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COHOMOLOGY OF CRYSTALLINE REPRESENTATIONS

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1. Introduction. In studying the arithmetic geometry of a scheme X defined over a number field K, it is fundamental to consider the $\mathrm{Gal}(\overline{K}/K)$-module structure of the étale cohomology groups $H^i(X \otimes \overline{K}, \cdot)$ with diverse coefficients. In particular, it is sometimes unavoidable to consider, say, $H^i(X \otimes \overline{K}, \mathbf{Z}/p^n)$ for a *fixed* p, as a global Galois-module (e.g. [16]). In this case, although the action of the decomposition groups D_v for v not dividing p is relatively well understood, analyzing the action of D_v for primes *v dividing p* poses difficult problems, which must be dealt with on a quite different footing.

Consider the representations of D_v's on p-adic vector spaces: if $v|p$, call them p-adic; otherwise call them l-adic. The theory of p-adic representations has been developed extensively in recent years by Fontaine, Messing, Faltings, and many others. By associating to a p-adic representation a crystal—a certain filtered module with a Frobenius, Fontaine [7] has distinguished a full subcategory of p-adic representations, *crystalline* representations. They are the p-adic analogue of unramified l-adic representatons. In fact, the representations arising from the p-adic étale cohomology of proper and smooth schemes over K with good reduction at v, known to be unramified for v not dividing p, were shown ([11], [4]) to be crystalline for $v|p$. Moreover, in that case, the theory of p-adic periods (a nice overview of which the reader can find in [9], [14]) yields that the associated crystals are canonically isomorphic to the de Rham cohomology of the scheme, endowed with its Hodge filtration and crystalline Frobenius—the Frobenius coming from the one acting on the crystalline cohomology of the special fiber of a proper, smooth model of the scheme over $\mathrm{Spec}(\mathcal{O}_v)$, where $\mathcal{O}_v$ is the ring of integers of the completion of K at v, via the canonical isomorphism between de Rham and crystalline cohomology. There is also a theory treating p-torsion and, most interestingly, certain types of local systems.

Let now K be a number field, S a set of primes of K containing all primes above ∞ and p, $G_S = \mathrm{Gal}(K_S/K)$, where K_S is the maximal extension of K unramified outside of S, and $\mathrm{Rep}(G_S)$ the category of p-adic representatons of G_S: finite-dimensional $\mathbf{Q}_p$-vector spaces endowed with a linear and continuous action of G_S.

Consider the full subcategory $\mathrm{Rep}_{cr}(G_S)$ of $\mathrm{Rep}(G_S)$ consisting of representations crystalline at p, i.e., such that the action of a decompositon group D_v at v is crystalline if $v|p$ (see [7]). In [3] the groups of one-term extensions of $\mathbf{Q}_p$ by representations from $\mathrm{Rep}_{cr}(G_S)$ were introduced and studied in an attempt to extract the "geometric" part of Galois cohomology, and in relation to special values of L-functions.

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In this paper, we define a simple cohomological functor $H_f^*(G_S, \cdot)$ from the category $\mathrm{Rep}_{cr}(G_S)$ to $\mathbf{Q}_p$-vector spaces which have in degree 1 the Bloch-Kato groups. The construction uses the fact that the theory of Fontaine supplies us with rings giving "good" resolutions of local crystalline representations. The functor $H_f^*(G_S, \cdot)$ is not the universal extension of $H_f^1(G_S, \cdot)$ to a cohomological functor (i.e., it is not isomorphic to the Yoneda-Ext groups), but it is much simpler to study and hopefully can be used to understand $H_f^1(G_S, \cdot)$. We show that it carries a product structure giving a natural duality.

We also consider all of the above in the setting of the theory of Fontaine-Laffaille [10] of integral crystalline representations (i.e., we treat p-torsion). Here K is absolutely unramified above p and $\mathrm{Rep}(G_S)$ is the category of $\mathbf{Z}_p$-modules of finite type endowed with a linear continuous action of G_S. Fix $a \leqslant 0, b \geqslant 0, b - a \leqslant p - 2$, and let $\mathrm{Rep}_{f,a,b}(G_S) \subset \mathrm{Rep}(G_S)$ be the category of representations T such that, for every $v|p$, T as a representation of D_v is crystalline, associated to a crystal M with filtration in the range $[a, b]$.

Then, as above, we define a cohomological functor $H^*_{f,a,b}(G_S, \cdot)$ from the category $\mathrm{Rep}_{f,a,b}(G_S)$ to $\mathbf{Z}_p$-modules having properties analogous to those of $H_f^*(G_S, \cdot)$. In addition, for $p > 2$ and S equal to the set of primes of K above ∞ and p, we prove both that the functor $H^*_{f,-1,0}(G_S, \cdot)$ restricted to representations coming from the generic fiber of finite flat group schemes of p-power order over the ring of integers of K computes flat cohomology with coefficients in these group schemes and that, under certain additional restrictions on the range of the filtration, the cup product defines a perfect duality between $H^i_{f,a,b}(G_S, T)$ and $H^{3-i}_{f,-b-1,-a-1}(G_S, T^*(1))$, which generalizes global flat duality for T coming from a flat group scheme.

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Throughout this paper p will be a fixed prime, and for a field K, $\overline{K}$ will denote a fixed algebraic closure of K.

2. Fontaine's theory of p-adic representations. We will briefly recall prerequisites from the theory of Fontaine. For more details we refer the reader to the six papers of Fontaine listed in the bibliography, as well as [3] and [4].

Let K be a complete valuation field of characteristic 0 with finite residue field k of characteristic p, $\mathcal{O}_K$ its ring of integers, $W(k)$ the ring of Witt vectors with coefficients in k, and K_0 its fraction field. Set $G_K = \mathrm{Gal}(\overline{K}/K)$ and let σ be the absolute Frobenius on $W(\bar{k})$.

The p-adic representations of $\mathrm{Gal}(\overline{K}/K)$ that we are interested in are defined by means of certain rings $B^+_{cris}, B_{cris}, B_{dR}$. Consider the ring $R = \mathrm{proj\,lim}\, \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$, where $\mathcal{O}_{\overline{K}}$ is the integral closure of $\mathcal{O}_K$ in $\overline{K}$ and the maps in the projective system are the pth power maps. Take its ring of Witt vectors $W(R)$. Then B^+_{cris} is the p-adic completion of the divided power envelope $D_\xi(W(R))$ of the ideal $\xi W(R)$ in $W(R)$. Here $\xi = [(p)] + p[(-1)]$, where $(p), (-1) \in R$ are reductions mod p of sequences of p-roots of p and -1 respectively, and for $x \in R$, $[x] = [x, 0, 0, \dots] \in W(R)$ is its Teichmüller representative. The ring B^+_{cris} is a topological $W(k)$-module having the following properties:

(1) $W(\bar{k})$ is embedded as a subring of B^+_{cris} and σ extends naturally to a Frobenius ϕ on B^+_{cris};
(2) B^+_{cris} is equipped with a decreasing separated filtration $F^n B^+_{cris}$ such that $\phi(F^n B^+_{cris}) \subset p^n B^+_{cris}$ (in fact, $F^n B^+_{cris} = \{x \in (\xi B^+_{cris})^{[n]} | \phi(x) \in p^n B^+_{cris}\}$, where $(\xi B^+_{cris})^{[n]}$ is the nth divided power of the ideal ξB^+_{cris} (compare [4]));
(3) G_K acts on B^+_{cris}; the action is $W(\bar{k})$-semilinear, continuous, commutes with ϕ and preserves the filtration;
(4) there exists an element $t \in F^1 B^+_{cris}$ such that $\phi(t) = pt$ and G_K acts on t via the cyclotomic character ($t = \beta(\zeta)$, where β is the G_K-linear map $\mathbf{Z}_p(1) \to F^1 B^+_{cris}$ (see [4]) and ζ is a fixed generator of $\mathbf{Z}_p(1)$).

Note that this B^+_{cris} is not the B^+_{cris} of Fontaine.

B_{cris} is defined as the ring $B^+_{cris}[p^{-1}, t^{-1}]$ with the induced topology, filtration, Frobenius and the Galois action.

Let $W_{K_0}(R) = K_0 \otimes_{W(k)} W(R)$. The ring B^+_{dR} is the completion of $W_{K_0}(R)$ with respect to $\xi W_{K_0}(R)$-adic topology. If we give $W(R)$ product topology and W_{K_0} tensor topology, then B^+_{dR} equipped with projective limit topology (where $W_{K_0}/(\xi W_{K_0}(R))^n$ has quotient topology) becomes a topological K-algebra. It has a natural filtration defined by the closures of the powers of the ideal $\xi W_{K_0}(R) = t W_{K_0}(R)$ and a continuous Galois action. The ring B_{dR} is defined as the fraction field of B^+_{dR} with induced structures. The map

$$D_\xi(W(R)) \to W_{K_0}(R) \to B^+_{dR}$$

can be extended to a continuous injective map from B^+_{cris} to B^+_{dR}, which preserves filtration and Galois action.

[11] and [3] show the existence of the exact sequences of Galois modules

$$0 \to \mathbf{Q}_p \xrightarrow{\Delta} B_{cris} \oplus B^+_{dR} \xrightarrow{\theta} B_{cris} \oplus B_{dR} \to 0,$$

where $\Delta(x) = (x, x)$ and $\theta(x, y) = (x - \phi(x), x - y)$, and

$$0 \to p^{-c}\mathbf{Z}_p(r) \to F^r B^+_{cris} \xrightarrow{1-p^{-r}\phi} B^+_{cris} \to 0, \qquad \text{for } r \geqslant 0.$$

Here $c = c(r) = \sum_{i=0}^{\infty} r/(p-1)p^i$, and so for $0 \leqslant r < p-1$ we get $c = c(r) = 0$ and

$$0 \to \mathbf{Z}_p(r) \xrightarrow{\beta r} F^r B^+_{cris} \xrightarrow{1-p^{-r}\phi} B^+_{cris} \to 0.$$

For a finite-dimensional continuous $\mathbf{Q}_p[G_K]$-module V, set

$$\mathbf{D}_{cr}(V) := (V \otimes B_{cris})^{G_K}, \qquad \mathbf{D}_{dR}(V) := (V \otimes B_{dR})^{G_K}.$$

$\mathbf{D}_{cr}(V)$ is a K_0-vector space with a Frobenius induced from that on B_{cris}, $\mathbf{D}_{dR}(V)$ is a K-vector space with a filtration induced from that on B_{dR} and always $\mathbf{D}_{cr}(V) \otimes_{K_0} K \hookrightarrow \mathbf{D}_{dR}(V)$. If $\dim_{K_0}(\mathbf{D}_{cr}(V)) = \dim_{\mathbf{Q}_p} V$ (resp. $\dim_K(\mathbf{D}_{dR}(V)) = \dim_{\mathbf{Q}_p} V$), then V is called *crystalline* (resp. *de Rham*). Crystalline implies de Rham.

Now assume $\mathcal{O}_K = W(k)$. For the integral theory we will need the following abelian categories:

(1) $\mathcal{MF}_{big}(\mathcal{O}_K)$—an object is given by a p-torsion $\mathcal{O}_K$-module M and a family of p-torsion $\mathcal{O}_K$-modules F^iM together with $\mathcal{O}_K$-linear maps $F^iM \to F^{i-1}M$, $F^iM \to M$ and σ-semilinear maps ϕ^i: $F^iM \to M$ satisfying certain compatibility conditions (see [10]);

(2) $\mathcal{MF}(\mathcal{O}_K)$—the full subcategory of $\mathcal{MF}_{big}(\mathcal{O}_K)$ with objects finite $\mathcal{O}_K$-modules M such that $F^iM = 0$ for $i \gg 0$, the maps $F^i(M) \to M$ are injective, and $\sum \operatorname{Im} \phi^i = M$;

(3) $\mathcal{MF}_{[a,b]}(\mathcal{O}_K)$—the full subcategory of objects M of $\mathcal{MF}(\mathcal{O}_K)$ such that $F^aM = M$ and $F^{b+1}M = 0$.

