p-adic motivic cohomology in arithmetic geometry

Wiesława Nizioł

Motivic cohomology

• **over fields:**

$\mathcal{DM}_k$ - the derived category of mixed motives over a field k; constructions by Levine, Voevodsky (1990's)

$$\begin{aligned} H^i_{\mathcal{M}}(X, \mathbf{Z}(n)) &= \mathrm{Ext}^i_{\mathcal{MM}_k}(\mathbf{Z}(0), h(X) \otimes \mathbf{Z}(n)) \\ &= \mathrm{Hom}_{\mathcal{DM}_k}(\mathbf{Z}(0), \mathbf{Z}_X(n)[i]) \\ &\simeq H^i(X, \mathbf{Z}(n)) \end{aligned}$$

$H^i(X, \mathbf{Z}(n))$ - **Bloch higher Chow groups**

Rationally:

$$H^i_{\mathcal{M}}(X, \mathbf{Z}(n)) \otimes \mathbf{Q} \simeq \mathrm{gr}^n_\gamma K_{2n-i}(X) \otimes \mathbf{Q}$$

- γ-graded pieces of K-theory

• **over Dedekind domains:**

fundations are still missing; use Bloch higher Chow groups

Motivic vs arithmetic cohomologies

• **Zariski vs étale motivic cohomology**

Motivic cohomology can be defined both in the Zariski and the étale topology. The change of topology map:

$$\rho_{i,n} : H^i(X, \mathbf{Z}/m(n)) \to H^i(X_{\text{ét}}, \mathbf{Z}/m(n))$$

Beilinson, Lichtenbaum, Quillen, Thomason:

$\rho_{i,n}$ is an isomorphism for large n

• **Étale motivic vs arithmetic cohomologies**

Realizations:

$$\begin{array}{ccccc} \text{étale} & \xleftarrow{\sim} & H^i(X_{\text{ét}}, \mathbf{Z}/m(n)) & \xrightarrow{\sim} & \text{syntomic} \\ & & \downarrow \wr & & \\ & & \text{log de Rham-Witt} & & \end{array}$$

Milnor K-theory

k - a field, char(k) $= p$; the Milnor K groups:

$$K_*^M(k) = T_{\mathbf{Z}}^*(k^\times)/(x \otimes (1-x) | x \in k - \{0,1\})$$

Milnor K_*^M/m = Galois cohomology:

- $(m,p) = 1$; Kummer theory:

$$K_1^M(k)/m \simeq k^*/k^{*m} \xrightarrow{\sim} H^1(k_{\text{ét}}, \mu_m).$$

Cup product gives the Galois symbol map

$$K_n^M(k)/m \to H^n(k_{\text{ét}}, \mu_m^{\otimes n}).$$

Conjecture (Bloch-Kato) *The Galois symbol map is an isomorphism.*

Voevodsky (2001, 2002): true for $m = 2^n$; conditionally for general m

Milnor K_*^M/m = Galois cohomology:

- $m = p^r > 0$; **Bloch-Gabber-Kato:**

$$\mathrm{dlog} : K_n^M(k)/p^r \xrightarrow{\simeq} H^0(k_{\text{ét}}, \nu_r^n).$$

The (étale) logarithmic de Rham-Witt sheaf:

$$\nu_r^n = \langle x \in W_r\Omega_X^n | x = \mathrm{dlog}\, \overline{x}_1 \wedge \ldots \wedge \mathrm{dlog}\, \overline{x}_n \rangle$$

$$0 \to \nu_\cdot^n \to W_\cdot\Omega_X^n \xrightarrow{F-1} W_\cdot\Omega_X^n \to 0$$

Motivic cohomology

X - a separated scheme over a field;

Bloch higher Chow groups:

$$H^i(X, \mathbf{Z}(n)) = H^i(\mathbf{Z}(n)(X))$$
$$H^i(X, \mathbf{Z}/m(n)) = H^i(\mathbf{Z}(n)(X) \otimes \mathbf{Z}/m)$$

