_K) \\
&\quad \oplus \mathrm{Hom}^{i-1}_{\mathscr{D}^b(\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}})}(F_0(M_0), F_{\mathrm{dR}}(N_K)).
\end{aligned} \tag{5.1}$$

A (closed) morphism $(a, b, c) \in \mathrm{Hom}_{\mathscr{D}_{\mathrm{pH}}}((M_0, M_K, a_M), (N_0, N_K, a_N))$ is a quasi-isomorphism if and only if so are the morphisms a and b.

By definition, we get a commutative square of dg categories over $\mathbf{Q}_p$:

$$
\begin{array}{ccc}
\mathscr{D}_{\mathrm{pH}} & \xrightarrow{T_{\mathrm{dR}}} & \mathscr{D}^b(F\,\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}}) \\
{\scriptstyle T_0}\downarrow & & \downarrow{\scriptstyle F_{\mathrm{dR}}} \\
\mathscr{D}^b(\mathrm{Mod}_{\mathbf{B}^+}(\varphi)) & \xrightarrow{F_0} & \mathscr{D}^b(\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}}).
\end{array}
\tag{5.2}
$$

As pointed out above, a morphism f of p-adic Hodge complexes is a quasi-isomorphism if and only if $T_{\mathrm{dR}}(f)$ and $T_0(f)$ are quasi-isomorphisms.

For $M \in C^b(\mathrm{DF}^+_K)$, we can define $\theta(M) \in \mathscr{D}_{\mathrm{pH}}$ to be the object

$$\theta(M) := (D, (\Lambda, F_0(D)), \mathrm{Id} : F_0(D) \simeq F_0(D)).$$

Since θ preserves quasi-isomorphisms, it induces a canonical functor

$$\theta : \quad \mathscr{D}^b(\mathrm{DF}^+_K) \to \mathscr{D}_{\mathrm{pH}}$$

that is compatible with the left t-structures.

Definition 5.1. We will say a p-adic Hodge complex $M = (M_0, M_K, a_M)$ is *geometric* if the complex M_0 is in the image of the canonical functor $C^b(\mathrm{Mod}_{F^{\mathrm{nr}}}(\varphi)) \to C^b(\mathrm{Mod}_{\mathbf{B}^+}(\varphi))$ and the complex M_K is strict with torsion-free finite rank cohomology groups (defined using the left t-structure in the category $C^b(\mathrm{FMod}'_{\mathbf{B}^+_{\mathrm{dR}}})$). Denote by $\mathscr{D}^g_{\mathrm{pH}}$ the full dg subcategory of $\mathscr{D}_{\mathrm{pH}}$ of geometric p-adic Hodge complexes.

Remark 5.2. We note that for a geometric p-adic Hodge complex $M = (M_0, M_K, a_M)$ its cohomology groups are

$$H^i(M) := (H^i(M_0), H^i(M_K), a_M : F_0 H^i(M_0) \simeq F_{\mathrm{dR}} H^i(M_K)) \in \mathrm{DF}^+_K, \quad i \geq 0.$$

Proposition 5.3. *The functor θ induces an equivalence of dg categories*

$$\theta : \quad \mathscr{D}^g(\mathrm{DF}^+_K) \xrightarrow{\sim} \mathscr{D}^g_{\mathrm{pH}},$$

where $\mathscr{D}^g(\mathrm{DF}^+_K)$ denotes the full dg subcategory of $\mathscr{D}^b(\mathrm{DF}^+_K)$ of geometric p-adic Hodge complexes.

Proof. First, we will show that θ is fully faithful. That is, that, given two complexes M, M' of $C^g(\mathrm{DF}^+_K)$, the functor θ induces a quasi-isomorphism:

$$\theta : \quad \mathrm{Hom}_{\mathscr{D}^b(\mathrm{DF}^+_K)}(M, M') \to \mathrm{Hom}_{\mathscr{D}^g_{\mathrm{pH}}}(\theta(M), \theta(M')).$$

By Proposition 3.3, since $F_0(M_0) = F_{\mathrm{dR}}(M_K), F_0(M_0') = F_{\mathrm{dR}}(M_K')$, we have the following sequence of quasi-isomorphisms:

$$\begin{aligned}
&\mathrm{Hom}_{\mathscr{D}^g_{\mathrm{pH}}}(\theta(M), \theta(M'))\\
&\quad = \mathrm{Hom}_{\mathscr{D}^g_{\mathrm{pH}}}((M_0, M_K, \mathrm{Id}_M), (M_0', M_K', \mathrm{Id}_{M'}))\\
&\quad \simeq (\mathrm{Hom}_{\mathscr{D}^b(\mathrm{Mod}_{\mathbf{B}^+}(\varphi))}(M_0, M_0') \xrightarrow{F_0} \mathrm{Hom}_{\mathscr{D}^b(\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}})}(F_0(M), F_{\mathrm{dR}}(M'))\\
&\qquad\qquad \xleftarrow{F_{\mathrm{dR}}} \mathrm{Hom}_{\mathscr{D}^b(F\,\mathrm{Mod}'_{\mathbf{B}^+_{\mathrm{dR}}})}(M_K, M_K'))\\
&\quad \simeq (\mathrm{Hom}_{\mathbf{B}^+,\varphi}(M_0, M_0') \xrightarrow{F_0} \mathrm{Hom}_{\mathbf{B}^+_{\mathrm{dR}}}(F_0(M), F_{\mathrm{dR}}(M')) \xleftarrow{F_{\mathrm{dR}}} \mathrm{Hom}^F(M_K, M_K'))\\
&\quad \simeq \mathrm{Hom}_{\mathscr{D}^b(\mathrm{DF}^+_{\overline{K}})}(M, M').
\end{aligned}$$

The third quasi-isomorphism follows from Section 3.1 and Lemma 3.2. The last quasi-isomorphism follows from Proposition 3.3.

Now, it remains to show that θ is essentially surjective. But, by the fact that θ is fully faithful as shown above, it suffices to show that the cohomology groups $H^i(M)$, $M \in C^g_{\mathrm{pH}}$, are in the image of θ but this we know (see Remark 5.2). □

5.1.2. p-adic absolute Hodge cohomology via geometric p-adic Hodge complexes. Let X be a variety over $\overline{K}$. For $r \geq 0$, consider the following complex in $\mathscr{D}^g_{\mathrm{pH}}$:

$$\begin{aligned}
&\mathrm{R}\Gamma^g_{\mathrm{pH}}(X, r)\\
&\quad := (\mathrm{R}\Gamma_{\mathrm{HK}}(X, r)^\tau_{\mathbf{B}^+}, (\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}}, F^r), \iota_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X)^\tau_{\mathbf{B}^+} \otimes_{\mathbf{B}^+} \mathbf{B}^+_{\mathrm{dR}}\\
&\quad \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}}),
\end{aligned}$$

where the r-twist in $\mathrm{R}\Gamma_{\mathrm{HK}}(X, r)^\tau_{\mathbf{B}^+}$ refers to Frobenius divided by p^r. As explained at the beginning of Section 4.1, the p-adic Hodge complex $\mathrm{R}\Gamma^g_{\mathrm{pH}}(X, r)$ is geometric. We will call it *the geometric p-adic Hodge cohomology of X*. Set

$$\mathrm{R}\Gamma_{\mathrm{DF}_{\overline{K}}}(X, r) := \theta^{-1}\mathrm{R}\Gamma^g_{\mathrm{pH}}(X, r) \in \mathscr{D}^g(\mathrm{DF}^+_{\overline{K}}).$$