Consider the category $\mathcal{MF}_{[a,b]}(\mathcal{O}_K)$ with $b - a \leqslant p - 2$. There exists a functor (exact and fully faithful) from $\mathcal{MF}_{[a,b]}(\mathcal{O}_K)$ to finite $\mathbf{Z}_p$-Galois representations. Its essential image is called the category of *crystalline representations of weight between a and b*. An exact quasi-inverse $\mathbf{D}_{cr,a,b}(T)$ together with an injection $\mathbf{D}_{cr,a,b}(T) \to (T \otimes B^+_{cris}\{a\}(a))^{G_K}$ is defined as inj lim$\{M \to (T \otimes B^+_{cris}\{a\}(a))^{G_K}\}$, where $T \otimes B^+_{cris}$ belongs to $\mathcal{MF}_{big}(\mathcal{O}_K)$ with the filtration induced from that on B^+_{cris}, and $\phi^i = \mathrm{id} \otimes (\phi/p^i)$. The limit is over all pairs consisting of $M \in \mathcal{MF}_{[a,b]}(\mathcal{O}_K)$ and a $\mathcal{MF}_{big}(\mathcal{O}_K)$-map $M \to (T \otimes B^+_{cris}\{a\}(a))^{G_K}$, the morphisms being the obvious ones. Here (a), $\{a\}$ are the Tate-twist and the $\mathcal{MF}$-twist respectively. As an $\mathcal{O}_K$-module, $T \otimes_{\mathbf{Z}_p} \mathcal{O}_K$ is (noncanonically) isomorphic to $\mathbf{D}_{cr,a,b}(T)$.

A finitely generated $\mathbf{Z}_p$-Galois representation T is called a *crystalline of weight between a and b* if every T/p^rT is crystalline of weight between a and b. The exact functor $\mathbf{D}_{cr,a,b}(T) = \text{proj lim}\, \mathbf{D}_{cr,a,b}(T/p^rT)$ defines an equivalence of categories between finitely generated $\mathbf{Z}_p$-crystalline representations of weight between a and b and finitely generated $\mathcal{O}_K$-modules M equipped with a decreasing filtration and σ-semilinear maps ϕ^i: $F^iM \to M$ such that every M/p^rM is in $\mathcal{MF}_{[a,b]}(\mathcal{O}_K)$. It preserves ranks. If T is a finitely generated crystalline representation of weight between a and b, then $T \otimes \mathbf{Q}_p$ is a rational crystalline representation, and the map ω: $B^+_{cris}\{a\}(a) \to B_{cris}$ induced by β^a yields a canonical isomorphism

$$\mathbf{D}_{cr,a,b}(T) \otimes K \tilde{\to} \mathbf{D}_{cr}(T \otimes \mathbf{Q}_p)$$

respecting filtrations and Frobenius.

Let Rep_{cr} (resp. $\mathrm{Rep}_{cr,a,b}$) denote the categories of rational (resp. integral of weight between a and b) crystalline representations. The category Rep_{cr} is closed under taking tensor products and duals. In the integral case, if $T \in \mathrm{Rep}_{cr,a,b}$ for $b - a \leqslant p - 2$, and $U \in \mathrm{Rep}_{cr,c,d}$ for $d - c \leqslant p - 2$, and $b + d - a - c \leqslant p - 2$, then $T \otimes U \in \mathrm{Rep}_{cr,a+c,b+d}$. Moreover if $T \in \mathrm{Rep}_{cr,a,b}$, $b - a \leqslant p - 2$, is a torsion representation, then the representation $T^* = \mathrm{Hom}(T, \mathbf{Q}/\mathbf{Z})$ belongs to $\mathrm{Rep}_{cr,-b,-a}$.

The above definitions are independent of a, b in the following sense: for $\mathrm{Rep}_{cr,a',b'}$ such that $a \leqslant a' \leqslant b' \leqslant b$, there is a natural equivalence of functors

$$\psi\colon \mathbf{D}_{cr,a',b'} \to \mathbf{D}_{cr,a,b}|_{\mathrm{Rep}_{cr,a',b'}}$$

given by "multiplication" by $t^{a'-a} \otimes \zeta^{-a'+a}$. In what follows, thus we will often omit the subscripts if this does not cause confusion.

3. Multiplication on cones. We will often use the following construction of multiplication on cones, which we have modeled after that in [2].

PROPOSITION 3.1. *Let R be a commutative ring. For $i = 1, 2, 3$, let $A_i^{\bullet}$, $C_i^{\bullet}$ be complexes of R-modules, and $f_i, g_i\colon A_i^{\bullet} \to C_i^{\bullet}$ maps of complexes of R-modules. Assume that, for every $\alpha \in R$, there are given maps $\cup_\alpha\colon A_1^{\bullet} \otimes A_2^{\bullet} \to A_3^{\bullet}$ and $\cup_\alpha\colon C_1^{\bullet} \otimes C_2^{\bullet} \to C_3^{\bullet}$, such that*

(i) *$\cup_\alpha$'s are maps of complexes;*
(ii) *$\cup_\alpha$'s commute with f_i's and g_i's;*
(iii) *all $\cup_\alpha$'s are homotopic;*
(iv) *the homotopies in (iii) can be chosen to commute with f_i's and g_i's.*

Set

$$D_i^{\bullet} := \mathrm{Cone}(A_i^{\bullet} \xrightarrow{f_i - g_i} C_i^{\bullet})[-1]$$

and define, for every $\alpha \in R$, the maps

$$\cup_\alpha\colon D_1^{\bullet} \otimes D_2^{\bullet} \to D_3^{\bullet}$$

as follows: represent the cochains $\gamma_i \in D_i^{\bullet}$ by the pairs (a_i, c_i) and set

$$\gamma_1 \cup_\alpha \gamma_2 = (a_1 \cup_\alpha a_2,\ c_1 \cup_\alpha w_\alpha(a_2) + (-1)^{\deg a_1} w_{1-\alpha}(a_1) \cup_\alpha c_2),$$

where, for $\beta \in R$, $w_\beta(a_i) = \beta f_i(a_i) + (1 - \beta) g_i(a_i)$. Then

(i) *$\cup_\alpha\colon D_1^{\bullet} \otimes D_2^{\bullet} \to D_3^{\bullet}$ is a map of complexes;*
(ii) *$\cup_\alpha$ commutes with the projections $D_1^{\bullet} \to A_i^{\bullet}$, $i = 1, 2, 3$;*
(iii) *$\cup_\alpha$ is canonical, i.e., if there is given another set of data $\tilde{A}_i^{\bullet}$, $\tilde{C}_i^{\bullet}$, $\tilde{f}_i$, $\tilde{g}_i$, $\tilde{\cup}_\alpha$ as above and maps $A_i^{\bullet} \to \tilde{A}_i^{\bullet}$, $C_i^{\bullet} \to \tilde{C}_i^{\bullet}$ commuting with the products $\cup_\alpha$, $\tilde{\cup}_\alpha$, then the induced maps on cones $D_i^{\bullet} \to \tilde{D}_i^{\bullet}$ commute with the products $\cup_\alpha$, $\tilde{\cup}_\alpha$ defined above;*
(iv) *all $\cup_\alpha$'s are homotopic;*
(v) *if $g_i = 0$, $i = 1, 2, 3$, then there are compatible pairings*

$$\begin{array}{ccc} A_1^{\bullet} \otimes D_2^{\bullet} & \xrightarrow{\tilde{\cup}_0} & D_3^{\bullet} \\ \uparrow & & \| \\ D_1^{\bullet} \otimes D_2^{\bullet} & \xrightarrow{\cup_0} & D_3^{\bullet} \end{array} \qquad \begin{array}{ccc} D_2^{\bullet} \otimes A_1^{\bullet} & \xrightarrow{\tilde{\cup}_1} & D_3^{\bullet}, \\ \uparrow & & \| \\ D_2^{\bullet} \otimes D_1^{\bullet} & \xrightarrow{\cup_1} & D_3^{\bullet}, \end{array}$$

where the maps of complexes $\tilde{\cup}_0$, $\tilde{\cup}_1$ *are induced from the maps* $\cup_0, \cup_1 \colon D_1^\bullet \otimes D_2^\bullet \to D_3^\bullet$ *via the projection* $D_1^\bullet \to A_1^\bullet$. *Explicitly, for* $a_i \in A_i^\bullet$, $c_i \in C_i^\bullet$,

$$a_1 \tilde{\cup}_0 (a_2, c_2) = (a_1 \cup_0 a_2, (-1)^{\deg a_1} f_1(a_1) \cup_0 c_2),$$

$$(a_2, c_2) \tilde{\cup}_1 a_1 = (a_2 \cup_1 a_1, c_2 \cup_1 f_1(a_1)).$$

Proof. The properties (ii), (iii), and (v) are straightforward. Set $t_i := f_i - g_i$. Henceforth, we will omit the subscripts of f, g, and t if that does not cause confusion.

To prove (i) we need to show that, for any two cochains $\gamma_1 \in D_1^\bullet$, $\gamma_2 \in D_2^\bullet$,

$$d(\gamma_1 \cup_\alpha \gamma_2) = d\gamma_1 \cup_\alpha \gamma_2 + (-1)^{\deg \gamma_1} \gamma_1 \cup_\alpha d\gamma_2. \tag{1}$$

Since $d(a_i, c_i) = (da_i, -t_i(a_i) - dc_i)$, we have

$$d(\gamma_1 \cup_\alpha \gamma_2)$$
$$= (d(a_1 \cup_\alpha a_2), -t(a_1 \cup_\alpha a_2) - d[c_1 \cup_\alpha w_\alpha(a_2) + (-1)^{\deg a_1} w_{1-\alpha}(a_1) \cup_\alpha c_2]),$$

and

$$\begin{aligned} d\gamma_1 \cup_\alpha \gamma_2 + (-1)^{\deg \gamma_1} \gamma_1 \cup_\alpha d\gamma_2 = {} & (da_1 \cup_\alpha a_2 + (-1)^{\deg a_1} a_1 \cup_\alpha da_2, \\ & (-t(a_1) - dc_1) \cup_\alpha w_\alpha(a_2) - (-1)^{\deg a_1} w_{1-\alpha}(da_1) \\ & \cup_\alpha c_2 - (-1)^{\deg c_1} c_1 \cup_\alpha w_\alpha(da_2) + w_{1-\alpha}(a_1) \\ & \cup_\alpha (-t(a_2) - dc_2)). \end{aligned}$$

The equality (1) follows now from the properties (i) and (ii) of the original $\cup_\alpha$'s.

To prove (iv), fix $\alpha, \beta \in R$, $\alpha \neq \beta$. The homotopy

$$h \colon (D_1^\bullet \otimes D_2^\bullet)^k \to D_3^{k-1}$$

between $\cup_\alpha$ and $\cup_\beta$ is given by

$$\begin{aligned} h(\gamma_1 \otimes \gamma_2) = (& h_A(a_1 \otimes a_2), \\ & h_C((-1)^{\deg c_1} w_{1-\beta}(a_1) \otimes c_2 - c_1 \otimes w_\beta(a_2)) \\ & + (-1)^{\deg c_1}(\alpha - \beta) c_1 \cup_\alpha c_2), \end{aligned}$$

where h_A, h_C are the homotopies on $A^\bullet$, $C^\bullet$ respectively satisfying the property (iv).