$\mathbf{Z}(n)(X) := z^n(X, 2n - *)$ is the complex:

- algebraic r-simplex:

$$\triangle^r \simeq \mathbf{A}^r_{\mathbf{Z}} = \mathrm{Spec}\, \mathbf{Z}[t_0, \ldots, t_r]/(\sum t_i - 1)$$

- $z^n(X, i) \subset z^n(X \times \triangle^i)$ - the free abelian group generated by irreducible codimension n subvarieties of $X \times \triangle^i$ meeting all faces properly.
- the chain complex

$$z^n(X, *) : z^n(X, 0) \leftarrow z^n(X, 1) \leftarrow z^n(X, 2) \leftarrow$$

boundaries: restrictions of cycles to faces

- $H^i(X, \mathbf{Z}(n)) = 0$ for $i > \min\{2n, n + \dim X\}$

- **Beilinson-Soulé conjecture** (still open):

$$H^i(X, \mathbf{Z}(n)) = 0, \quad i < 0$$

- $H^{2n}(X, \mathbf{Z}(n)) \simeq CH^n(X)$, Chow group

- for a field k, $H^n(k, \mathbf{Z}(n)) \simeq K_n^M(k)$

- localization (difficult): for $Z \overset{c}{\hookrightarrow} X \hookleftarrow U$

$$\begin{aligned} \to & H^{i-2c}(Z, \mathbf{Z}(n-c)) \to H^i(X, \mathbf{Z}(n)) \to \\ & H^i(U, \mathbf{Z}(n)) \to H^{i+1-2c}(Z, \mathbf{Z}(n-c)) \to \end{aligned}$$

- higher cycle classes (difficult):

$$c_{i,n} : H^i(X, \mathbf{Z}(n)) \to H^i(X, n)$$

needed: weak purity and homotopy property

Question *How to define cycle classes into syntomic cohomology ?*

Étale motivic cohomology

$X \mapsto \mathbf{Z}(n)(X) := z^n(X, 2n - *)$: sheaf in the étale topology; X separated, noetherian :

$$H^i(X, \mathbf{Z}(n)) \simeq H^i(X_{\text{Zar}}, \mathbf{Z}(n))$$

• X/k - smooth, k-perfect, $\text{char}(k) = p > 0$,
Geisser-Levine (2000):

$$H^{i+n}(X_{\text{Zar}}, \mathbf{Z}/p^r(n)) \simeq H^i(X_{\text{Zar}}, \nu_r^n)$$
$$H^{i+n}(X_{\text{ét}}, \mathbf{Z}/p^r(n)) \simeq H^i(X_{\text{ét}}, \nu_r^n)$$

• X smooth, m invertible on X; cycle class

$$c_{i,n}^{\text{ét}} : H^i(X_{\text{ét}}, \mathbf{Z}/m(n)) \xrightarrow{\sim} H^i(X_{\text{ét}}, \mu_m^{\otimes n})$$

Change of topology map:

$$\rho_{i,n} : H^i(X_{\text{Zar}}, \mathbf{Z}/m(n)) \to H^i(X_{\text{ét}}, \mathbf{Z}/m(n))$$

Example

$$\rho_{2n,n} = cl^{\text{ét}} : CH^n(X)/m \to H^{2n}(X_{\text{ét}}, \mu_m^{\otimes n})$$

– neither injective nor surjective

Conjecture (Beilinson-Lichtenbaum)

$$\rho_{i,n} : H^i(X, \mathbf{Z}/m(n)) \xrightarrow{\simeq} H^i(X_{\text{ét}}, \mathbf{Z}/m(n)),\ i \leq n$$

- **Suslin (2000):** true for $k = \overline{k}$, $n \geq \dim X$
- **Suslin-Voevodsky, Geisser-Levine (2000):**