The *p-adic absolute Hodge cohomology of X* is defined as

$$\mathrm{R}\Gamma_{\mathscr{H}}(X, r) := \mathrm{Hom}_{\mathscr{D}^g_{\mathrm{pH}}}(\mathbb{1}, \mathrm{R}\Gamma^g_{\mathrm{pH}}(X, r)). \tag{5.3}$$

By Proposition 5.3, we have

$$\mathrm{R}\Gamma_{\mathscr{H}}(X, r) \simeq \mathrm{Hom}_{\mathscr{D}^b(\mathrm{DF}^+_{\overline{K}})}(\mathbb{1}, \mathrm{R}\Gamma_{\mathrm{DF}_{\overline{K}}}(X, r)).$$

Theorem 5.4. *There exists a natural quasi-isomorphism (in the classical derived category)*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, r) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathscr{H}}(X, r), \quad r \geq 0.$$

Proof. We can write

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,r) \xrightarrow{\sim} [\mathrm{R}\Gamma_{\mathrm{HK}}(X)^{\tau,\varphi=p^r}_{\mathbf{B}^+} \oplus F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}}) \xrightarrow{\iota_{\mathrm{dR}}\otimes\iota-\mathrm{can}} \mathrm{R}\Gamma_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}}\ ].$$

By Proposition 5.3, we have

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{syn}}(X,r) &\simeq \mathrm{Hom}_{\mathscr{D}^g_{\mathrm{pH}}}(\mathbb{1},\mathrm{R}\Gamma^g_{\mathrm{pH}}(X,r))\\ &\simeq \mathrm{Hom}_{\mathscr{D}^b(\mathrm{DF}^+_{\overline{K}})}(\mathbb{1},\mathrm{R}\Gamma_{\mathrm{DF}_{\overline{K}}}(X,r)) \simeq \mathrm{R}\Gamma_{\mathscr{H}}(X,r),\end{aligned}$$

as wanted. □

5.2. Via Beilinson's Basic Lemma. Using Beilinson's Basic Lemma as in [9] we can associate to every $X\in\mathscr{V}ar_{\overline{K}}$ (more generally, to any finite diagram of such X) the following functorial data:

(1) $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X)$, a complex of finite (φ,N)-modules over F^{nr} representing $\mathrm{R}\Gamma_{\mathrm{HK}}(X)$;
(2) $\mathrm{R}\Gamma^B_{\mathrm{dR}}(X)$, a complex of finite filtered $\overline{K}$-vector spaces representing $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$;
(3) the Hyodo-Kato isomorphism of complexes

$$\iota_{\mathrm{dR}}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\overline{K}\xrightarrow{\sim}\mathrm{R}\Gamma^B_{\mathrm{dR}}(X)$$

representing the original Hyodo-Kato map ι_{dR}.

They yield a functorial complex $\mathrm{R}\Gamma^B_{\mathrm{DF}_{\overline{K}}}(X,r)$, $r\geq 0$, of effective filtered φ-modules $(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X)^{\tau}_{\mathbf{B}^+},(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}},F^r),\iota_{\mathrm{dR}})$. By construction, for every good pair (X,Y,i), the associated complex is

$$\begin{aligned}\mathrm{R}\Gamma^B_{\mathrm{DF}_{\overline{K}}}(X,Y,r) &\simeq H^i_{\mathrm{DF}_{\overline{K}}}(X,Y,r)\\ &:= (H^i_{\mathrm{HK}}(X,Y,r)^{\tau}_{\mathbf{B}^+},(H^i_{\mathrm{dR}}(X,Y)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}},F^r),\iota_{\mathrm{dR}}).\end{aligned}$$

We note that the complex $\mathrm{R}\Gamma^B_{\mathrm{DF}_{\overline{K}}}(X,r)$ is geometric. Indeed, Hodge Theory yields that "geometric" morphisms between de Rham cohomology groups are strict for the Hodge-Deligne filtration. Hence the complex $\mathrm{R}\Gamma^B_{\mathrm{dR}}(X)$ has strict differentials. This implies that this is also the case for the complex $\mathrm{R}\Gamma^B_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}}$. Moreover, the cohomology groups $H^i\mathrm{R}\Gamma^B_{\mathrm{DF}_{\overline{K}}}(X,r)$ are effective filtered φ-modules. Hence $\mathrm{R}\Gamma^B_{\mathrm{DF}_{\overline{K}}}(X,r)\in\mathscr{D}^g(\mathrm{DF}^+_{\overline{K}})$.

We set

$$\mathrm{R}\Gamma^B_{\mathscr{H}}(X,r)=\mathrm{R}\Gamma^B_{\mathrm{syn}}(X,r):=\mathrm{R}\Gamma_+(\overline{K},\mathrm{R}\Gamma^B_{\mathrm{DF}_{\overline{K}}}(X,r)).$$

The two syntomic complexes described in Sections 5.1.2 and 5.2 are naturally quasi-isomorphic.

Theorem 5.5.

(1) *There is a canonical quasi-isomorphism in* $\mathscr{D}^b(\mathrm{DF}^+_{\overline{K}})$

$$\mathrm{R}\Gamma_{\mathrm{DF}_{\overline{K}}}(X,r) \xrightarrow{\sim} \mathrm{R}\Gamma^B_{\mathrm{DF}_{\overline{K}}}(X,r), \qquad r \in \mathbf{N}.$$

(2) *There is a canonical quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathscr{H}}(X,r) \simeq \mathrm{R}\Gamma^B_{\mathscr{H}}(X,r), \quad r \in \mathbf{N}.$$

Proof. The second statement follows immediately from the first one and Theorem 5.4. To prove the first statement, we follow the proof of Corollary 3.7 from [9] and just briefly describe the argument here. Consider the complex $\mathrm{R}\Gamma^B_{\mathrm{pH}}(X,r)$ in $\mathscr{D}^g_{\mathrm{pH}}$ defined using Beilinson's Basic Lemma starting with $\mathrm{R}\Gamma_{\mathrm{pH}}(X,r)$. We note that, for a good pair (X,Y,j), we have

$$\begin{aligned}&\mathrm{R}\Gamma^g_{\mathrm{pH}}((X,Y,j),r)\\ &\simeq (H^j_{\mathrm{HK}}(X,Y,r)^{\tau}_{\mathbf{B}^+}, (H^j_{\mathrm{dR}}(X,Y)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}}, F^r), \iota_{\mathrm{dR}} : H^j_{\mathrm{HK}}(X,Y)^{\tau}_{\mathbf{B}^+}\otimes_{\mathbf{B}^+}\mathbf{B}^+_{\mathrm{dR}}\\ &\xrightarrow{\sim} H^j_{\mathrm{dR}}(X,Y)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}}).\end{aligned}$$