We claim that $\gamma_1 \cup_\alpha \gamma_2 - \gamma_1 \cup_\beta \gamma_2 = (hd + dh)(\gamma_1 \otimes \gamma_2)$. Indeed, we have

$$\begin{aligned}
\gamma_1 \cup_\alpha \gamma_2 - \gamma_1 \cup_\beta \gamma_2 = {} & (a_1 \cup_\alpha a_2 - a_1 \cup_\beta a_2, \\
& c_1 \cup_\alpha w_\alpha(a_2) - c_1 \cup_\beta w_\beta(a_2) + (-1)^{\deg a_1} w_{1-\alpha}(a_1) \\
& \cup_\alpha c_2 - (-1)^{\deg a_1} w_{1-\beta}(a_1) \cup_\beta c_2) \\
= {} & (a_1 \cup_\alpha a_2 - a_1 \cup_\beta a_2, \\
& c_1 \cup_\alpha w_\beta(a_2) - c_1 \cup_\beta w_\beta(a_2) \\
& + (-1)^{\deg a_1}[w_{1-\beta}(a_1) \cup_\alpha c_2 - w_{1-\beta}(a_1) \cup_\beta c_2] + (\alpha - \beta)c_1 \\
& \cup_\alpha t(a_2) - (-1)^{\deg a_1}(\alpha - \beta)t(a_1) \cup_\alpha c_2) \\
= {} & ((h_A d + dh_A)(a_1 \otimes a_2), \\
& (h_C d + dh_C)[(c_1 \otimes w_\beta(a_2) + (-1)^{\deg a_1} w_{1-\beta}(a_1)) \otimes c_2] \\
& + (\alpha - \beta)c_1 \cup_\alpha t(a_2) - (-1)^{\deg a_1}(\alpha - \beta)t(a_1) \cup_\alpha c_2)
\end{aligned}$$

and

$$\begin{aligned}
(hd + dh)(\gamma_1 \otimes \gamma_2) = {} & h(d\gamma_1 \otimes \gamma_2) + (-1)^{\deg \gamma_1} h(\gamma_1 \otimes d\gamma_2) + dh(\gamma_1 \otimes \gamma_2) \\
= {} & ((h_A d + dh_A)(a_1 \otimes a_2), \\
& h_C[(-1)^{\deg a_1} w_{1-\beta}(da_1) \otimes c_2 + (t(a_1) + dc_1) \otimes w_\beta(a_2)] \\
& - (-1)^{\deg a_1}(\alpha - \beta)(t(a_1) + dc_1) \cup_\alpha c_2 \\
& + h_C[w_{1-\beta}(a_1) \otimes (t(a_2) + dc_2) + (-1)^{\deg c_1} c_1 \otimes w_\beta(da_2)] \\
& + (\alpha - \beta)c_1 \cup_\alpha (t(a_2) + dc_2) - t(h_A(a_1 \otimes a_2)) \\
& + dh_C[(-1)^{\deg a_1} w_{1-\beta}(a_1) \otimes c_2 + c_1 \otimes w_\beta(a_2)] \\
& - (-1)^{\deg c_1}(\alpha - \beta)d(c_1 \cup_\alpha c_2)).
\end{aligned}$$

All terms of $\gamma_1 \cup_\alpha \gamma_2 - \gamma_1 \cup_\beta \gamma_2$ can be found easily among the terms of $(hd + dh)(\gamma_1 \otimes \gamma_2)$, the remaining of which give the sum

$$\begin{aligned}
& h_C(t(a_1) \otimes w_\beta(a_2)) + h_C(w_{1-\beta}(a_1) \otimes t(a_2)) - t(h_A(a_1 \otimes a_2)) \\
& \quad = h_C(f(a_1) \otimes f(a_2)) - f(h_A(a_1 \otimes a_2)) - h_C(g(a_1) \otimes g(a_2)) + g(h_A(a_1 \otimes a_2)),
\end{aligned}$$

which is zero as desired due to the compatibility of the homotopies h_A and h_C. □

4. Extensions in the category of local crystalline representations. Let now $K = K_0$. Consider the category $\mathrm{Rep}_{cr,a,b}$ for $a \leqslant 0$, $b \geqslant 0$, $b - a \leqslant p - 2$. For $T \in \mathrm{Rep}_{cr,a,b}$, set

$$B^{\bullet}_{T,a,b} := F^0\mathbf{D}_{cr}(T) \xrightarrow{1-\phi^0} \mathbf{D}_{cr}(T), \qquad H^{\bullet}_{f,a,b}(G_K, T) := H^*(B^{\bullet}_{T,a,b}).$$

This defines a cohomological functor, which was shown by [3, 4.4] to be isomorphic to the Yoneda-Ext-groups $\mathrm{Ext}^*(\mathbf{Z}_p, T)$ in the category $\mathrm{Rep}_{cr,a,b}$.

We now study its relation with Galois cohomology. By the definition of the functor $\mathbf{D}_{cr}(\cdot)$, we have the commutative diagram of Galois modules

$$\begin{array}{ccc} F^0\mathbf{D}_{cr}(T) & \xrightarrow{1-\phi^0} & \mathbf{D}_{cr}(T) \\ \downarrow & & \downarrow \\ T \otimes F^0B^+_{cris}\{a\}(a) & \xrightarrow{1\otimes(1-\phi^0)} & T \otimes B^+_{cris}\{a\}(a) \\ \| & & \| \\ T \otimes F^{(-a)}B^+_{cris}(a) & \xrightarrow{1\otimes(1-p^a\phi)} & T \otimes B^+_{cris}(a). \end{array}$$

Thus, if we set

$$E^{\bullet}_{T,a,b} := T \otimes F^{(-a)}B^+_{cris}(a) \xrightarrow{1\otimes(1-p^a\phi)} T \otimes B^+_{cris}(a),$$

we get an injective map $B^{\bullet}_{T,a,b} \to E^{\bullet}_{T,a,b}$. The complex $E^{\bullet}_{T,a,b}$ is a resolution of the representation T (cf. the exact sequences from Section 2).

Recall that $B^{\bullet}_{T,a,b}$, $H^*_{f,a,b}(G_K, T)$, and $E^{\bullet}_{T,a,b}$ do not really depend on the a and b chosen (restricted to the usual conditions). In what follows thus, we can and often will omit the subscripts a, b in $B^{\bullet}_{T,a,b}$, $H^*_{f,a,b}(G_K, T)$, and $E^{\bullet}_{T,a,b}$ if that does not cause confusion.

For a profinite group G and a topological G-module M, let $C^i(G, M)$ denote the group of the ith continuous cochains with values in M: continuous maps from G^{i+1} to M. The complex $C^{\bullet}(G, M)$ gives the so-called standard resolution of M, which we will use to compute the continuous Galois cohomology of M. For a complex $M^{\bullet}$ of topological G-modules, $C^{\bullet}(G, M^{\bullet})$ will denote the simple complex of the double complex formed from the standard resolutions of the members of $M^{\bullet}$.

Returning to crystalline representations, set

$$S^{\bullet}_T = C^{\bullet}(G_K, T), \qquad A^{\bullet}_T = C^{\bullet}(G_K, E^{\bullet}_T).$$

We claim that the complex $A^{\bullet}_T$ is an acyclic resolution of T. Indeed $E^{\bullet}_T$ resolves T, and we can easily find a continuous section (just a map of topological spaces) of the map

$$F^rB^+_{cris} \xrightarrow{1-p^{-r}\phi} B^+_{cris}, \qquad r \geqslant 0,$$

write every element $b \in B^+_{cris}$ as a unique convergent sum

$$b = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p^n [x_{n,m}] \delta_m,$$

where $x_{n,m} \in R$, $\{\delta_m, m \in \mathbf{N}\}$ is the natural choice of bases of the free R-module

$$B^+_{cris}/p \cong R[\xi_0, \xi_1, \ldots]/(\xi_0^p, \xi_1^p, \ldots), \qquad \xi_i = \xi^{p^i}/(p^i)!,$$

and for every n, all but a finite number of $x_{n,m}$'s vanish and arbitrarily lift the elements $[x]\delta_m$, for $x \in R$, $m \in \mathbf{N}$.

Consider now the canonical maps

$$f\colon S_T^{\cdot G_K} \to A_T^{\cdot G_K}, \qquad g\colon E_T^{\cdot G_K} \to A_T^{\cdot G_K}, \qquad t\colon B_T^\cdot \to E_T^{\cdot G_K} \xrightarrow{g} A_T^{\cdot G_K}.$$

The map f being a quasi isomorphism, we get a map (in the derived category) $l\colon B_T^\cdot \to S_T^{\cdot G_K}$ inducing the canonical map

$$l\colon H^*_{f,a,b}(G_K, T) \to H^*(G_K, T)$$

from crystalline to unrestricted extensions.

We will now treat products.

PROPOSITION 4.1. *If $T \in \mathrm{Rep}_{cr,a,b}$, $S \in \mathrm{Rep}_{cr,c,d}$, then there exists a canonical product*

$$\cup\colon H^p_{f,a,b}(G_K, T) \otimes H^q_{f,c,d}(G_K, S) \to H^{p+q}_{f,a+c,b+d}(G_K, T \otimes S),$$

which is anticommutative and associative.

Proof. In fact, we will prove that, for every $\alpha \in \mathbf{Z}_p$, there exist canonical mappings

$$B^\cdot_{T,a,b} \otimes B^\cdot_{S,c,d} \xrightarrow{\cup_\alpha} B^\cdot_{T\otimes S,a+c,b+d}, \qquad E^\cdot_{T,a,b} \otimes E^\cdot_{S,c,d} \xrightarrow{\cup_\alpha} E^\cdot_{T\otimes S,a+c,b+d},$$

such that

(i) $\cup_\alpha$ commutes with the map $B^\cdot \to E^\cdot$;
(ii) $\cup_\alpha$ is a map of complexes;
(iii) $\cup_0$, $\cup_1$ are associative;
(iv) all $\cup_\alpha$'s are homotopic;
(v) $\cup_\alpha$ anticommutes with $\cup_{1-\alpha}$, i.e., $\gamma \cup_\alpha \gamma' = (-1)^{\deg \gamma \cdot \deg \gamma'} \gamma' \cup_{1-\alpha} \gamma$.

Write

$$E_T^\cdot = \mathrm{Cone}(T \otimes F^{(-a)} B^+_{cris}(a) \xrightarrow{1 \otimes p^a \phi - 1} T \otimes B^+_{cris}(a))[-1].$$

If we define products on the complexes $T \otimes F^{(-a)}B^+_{cris}(a)$ and $T \otimes B^+_{cris}(a)$ (concentrated in degree 0) via tensor products followed by multiplication in B^+_{cris}, Proposition 3.1 gives us, for every $\alpha \in \mathbf{Z}_p$, a map $\cup_\alpha: E^{\cdot}_T \otimes E^{\cdot}_S \to E^{\cdot}_{T\otimes S}$ satisfying (ii) and (iv). Explicitly,

$$x \cup_\alpha y = \begin{cases} xy & \text{if } x \in E^0_T, y \in E^0_S \\ ((1-\alpha)p^a\phi(x) + \alpha x)y & \text{if } x \in E^0_T, y \in E^1_S \\ x(\alpha p^c \phi(y) + (1-\alpha)y) & \text{if } x \in E^1_T, y \in E^0_S \\ 0 & \text{otherwise.} \end{cases}$$

Define $\cup_\alpha: B^{\cdot}_T \otimes B^{\cdot}_S \to B^{\cdot}_{T\otimes S}$ via the injections $B^{\cdot}_Y \to E^{\cdot}_Y$, $Y = T, S, T \otimes S$. This can be done because the multiplication map

$$T \otimes B^+_{cris}(a) \otimes S \otimes B^+_{cris}(c) \to T \otimes S \otimes B^+_{cris}(a + c)$$

is in $\mathcal{MF}_{big}(\mathcal{O}_K)$ for finite T and S and so induces a map

$$\mathbf{D}_{cr}(T) \otimes \mathbf{D}_{cr}(S) \to \mathbf{D}_{cr}(T \otimes S).$$

It is easy to see that so-defined $\cup_\alpha$'s also have properties (ii) and (iv).

Points (i), (iii), and (v) are straightforward. □

The map $H^*_f(G_K, T) \xrightarrow{l} H^*(G_K, T)$ commutes with cup products. To see this, note that, for any $\alpha \in \mathbf{Z}_p$, the maps f and g commute with the following products: the standard product on $S^{\cdot}_T$, $\cup_\alpha$ on $E^{\cdot}_T$ and $\cup_\alpha$ on $A^{\cdot}_T$. The last one is the product induced from $\cup_\alpha$ on $E^{\cdot}_T$ and the standard product. It is a map of complexes and $\cup_0$, $\cup_1$ are associative.