Bloch-Kato $\Leftrightarrow$ Beilinson-Lichtenbaum

- use Gersten resolution to pass from fields to schemes:

$$0 \to \mathcal{H}^p(\mathbf{Z}(n)) \to \bigoplus_{x \in X^{(0)}} (i_x)_* H^p(k_x, \mathbf{Z}(n)) \to \bigoplus_{x \in X^{(1)}} (i_x)_* H^{p-1}(k_x, \mathbf{Z}(n-1)) \to$$

$X^{(s)}$ - points in X of codimension s

• V - complete dvr, mixed characteristic $(0, p)$, perfect residue field; X/V - smooth scheme

Geisser (2004): Bloch-Kato mod $p \Rightarrow$

(1) Beilinson-Lichtenbaum mod p is true:

$$\rho_{i,n} : H^i(X, \mathbf{Z}/p^r(n)) \xrightarrow{\sim} H^i(X_{\text{ét}}, \mathbf{Z}/p^r(n)),\ i \leq n$$

(2) for X proper and $i \leq r < p-1$:

$$c^{\text{syn}}_{i,r} : H^i(X_{\text{ét}}, \mathbf{Z}/p^n(r)) \xrightarrow{\sim} H^i(X, S_n(r))$$

– uses p-adic Hodge theory !

$S_n(r)$ - the syntomic complex:

$$\to H^i(X_n, S_n(r)) \to F^r H^i_{dR}(X_n) \xrightarrow{1-\phi/p^r} H^i_{dR}(X_n)$$

Algebraic K-theory

X- noetherian scheme; $\mathcal{M}_X$ (resp. $\mathcal{P}_X$) – the category of coherent (resp. locally free sheaves) on X. $\mathcal{M}_X \to \mathcal{K}'_X$, $\mathcal{P}_X \to \mathcal{K}_X$: certain associated simplicial spaces.

The algebraic K and K' groups of X:

$$K_i(X) = \pi_i(\mathcal{K}_X), \quad K_i(X, \mathbf{Z}/m) = \pi_i(\mathcal{K}_X, \mathbf{Z}/m),$$
$$K'_i(X) = \pi_i(\mathcal{K}'_X), \quad K'_i(X, \mathbf{Z}/m) = \pi_i(\mathcal{K}'_X, \mathbf{Z}/m)$$

- $K_0(X)$ and $K'_0(X)$ are the Grothendieck groups of vector bundles and coherent sheaves

- k - field, $K_1^M(k) = K_1(k) = k^*$; product:

  $$K_n^M(k) \to K_n(k)$$

  that is an isomorphism for $n \leq 2$

- γ filtration (exterior powers of vector bundles): $F_\gamma^* K_*(X)$

  $$\mathrm{gr}_\gamma^j K_{2j-i}(X) \otimes \mathbf{Q} \simeq H^i(X, \mathbf{Q}(j))$$

  In fact, modulo small torsion

- Poincaré duality: if X is regular then

  $$K_i(X) \xrightarrow{\sim} K'_i(X)$$

- localization (easy): for $Z \hookrightarrow X \hookleftarrow U$,

  $$\to K'_{i+1}(U) \to K'_i(Z) \to K'_i(X) \to K'_i(U) \to$$

- higher Chern classes (easy):

  $$c_{i,n} : \mathrm{gr}_\gamma^n K_{2n-i}(X) \to H^i(X, n)$$

  Needed: the cohomology of $\mathbf{BGL}$ is the right one (which holds for most p-adic cohomologies)

Étale K-theory

$X \mapsto \mathcal{K}(X)$, $X \mapsto \mathcal{K}/m(X)$: presheaves of simplicial spaces. Assume X is regular.

$$K_*(X, \mathbf{Z}/m) \simeq H^{-*}(X_{\text{Zar}}, \mathcal{K}/m).$$

If m is invertible on X then the étale K-theory of Dwyer-Friedlander

$$K_j^{\text{ét}}(X, \mathbf{Z}/m) \simeq H^{-j}(X_{\text{ét}}, \mathcal{K}/m).$$