Hence $\mathrm{R}\Gamma^B_{\mathrm{pH}}(X,r)$ is isomorphic to $\theta\mathrm{R}\Gamma^B_{\mathrm{DF}_{\overline{K}}}(X,r)$. Moreover, we get a functorial quasi-isomorphism in $\mathscr{D}^b(\mathrm{DF}^+_{\overline{K}})$:

$$\kappa_X : \quad \mathrm{R}\Gamma^B_{\mathrm{pH}}(X,r) \simeq \theta\mathrm{R}\Gamma_{\mathrm{DF}_{\overline{K}}}(X,r).$$

It follows that $\mathrm{R}\Gamma^B_{\mathrm{DF}_{\overline{K}}}(X,r) \simeq \mathrm{R}\Gamma_{\mathrm{DF}_{\overline{K}}}(X,r)$, as wanted. ∎

6. Syntomic cohomology and Banach-Colmez spaces

We will show in this section that a syntomic complex (of an algebraic variety) can be realized as a complex of Banach-Colmez spaces. Similarly for the fundamental exact sequence and the syntomic period map: both of them can be canonically lifted to the category of Banach-Colmez spaces. We will also discuss an analog for formal schemes.

6.1. Syntomic complex as a complex of Banach-Colmez spaces.

6.1.1. Algebraic varieties. Let X be a variety over $\overline{K}$.

Theorem 6.1.

(1) *There exists a complex* $\mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r) \in \mathscr{D}^b(\mathscr{BC})$ *such that*

$$\mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r)(C) = \mathrm{R}\Gamma^B_{\mathrm{syn}}(X,r)$$

and the distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,r) \to \mathrm{R}\Gamma_{\mathrm{HK}}(X)^{\tau,\varphi=p^r}_{\mathbf{B}^+} \xrightarrow{\iota_{\mathrm{dR}}} (\mathrm{R}\Gamma_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^r$$

can be lifted canonically to a distinguished triangle of Banach-Colmez spaces.

(2) *The syntomic period map ρ_{syn} can be lifted canonically to a map of Banach-Colmez spaces; i.e., we have a map*

$$\rho_{\mathrm{syn}} : \mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r) \to \mathbb{R}\Gamma^B_{\text{ét}}(X,\mathbf{Q}_p(r)) \tag{6.1}$$

such that the induced map on C-points is the classical syntomic period map. Here the complex $\mathbb{R}\Gamma^B_{\text{ét}}(X,\mathbf{Q}_p(r))$ is a a complex of finite-rank $\mathbf{Q}_p$-vector spaces defined using Beilinson's Basic Lemma.

(3) *We have a canonical spectral sequence*

$$E_2^{i,j} := H^i_+(\overline{K}, H^j_{\mathrm{DF}_{\overline{K}}}(X,r)) \Rightarrow H^{i+j}_{\mathrm{syn}}(X,r)$$

that degenerates at E_2. It can be canonically lifted to the category of Banach-Colmez spaces.

Proof. For the first claim, define

$$\mathrm{R}\Gamma^B_{\mathrm{syn}}(X,r) = [\mathrm{R}\Gamma^B_{\mathrm{HK}}(X)^{\tau,\varphi=p^r}_{\mathbf{B}^+} \xrightarrow{\iota_{\mathrm{dR}}} (\mathrm{R}\Gamma^B_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^r]. \tag{6.2}$$

By Section 2.2.2,

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(X)^{\tau,\varphi=p^r}_{\mathbf{B}^+} = \mathbb{X}^r_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X))(C),$$
$$(\mathrm{R}\Gamma^B_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^r = \mathbb{X}^r_{\mathrm{dR}}(\mathrm{R}\Gamma^B_{\mathrm{dR}}(X))(C),$$

and the map ι_{dR} can be lifted canonically to a map between complexes of Banach-Colmez spaces. We can now define the following complex of Banach-Colmez spaces:

$$\mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r) := [\mathbb{X}^r_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X)) \xrightarrow{\iota_{\mathrm{dR}}} \mathbb{X}^r_{\mathrm{dR}}(\mathrm{R}\Gamma^B_{\mathrm{dR}}(X))]. \tag{6.3}$$

We have $\mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r)(C) = \mathrm{R}\Gamma^B_{\mathrm{syn}}(X,r)$ and hence $H^i(\mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r))(C) = H^i\mathrm{R}\Gamma^B_{\mathrm{syn}}(X,r)$.

To define the lift (6.1), it suffices to inspect the sequence of quasi-isomorphisms (4.2) defining the syntomic period map and to

(1) replace the complexes $\mathrm{R}\Gamma_{\mathrm{HK}}(X)$, $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$, and $\mathrm{R}\Gamma_{\text{ét}}(X,\mathbf{Q}_p)$ by their analogs $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X)$, $\mathrm{R}\Gamma^B_{\mathrm{dR}}(X)$, and $\mathrm{R}\Gamma^B_{\text{ét}}(X,\mathbf{Q}_p)$ defined using Beilinson's Basic Lemma;

(2) note that $(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}})^{N=0,\varphi=p^r}$ and $(\mathrm{R}\Gamma^B_{\mathrm{dR}}(X)\otimes_{\overline{K}}\mathbf{B}_{\mathrm{dR}})/F^r$ are C-points of complexes of Ind-Banach-Colmez spaces, and similarly for $\mathrm{R}\Gamma^B_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}^{N=0,\varphi=p^r}_{\mathrm{st}}$ and $\mathrm{R}\Gamma^B_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}/F^r$;

(3) note that for a pst-pair (D,V), the period isomorphisms

$$(D\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}})^{N=0,\varphi=p^r} \simeq V\otimes_{\mathbf{Q}_p}\mathbf{B}^{N=0,\varphi=p^r}_{\mathrm{st}},\ (D_K\otimes_K\mathbf{B}_{\mathrm{dR}})/F^r \simeq V\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}/F^r$$

can be lifted canonically to the category of Ind-Banach-Colmez spaces.