For good duality statements we will need cohomology with support on the closed point x of $\mathrm{Spec}(\mathcal{O}_K)$. For $T \in \mathrm{Rep}_{cr,a,b}$ define

$${}_xB^{\cdot}_{T,a,b} := \mathrm{Cone}(B^{\cdot}_{T,a,b} \xrightarrow{l} A^{\cdot G_K}_T)[-1], \qquad {}_xH^{\cdot}_{f,a,b}(G_K, T) := H^*({}_xB^{\cdot}_{T,a,b}).$$

Again, we will omit the subscripts a, b wherever it is possible. From the definition of ${}_xB^{\cdot}_{T,a,b}$ we get the long exact sequence

$$\xrightarrow{l} H^{i-1}(G_K, T) \to {}_xH^i_{f,a,b}(G_K, T) \to H^i_f(G_K, T) \xrightarrow{l} H^i(G_K, T) \to,$$

from which it follows that ${}_xH^i_f(G_K, T) = 0$, for $i \neq 2, 3$, and

$${}_xH^2_f(G_K, T) \xleftarrow{\sim} H^1(G_K, T)/H^1_f(G_K, T), \qquad {}_xH^3_f(G_K, T) \xleftarrow{\sim} H^2(G_K, T).$$

PROPOSITION 4.2. *For $T \in \mathrm{Rep}_{cr,a,b}$ and $S \in \mathrm{Rep}_{cr,c,d}$, there exist canonical products*

$$\cup: H^p_{f,a,b}(G_K, T) \otimes {}_xH^q_{f,c,d}(G_K, S) \to {}_xH^{p+q}_{f,a+c,b+d}(G_K, T \otimes S),$$

$$\tilde{\cup}: {}_xH^q_{f,c,d}(G_K, S) \otimes H^p_{f,a,b}(G_K, T) \to {}_xH^{p+q}_{f,a+c,b+d}(G_K, T \otimes S).$$

$\cup$ anticommutes with $\tilde{\cup}$ and both products commute with projections into $H^*_f(G_K, \cdot)$.

Proof. In the context of Proposition 3.1, take $A^\bullet_i = B^\bullet_{Y_i}$, $C^\bullet_i = A^{\bullet G_K}_{Y_i}$, $f_i = t$, $g_i = 0$, where $Y_1 = T$, $Y_2 = S$, $Y_3 = T \otimes S$. The maps $\tilde{\cup}_0$, $\tilde{\cup}_1$ defined there give us the maps of complexes

$$\cup: B^\bullet_T \bigotimes {}_xB^\bullet_S \to {}_xB^\bullet_{T\otimes S}, \qquad \tilde{\cup}: {}_xB^\bullet_S \bigotimes B^\bullet_T \to {}_xB^\bullet_{S\otimes T}$$

respectively commuting with the projections ${}_xB^\bullet \to B^\bullet$.

We claim that there exists a transposition operator $\mathscr{T}: {}_xB^\bullet_T \to {}_xB^\bullet_T$ which is homotopic to the identity and such that, for any two cochains $\gamma \in B^\bullet_T$, $\gamma' \in {}_xB^\bullet_S$,

$$\gamma \cup \mathscr{T}\gamma' = (-1)^{\deg\gamma\cdot\deg\gamma'}\mathscr{T}(\gamma' \tilde{\cup} \gamma).$$

To see it, recall that the canonical equivariant transposition operator τ on standard resolutions (if $f \in C^k(G, T)$, then $(\tau f)(g_0, \ldots, g_k) = (-1)^{(1/2)k(k+1)} f(g_k, \ldots, g_0)$) is homotopic to the identity via a canonical equivariant homotopy h'. Thus, if $\mathscr{T}$ is the transposition operator on ${}_xB^\bullet_T$ given by the action of τ on the second coordinate, $\mathscr{T}$ is also homotopic to the identity (via the homotopy equal to 0 on the first coordinate and $-h'$ on the second). Finally, since for standard cochains γ, γ', τ satisfies $\tau\gamma \cup \tau\gamma' = (-1)^{\deg\gamma\cdot\deg\gamma'}\tau(\gamma' \cup \gamma)$ and $\cup_0$ anticommutes with $\cup_1$ on the complexes $B^\bullet$, we get the required formula. □

Everything works in an analogous way for rational crystalline representations. The Yoneda-Ext-groups are computed by the complex

$$\tilde{B}^\bullet_V: \mathbf{D}_{cr}(V) \oplus F^0\mathbf{D}_{dR}(V) \xrightarrow{\theta} \mathbf{D}_{cr}(V) \oplus \mathbf{D}_{dR}(V);$$

cf. [3, 3]. The complex

$$\tilde{E}^\bullet_V: V \otimes B_{cris} \oplus V \otimes B^+_{dR} \xrightarrow{1\otimes\theta} V \otimes B_{cris} \oplus V \otimes B_{dR}$$

gives us a resolution of V and the complex $\tilde{A}^\bullet_V = C^\bullet(G_K, \tilde{E}^\bullet_V)$—an acyclic resolution of V. To see this last fact, note that the surjectivity of the map

$$C^n(G_K, \tilde{E}^0_V) \xrightarrow{1\otimes\theta} C^n(G_K, \tilde{E}^1_V)$$

follows from the existence of a continuous section of the surjection

$$t^{-k}B^+_{cris}\left[\frac{1}{p}\right] \oplus B^+_{dR} \xrightarrow{\theta} t^{-k}B^+_{cris}\left[\frac{1}{p}\right] \oplus t^{-k}B^+_{dR}, \qquad k \geqslant 0,$$

which, in turn, follows from the fact that the surjections

$$\left(t^{-k}B^+_{cris}\left[\frac{1}{p}\right]\right)^{\phi=1} \oplus B^+_{dR} \to t^{-k}B^+_{dR}, \qquad (x, y) \to x - y,$$

$$t^{-k}F^kB^+_{cris}\left[\frac{1}{p}\right] \xrightarrow{1-p^{-k}\phi} t^{-k}B^+_{cris}\left[\frac{1}{p}\right],$$

admit continuous sections (cf. [3, 1.18] and the beginning of this section respectively). Finally, we set

$$H^*_f(G_K, V) := H^*(\tilde{B}^\bullet_V), \qquad {}_xH^*_f(G_K, V) := H^*(\mathrm{Cone}(\tilde{B}^\bullet_V \xrightarrow{t} \tilde{A}^{\bullet G_K}_V)[-1]).$$

Product structures are given by a family of mappings

$$\tilde{E}^\bullet_V \otimes \tilde{E}^\bullet_W \xrightarrow{\cup_\alpha} \tilde{E}^\bullet_{V\otimes W}$$

having properties (i)–(v) from Proposition 4.1. More explicitly, writing $\gamma \in \tilde{E}^\bullet_V$, $\gamma' \in \tilde{E}^\bullet_W$ as $\gamma = (x, w)$, $\gamma' = (y, z)$, we have

$$\gamma \cup_\alpha \gamma' = \begin{cases} (xy, wx) & \text{if} \deg \gamma = \deg \gamma' = 0 \\ ([(1-\alpha)\phi(x) + \alpha x]y, [(1-\alpha)w + \alpha x]z) & \text{if} \deg \gamma = 0, \deg \gamma' = 1 \\ (x[\alpha\phi(y) + (1-\alpha)y], w[\alpha z + (1-\alpha)y]) & \text{if} \deg \gamma = 1, \deg \gamma' = 0 \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 4.3. *Let $T \in \mathrm{Rep}_{cr,a,b}$. Then*

(1) $H^*_{f,a,b}(G_K, T) \xrightarrow{\sim} \text{proj lim } H^*_{f,a,b}(G_K, T/p^rT)$;
(2) *there exists a canonical isomorphism*

$$\eta\colon H^*_{f,a,b}(G_K, T) \otimes \mathbf{Q} \xrightarrow{\sim} H^*_f(G_K, T \otimes \mathbf{Q}_p)$$

which commutes with the products and the maps into the Galois cohomology.

Proof. Set $T_{\mathbf{Q}_p} = T \otimes \mathbf{Q}_p$. Since T/p^rT is finite, the first assertion is straightforward. To prove the second consider the commutative diagram

$$\begin{array}{ccccc} T & \xrightarrow{\mu} & T \otimes F^{(-a)}B^+_{cris}(a) & \xrightarrow{1-1\otimes p^a\phi} & T \otimes B^+_{cris}(a) \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \eta_0} & & \downarrow{\scriptstyle \eta_1} \\ T_{\mathbf{Q}_p} & \xrightarrow{\Delta} & T_{\mathbf{Q}_p} \otimes B_{cris} \oplus T_{\mathbf{Q}_p} \otimes B^+_{dR} & \xrightarrow{\theta} & T_{\mathbf{Q}_p} \otimes B_{cris} \oplus T_{\mathbf{Q}_p} \otimes B_{dR}, \end{array}$$

where $\mu(v) = v \otimes t^{-a} \otimes \zeta^a$, $\Delta(v) = (v \otimes 1, v \otimes 1)$, $\eta_0(v \otimes x) = (v \otimes \omega(x), v \otimes \omega(x))$, and $\eta_1(v \otimes x) = (v \otimes \omega(x), 0)$. The maps η_0, η_1 induce a map of complexes η: $B^\cdot_{T,a,b} \to \tilde{B}^\cdot_{T_{\mathbf{Q}_p}}$. Tensoring with $\mathbf{Q}$ yields a commutative diagram

$$\begin{array}{cccc} B^\cdot_{T,a,b}: & \mathbf{D}^0_{cr}(T) \otimes \mathbf{Q} & \xrightarrow{1-\phi_0} & \mathbf{D}_{cr} \otimes \mathbf{Q} \\ & \downarrow{\scriptstyle \eta_0} & & \downarrow{\scriptstyle \eta_1} \\ \tilde{B}^\cdot_{T_{\mathbf{Q}_p}}: & \mathbf{D}_{cr}(T_{\mathbf{Q}_p}) \oplus \mathbf{D}^0_{dR}(T_{\mathbf{Q}_p}) & \xrightarrow{\theta} & \mathbf{D}_{dR}(T_{\mathbf{Q}_p}) \oplus \mathbf{D}_{dR}(T_{\mathbf{Q}_p}), \end{array}$$

from which it is clear, since $\mathbf{D}_{cr}(T_{\mathbf{Q}_p}) \tilde{\to} \mathbf{D}_{dR}(T_{\mathbf{Q}_p})$, that η gives the required isomorphism. The remaining assertions are easy to verify. □

The same statements are true for the cohomology with support on the closed point x.

5. Extensions in the category of global crystalline representations. Let K be a number field absolutely unramified above p, S a finite set of primes containing all primes above p and ∞, $G = \mathrm{Gal}(\bar{K}/K)$, and $G_S = \mathrm{Gal}(K_S/K)$, where K_S is the maximal subfield of $\bar{K}$ unramified outside of S. For every prime v of K, let K_v be the completion of K at v and $D_v = \mathrm{Gal}(\bar{K}_v/K_v)$. Fix also an embedding $\bar{K} \hookrightarrow \bar{K}_v$. This induces a map $D_v \to G \to G_S$ which determines the restriction res_v: $H^i(G_S, V) \to H^i(D_v, V)$. Let $\mathrm{Rep}_{cr,S,a,b}$ be the category of finitely generated $\mathbf{Z}_p[G_S]$-modules which are crystalline of weight between a and b at every $v|p$, for $a \leqslant 0$, $b \geqslant 0$, $b - a \leqslant p - 2$.