Conjecture (Quillen-Lichtenbaum)

$$\rho_j : K_j(X, \mathbf{Z}/m) \xrightarrow{\simeq} K_j^{\text{ét}}(X, \mathbf{Z}/m), j \geq \text{cd}_m X_{\text{ét}}$$

Atiyah-Hirzebruch spectral sequence

motivic cohomology $\Rightarrow$ K-theory

X/k - smooth; Bloch-Lichtenbaum, Friedlander-Suslin, Levine; Grayson-Suslin (1995-2003):

$$E_2^{s,t} = H^{s-t}(X, \mathbf{Z}/m(-t)) \Rightarrow K_{-s-t}(X, \mathbf{Z}/m)$$

• m invertible on X; **Levine (1999)**:

$$E_2^{s,t} = H^{s-t}(X_{\text{ét}}, \mathbf{Z}/m(-t)) \Rightarrow K^{\text{ét}}_{-s-t}(X, \mathbf{Z}/m)$$

Degenerates at E_2 modulo small torsion $\Rightarrow$

$$c^{\text{ét}}_{i,j} : \quad \mathrm{gr}^j_\gamma K^{\text{ét}}_{2j-i}(X, \mathbf{Z}/m) \xrightarrow{\sim} H^i(X_{\text{ét}}, \mu_m^{\otimes j})$$

Beilinson-Lichtenbaum $\Rightarrow$ Quillen-Lichtenbaum

• k - perfect, $\mathrm{char}(k) = p > 0$, **Geisser-Levine:**

$$K_n(X, \mathbf{Z}/p^r) = 0 \quad \text{for } n > \dim X$$

Application: p-adic Hodge theory (Niziol)

K - complete dvf, char = $(0, p)$, V - ring of integers, k - perfect residue field; $\overline{K}$ - an algebraic closure of K, G_K - its Galois group
X/V - proper variety, X_K - smooth

p-adic Hodge theory:

étale: $H^i(X_{\overline{K}}, \mathbf{Q}_p)$ + G_K-action

p-adic period map: $\Updownarrow$

de Rham : $H^i_{dR}(X_K/K)$ + F^*, ϕ, N

Methods:

- syntomic; Fontaine-Messing, Kato, Tsuji (1987-1998): period map = the cospecialization map on the syntomic-étale site
- almost-étale; Faltings (1989-2002)
- motivic; Niziol (1998-2006): period map = the localization map in motivic cohomology

The good reduction case

$V = W$ - Witt vectors of k, X/V - smooth, proper, relative dimension d

Crystalline conjecture: For $i \leq r < p-1$, there exists a G_K-equivariant period isomorphism

$$H^i(X_{\overline{K}}, \mu_n^{\otimes r}) \xrightarrow{\sim} F^r(H^i_{dR}(X_n/V_n) \otimes B^+_{\mathrm{cr},n})^{p^r=\phi}$$
$$\simeq H^i(X_{\overline{V}}, S_n(r))$$

- crystalline ring of periods:

$$B^+_{\mathrm{cr},n} = H^*_{\mathrm{cr}}(\mathrm{Spec}(\overline{V}_n)/W_n);\ F^*, \phi, G_K$$