The spectral sequence in the third claim is constructed from (6.2) and (6.3) and uses the computations of extensions in effective filtered φ-modules from (3.6). □

6.1.2. Syntomic Euler characteristic. Let X be a variety over $\overline{K}$. By Section 2.2.2, the syntomic Euler characteristic of X is equal to

$$\begin{aligned}
&\chi(\mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r))\\
&\quad=\sum_{i\geq 0}(-1)^i\mathrm{Dim}\,\mathbb{X}^r_{\mathrm{st}}(D^i)-\sum_{i\geq 0}(-1)^i\mathrm{Dim}\,\mathbb{X}^r_{\mathrm{dR}}(D^i)\\
&\quad=\sum_{i\geq 0}(-1)^i\sum_{r'_i\leq r}(r-r'_i,1)-\sum_{i\geq 0}(-1)^i(r\dim_{\overline{K}}D^i_{\overline{K}}-\sum_{j=1}^{r}\dim F^jD^i_{\overline{K}},0),
\end{aligned}$$

where $D^i=(D^i,D^i_{\overline{K}})=(H^i_{\mathrm{HK}}(X),H^i_{\mathrm{dR}}(X))$. For r large enough this stabilizes. That is, if $F^{r+1}D^i_{\overline{K}}=0$, $i\geq 0$, and if all r'_i's are $\leq r$, then

$$\begin{aligned}
&\chi(\mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r))\\
&\quad=\sum_{i\geq 0}(-1)^i(r\dim_{F^{\mathrm{nr}}}D^i-t_N(D^i),\dim_{F^{\mathrm{nr}}}D^i)\\
&\qquad-\sum_{i\geq 0}(-1)^i(r\dim_K D^i_K-t_H(D^i_K),0)\\
&\quad=\sum_{i\geq 0}(-1)^i(t_H(D^i_K)-t_N(D^i),\dim_{F^{\mathrm{nr}}}D^i)\\
&\quad=(\sum_{i\geq 0}(-1)^i(t_H(D^i_K)-t_N(D^i)),\chi(\mathrm{R}\Gamma_{\mathrm{dR}}(X))).
\end{aligned}$$

Since the filtered (φ,N,G_K)-module $D^i=(D^i,D^i_{\overline{K}})$ is admissible, we have $t_N(D^i)=t_H(D^i_K)$ and $\chi(\mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r))=(0,\chi(\mathrm{R}\Gamma_{\mathrm{dR}}(X)))$.

6.1.3. Formal schemes. Part of Theorem 6.1 has an analog for formal schemes. Let $\mathscr{X}$ be a quasi-compact (p-adic) formal scheme over $\mathscr{O}_K$ with strict semistable reduction. As shown in [8], for $r\geq 0$, one can associate to $\mathscr{X}$ a functorial syntomic cohomology complex $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\overline{K}},r)$ defined by a formula analogous to (4.1). We see $\mathscr{X}$ as a log-scheme and the Hyodo-Kato and the de Rham cohomologies used are logarithmic.

Proposition 6.2. *Assume that $\mathscr{X}$ is proper.*

(1) *Then there exists a complex $\mathbb{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\overline{K}},r)\in\mathscr{D}^b(\mathscr{BC})$ such that*

$$\mathbb{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\overline{K}},r)(C)=\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\overline{K}},r)$$

and the distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\overline{K}},r)\to\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X})^{\tau,\varphi=p^r}_{\mathbf{B}^+}\xrightarrow{\iota_{\mathrm{dR}}}(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_{\overline{K}})\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^r$$

can be lifted canonically to a distinguished triangle of Banach-Colmez spaces.

(2) *We have a canonical spectral sequence*

$$E_2^{i,j} := H^i_+(\overline{K}, H^j_{\mathrm{DF}_{\overline{K}}}(\mathscr{X}_{\overline{K}}, r)) \Rightarrow H^{i+j}_{\mathrm{syn}}(\mathscr{X}_{\overline{K}}, r)$$

that degenerates at E_2. It can be canonically lifted to the category of Banach-Colmez spaces.

Proof. Consider the following complex in $\mathscr{D}^g_{\mathrm{pH}}$:

$$\begin{aligned}&\mathrm{R}\Gamma^g_{\mathrm{pH}}(\mathscr{X}_{\overline{K}}, r)\\ &\quad := (\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}, r)^{\tau}_{\mathbf{B}^+}, (\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}}, F^r), \iota_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X})^{\tau}_{\mathbf{B}^+} \otimes_{\mathbf{B}^+} \mathbf{B}^+_{\mathrm{dR}}\\ &\quad \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}}),\end{aligned}$$

where the r-twist in $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}, r)^{\tau}_{\mathbf{B}^+}$ refers to Frobenius divided by p^r. The p-adic Hodge complex $\mathrm{R}\Gamma^g_{\mathrm{pH}}(\mathscr{X}_{\overline{K}}, r)$ is geometric: the explanation at the beginning of Section 4.1 goes through once we have the degeneration of the Hodge-de Rham spectral sequence, and this was proved in [20, Theorem 8.4]. Set

$$\mathrm{R}\Gamma_{\mathrm{DF}_{\overline{K}}}(\mathscr{X}_{\overline{K}}, r) := \theta^{-1}\mathrm{R}\Gamma^g_{\mathrm{pH}}(\mathscr{X}_{\overline{K}}, r) \in \mathscr{D}^g(\mathrm{DF}^+_{\overline{K}}).$$

We set $H^i_{\mathrm{DF}_{\overline{K}}}(\mathscr{X}_{\overline{K}}, r) := H^i\mathrm{R}\Gamma_{\mathrm{DF}_{\overline{K}}}(\mathscr{X}_{\overline{K}}, r)$.

By Proposition 5.3, we have

$$\mathrm{Hom}_{\mathscr{D}^g_{\mathrm{pH}}}(\mathbb{1}, \mathrm{R}\Gamma^g_{\mathrm{pH}}(\mathscr{X}_{\overline{K}}, r)) \simeq \mathrm{Hom}_{\mathscr{D}^b(\mathrm{DF}^+_{\overline{K}})}(\mathbb{1}, \mathrm{R}\Gamma_{\mathrm{DF}_{\overline{K}}}(\mathscr{X}_{\overline{K}}, r)).$$

Since

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\overline{K}}, r) \xrightarrow{\sim}\ &[\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X})^{\tau,\varphi=p^r}_{\mathbf{B}^+} \oplus F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})\\ &\qquad \xrightarrow{\iota_{\mathrm{dR}}\otimes\iota-\mathrm{can}} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}}]\\ &\simeq \mathrm{Hom}_{\mathscr{D}^g_{\mathrm{pH}}}(\mathbb{1}, \mathrm{R}\Gamma^g_{\mathrm{pH}}(\mathscr{X}_{\overline{K}}, r)),\end{aligned}$$

we get that

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\overline{K}}, r) \simeq \mathrm{Hom}_{\mathscr{D}^b(\mathrm{DF}^+_{\overline{K}})}(\mathbb{1}, \mathrm{R}\Gamma_{\mathrm{DF}_{\overline{K}}}(\mathscr{X}_{\overline{K}}, r)).$$

The first claim of our proposition follows now from Proposition 3.3 and (3.7). The second just as in the proof of Theorem 6.1. □

6.2. Identity component of syntomic cohomology. We will show that the image of the syntomic cohomology Banach-Colmez space in the étale cohomology is the $\mathbf{Q}_p$-vector space of its connected components.