We will define certain cohomological functors on the category $\mathrm{Rep}_{cr,S,a,b}$. For $T \in \mathrm{Rep}_{cr,S,a,b}$, set

$$D^\cdot_{T,S,a,b} := \mathrm{Cone}\left(S^{\cdot G_S}_T \bigoplus \oplus_{v|p} B^\cdot_{T,v} \xrightarrow{r-t} \oplus_{v|p} A^{\cdot D_v}_{T,v}\right)[-1],$$

$$H^*_{f,a,b}(G_S, T) := H^*(D^\cdot_{T,S,a,b}),$$

where

— $B^\cdot_{T,v}$, $E^\cdot_{T,v}$, and $A^\cdot_{T,v}$ are the complexes $B^\cdot_T$, $E^\cdot_T$, and $A^\cdot_T$ for K_v;
— $S^\cdot_T = C^\cdot(G_S, T)$ with the standard cup product;

— $S^{\bullet}_{T,v} = C^{\bullet}(D_v, T)$ with the standard cup product;
— the maps r, t are

$$r\colon S_T^{\bullet G_S} \xrightarrow{\oplus \mathrm{res}_v} \oplus_{v|p} S_{T,v}^{\bullet D_v} \xrightarrow{\oplus f_v} \oplus_{v|p} A_{T,v}^{\bullet D_v},$$

$$t\colon \oplus_{v|p} B^{\bullet}_{T,v} \to \oplus_{v|p} E_{T,v}^{\bullet D_v} \xrightarrow{\oplus g_v} \oplus_{v|p} A_{T,v}^{\bullet D_v}.$$

We will often write $D^{\bullet}_T$ and $H^*_f(G_S, T)$, for short. $H^*_{f,a,b}(G_S, \cdot)$ is a cohomological functor such that

$$H^0_f(D_S, T) = T^{G_S}, \qquad H^1_f(G_S, T) = \mathrm{Ext}^1_{\mathrm{Rep}_{cr,S,a,b}}(\mathbf{Z}_p, T),$$

$$H^i_f(G_S, T) \xrightarrow{\sim} H^i(G_S, T), \qquad \text{for } i \geqslant 4.$$

Let $R = \{v|p\}$, $S' = S_{fin} - R$. Set

$$H^i_c(G_{S/S'}, T) := H^i(\mathrm{Cone}(S_T^{\bullet G_S} \xrightarrow{\mathrm{res}} \oplus_{v|p} A_{T,v}^{\bullet D_v})[-1]).$$

Then, writing $D^{\bullet}_{T,S,a,b}$ in three different ways

$$\begin{aligned} D^{\bullet}_{T,S,a,b} &= \mathrm{Cone}(S_T^{\bullet G_S} \bigoplus \oplus_{v|p} B^{\bullet}_{T,v} \xrightarrow{r-t} \oplus_{v|p} A_{T,v}^{\bullet D_v})[-1]) \\ &= \mathrm{Cone}(S_T^{\bullet G_S} \xrightarrow{r} \oplus_{v|p} \mathrm{Cone}(B^{\bullet}_{T,v} \xrightarrow{-t} A_{T,v}^{\bullet D_v}))[-1] \\ &= \mathrm{Cone}(\oplus_{v|p} B^{\bullet}_{T,v} \xrightarrow{-t} \mathrm{Cone}(S_T^{\bullet G_S} \xrightarrow{r} \oplus_{v|p} A_{T,v}^{\bullet D_v}))[-1], \end{aligned}$$

we get three long exact sequences

$$(2) \qquad \begin{aligned} &\to H^i_f(G_S, T) \to H^i(G_S, T) \bigoplus \oplus_{v|p} H^i_f(D_v, T) \xrightarrow{\mathrm{res}-l} \oplus_{v|p} H^i(D_v, T) \to, \\ &\to H^{i-1}(G_S, T) \xrightarrow{r} \oplus_{v|p}\, {}_vH^i_f(D_v, T) \to H^i_f(G_S, T) \to H^i(G_S, T) \xrightarrow{r}, \\ &\to \oplus_{v|p} H^{i-1}_f(D_v, T) \xrightarrow{-t} H^i_c(G_{S/S'}, T) \to H^i_f(G_S, T) \to \oplus_{v|p} H^i_f(D_v, T) \xrightarrow{-t}. \end{aligned}$$

Here, the map ${}_vH^*_f(D_v, T) \to H^*_f(G_S, T)$ is induced by multiplication by -1 on $B^{\bullet}_{T,v}$.

THEOREM 5.1. *If $T \in \mathrm{Rep}_{cr,S,a,b}$, $U \in \mathrm{Rep}_{cr,S,c,d}$, then there exists a canonical product*

$$\cup\colon H^p_{f,a,b}(G_S, T) \otimes H^q_{f,c,d}(G_S, U) \to H^{p+q}_{f,a+c,b+d}(G_S, T \otimes U)$$

which is anticommutative and associative. It commutes with the projections into $H^*(G_S, T)$ and $H^*_f(D_v, T)$, $v|p$.

Proof. The product $\cup$ is defined by the family of canonical mappings

$$\cup_\alpha\colon D^\bullet_{T,S,a,b} \otimes D^\bullet_{U,S,c,d} \to D^\bullet_{T\otimes U,S,a+c,b+d}$$

for every $\alpha \in \mathbf{Z}_p$, constructed below. Associativity, anticommutativity, and commutation with projections follow.

Define $\cup_\alpha$ as follows: represent $\gamma, \gamma' \in D^\bullet_T, D^\bullet_U$ as

$$\gamma = (a_T, \oplus_{v|p} b_{T,v}, \oplus_{v|p} c_{T,v}) \qquad a_T \in S^\bullet_T,\ b_{T,v} \in B^\bullet_{T,v},\ c_{T,v} \in A^\bullet_{T,v},$$

$$\gamma' = (a_U, \oplus_{v|p} b_{U,v}, \oplus_{v|p} c_{U,v}) \qquad a_U \in S^\bullet_U,\ b_{U,v} \in B^\bullet_{U,v},\ c_{U,v} \in A^\bullet_{U,v},$$

and set

$$\gamma \cup_\alpha \gamma' = (a_T \cup_\alpha a_U, \oplus_{v|p} b_{T,v} \cup_\alpha b_{U,v},$$
$$\oplus_{v|p}\{(-1)^{\deg a_T}[(1-\alpha)\tilde{a}_T + \alpha \tilde{b}_{T,v}] \cup_\alpha c_{U,v} + c_{T,v}$$
$$\cup_\alpha [\alpha \tilde{a}_U + (1-\alpha)\tilde{b}_{U,v}]\}),$$

where $\sim$ is r_v or t_v and $\cup_\alpha$ is $\cup_\alpha$ on $B^\bullet_{T,v}$ or on $A^\bullet_{T,v}$ or the standard product on $S^\bullet_T$ depending on the context. We claim that the maps $\cup_\alpha$'s have the following properties:

(i) $\cup_\alpha$'s are maps of complexes;
(ii) $\cup_0$, $\cup_1$ are associative;
(iii) all $\cup_\alpha$'s are homotopic;
(iv) there exists a transposition operator $\mathscr{T}\colon D^\bullet_T \to D^\bullet_T$ which is homotopic to the identity and such that, for any two cochains γ, γ',

$$\mathscr{T}\gamma \cup_\alpha \mathscr{T}\gamma' = (-1)^{\deg \gamma \cdot \deg \gamma'} \mathscr{T}(\gamma' \cup_{1-\alpha} \gamma).$$

Indeed, let $\alpha, \beta \in \mathbf{Z}_p$, $\alpha \neq \beta$, and fix $v|p$. Recall that Proposition 4.1 supplies us with commuting homotopies h_{B_v} and h_{E_v} (for complexes $B^\bullet_v$ and $E^\bullet_v$ respectively) between the maps $\cup_\alpha$ and $\cup_\beta$. Define the homotopy

$$h_{A_v}\colon (A^\bullet_{T,v} \otimes A^\bullet_{U,v})^k \to A^{k-1}_{T\otimes U,v}$$

between $\cup_\alpha$ and $\cup_\beta$ as follows: if $\gamma \in C^a(G_S, E^b_{T,v})$, $\gamma' \in C^e(G_S, E^f_{U,v})$, then

$$h_{A_v}(\gamma \otimes \gamma')(d_0, \ldots, d_{a+e}) = (-1)^{be+(a+e)} h_{E_v}(\gamma(d_0, \ldots, d_a) \otimes \gamma'(d_a, \ldots, d_{a+e})).$$

That h_{A_v} is a homotopy follows from the fact that h_{E_v} is one. It commutes with the homotopy h_{E_v} and the trivial homotopy on $S^\bullet_T$. All homotopies are Galois equivariant. Now the pairings $\cup_\alpha$'s on $S^\bullet_T$, $B^\bullet_{T,v}$ and $A^\bullet_{T,v}$ satisfy the assumptions of

Proposition 3.1. The maps $\cup_\alpha$ on $D_T^\cdot$ defined above are equal to the ones described in this proposition. Consequently, they have properties (i) and (iii).

To prove (iv), set $\mathcal{T}$ equal to the action of the canonical equivariant transposition operator τ for standard resolutions (cf. the proof of Proposition 4.2) on the first and the third coordinates of $D_T^\cdot$. $\mathcal{T}$ is homotopic to the identity via the homotopy equal to h' (again, see the proof of Proposition 4.2) on the first coordinate, 0 on the second, and $-h'$ on the third. Finally, since for standard cochains γ, γ', τ satisfies $\tau\gamma \cup \tau\gamma' = (-1)^{\deg\gamma\cdot\deg\gamma'}\tau(\gamma' \cup \gamma)$ and $\cup_\alpha$ anticommutes with $\cup_{1-\alpha}$ on the complexes $B_{T,v}^\cdot$ we get the required formula.

Assertion (ii) is straightforward. □

For K any number field and V any rational crystalline representation, we can define the groups $H_f^*(G_S, V)$ in a similar way. Having the definition of the local products, the definition of the global rational product is a formal analogue of the integral one.

Remark 1. The groups $H_f^*(G_S, V)$ are equal to the groups $H^*_{f,\mathrm{Spec}(\mathcal{O}_{S'})}(K, V)$ defined by Bloch and Kato [3, 5.1].

Again, we will need cohomology with compact support. For $T \in \mathrm{Rep}_{cr,S,a,b}$, define

$$_!D_{T,S,a,b}^\cdot := \mathrm{Cone}\left(S_T^{\cdot G_S} \bigoplus \oplus_{v|p} B_{T,v}^\cdot \xrightarrow{r+\mathrm{res}-t} \oplus_{v|p} A_{T,v}^{\cdot D_v} \bigoplus \oplus_{v\in S'} S_{T,v}^{\cdot D_v}\right)[-1],$$

$$_!H_{f,a,b}^*(G_S, T) := H^*(_!D_{T,S,a,b}^\cdot).$$

Here $(r + \mathrm{res} - t)(s, b) = (r(s), \mathrm{res}(s) - t(b))$ for $s \in S_T^{\cdot G_S}$ and $b \in \oplus_{v|p} B_{T,v}^\cdot$. We will often write $_!D_T^\cdot$, $_!H_f^*(G_S, T)$, for short. From the definition of $_!H_{f,a,b}^*(G_S, \cdot)$ we get the long exact sequence

$$\to \oplus_{v\in S'} H^{i-1}(D_v, T) \to {_!H_f^i}(G_S, T) \to H_f^i(G_S, T) \to \oplus_{v\in S'} H^i(D_v, T) \to .$$

Set

$$H_c^*(G_S, T) := H^*(\mathrm{Cone}(S_T^{\cdot G_S} \xrightarrow{\mathrm{res}} \oplus_{v\in S_{fin}} S_{T,v}^{\cdot D_v}))[-1],$$

$$H_c^*(G_{S/R}, T) := H^*(\mathrm{Cone}(S_T^{\cdot G_S} \xrightarrow{\mathrm{res}} \oplus_{v\in S'} S_{T,v}^{\cdot D_v}))[-1].$$

Then writing $_!D_{T,S,a,b}^\cdot$ in three different ways

$$\begin{aligned} _!D_{T,S,a,b}^\cdot &= \mathrm{Cone}\left(S_T^{\cdot G_S} \bigoplus \oplus_{v|p} B_{T,v}^\cdot \xrightarrow{r+\mathrm{res}-t} \oplus_{v|p} A_{T,v}^{\cdot D_v} \bigoplus \oplus_{v\in S'} S_{T,v}^{\cdot D_v}\right)[-1] \\ &= \mathrm{Cone}(\mathrm{Cone}(S_T^{\cdot G_S} \xrightarrow{\mathrm{res}} \oplus_{v\in S'} S_{T,v}^{\cdot D_v})[-1] \xrightarrow{r} \oplus_{v|p}\mathrm{Cone}(B_{T,v}^\cdot \xrightarrow{-t} A_{T,v}^{\cdot D_v}))[-1] \\ &= \mathrm{Cone}\left(\oplus_{v|p} B_{T,v}^\cdot \xrightarrow{-t} \mathrm{Cone}\left(S_T^{\cdot G_S} \xrightarrow{r+\mathrm{res}} \oplus_{v|p} A_{T,v}^{\cdot D_v} \bigoplus \oplus_{v\in S'} S_{T,v}^\cdot\right)\right)[-1], \end{aligned}$$

we get three long exact sequences

$$(3)\quad \to {}_!H^i_f(G_S, T) \to H^i(G_S, T) \bigoplus \oplus_{v|p} H^i_f(D_v, T) \xrightarrow{r+\mathrm{res}-t} \oplus_{v\in S_{fin}} H^i(D_v, T) \to,$$

$$\to H^{i-1}_c(G_{S/R}, T) \xrightarrow{r} \oplus_{v|p}\, {}_vH^i(D_v, T) \to {}_!H^i_f(G_S, T) \to H^i_c(G_{S/R}, T) \xrightarrow{r},$$

$$\to \oplus_{v|p} H^{i-1}_f(D_v, T) \xrightarrow{-l} H^i_c(G_S, T) \to {}_!H^i_f(G_S, T) \to \oplus_{v|p} H^i_f(D_v, T) \xrightarrow{-l}.$$

Moreover, for $i \geqslant 3$, $H^i_c(G_S, T) \xrightarrow{\sim} {}_!H^i_f(G_S, T)$.

