p-adic Hodge Theory, from algebraic to analytic varieties

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An Archimedean comparison theorem

$X/\mathbf{Q}$ – algebraic variety, smooth, projective. Classical de Rham theorem: there exists a nondegenerate pairing

$$H^n_{\mathrm{dR}}(X_{\mathbf{C}}) \times H_n(X(\mathbf{C}), \mathbf{C}) \to \mathbf{C}, \quad (\omega, \gamma) \mapsto \int_\gamma \omega.$$

$H_n(X(\mathbf{C}), \mathbf{C})$ – singular homology, de Rham cohomology:

$$H^n_{\mathrm{dR}}(X_{\mathbf{C}}) := H^n(X_C, \mathscr{O}_{X_{\mathbf{C}}} \to \Omega^1_{X_{\mathbf{C}}/\mathbf{C}} \to \Omega^2_{X_{\mathbf{C}}/\mathbf{C}} \to \cdots)$$

$\mathbf{C}$ *contains periods for all varieties* ! Example of periods:

$$\int_\gamma \frac{\mathrm{d}z}{z} = 2\pi i, \text{ or } \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = 2\int_1^{+\infty} \frac{dx}{\sqrt{x^3 - x}}$$

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Note, Archimedean completion:

$$\mathbf{Q} \mapsto \widehat{\mathbf{Q}} \simeq \mathbf{R} \hookrightarrow \mathbf{C} \simeq \overline{\mathbf{R}}$$

But, we also have non-Archimedean completions:

$$\mathbf{Q} \mapsto \widehat{\mathbf{Q}} \simeq \mathbf{Q}_p \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}_p = \widehat{\overline{\mathbf{Q}}}_p$$

History of p-adic Hodge Theory

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Digression: non-Archimedean completion

(i) p-prime number, $|\bullet| = |\bullet|_p$ – p-adic norm on $\mathbf{Q}$, normalized with $|p| = p^{-1}$. Have $|xy| = |x||y|$ and $|x+y| \leq \max(|x|, |y|)$.

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(*ii*) $\mathbf{Q}_p$ – completion of $\mathbf{Q}$ for the p-adic norm $|\bullet|$,

$$\mathbf{Z}_p := \{x \in \mathbf{Q}_p \,|\, |x| \leq 1\}, \quad \mathbf{Z}_p \simeq \varprojlim_n \mathbf{Z}/p^n,$$

$$\mathbf{Z}_p\text{''} = \text{''}\{0, 1, \ldots, p-1\}[[p]],$$

$$\mathbf{Q}_p = \mathbf{Z}_p[1/p], \quad x \in \mathbf{Q}_p, x = \sum_{n \geq n_0} x_n p^n, x_n \in \{0, \ldots, p-1\}.$$

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(iii) $\overline{\mathbf{Q}}_p$ – algebraic closure of $\mathbf{Q}_p$, $|\bullet|$ extends uniquely to $\overline{\mathbf{Q}}_p$, $G_{\mathbf{Q}_p} := \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ acts via isometries. $\overline{\mathbf{Q}}_p$ is not complete for $|\bullet|$: $\overline{\mathbf{Q}}_p$ is infinite dimensional ($x^n - p$ is irreducible in $\mathbf{Q}_p[x]$).

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(iv) Let $\mathbf{C}_p$ be the completion of $\overline{\mathbf{Q}}_p$. $G_{\mathbf{Q}_p} = \mathrm{Aut}_{\mathrm{cont}}(\mathbf{C}_p)$. $\dim_{\mathbf{Q}_p} \mathbf{C}_p$ is not countable. $\mathbf{C}_p \simeq \mathbf{C}$ as an abstract field.

Étale cohomology

Back to the nondegenerate pairing:

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Dually:

$$H^n_{\mathrm{dR}}(X) \otimes_K \mathbf{C} \simeq H^n_B(X(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$$

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Fix prime p. For p-adic coefficients, we have

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1. locally: $H^n_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p) \simeq H^n(\pi(X_{\overline{\mathbf{Q}}_p}), \mathbf{Q}_p)$, $\pi(X_{\overline{\mathbf{Q}}_p})$ – algebraic fundamental group=profinite completion of the classical one,

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2. finite rank over $\mathbf{Q}_p$,
3. continuous action of $G_{\mathbf{Q}_p}$; it carries information about:
    - 3.1 finite extensions of $\mathbf{Q}_p$,
    - 3.2 the arithmetic of X, for example its rational points $X(\mathbf{Q})$.