Proposition 6.3. *For a variety $X \in \mathscr{V}ar_{\overline{K}}$ and $i \geq 0$, we have the natural isomorphisms*

$$\ker(\rho^i_{\mathrm{syn}}) \simeq H^1_+(\overline{K}, H^{i-1}_{\mathrm{DF}_{\overline{K}}}(X,r)), \quad \mathrm{Im}(\rho^i_{\mathrm{syn}}) \simeq H^0_+(\overline{K}, H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r)),$$

where $\rho^i_{\mathrm{syn}} : H^i_{\mathrm{syn}}(X,r) \to H^i_{\text{ét}}(X, \mathbf{Q}_p(r))$ is the syntomic period map.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & \longrightarrow & H^i_{\mathrm{syn}}(X,r) & \longrightarrow & H^i_{\mathrm{cr}}(X)^{\varphi=p^r} & \xrightarrow{\gamma_i} & (H^i_{\mathrm{dR}}(X)\otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r & \longrightarrow & \\
 & & \downarrow{\scriptstyle \rho^i_{\mathrm{syn}}} & & \downarrow{\scriptstyle \rho^i_{\mathrm{cr}}} & & \downarrow{\scriptstyle \rho^i_{\mathrm{dR}}} & & \\
0 & \longrightarrow & H^i_{\text{ét}}(X,\mathbf{Q}_p(r)) & \longrightarrow & H^i_{\text{ét}}(X,\mathbf{Q}_p)\otimes \mathbf{B}_{\mathrm{cr}}^{\varphi=p^r} & \longrightarrow & H^i_{\text{ét}}(X,\mathbf{Q}_p)\otimes_{\overline{K}} \mathbf{B}_{\mathrm{dR}}/F^r & \longrightarrow & 0.
\end{array}$$

It follows that ρ^i_{syn} factors through $H^0_+(\overline{K}, H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r))$ and that

$$H^0_+(\overline{K}, H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r)) \hookrightarrow H^i_{\text{ét}}(X, \mathbf{Q}_p(r)).$$

Hence $H^1_+(\overline{K}, H^{i-1}_{\mathrm{DF}_{\overline{K}}}(X,r)) \simeq \ker(\rho^i_{\mathrm{syn}})$ and $H^0_+(\overline{K}, H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r)) \simeq \mathrm{Im}(\rho^i_{\mathrm{syn}})$, as wanted. $\square$

Corollary 6.4. *For $i \leq r$ or $r \geq d$, there are natural isomorphisms*

$$H^i_{\mathrm{syn}}(X,r) \xrightarrow{\sim} H^0_+(\overline{K}, H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r)) \xrightarrow{\rho^i_{\mathrm{syn}}} H^i_{\text{ét}}(X, \mathbf{Q}_p(r)).$$

Proof. The first isomorphism follows from Lemma 4.6. The second, from the above proposition and the fact that, for $i \leq r$ or $r \geq d$, the pair $(H^i_{\mathrm{HK}}(X), H^i_{\mathrm{dR}}(X))$ comes from an admissible filtered (φ, N, G_L)-module (for a finite extension L/K); hence we can revoke [17, Proposition 2.10]. $\square$

The exact sequence of Banach-Colmez spaces

$$0 \to H^1_+(\overline{K}, H^{i-1}_{\mathrm{DF}_{\overline{K}}}(X,r)) \to H^i\mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r) \to H^0_+(\overline{K}, H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r)) \to 0$$

has the following interpretation. The Banach-Colmez space

$$H^1_+(\overline{K}, H^{i-1}_{\mathrm{DF}_{\overline{K}}}(X,r))$$

is connected (as a quotient of the connected Space $\mathbb{X}^r_{\mathrm{dR}}(H^{i-1}_{\mathrm{dR}}(X))$) and the Space $H^0_+(\overline{K}, H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r))$ is a finite rank $\mathbf{Q}_p$-vector space. Hence we see that $H^1_+(\overline{K}, H^{i-1}_{\mathrm{DF}_{\overline{K}}}(X,r))$ is the identity component of $H^i\mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r)$ and

$$H^0_+(\overline{K}, H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r)) = \pi_0(H^i\mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r)).$$

The projection of Banach-Colmex spaces

$$H^i\mathbb{R}\Gamma^B_{\mathrm{syn}}(X,r) \to H^0_+(\overline{K}, H^{i}_{\mathrm{DF}_{\overline{K}}}(X,r))$$

has a noncanonical section.

7. Computations

In this section we will present several computations of geometric syntomic cohomology groups. They show that these groups yield interesting invariants of algebraic varieties.

7.1. Ordinary varieties. We will show that geometric syntomic cohomology of ordinary varieties is a finite rank $\mathbf{Q}_p$-vector space. For varying twists r, it yields a Galois equivariant filtration of the étale cohomology.

Recall the following terminology from [18]. A p-adic G_K-representation V is called *ordinary* if it admits an equivariant increasing filtration $V_j, j \in \mathbf{Z}$, such that an open subgroup of the inertia group acts on V_j/V_{j-1} by χ^{-j}, χ being the cyclotomic character.

A filtered (φ, N, G_K)-module is called *ordinary* if its Newton and Hodge polygons agree. Such a module admits an increasing filtration by filtered (φ, N, G_K)-submodules, $D_j, j \in \mathbf{Z}$, such that $D_j/D_{j-1} = D_{[-j]}$. Here we set $D_{[-j]} := (D_{[-j]}, D_{[-j],\overline{K}})$, where $D_{[-j]} = D \cap (D \otimes_{F^{\mathrm{nr}}} W(\overline{k}))_{[-j]}$ for $(D \otimes_{F^{\mathrm{nr}}} W(\overline{k}))_{[-j]}$–a submodule of $D \otimes_{F^{\mathrm{nr}}} W(\overline{k})$ generated by x such that $\varphi(x) = p^{-j}x$. It is equipped with the trivial monodromy, and the filtration on $D_{[-j],K} := D^{G_K}_{[-j],\overline{K}}$ is given by $F^{-j}D_K = D_K, F^{-j+1}D_K = 0$. The categories of ordinary Galois representations and ordinary filtered (φ, N, G_K)-modules are equivalent. We have $V_{pst}(D_{[-j]}) \simeq V_{-j}$.

A variety X over K is *ordinary* if so are the pairs $(H^i_{\mathrm{HK}}(X), H^i_{\mathrm{dR}}(X))$, $i \geq 0$. A variety X over $\overline{K}$ is *ordinary* if it comes from an ordinary variety defined over a finite extension of K.

Example 7.1. For $r \geq 0$ and $i \geq 0$, we have

$$H^i_{\mathrm{syn}}(X, r) = H^0_+(\overline{K}, H^i_{\mathrm{DF}_{\overline{K}}}(X, r)) \xrightarrow{\sim} H^i_{\mathrm{\acute{e}t}}(X, \mathbf{Q}_p)_r(r) \hookrightarrow H^i_{\mathrm{\acute{e}t}}(X, \mathbf{Q}_p(r)).$$

In particular, it is a finite rank $\mathbf{Q}_p$-vector space.

Proof. For $k \geq 0$, let $D = F^{\mathrm{nr}}e, \varphi(e) = p^k e, F^k D_K = D_K, F^{k+1}D_K = 0$. Equip it with the trivial action of the monodromy and G_K. The pair $(D, D_{\overline{K}})$ forms an admissible filtered (φ, N, G_K)-module whose associated Galois representation is $\mathbf{Q}_p(-k)$.