Remark 2. The following remark gives some insight into the above long exact sequences. Class field theory yields the isomorphism [17, II.2.9]

$$H^i(G_S, T) \cong H^i_{\acute{e}t}(U, \mathscr{T}),$$

where $U = \mathrm{Spec}(\mathscr{O}_K) - S_{fin}$, and $\mathscr{T}$ is the étale sheaf associated to T. Using that and [17, II.1.1], we find the isomorphisms

$$H^i_c(G_S, T) \cong H^i_{\acute{e}t}(\mathrm{Spec}(\mathscr{O}_K), j_!\mathscr{T}),$$

$$H^i_c(G_{S/S'}, T) \cong H^i_{\acute{e}t}(\mathrm{Spec}(\mathscr{O}_K) - S', k_!\mathscr{T}),$$

$$H^i_c(G_{S/R}, T) \cong H^i_{\acute{e}t}(\mathrm{Spec}(\mathscr{O}_K) - \{v|p\}, l_!\mathscr{T}),$$

where $j\colon U \hookrightarrow \mathrm{Spec}(\mathscr{O}_K)$, $k\colon U \hookrightarrow \mathrm{Spec}(\mathscr{O}_K) - S'$, $l\colon U \hookrightarrow \mathrm{Spec}(\mathscr{O}_K) - \{v|p\}$.

THEOREM 5.2. *For $T \in \mathrm{Rep}_{cr,S,a,b}$ and $U \in \mathrm{Rep}_{cr,S,c,d}$, there exist canonical products*

$$\cup\colon H^p_{f,a,b}(G_S, T) \otimes {}_!H^q_{f,c,d}(G_S, U) \to {}_!H^{p+q}_{f,a+c,b+d}(G_S, T \otimes U),$$

$$\tilde{\cup}\colon {}_!H^q_{f,c,d}(G_S, U) \otimes H^p_{f,a,b}(G_S, T) \to {}_!H^{p+q}_{f,a+c,b+d}(G_S, T \otimes U).$$

$\cup$ anticommutes with $\tilde{\cup}$, and they both commute with the projections into $H^*(G_S, T)$ and $H^*_f(D_v, T)$, $v|p$. Moreover, the diagrams

$$\begin{array}{ccc} {}_xH^q_f(D_v, U) & \xrightarrow{\ \cup\ } & \mathrm{Hom}(H^p_f(D_v, T), {}_xH^{p+q}_f(D_v, T \otimes U)) \\ \downarrow & & \downarrow \\ H^q_f(G_S, U) & \xrightarrow{\ \tilde{\cup}\ } & \mathrm{Hom}({}_!H^p_f(G_S, T), {}_!H^{p+q}_f(G_S, T \otimes U)), \end{array} \qquad \textit{for } v|p,$$

$$\begin{array}{ccc} {}_!H^{q+1}_f(G_S, T) & \xrightarrow{\ \cup\ } & \mathrm{Hom}(H^p_f(G_S, U), {}_!H^{p+q+1}_f(G_S, T \otimes U)) \\ \uparrow & & \uparrow \\ H^q(D_v, T) & \xrightarrow{\ \cup\ } & \mathrm{Hom}(H^p(D_v, T), H^{p+q}(D_v, T \otimes U)), \end{array} \qquad \textit{for } v \in S'$$

commute (the second one only up to multiplicaton by $(-1)^p$).

Proof. We construct the products $\cup$ and $\tilde{\cup}$ in a way analogous to the local setting (Proposition 4.2). The commutativity of the diagrams easily follows. □

The above cohomology groups (with or without compact support) are independent of the range of the weight chosen: for two different choices, the natural transformation ψ defined in Section 2 gives a δ-isomorphism of the corresponding cohomological functors which is easily seen to preserve product structures.

The definitions are similiar in the rational case. An analogue of Proposition 4.3 holds for both functions $H_f^*(G_S, \cdot)$ and ${}_!H_f^*(G_S, \cdot)$.

6. Dualities. We keep the notation from the previous sections. In addition, if V is a rational Galois representation, then set $V^D = \mathrm{Hom}(V, \mathbf{Q}_p(1))$, and if T is a finite Galois representation, then set $T^D = \mathrm{Hom}(T, \mathbf{Q}/\mathbf{Z}(1))$. Returning to the local setting we quote from [3] the following statement.

PROPOSITION 6.1. *If V is a rational crystalline representation, then $H_f^1(G_K, V^D)$ is the exact annihilator of $H_f^1(G_K, V)$ in the perfect pairing*

$$H^1(G_K, V) \times H^1(G_K, V^D) \to H^2(G_K, \mathbf{Q}_p(1)) \tilde{\to} \mathbf{Q}_p.$$

Similarly, we have the following statement.

PROPOSITION 6.2. *Let $a \leqslant -1$, $b \geqslant 0$, $b - a \leqslant (p-2)/2$. Then, for any finite Galois representation T from $\mathrm{Rep}_{cr,a,b}$, $H^1_{f,-b-1,-a-1}(G_K, T^D)$ is the exact annihilator of $H^1_{f,a,b}(G_K, T)$ in the perfect pairing*

$$H^1(G_K, T) \times H^1(G_K, T^D) \to H^2(G_K, \bar{K}^*) \cong \mathbf{Q}/\mathbf{Z}.$$

Proof. The commutative diagram

$$\begin{array}{ccc} H_f^1(G_K, T) \times H_f^1(G_K, T^D) & \longrightarrow & H_f^2(G_K, T \otimes T^D) = 0 \\ \downarrow{\scriptstyle \iota \times \iota} & & \downarrow{\scriptstyle \iota} \\ H^1(G_K, T) \times H^1(G_K, T^D) & \longrightarrow & H^2(G_K, T \otimes T^D) \end{array}$$

shows that $H_f^1(G_K, T)$ and $H_f^1(G_K, T^D)$ annihilate each other. It remains to compare their $\mathbf{Z}_p$-lengths. Let $M = \mathbf{D}_{cr}(T)$. Then $M^*\{-1\} \cong \mathbf{D}_{cr}(T^D)$, where M^* denotes the $\mathcal{MF}$-dual [4]. Moreover, $\lg(F^0M) + \lg(F^0M^*\{-1\}) = \lg(M)$, and therefore by local Tate duality

$$\begin{aligned} \lg(H_f^1(G_K, T)) + \lg(H_f^1(G_K, T^D)) &= \lg(M) - \lg(F^0M) + \lg(H_f^0(G_K, T)) \\ &\quad + \lg(M^*\{-1\}) - \lg(F^0M^*\{-1\}) \\ &\quad + \lg(H_f^0(G_K, T)) \end{aligned}$$

$$\begin{aligned}&= \lg(M) + \lg(M^*\{-1\}) - \lg(F^0M) \\ &\quad - \lg(F^0M^*\{-1\}) + \lg(H^0(G_K, T)) \\ &\quad + \lg(H^2(G_K, T)) \\ &= \lg(M^*\{-1\}) - [K\colon \mathbf{Q}_p] \lg(T) + \lg(H^1(G_K, T)) \\ &= \lg(H^1(G_K, T)),\end{aligned}$$

as desired. □

COROLLARY 6.1. *Assume $p > 2$.*

(a) *Let $a \leqslant -1$, $b \geqslant 0$, $b - a \leqslant p - 2$. Then, for every n, there exists an isomorphism ${}_xH^3_{f,a,b}(G_K, \mu_{p^n}) \tilde{\to} \mathbf{Z}/p^n$.*

(b) *Let $a \leqslant -1$, $b \geqslant 0$, $b - a \leqslant (p-2)/2$. Then, for any Galois representation T from $\mathrm{Rep}_{cr,a,b}$ annihilated by p^n, the cup product gives a perfect pairing*

$$ {}_xH^i_{f,a,b}(G_K, T) \times H^{3-i}_{f,-b-1,-a-1}(G_K, T^D) \to {}_xH^3_{f,a-b-1,b-a-1}(G_K, \mu_{p^n}) \tilde{\to} \mathbf{Z}/p^n. $$

Proof. Define the trace as the composition

$$ {}_xH^3_f(G_K, \mu_{p^n}) \tilde{\to} H^2(G_K, \mu_{p^n}) \underset{\sim}{\overset{\mathrm{inv}}{\to}} \mathbf{Z}/p^n. $$

We easily find the commutative diagram

$$\begin{array}{ccc} {}_xH^i_f(G_K, T) & \xrightarrow{\ \cup\ } & \mathrm{Hom}(H^{3-i}_f(G_K, T^D), {}_xH^3_f(G_K, T \otimes T^D)) \\ \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta \cdot l^*} \\ H^{i-1}(G_K, T) & \xrightarrow{\ \cup\ } & \mathrm{Hom}(H^{3-i}(G_K, T^D), H^2(G_K, T \otimes T^D)). \end{array}$$

It shows that, for $i = 3$, when δ and l are isomorphisms, assertion (b) follows from local Tate duality, and for $i = 2$, when ${}_xH^2_f(G_K, T) \tilde{\leftarrow} H^1(G_K, T)/H^1_f(G_K, T)$, it follows from the previous theorem. For $i \neq 2, 3$, assertion (b) is trivial. □

COROLLARY 6.2. *Assume $p > 2$.*

(a) *There exists an isomorphism ${}_xH^3_f(G_K, \mathbf{Q}_p(1)) \tilde{\to} \mathbf{Q}_p$.*

(b) *For any rational crystalline representation V, the cup product gives a perfect pairing*

$$ {}_xH^i_f(G_K, V) \times H^{3-i}_f(G_K, V^D) \to {}_xH^3_f(G_K, \mathbf{Q}_p(1)) \tilde{\to} \mathbf{Q}_p. $$

We turn now to the global setting. The next two propositions will establish the existence of traces.

PROPOSITION 6.3. *Assume $p > 2$. Let $a \leqslant -1, b \geqslant 0, b - a \leqslant p - 2$. For every n, there exist isomorphisms*

$$H^3_c(G_S, \mu_{p^n}) \tilde{\to} \mathbf{Z}/p^n, \qquad {}_!H^3_{f,a,b}(G_S, \mu_{p^n}) \tilde{\to} \mathbf{Z}/p^n.$$

Proof. We have found earlier the isomorphism $\varphi: H^3_c(G_S, \mu_{p^n}) \tilde{\to} {}_!H^3_f(G_S, \mu_{p^n})$ and the exact sequence

$$H^2(G_S, \mu_{p^n}) \xrightarrow{\oplus \mathrm{res}_v} \oplus_{v \in S_{fin}} H^2(D_v, \mu_{p^n}) \to H^3_c(G_S, \mu_{p^n}) \to 0.$$

On the other hand, global Tate duality yields the exact sequence

$$H^2(G_S, \mu_{p^n}) \xrightarrow{\oplus \mathrm{res}_v} \oplus_{v \in S_{fin}} H^2(D_v, \mu_{p^n}) \xrightarrow{\oplus \mathrm{inv}_v} \mathbf{Z}/p^n \to 0.$$

This gives us one of the required isomorphisms $\psi: H^3_c(G_S, \mu_{p^n}) \tilde{\to} \mathbf{Z}/p^n$. Define the other one as the composition ${}_!H^3_f(G_S, \mu_{p^n}) \xrightarrow{\varphi^{-1}} H^3_c(G_S, \mu_{p^n}) \xrightarrow{\psi} \mathbf{Z}/p^n$. □

We find easily that the maps

$${}_vH^3_f(D_v, \mu_{p^n}) \to {}_!H^3_f(G_S, \mu_{p^n}), \qquad {}_vH^3_f(D_v, \mu_{p^n}) \to H^3_c(G_S, \mu_{p^n}),$$

commute with the trace

$${}_vH^3_f(D_v, \mu_{p^n}(\overline{K})) \tilde{\to} {}_vH^3_f(D_v, \mu_{p^n}(\overline{K}_v)) \tilde{\to} \mathbf{Z}/p^n$$

and the above defined traces.