.

Examples of Galois representations on $H^n_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$

(1) Tate twists: Cyclotomic character

$$\chi : G_{\mathbf{Q}_p} \to \mathbf{Z}_p^* : \sigma(e^{\frac{2\pi i}{p^n}}) = e^{\chi(\sigma)\frac{2\pi i}{p^n}}.$$

If $i \in \mathbf{Z}$, $\mathbf{Q}_p(i)$ is $\mathbf{Q}_p$ with action of $G_{\mathbf{Q}_p}$ via χ^i.

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$$V_pE \simeq H^1_{\text{ét}}(E_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)^*, \quad \dim_{\mathbf{Q}_p} V_pE = 2.$$

Main question

Does there exist a period ring B (that contains periods of all varieties over $\mathbf{Q}_p$) and a pairing $(\omega, \gamma) \mapsto \int_\gamma \omega \in B$ such that

1. $H^n_{\mathrm{dR}}(X) \otimes_{\mathbf{Q}_p} B \simeq H^n_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B$
2. we can recover the Galois representation $H^n_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$ from $H^n_{\mathrm{dR}}(X)$.

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2. $2\pi i$ period of $\mathbb{P}^1$ and $H^2_{\text{ét}}(\mathbb{P}^1_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)^* \cong \mathbf{Q}_p(1)$, so need $\sigma(2\pi i) = \chi(\sigma) 2\pi i, \quad \forall \sigma \in G_{\mathbf{Q}_p}$. But Tate:

$$\{x \in \mathbf{C}_p \mid \sigma(x) = \chi(\sigma)x, \forall \sigma \in G_{\mathbf{Q}_p}\} = 0$$

Period ring $\mathbf{B}_{\mathrm{dR}}$

Fontaine ('80) constructed a ring

$$\mathbf{B}_{\mathrm{dR}}^{+}, \quad 2\pi i = t \in \mathbf{B}_{\mathrm{dR}}^{+}, \quad \sigma(t) = \chi(\sigma)t, \sigma \in G_{\mathbf{Q}_p}.$$

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Properties of $\mathbf{B}_{\mathrm{dR}}^{+}$:

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3. Colmez: $\mathbf{B}_{\mathrm{dR}}^{+} \simeq \widehat{\overline{\mathbf{Q}}}_p$, a completion involving "higher derivatives".

Example to illustrate (3)

Define the norm

$$x \in \overline{\mathbf{Q}}_p, \quad |x|_{p,1} := \sup(|x|_p, |\tfrac{dx}{dp}|_p).$$

It is submultiplicative: $|xy|_{p,1} \leq |x|_{p,1}|y|_{p,1}$

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$$\begin{array}{ccccccccc} 0 & \longrightarrow & t\mathbf{B}_{\mathrm{dR}}^+/t^2\mathbf{B}_{\mathrm{dR}}^+ & \longrightarrow & \mathbf{B}_{\mathrm{dR}}^+/t^2\mathbf{B}_{\mathrm{dR}}^+ & \longrightarrow & \mathbf{B}_{\mathrm{dR}}^+/t\mathbf{B}_{\mathrm{dR}}^+ & \longrightarrow & 0 \\ & & \downarrow \wr & & & & \downarrow \wr & & \\ & & \mathbf{C}_p & & & & \mathbf{C}_p & & \end{array}$$

So $\mathbf{B}_{\mathrm{dR}}^+/t^2\mathbf{B}_{\mathrm{dR}}^+$ looks like a $\mathbf{C}_p$-vector space of dimension 2 (more generally, $\mathbf{B}_{\mathrm{dR}}^+/t^n\mathbf{B}_{\mathrm{dR}}^+ \sim \mathbf{C}_p^n$).

p-adic comparison theorems

Define $\mathbf{B}_{\mathrm{dR}} := \mathbf{B}_{\mathrm{dR}}^{+}[1/t]$.