- de Rham cohomology:

$$H^i_{dR}(X_n/V_n) \simeq H^*_{\mathrm{cr}}(X_0/V_n),\ F^*, \phi$$

Bloch-Kato ⇒ Crystalline conjecture

Assume that we have syntomic cycle maps

$$\begin{array}{ccc}
H^i(X_{\overline{V}}, \mathbf{Z}/p^n(r)) & \xrightarrow[\sim]{j^*} & H^i(X_{\overline{K}}, \mathbf{Z}/p^n(r)) \\
\downarrow \rho_{i,r} & & \downarrow \rho^K_{i,r} \\
H^i(X_{\overline{V},\text{ét}}, \mathbf{Z}/p^n(r)) & \xrightarrow{j^*} & H^i(X_{\overline{K},\text{ét}}, \mathbf{Z}/p^n(r)) \\
\downarrow c^{\text{syn}}_{i,r} & & \wr\downarrow c^{\text{ét}}_{i,r} \\
H^i(X_{\overline{V}}, S_n(r)) & \xleftarrow{\alpha^{\text{cr}}_{i,r}} & H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes r})
\end{array}$$

- **the key point:** j^* is an isomorphism: by localization, the kernel and cokernel are controlled by $H^i(X_{\overline{k}}, \mathbf{Z}/p^n(r))$, which is killed by totally ramified extensions of V of degree p^n
- $\rho^K_{i,r}$ is an isomorphism for $r \geq i$: by Beilinson-Lichtenbaum
- define the map $\alpha^{\text{cr}}_{i,r}$ to make this diagram commute.
- Notice: all the maps in the above diagram are isomorphisms.

Suslin $\Rightarrow$ Crystalline conjecture

$$\begin{array}{ccc}
\mathrm{gr}^r_\gamma K_{2r-i}(X_{\overline{V}}, \mathbf{Z}/p^n) & \xrightarrow[j^*]{\sim} & \mathrm{gr}^r_\gamma K_{2r-i}(X_{\overline{K}}, \mathbf{Z}/p^n) \\
\downarrow \rho_{2r-i} & & \downarrow \rho^K_{2r-i} \\
\mathrm{gr}^r_\gamma K^{\text{ét}}_{2r-i}(X_{\overline{V}}, \mathbf{Z}/p^n) & \xrightarrow[j^*]{} & \mathrm{gr}^r_\gamma K^{\text{ét}}_{2r-i}(X_{\overline{K}}, \mathbf{Z}/p^n) \\
\downarrow c^{\text{syn}}_{i,r} & & \downarrow c^{\text{ét}}_{i,r} \\
H^i(X_{\overline{V}}, S_n(r)) & \xleftarrow{\alpha^{\text{cr}}_{i,r}} & H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes r}).
\end{array}$$

- **the key point:** j^* is an isomorphism (as before)
- ρ^K_{2r-i} is an isomorphism modulo small torsion for $2r - i \geq 2d$: by Suslin
- $c^{\text{ét}}_{i,r}$ is an isomorphism modulo small torsion: by the Atiyah-Hirzebruch spectral sequence
- define $\alpha^{\text{cr}}_{i,r}$ to make this diagram commute

The semistable reduction case

$X^{\times}/V^{\times}$ - proper, semistable reduction, no multiplicities (more generally: log-smooth=toroidal)

Semistable conjecture There exists

$\alpha_{st} : H^*(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{st}} \simeq H^*_{\mathrm{cr}}(X_0^{\times}/W^0) \otimes_W B_{\mathrm{st}}$
preserving G_K-action, N, F^*, ϕ

- the period ring B_{st}: G_K, ϕ, N, F^*
- $H^*_{\mathrm{cr}}(X_0^{\times}/W^0)[1/p]$ (analogue of limit Hodge structures); ϕ, N, F^*:

$$K \otimes_W H^*_{\mathrm{cr}}(X_0^{\times}/W^0) \simeq H^*_{dR}(X_K/K)$$

Corollary

$H^*(X_{\overline{K}}, \mathbf{Q}_p) \simeq F^0(H^*_{\mathrm{cr}}(X_0^{\times}/W^0) \otimes B_{\mathrm{st}})^{N=0,\phi=1}$

Have: for $i \leq r$

$$F^r(H^i_{\mathrm{cr}}(X_0^{\times}/W^0) \otimes B_{\mathrm{st}})^{N=0,\phi=p^r} \simeq H^i(X^{\times}_{\overline{V}}, S_{\mathbf{Q}_p}(r))$$