If $r < k$, we have

$$(D \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{cr}})^{\varphi = p^r} = (\mathbf{B}^+_{\mathrm{cr}})^{\varphi = p^{r-k}} = 0, \quad (D \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{dR}})/F^r = \mathbf{B}^+_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}} = 0.$$

Hence $H^0_+(\overline{K}, D) = H^1_+(\overline{K}, D) = 0$.

If $r \geq k$, we have

$$\begin{aligned}[] [(D \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{cr}})^{\varphi = p^r} &\to (D \otimes_{F^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{dR}})/F^r] \\ &= [(\mathbf{B}^+_{\mathrm{cr}})^{\varphi = p^{r-k}} \to \mathbf{B}^+_{\mathrm{dR}}/F^{r-k}] \xleftarrow{\sim} \mathbf{Q}_p(r-k). \end{aligned}$$

Hence $H^0_+(\overline{K},D)=\mathbf{Q}_p(r-k)$ and $H^1_+(\overline{K},D)=0$.

By devissage, it follows that, for any ordinary filtered (φ,N,G_K)-module D, we have $H^1_+(\overline{K},D)=0$ and $H^0_+(\overline{K},D)\xrightarrow{\sim} V_{\mathrm{st}}(D_r)(r)=V_r(r)$, as wanted. □

We note that in the étale cohomology group $H^i_{\text{ét}}(X,\mathbf{Q}_p)$ we see the cyclotomic characters $\chi^j, 0\geq j\geq -i$; hence in the group $H^i_{\text{ét}}(X,\mathbf{Q}_p(r))$ we see the characters $\chi^j, r\geq j\geq -i+r$. The above example shows that syntomic cohomology $H^i_{\mathrm{syn}}(X,r)$, $r\geq 0$, picks up the twists $\mathbf{Q}_p(j)$, for $r\geq j\geq 0$, in the étale cohomology $H^i_{\text{ét}}(X,\mathbf{Q}_p(r))$. If r is large, $r\geq i$, we pick up the whole étale cohomology group.

7.2. Curves. Let X be a proper smooth curve over $\overline{K}$. We will show that the geometric syntomic cohomology of X recovers the "étale p-rank of the special fiber of the Néron model of the Jacobian of the curve".

By Lemma 4.6,

$$H^i_{\mathrm{syn}}(X,r)=H^0_+(\overline{K},H^i_{\mathrm{DF}_{\overline{K}}}(X,r)),\quad r\geq 1, i\leq r+1,$$

is a finite rank vector space over $\mathbf{Q}_p$ that is a subspace of $H^i_{\text{ét}}(X,\mathbf{Q}_p(r))$. Moreover,

$$\rho^i_{\mathrm{syn}}: H^i_{\mathrm{syn}}(X,r)\xrightarrow{\sim} H^i_{\text{ét}}(X,\mathbf{Q}_p(r)),\quad r\geq 1, i\leq r.$$

For $r=0$, we have

$$H^0_{\mathrm{syn}}(X,0)\xrightarrow{\sim} H^0_{\text{ét}}(X,\mathbf{Q}_p(0))\simeq \mathbf{Q}_p.$$

Since $F^0(H^i_{\mathrm{dR}}(X)\otimes \mathbf{B}^+_{\mathrm{dR}})=H^i_{\mathrm{dR}}(X)\otimes \mathbf{B}^+_{\mathrm{dR}}$, we have

$$H^i_{\mathrm{syn}}(X,0)\simeq (H^i_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{cr}})^{\varphi=1}\hookrightarrow H^i_{\text{ét}}(X,\mathbf{Q}_p(0)).$$

Since $H^2_{\mathrm{HK}}(X)\simeq F^{\mathrm{nr}}$ with Frobenius $p\varphi$, we have $H^2_{\mathrm{syn}}(X,0)=0$ since

$$H^2_{\mathrm{syn}}(X,0)\hookrightarrow (\mathbf{B}^+_{\mathrm{cr}})^{\varphi=p^{-1}}=0.$$

It remains to understand the inclusion $H^1_{\mathrm{syn}}(X,0)\hookrightarrow H^1_{\text{ét}}(X,\mathbf{Q}_p)$. Since the Frobenius slopes on $H^1_{\mathrm{HK}}(X)$ are ≥ 0, we have

$$\begin{aligned}H^1_{\mathrm{syn}}(X,0)\simeq (H^1_{\mathrm{HK}}(X)\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{cr}})^{\varphi=1}&=(H^1_{\mathrm{HK}}(X)_{[0]}\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{cr}})^{\varphi=1}\\&=(H^1_{\mathrm{HK}}(X)_{[0]}\otimes_{F^{\mathrm{nr}}}W(\overline{k}))^{\varphi=1}.\end{aligned}$$

This is a $\mathbf{Q}_p$-vector space, of the same rank as the F^{nr}-rank of $H^1_{\mathrm{HK}}(X)_{[0]}$, that injects via the period map into $H^1_{\text{ét}}(X,\mathbf{Q}_p)$. Since $H^1_{\text{ét}}(X,\mathbf{Q}_p)\simeq V_p(\mathrm{Jac}(X))^*$ (the $\mathbf{Q}_p$-dual), this rank is the "étale p-rank of the special fiber of the Néron model of the Jacobian of the curve".

7.3. Product of elliptic curves. Varieties that are not ordinary give rise to geometric syntomic cohomology groups that have nontrivial C-dimension. We will start with some preliminary computations.

7.3.1. The group $H^0_+(\cdot)$. Let D be an admissible filtered (φ, N, G_K)-module with $F^0D_K = D_K$ (this implies that the slopes of Frobenius are ≥ 0). Then the group $H^0_+((D\otimes_{F^{\rm nr}}\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^r}, F^r(D_K\otimes_K \mathbf{B}^+_{\rm dR}))$ is the largest subrepresentation V_r of $V_{\rm st}(D)(r)$ with Hodge weights in $[0,r]$. It corresponds to the largest (weakly) admissible submodule D_r of D with $F^{r+1}D_{r,K}=0$ (note that this is equivalent to all Hodge weights being less than or equal to r and it implies that all Frobenius slopes have the same property).

7.3.2. Trace map. Let X be a variety over $\overline{K}$ of dimension d. We claim that we have a canonical isomorphism $H^{2d}_{\rm syn}(X,d)\simeq \mathbf{Q}_p$. Indeed, by Lemma 4.6, we have $H^{2d}_{\rm syn}(X,d) = H^0_+(\overline{K}, H^{2d}_{{\rm DF}_{\overline{K}}}(X,d))$ and

$$\begin{aligned} &H^0_+(\overline{K}, H^{2d}_{{\rm DF}_{\overline{K}}}(X,d))\\ &\qquad = \operatorname{Ker}((H^{2d}_{\rm HK}(X)\otimes_{F^{\rm nr}}\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^d}\to (H^{2d}_{\rm dR}(X)\otimes_{\overline{K}}\mathbf{B}^+_{\rm dR})/F^d)\\ &\qquad = (H^{2d}_{\rm HK}(X)\otimes_{F^{\rm nr}}\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^d}. \end{aligned}$$

The last equality holds because $F^dH^{2d}_{\rm dR}(X) = H^{2d}_{\rm dR}(X)$. Since $H^{2d}_{\rm HK}(X)\simeq F^{\rm nr}$ with Frobenius $p^d\varphi$, we have $(H^{2d}_{\rm HK}(X)\otimes_{F^{\rm nr}}\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^d} = \mathbf{B}^{+,\varphi=1}_{\rm cr} = \mathbf{Q}_p$, as wanted.