Remark 3. Both local and global traces are compatible with the change of the range of the weight.

In the rational case we have similarly the following proposition.

PROPOSITION 6.4. *Assuming $p > 2$, there exist isomorphisms*

$$H^3_c(G_S, \mathbf{Q}_p(1)) \tilde{\to} \mathbf{Q}_p, \qquad {}_!H^3_f(G_S, \mathbf{Q}_p(1)) \tilde{\to} \mathbf{Q}_p,$$

compatible with the local traces.

Combining local and global standard products, we get two anticommuting canonical products (apply Proposition 3.1 with $A^{\cdot}_i = S^{\cdot G_S}_{Y_i}$, $C^{\cdot}_i = \oplus_{v \in S_{fin}} S^{\cdot D_v}_{Y_i, v}$, $f_i =$ res, $g_i = 0$, where $Y_1 = V$, $Y_2 = W$, $Y_3 = V \otimes W$, together with the argument from the proof of Proposition 4.2)

$$\cup: H^i(G_S, V) \otimes H^k_c(G_S, W) \to H^{i+k}_c(G_S, V \otimes W),$$

$$\tilde{\cup}: H^k_c(G_S, W) \otimes H^i(G_S, V) \to H^{i+k}_c(G_S, W \otimes V).$$

It is a question of unwinding the definitions to check that these products are compatible with the local and global crystalline cup products, i.e., that the diagrams

$$\begin{array}{ccc} {}_!H^p_f(G_S, V) & \xrightarrow{\cup} & \mathrm{Hom}(H^q_f(G_S, W), {}_!H^{p+q}_f(G_S, V \otimes W)) \\ \uparrow & & \uparrow \\ H^p_c(G_S, V) & \xrightarrow{\cup} & \mathrm{Hom}(H^q(G_S, W), H^{p+q}_c(G_S, V \otimes W)), \end{array}$$

$$\begin{array}{ccc} H^{p+1}_c(G_S, V) & \xrightarrow{\cup} & \mathrm{Hom}(H^{q-1}(G_S, W), H^{p+q}_c(G_S, V \otimes W)) \\ {\scriptstyle -l}\uparrow & & \uparrow{\scriptstyle r^*} \\ H^p_f(D_v, V) & \xrightarrow{\cup} & \mathrm{Hom}({}_vH^q_f(D_v, W), {}_vH^{p+q}_f(D_v, V \otimes W)) \end{array}$$

commute (the latter one only up to multiplication by $(-1)^{p+1}$).

We will need the following version of global Tate duality.

LEMMA 6.1. *If $p > 2$, $S \supset \{v|p, v|\infty\}$ and T is a finite G_S-representation annihilated by p^n, then the pairing*

$$H^r(G_S, T) \times H^{3-r}_c(G_S, T^D) \xrightarrow{\cup} H^3_c(G_S, T \otimes T^D) \to H^3_c(G_S, \mu_{p^n}) \xrightarrow{\sim} \mathbf{Z}/p^n$$

is perfect.

Proof. Because of the relation of the above groups with étale cohomology groups, we could try to compare the étale cup product with the Galois one and evoke Artin-Verdier duality, but we prefer to use the following Galois-theoretical argument (all details can be found in [17, II.3.1]):

— the product gives us the map of cohomological functors

$$\gamma^r(T)\colon H^r(G_S, T) \to H^{3-r}_c(G_S, T^D)^*;$$

— for all $r < 0$ and all T, both $H^r(G_S, T)$ and $H^{3-r}_c(G_S, T^D)$ vanish, and thus $\gamma^r(T)$ is an isomorphism;

— $\gamma^0(T)$ is an isomorphism for all T;

— assuming $r_0 \geqslant 1$, we see that if $\gamma^r(T)$ is an isomorphism for all $r < r_0$ and all T, then $\gamma^{r_0}(T)$ is injective;

— if $\gamma^{r_0}(T)$ is injective, then it is, in fact an isomorphism by finiteness of all the groups involved and by the global Tate duality sequence [17, I.4.10];

— by repeating the above, we get that $\gamma^r(T)$ is an isomorphism for all r. □

Using the finiteness of the groups involved and the compatibility of integral and rational traces, we get the following statement.

LEMMA 6.2. *If* $p > 2$, $S \supset \{v|p, v|\infty\}$ *and* T *is a finite-dimensional continuous* $\mathbf{Q}_p[G_S]$*-module, then the pairing*

$$H^r(G_S, T) \times H_c^{3-r}(G_S, T^D) \xrightarrow{\cup} H_c^3(G_S, T \otimes T^D) \to H_c^3(G_S, \mathbf{Q}_p(1)) \to \mathbf{Q}_p$$

is perfect.

THEOREM 6.1. *Let* $a \leqslant -1, b \geqslant 0, b - a \leqslant (p-2)/2$. *Then for any Galois representation* T *from* $\mathrm{Rep}_{cr,S,a,b}$ *annihilated by* p^n, *the cup product gives a perfect pairing*

$$_!H_f^i(G_S, T) \times H_f^{3-i}(G_S, T^D) \to {}_!H_f^3(G_S, \mu_{p^n}) \xrightarrow{\sim} \mathbf{Z}/p^n.$$

Proof. Using the long exact sequences 2 and 3, we get the following diagram with exact rows.

$$\begin{array}{ccccccccc}
\longrightarrow & H_c^i(G_S, T) & \longrightarrow & {}_!H_f^i(G_S, T) & \longrightarrow & \oplus_{v|p} H_f^i(D_v, T) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H^{3-i}(G_S, T^D)^* & \longrightarrow & H_f^{3-i}(G_S, T^D)^* & \longrightarrow & \oplus_{v|p\,v} H_f^{3-i}(D_v, T)^* & \longrightarrow .
\end{array}$$

The maps are defined via cup products and traces. Compatibilities of products and traces listed before yields that all squares either commute or anticommute. Now local duality (Corollary 6.1) and global Tate duality (Lemma 6.1) give that the maps

$$_!H_f^i(G_S, T) \to H_f^{3-i}(G_S, T^D)^*$$

are isomorphisms as desired. □

Again, similarly we have the following result.

THEOREM 6.2. *Let* $p > 2$. *Then for any rational crystalline representation* V, *the cup product gives a perfect pairing*

$$_!H_f^i(G_S, V) \otimes H_f^{3-i}(G_S, V^D) \to H_f^3(G_S, \mathbf{Q}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p.$$

For a torsion-free integral crystalline representation T, denote by $H_f^*(G_S, T_{div})$ the group inj lim $H_f^*(G_S, T/p^n)$. We will use similar notation for local cohomology groups and cohomology with compact support. Set $T^D := \mathrm{Hom}(T, \mathbf{Z}_p(1))$. Define

$$D(T) = \mathrm{Im}({}_!H_f^1(G_S, T_{div}) \to H_f^1(G_S, T_{div}))$$

$$\cong \ker(H_f^1(G_S, T_{div}) \to \oplus_{v \in S'} H^1(D_v, T_{div})).$$

COROLLARY 6.3. *If* $T \in \mathrm{Rep}_{cr,S,a,b}$ *is a torsion-free integral crystalline representation such that* $a \leqslant -1, b \geqslant 0, b - a \leqslant (p-2)/2$ *and* $(T \otimes \mathbf{Q}_p)^{D_v} = (T^D \otimes \mathbf{Q}_p)^{D_v} = 0$

for all $v \in S'$, *then there is a canonical pairing*

$$D(T^D) \times D(T) \to \mathbf{Q}_p/\mathbf{Z}_p$$

annihilating exactly the divisible subgroups.

Proof. Since, by assumption, $H^1(D_v, T \otimes \mathbf{Q}_p) = H^1(D_v, T^D \otimes \mathbf{Q}_p) = 0$ for all $v \in S'$, we have the commutative diagram (see Theorem 5.2)

$$\begin{array}{ccccccc}
0 \longrightarrow & D(T^D)/D(T^D)_{div} & \longrightarrow & \dfrac{H^1_f(G_S, T^D_{div})}{H^1_f(G_S, T^D) \otimes \mathbf{Q}_p/\mathbf{Z}_p} & \longrightarrow & \oplus_{v \in S'} H^1(D_v, T^D_{div}) \\
& & & \downarrow \wr & & \downarrow \wr \\
0 \longrightarrow & W(T)^* & \longrightarrow & {}_!H^2_f(G_S, T)(p)^* & \longrightarrow & \oplus_{v \in S'} H^1(D_v, T)(p)^*,
\end{array}$$

where

$$\begin{aligned} W(T) &= \operatorname{coker}(\oplus_{v \in S'} H^1(D_v, T)(p) \to {}_!H^2_f(G_S, T)(p)) \\ &\cong \ker(H^2_f(G_S, T)(p) \to \oplus_{v \in S'} H^2(D_v, T)(p)), \end{aligned}$$

yielding an isomorphism $D(T^D)/D(T^D)_{div} \xrightarrow{\sim} W(T)^*$. Moreover, the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & D(T)/D(T)_{div} & \longrightarrow & \dfrac{H^1_f(G_S, T_{div})}{H^1_f(G_S, T) \otimes \mathbf{Q}_p/\mathbf{Z}_p} & \longrightarrow & \oplus_{v \in S'} H^1(D_v, T_{div}) \\
& & & \downarrow \wr & & \downarrow \wr \\
0 \longrightarrow & W(T) & \longrightarrow & H^2_f(G_S, T)(p) & \longrightarrow & \oplus_{v \in S'} H^2(D_v, T)(p)
\end{array}$$

demonstrates that there is an isomorphism $D(T)/D(T)_{div} \xrightarrow{\sim} W(T)$. □

Under the assumptions of the corollary, the group $D(T)/D(T)_{div}$ is isomorphic to the Shafarevich group of T, Ш(T), defined in [3]. The above gives thus a nondegenerate pairing of finite groups

$$Ш(T^D) \times Ш(T) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

We hope that it agrees with the pairing defined in [5].

Remark 4. The definitions of the groups H^*_f (local and global) and the cup products make sense for all Galois representations (unramified outside of S in the global case). However, one does not get, in general, cohomological functors.

In particular, if the rational Galois representations considered are de Rham, then the local groups $H^1_f(G_K, \cdot)$ are the same as that defined by Bloch and Kato [3, 3.8.4], and the global groups $H^1_f(G_S, \cdot)$ correspond to their $H^1_{f,\mathrm{Spec}(\mathcal{O}_{S'})}(K, \cdot)$ (see [3, 5.1]). Moreover, the local and global dualities still hold—use the above proofs. (Note that the local one was already proved in [3, 3.8].)