Theorem (De Rham comparison, Faltings '89) X – proper, smooth over K, $[K:\mathbf{Q}_p] < \infty$. There exists a period isomorphism

$$\alpha_{\mathrm{dR}}: \quad H^n_{\mathrm{dR}}(X) \otimes_K \mathbf{B}_{\mathrm{dR}} \simeq H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} \tag{1.2}$$

compatible with Galois action and filtration, where
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2. take G_K-fixed points of (1.2):

$$H^n_{\mathrm{dR}}(X) \simeq (H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}})^{G_K}, \quad + \text{ Fil.}$$

Can not go the other way !

Need more refined period rings (Fontaine):

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History: Fontaine-Messing, Hyodo, Kato, Faltings, Tsuji, Nizioł ('85-2005); Beilinson, Bhatt, Scholze (2010+).

Digression: Banach-Colmez spaces

What structure does
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Remark The category of topological vector spaces is not good: $\mathbf{C}_p \oplus \mathbf{Q}_p \simeq \mathbf{C}_p$!

Theorem (Colmez, Fontaine) There exists an abelian category of Banach-Colmez vector spaces $\mathbb{W}$ which are finite dimensional $\mathbf{C}_p$-vector spaces $\pm$ finite dimensional $\mathbf{Q}_p$-vector spaces. We have

1. $\mathrm{Dim}(\mathbb{W}) := (\dim_{\mathbf{C}_p} \mathbb{W}, \dim_{\mathbf{Q}_p} \mathbb{W})$
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3. $\mathbf{C}_p/\mathbf{Q}_p$ has $\mathrm{Dim} = (1, -1)$.

Applications

(1) Categories of (abstract) Galois representations (Fontaine):

$$\mathrm{Rep}(G_K) \supset \mathrm{Rep}_{\mathrm{HT}}(G_K) \supset \mathrm{Rep}_{\mathrm{dR}}(G_K) \supsetneq \mathrm{Rep}_{\mathrm{geometric}}(G_K)$$

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Conjecture (Fontaine-Mazur, '90) Suppose that $\rho : G_{\mathbf{Q}} \to \mathrm{GL}(V)$ is an irreducible p-adic representation which is unramified at all but finitely many primes and $\rho|G_{\mathbf{Q}_p}$ is de Rham. Then there is a smooth projective variety $X/\mathbf{Q}$ and integers $i \geq 0$ and j such that V is a subquotient of $H^i(X(\mathbf{C}), \overline{\mathbf{Q}}_p(j))$.

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Known: basically in dimension 2 by Emerton, Kisin, Lue Pan.
geometric $\Rightarrow$ automorphic $\Rightarrow$ related to harmonic analysis

Rigid analytic varieties

Local behaviour very different from the algebraic case

Example

1. $\mathbb{D}$ open unit disk over $\mathbf{C}_p$. We have
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$\bigwedge^r \mathbf{Q}^d_p = \oplus_{i_1 < \cdots < i_r} \operatorname{dlog} z_{i_1} \wedge \cdots \wedge \operatorname{dlog} z_{i_r} \mathbf{Q}_p$, $\mathrm{Dim} = (\infty, \binom{d}{r})$.

(iii) Drinfeld half plane $\Omega_K := \mathbb{P}_K \setminus \mathbb{P}(K)$:

$$0 \to \mathscr{O}(\Omega_{\mathbf{C}_p})/\ker d \to H^1_{\text{proét}}(\Omega_{\mathbf{C}_p}, \mathbf{Q}_p(1)) \to \mathrm{Sp}(\mathbf{Q}_p)^* \to 0$$

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Theorem (CDN) $r \geq 0$, X - Stein analytic variety over K. There exists a G_K-equivariant exact sequence:

$$0 \to H^r_{\text{proét}}(X_{\mathbf{C}_p}, \mathbf{Q}_p(r)) \to \Omega^r(X_{\mathbf{C}_p})^{d=0} \oplus (H^r_{\text{HK}}(X_{\mathbf{C}_p})\widehat{\otimes}_{K^{\text{nr}}} \mathbf{B}^+_{\text{st}})^{N=0,\varphi=p^r} \to H^r_{\text{dR}}(X_{\mathbf{C}_p}) \to 0$$

Pro-étale cohomology can be recovered from de Rham data !