7.3.3. Product of elliptic curves. We are now ready to do computations for products of certain elliptic curves.

Example 7.2. Let E be an elliptic curve over K with a supersingular reduction. Let $X = E\times E$.

(1) We have the following exact sequence of Galois representations:

$$0\to H^2_{\rm syn}(X_{\overline{K}},1)\to H^2_{\rm \acute{e}t}(X_{\overline{K}},\mathbf{Q}_p(1))\to C(-1)\to H^3_{\rm syn}(X_{\overline{K}},1)\to 0.$$

(2) If $\operatorname{Sym}^2 H^1_{\rm \acute{e}t}(E_{\overline{K}})$ is an irreducible Galois representation, then

$$\begin{aligned} H^2_{\rm \acute{e}t}(X_{\overline{K}},\mathbf{Q}_p(1))/H^2_{\rm syn}(X_{\overline{K}},1) &\simeq \operatorname{Sym}^2 H^1_{\rm \acute{e}t}(E_{\overline{K}},\mathbf{Q}_p)(1),\\ H^3_{\rm syn}(X_{\overline{K}},1) &\simeq C(-1)/\operatorname{Sym}^2 H^1_{\rm \acute{e}t}(E_{\overline{K}},\mathbf{Q}_p)(1). \end{aligned}$$

Proof. We note that we have the following exact sequence of Galois representations:

(7.1)

$$\begin{aligned} 0\to H^2_{\rm syn}(X_{\overline{K}},1)\to (H^2_{\rm HK}(X)\otimes_F\mathbf{B}^+_{\rm cr})^{\varphi=p} &\xrightarrow{\iota_{\rm dR}} (H^2_{\rm dR}(X)\otimes_K\mathbf{B}^+_{\rm dR})/F^1\\ &\to H^3_{\rm syn}(X_{\overline{K}},1)\to 0. \end{aligned}$$

It is obtained from the fundamental long exact sequence. The exactness on the left follows from Lemma 4.6. For the exactness on the right, we note that using the Künneth formula in Hyodo-Kato cohomology and Section 7.3.2, we get that

$$(H^3_{\rm HK}(X)\otimes_F\mathbf{B}^+_{\rm cr})^{\varphi=p} = (H^1_{\rm HK}(E)\otimes_F\mathbf{B}^+_{\rm cr})^{\varphi=1}\oplus (H^1_{\rm HK}(E)\otimes_F\mathbf{B}^+_{\rm cr})^{\varphi=1}.$$

Using the fact that the slope of the Frobenius on $H^1_{\mathrm{HK}}(E)$ is $1/2$ we obtain

$$(H^3_{\mathrm{HK}}(X) \otimes_F \mathbf{B}^+_{\mathrm{cr}})^{\varphi=p} = (\mathbf{B}^{+,\varphi=p^{-1/2}}_{\mathrm{cr}})^4 = 0.$$

Consider now the following commutative diagram induced by the period maps:

$$\begin{array}{ccccccc}
H^2_{\mathrm{syn}}(X_{\overline{K}},1) & \hookrightarrow & (H^2_{\mathrm{HK}}(X)\otimes_F \mathbf{B}^+_{\mathrm{cr}})^{\varphi=p} & \to & (H^2_{\mathrm{dR}}(X)\otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^1 & \twoheadrightarrow & H^3_{\mathrm{syn}}(X_{\overline{K}},1) \\
\downarrow{\scriptstyle \rho_{\mathrm{syn}}} & & \downarrow{\scriptstyle \rho_{\mathrm{HK}}} & & \downarrow{\scriptstyle \rho_{\mathrm{dR}}} & & \\
H^2_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbf{Q}_p(1)) & \hookrightarrow & (H^2_{\mathrm{HK}}(X)\otimes_F \mathbf{B}_{\mathrm{cr}})^{\varphi=p} & \twoheadrightarrow & (H^2_{\mathrm{dR}}(X)\otimes_K \mathbf{B}_{\mathrm{dR}})/F^1 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
C(-1) & \hookrightarrow & \operatorname{coker}\rho_{\mathrm{HK}} & \stackrel{f}{\twoheadrightarrow} & \operatorname{coker}\rho_{\mathrm{dR}} & &
\end{array}$$

The rows and columns are exact. The fact that $\ker(f) \simeq F^2H^2_{\mathrm{dR}}(X) \otimes_K C(-1) \simeq C(-1)$ follows from the fundamental exact sequence

$$0 \to \mathbf{Q}_p t \to \mathbf{B}^{+,\varphi=p}_{\mathrm{cr}} \to \mathbf{B}^+_{\mathrm{dR}}/F^1 \to 0.$$

The snake lemma yields the following exact sequence:

$$0 \to H^2_{\mathrm{syn}}(X_{\overline{K}}, 1) \to H^2_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbf{Q}_p(1)) \to C(-1) \to H^3_{\mathrm{syn}}(X_{\overline{K}}, 1) \to 0,$$

as claimed in the first part of our example.

We argue now for the second part. Since de Rham cohomology satisfies the Künneth formula, the computations in Section 7.3.2 yield that $(H^2_{\mathrm{dR}}(X) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^1 \simeq C$. In a similar way, using the Künneth formula in Hyodo-Kato cohomology and Section 7.3.2, we get that

$$(H^2_{\mathrm{HK}}(X) \otimes_F \mathbf{B}^+_{\mathrm{cr}})^{\varphi=p} = (H^1_{\mathrm{HK}}(E)^{\otimes 2} \otimes_F \mathbf{B}^+_{\mathrm{cr}})^{\varphi=p} \oplus \mathbf{Q}^2_p$$

and the vector space $\mathbf{Q}^2_p$ is in the kernel of ι_{dR} (since $F^1H^2_{\mathrm{dR}}(E) = H^2_{\mathrm{dR}}(E)$). Hence we have the exact sequence

$$0 \to H^2_{\mathrm{syn}}(X_{\overline{K}}, 1) \to (H^1_{\mathrm{HK}}(E)^{\otimes 2} \otimes_F \mathbf{B}^+_{\mathrm{cr}})^{\varphi=p} \oplus \mathbf{Q}^2_p \to C \to H^3_{\mathrm{syn}}(X_{\overline{K}}, 1) \to 0.$$