$H^i(X^{\times}_{\overline{V}}, S_{\mathbf{Q}_p}(r))$ - log-syntomic cohomology

Semistable period maps: for r large enough a compatible family

$$\alpha_{i,r}^n : H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes r}) \to H^i(X_{\overline{V}}^{\times}, S_n(r))$$

Main difficulty: the model $X_{\overline{V}}$ is in general singular

- higher Chow groups behave badly
- the localization map j^* in K-theory is difficult to understand

Solution:

- each $X_{V'}^{\times}$, finite extension V'/V, can be desingularized by a log-blow-up $Y^{\times} \to X_{V'}^{\times}$, Y - regular
- log-blow-up does not change the log-syntomic cohomology, so to define the maps $\alpha_{i,r}^n$ we can work with the regular models $Y^{\times}$

Suslin $\Rightarrow$ Semistable conjecture

$$\begin{array}{ccc} \mathrm{gr}_\gamma^r K_{2r-i}(Y, \mathbf{Z}/p^n) & \xrightarrow[j^*]{\sim} & \mathrm{gr}_\gamma^r K_{2r-i}(Y_K, \mathbf{Z}/p^n) \\ \downarrow c_{i,r}^{\mathrm{syn}} \rho_{2r-i} & & \wr \downarrow c_{i,r}^{\text{ét}} \rho_{2r-i} \\ H^i(Y^\times, S_n(r)) & \xleftarrow{\alpha_{i,r}^n} & H^i(Y_K, \mu_{p^n}^{\otimes r}). \end{array}$$

- **the key point:** j^* is an isomorphism for $2r - i > d + 1$: use the localization sequence

$$\to K'_j(Y_k, \mathbf{Z}/p^n) \to K'_j(Y, \mathbf{Z}/p^n) \xrightarrow{j^*} K'_j(Y_K, \mathbf{Z}/p^n)$$

and Geisser-Levine: $K'_j(Y_k, \mathbf{Z}/p^n) = 0$ for $j > d$

Corollary (Niziol, 2006): there exists a unique period map

$$\alpha_{i,r}^{st} : H^i(X_{\overline{K}}, \mathbf{Q}_p(r)) \to H^i(X_{\overline{V}}^\times, S_{\mathbf{Q}_p}(r))$$

compatible with the étale and syntomic higher Chern classes. In particular, the syntomic, almost étale and motivic period maps are equal.

Ideal picture

Not yet in existence !

$$
\begin{array}{ccc}
H^i(Y^\times, \mathbf{Z}/p^n(r)) & \xrightarrow[\sim]{j^*} & H^i(Y_K, \mathbf{Z}/p^n(r)) \\
\downarrow{\scriptstyle \rho_{i,r}} & & \downarrow{\scriptstyle \rho^K_{i,r}} \\
H^i(Y^\times_{\text{ét}}, \mathbf{Z}/p^n(r)) & \xrightarrow{j^*} & H^i(Y_{K,\text{ét}}, \mathbf{Z}/p^n(r)) \\
\downarrow{\scriptstyle c^{\text{syn}}_{i,r}} & & \wr\downarrow{\scriptstyle c^{\text{ét}}_{i,r}} \\
H^i(Y^\times, S_n(r)) & \xleftarrow{\alpha^n_{i,r}} & H^i(Y, \mu_{p^n}^{\otimes r})
\end{array}
$$

Question *Can we define limit motivic cohomology $H^i(Y^\times, \mathbf{Z}/p^n(r))$ such that*

- *the localization map j^* is an isomorphism*
- *the change of topology map $\rho_{i,r}$ is an isomorphism for $r \geq i$*
- *the log-syntomic cycle classes $c^{\text{syn}}_{i,r}$ are isomorphisms for $p - 1 > r \geq i$*

Recent work: Levine (2005), Ayoub, Niziol (2006)