Using the fact that the slope of the Frobenius on $H^1_{\mathrm{HK}}(E)$ is $1/2$ and $\mathbf{B}^{+,\varphi=1}_{\mathrm{cr}} = \mathbf{Q}_p$, we compute that $(H^1_{\mathrm{HK}}(X)^{\otimes 2} \otimes_F \mathbf{B}^+_{\mathrm{cr}})^{\varphi=p} \simeq \mathbf{Q}^4_p$. Since $H^1_{\mathrm{HK}}(E)^{\otimes 2} = \operatorname{Sym}^2 H^1_{\mathrm{HK}}(E) \oplus F$, we have

$$(H^1_{\mathrm{HK}}(E)^{\otimes 2} \otimes_F \mathbf{B}^+_{\mathrm{cr}})^{\varphi=p} = (\operatorname{Sym}^2 H^1_{\mathrm{HK}}(E) \otimes_F \mathbf{B}^+_{\mathrm{cr}})^{\varphi=p} \oplus \mathbf{Q}_p,$$

and, by Section 7.3.1, $\mathbf{Q}_p$ is in the kernel of ι_{dR}. Hence we have the exact sequence

$$0 \to H^2_{\mathrm{syn}}(X_{\overline{K}}, 1) \to (\operatorname{Sym}^2 H^1_{\mathrm{HK}}(E) \otimes_F \mathbf{B}^+_{\mathrm{cr}})^{\varphi=p} \oplus \mathbf{Q}^3_p \to C \to H^3_{\mathrm{syn}}(X_{\overline{K}}, 1) \to 0.$$

It follows that if $\mathrm{Sym}^2 H^1_{\acute{e}t}(E_{\overline{K}},\mathbf{Q}_p)$ is an irreducible Galois representation, then $H^2_{\mathrm{syn}}(X_{\overline{K}},1)\simeq \mathbf{Q}_p^3$ and

$$H^2_{\acute{e}t}(X_{\overline{K}},\mathbf{Q}_p(1))/H^2_{\mathrm{syn}}(X_{\overline{K}},1)\simeq \mathrm{Sym}^2 H^1_{\acute{e}t}(E_{\overline{K}},\mathbf{Q}_p)(1),$$

as wanted. □

Remark 7.3.

(1) The above proof shows that $H^3_{\mathrm{syn}}(X_{\overline{K}},1)$ as a Galois representation is a quotient of $C(-1)$ and of C by a finite rank $\mathbf{Q}_p$-Galois representation. This type of curious phenomena was studied by Fontaine in [13].

(2) If X is a variety that is not ordinary, similar computations to the ones we have done above show that the syntomic cohomology of the products X^n acquires nontrivial C-dimension as n becomes large.

7.4. Galois descent. Let X be a variety over K. Then the syntomic complex $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K}},r)$ is equipped with a G_K-action. Set[14] $\mathrm{R}\Gamma^{\natural}_{\mathrm{syn}}(X,r):=\mathrm{R}\Gamma(G_K,\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K}},r))$. Since

$$\mathrm{R}\Gamma_{\acute{e}t}(X,\mathbf{Q}_p(r))=\mathrm{R}\Gamma(G_K,\mathrm{R}\Gamma_{\acute{e}t}(X_{\overline{K}},\mathbf{Q}_p(r))),$$

we have the induced syntomic period map

$$\rho^{\natural}_{\mathrm{syn}}:\mathrm{R}\Gamma^{\natural}_{\mathrm{syn}}(X,r)\to \mathrm{R}\Gamma_{\acute{e}t}(X,\mathbf{Q}_p(r)).$$

Proposition 7.4. *There is a canonical distinguished triangle*

(7.2)
$$\mathrm{R}\Gamma^{\natural}_{\mathrm{syn}}(X,r)\xrightarrow{\rho^{\natural}_{\mathrm{syn}}}\mathrm{R}\Gamma_{\acute{e}t}(X,\mathbf{Q}_p(r))\to \mathrm{R}\Gamma(G_K,(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}})\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{cr}}/\mathbf{B}^+_{\mathrm{cr}})^{\varphi=p^r}).$$

Proof. Consider the complex

$$C^+(X_{\overline{K}}):=[\,(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}})\otimes_{F^{\mathrm{nr}}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{dR}}\otimes\iota}(\mathrm{R}\Gamma^B_{\mathrm{dR}}(X_{\overline{K}})\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^r]$$

representing $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K}},r)$. Define $C(X_{\overline{K}})$ by omitting the superscript $+$ in the above definition. We have

$$C(X_{\overline{K}})/C^+(X_{\overline{K}})=[A(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}}))\xrightarrow{\iota_{\mathrm{dR}}\otimes\iota}B(\mathrm{R}\Gamma^B_{\mathrm{dR}}(X_{\overline{K}}))],$$

where

$$A(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}})):=(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}})\otimes_{F^{\mathrm{nr}}}\mathbf{B}_{\mathrm{st}}/\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r},$$
$$B(\mathrm{R}\Gamma^B_{\mathrm{dR}}(X_{\overline{K}})):=(\mathrm{R}\Gamma^B_{\mathrm{dR}}(X_{\overline{K}})\otimes_{\overline{K}}\mathbf{B}_{\mathrm{dR}})/F^r/(\mathrm{R}\Gamma^B_{\mathrm{dR}}(X_{\overline{K}})\otimes_{\overline{K}}\mathbf{B}^+_{\mathrm{dR}})/F^r.$$

[14]See [17, 4.2] for the necessary formalities concerning continuous Galois cohomology and Hochschild-Serre spectral sequence.

Hence

$$\begin{aligned}&\mathrm{R}\Gamma(G_K, C(X_{\overline{K}})/C^+(X_{\overline{K}}))\\&\quad= [\mathrm{R}\Gamma(G_K, A(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K}}))) \xrightarrow{\iota_{\mathrm{dR}}\otimes\iota} \mathrm{R}\Gamma(G_K, B(\mathrm{R}\Gamma^B_{\mathrm{dR}}(X_{\overline{K}})))],\end{aligned}$$

and it suffices to show that the complex $\mathrm{R}\Gamma(G_K, B(\mathrm{R}\Gamma^B_{\mathrm{dR}}(X_{\overline{K}})))$ is acyclic. Since we have a quasi-isomorphism $\mathrm{R}\Gamma^B_{\mathrm{dR}}(X)\otimes_K \overline{K} \xrightarrow{\sim} \mathrm{R}\Gamma^B_{\mathrm{dR}}(X_{\overline{K}})$, it suffices to show that, for a finite rank K-vector space D_K with a descending filtration such that $F^0D_K = D_K$, the complex $\mathrm{R}\Gamma(G_K, B(D_K))$ is acyclic. But the complex $B(D_K)$ has a natural filtration with graded pieces equal to copies of $C(j)$'s for strictly negative j's. Since $H^*(G_K, C(j)) = 0, j < 0$, the acyclicity of the complex $\mathrm{R}\Gamma(G_K, B(D_K))$ follows. □

Remark 7.5. It is not clear to us what the relation is between the distinguished triangle (7.2) and the Bloch-Kato (dual) exponential exact sequence.

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