7. Euler Characteristics. Consider now only finite or rational crystalline representations. Then since we assumed S to be finite, all involved cohomology groups are finite or finite-dimensional. The Euler characteristic is defined as the alternating sum of corresponding lengths or dimensions. In the local situation we have the obvious formulas

$$\chi_{cr}(K, V) = \begin{cases} \lg(F^0\mathbf{D}_{cr}(V)) - \lg(\mathbf{D}_{cr}(V)) & \text{if } V \text{ is finite} \\ -\dim_{\mathbf{Q}_p}(\mathbf{D}_{dR}(V)/F^0\mathbf{D}_{dR}(V)) & \text{if } V \text{ is rational;} \end{cases}$$

$$_x\chi_{cr}(K, V) = \begin{cases} \lg(F^0\mathbf{D}_{cr}(V)) & \text{if } V \text{ is finite} \\ \dim_{\mathbf{Q}_p}(F^0\mathbf{D}_{dR}(V)) & \text{if } V \text{ is rational.} \end{cases}$$

Thus, if we take V finite with $b - a \leqslant (p-2)/2$, or V rational and p odd, we get

$$\chi_{cr}(K, V) + {}_x\chi_{cr}(K, V^D) = 0.$$

In the global setting, if T is finite, we get

$$\chi_{cr}(G_S, T) = \sum_{v|\infty} \lg(T^{D_v}) + \sum_{v|p} \chi_{cr}(K_v, T),$$

$$_!\chi_{cr}(G_S, T) = \sum_{v|\infty} \lg(T^{D_v}) + \sum_{v|p} \chi_{cr}(K_v, T) - \left(\sum_{v\in S'} \deg K_v\right)\lg(T).$$

To see this use the exact sequence

$$\to H^i_f(G_S, T) \to H^i(G_S, T) \bigoplus \oplus_{v|p} H^i_f(D_v, T) \to \oplus_{v|p} H^i(D_v, T) \to$$

to derive

$$\begin{aligned} \chi_{cr}(G_S, T) &= \chi(G_S, T) + \sum_{v|p} (\chi_{cr}(K_v, T) - \chi(K_v, T)) \\ &= \sum_{v|\infty} (\lg(T^{D_v}) - \varepsilon(v)\lg(T)) + \sum_{v|p} [K_v: \mathbf{Q}_p]\lg(T) + \sum_{v|p} \chi_{cr}(K, T), \end{aligned}$$

where $\varepsilon(v) = 1$ or 2 depending on whether v is real or complex. If V is rational, we get similarly

$$\chi_{cr}(G_S, V) = \sum_{v|\infty} \dim_{\mathbf{Q}_p}(V^{D_v}) + \sum_{v|p} \chi_{cr}(K_v, V),$$

$$_!\chi_{cr}(G_S, V) = \sum_{v|\infty} \dim_{\mathbf{Q}_p}(V^{D_v}) + \sum_{v|p} \chi_{cr}(K_v, V) - \left(\sum_{v \in S'} \deg K_v\right) \dim_{\mathbf{Q}_p}(V).$$

If we now take V finite with $b - a \leqslant (p-2)/2$, or V rational and p odd, we find (by direct computations or by the duality results of the previous section) that

$$_!\chi_{cr}(G_S, V) + \chi_{cr}(G_S, V^D) = 0.$$

8. Finite flat group schemes. Assume that p is odd. Representations coming from the generic fiber of finite flat commutative p-group schemes over a ring of Witt vectors (or an open set of integers in a number field absolutely unramified above p) are crystalline (respectively crystalline at p and unramified outside of p).

We will construct a δ-isomorphism between our functors applied to these representations and flat cohomology of the corresponding group schemes. When $p \geqslant 5$, this isomorphism will turn out to be compatible with the duality theorems for crystalline representations proved earlier and flat duality theorems for the corresponding group schemes.

For any scheme X, let $\mathscr{G}_X$ denote the category of finite commutative flat group schemes over X annihilated by a power of p and $\mathscr{G}_X(p^n)$—the subcategory of groups annihilated by p^n. If $X = \operatorname{Spec}(A)$, we will also write $\mathscr{G}_A$. Let $X_{\acute{e}t}$ and X_{fl} denote the (small) étale respectively the (big) flat site of X. We set $H^*(X, \cdot) := H^*(X_{fl}, \cdot)$.

For computing flat cohomology we will use the canonical flasque (Godement) resolutions. For all the schemes and the maps of schemes involved, we choose in a compatible and functorial manner (small) conservative families of points of the associated flat topoi [13, XVII]. This assures the functoriality of Godement resolutions; i.e., if for a complex $\mathscr{F}^\cdot$ of flat sheaves on X, $G^\cdot(X_{fl}, \mathscr{F}^\cdot)$ denotes the simple complex associated to the Godement double complex of $\mathscr{F}^\cdot$, then for any map of schemes $f\colon Y \to X$. we have a canonical map $f^*\colon f^*G^\cdot(X_{fl}, \mathscr{F}^\cdot) \to G^\cdot(Y_{fl}, f^*\mathscr{F}^\cdot)$ commuting with the identity on $f^*\mathscr{F}^\cdot$ and behaving well with respect to compositions of maps of schemes.

Like in the classical case [12], here too, Godement resolutions can be used to define products: at first, one uses the explicit description of the category of points above a given one (for any map in a flat site) to find a representation of Godement cochains by functions mapping sequences of points of certain flat opens to the stalks of the given sheaf at these points; then one proceeds to define a canonical map of complexes

$$G^\cdot(X_{fl}, \mathscr{F}) \otimes G^\cdot(X_{fl}, \mathscr{G}) \xrightarrow{\cup} G^\cdot(X_{fl}, \mathscr{F} \otimes \mathscr{G})$$

formally following the definition given in [12]. This map induces the flat cup product on cohomology. There is also a version with supports. Moreover, if $f\colon Y \to X$ is a map of schemes, the canonical map f^* commutes with the product $\cup$. For a scheme X, we set $R^{\cdot}(X, \mathscr{F}^{\cdot}) := \Gamma(X, G^{\cdot}(X_{fl}, \mathscr{F}^{\cdot}))$. (In the case $X = \mathrm{Spec}(A)$, we will also write $R^{\cdot}(A, \mathscr{F}^{\cdot})$.)

Let W be the ring of Witt vectors with coefficients in a finite field k of characteristic p, K its fraction field, and $G_K = \mathrm{Gal}(\overline{K}/K)$. Then the category $\mathscr{G}_W$ is abelian, and the functor L sending a group N to the Galois representation $N_K(\overline{K})$ defines an equivalence of categories between $\mathscr{G}_W$ and $\mathrm{Rep}_{cr,-1,0}$; cf. [10]. Moreover, for the object $\mathbf{D}_{cr}(N_K(\overline{K}))$ of $\mathscr{MF}_{[-1,0]}(W)$ associated to $N_K(\overline{K})$, we have the natural isomorphisms

— $F^{-1}\mathbf{D}_{cr}(N_K(\overline{K})) \cong D(N^t_k)$, the Dieudonné module of the reduction of N^t (the Cartier dual of N) to the special fiber;

— $F^0\mathbf{D}_{cr}(N_K(\overline{K})) \cong \omega_{N^t/W}$, the W-module of invariant differentials on N^t.

THEOREM 8.1. *There are isomorphisms of δ-functors*

$$H^*(\mathrm{Spec}(W), \cdot) \overset{\alpha}{\underset{\sim}{\leftarrow}} H^*_{f,-1,0}(G_K, L(\cdot)), \qquad H^*_x(\mathrm{Spec}(W), \cdot) \overset{\alpha_x}{\underset{\sim}{\leftarrow}} {}_xH^*_{f,-1,0}(G_K, L(\cdot)),$$

which make the diagram with exact rows

$$\begin{array}{ccccccccc}
\longrightarrow & {}_xH^*_{f,-1,0}(G_K, L(N)) & \longrightarrow & H^*_{f,-1,0}(G_K, L(N)) & \longrightarrow & H^*(G_K, L(N)) & \longrightarrow \\
& \wr\downarrow \alpha_x V & & \wr\downarrow \alpha & & \wr\downarrow \omega & \\
\longrightarrow & H^*_x(\mathrm{Spec}(W), N) & \longrightarrow & H^*(\mathrm{Spec}(W), N) & \longrightarrow & H^*(\mathrm{Spec}(K)_{fl}, N) & \longrightarrow
\end{array}$$

commute, where ω is the canonical map. In particular, the isomorphism α_x commutes with the trace maps

$$H^3_x(\mathrm{Spec}(W), \mu_{p^n}) \tilde{\to} \mathbf{Z}/p^n, \qquad {}_xH^3_{f,-1,0}(G_K, \mu_{p^n}(\overline{K})) \tilde{\to} \mathbf{Z}/p^n.$$

Proof. Fix n and $N \in \mathscr{G}_W(p^n)$. Then $H^0(\mathrm{Spec}(W), N) = N(W)$, and

$$H^0_f(G_K, N_K(\overline{K})) = H^0(G_K, N_K(\overline{K})) = N_K(K) \tilde{\leftarrow} N(W).$$

Both functors $H^*(\mathrm{Spec}(W), N)$ and $H^*_{f,-1,0}(G_K, N_K(\overline{K}))$ are erasable since they are nontrivial only in degree 1 (for $H^*(\mathrm{Spec}(W), N)$ see [17, III.1.1]), where both groups $H^1(\mathrm{Spec}(W), N)$ and $H^1_f(G_K, N_K(\overline{K}))$ are isomorphic to the group $\mathrm{Ext}^1(\mathbf{Z}/p^n, N)$ in the category $\mathscr{G}_W$. This allows us to define the isomorphism α as the unique δ-homomorphism

$$H^*_{f,-1,0}(G_K, N_K(\overline{K})) \to H^*(\mathrm{Spec}(W), N)$$

inducing in degree 0 the natural isomorphism $N_K(K) \tilde{\to} N(W)$.

For the other statements of the theorem we need to define this isomorphism in the derived category. Set $j\colon \mathrm{Spec}(K) \to \mathrm{Spec}(W)$ and let $\mathscr{A}^{\bullet}_{N_K(\bar{K})}$ denote the complex of flat sheaves associated to the Galois complex $A^{\bullet}_{N_K(\bar{K})}$. Let the maps

$$a\colon B^{\bullet}_{N_K(\bar{K})} \xrightarrow{t} A^{\bullet G_K}_{N_K(\bar{K})} \xrightarrow{a_1} R^{\bullet}(K, \mathscr{A}^{\bullet}_{N_K(\bar{K})}),$$

$$b\colon R^{\bullet}(W, N) \xrightarrow{b_1} R^{\bullet}(K, N) \xrightarrow{b_2} R^{\bullet}(K, \mathscr{A}^{\bullet}_{N_K(\bar{K})}),$$

be the canonical ones. Since $\mathscr{A}^{\bullet}_{N_K(\bar{K})}$ is an acyclic resolution of N_K on $\mathrm{Spec}(K)$ and $A^{\bullet G_K}_{N_K(\bar{K})} \tilde{\to} \Gamma(\mathrm{Spec}(K), \mathscr{A}^{\bullet}_{N_K(\bar{K})})$, a_1 and b_2 are quasi isomorphisms. Define the complexes

$$C^{\bullet}_N := \mathrm{Cone}(R^{\bullet}(W, N) \oplus B^{\bullet}_{N_K(\bar{K})} \xrightarrow{b-a} R^{\bullet}(K, \mathscr{A}^{\bullet}_{N_K(\bar{K})}))[-1],$$

$$F^{\bullet}_N := C^{\bullet}_{N,\tau\leqslant 1}.$$

LEMMA 8.1. *$F^{\bullet}_N$ defines a δ-functor, and the projections*

$$p\colon F^{\bullet}_N \to R^{\bullet}(W, N), \qquad q\colon F^{\bullet}_N \to B^{\bullet}_{N_K(\bar{K})},$$

define isomorphisms of δ-functors.

Proof. Using the quasi isomorphism b_2, we construct the long exact sequence

$$\to H^i(C^{\bullet}_N) \to H^i(\mathrm{Spec}(W), N) \oplus H^i_f(G_K, N_K(\bar{K})) \xrightarrow{b_1 - b_2^{-1}\cdot a} H^i(\mathrm{Spec}(K), N) \to .$$

In degree 0, b_1 is the natural map $N(W) \to N_K(K)$ and $b_2^{-1}\cdot a$ is equal to the identity; in degree 1 we have

$$0 \to H^1(C^{\bullet}_N) \to H^1(\mathrm{Spec}(W), N) \oplus H^1_f(G_K, N_K(\bar{K})) \xrightarrow{b_1 - b_2^{-1}\cdot a} H^1(\mathrm{Spec}(K), N).$$

Now b_1 and $b_2^{-1}\cdot a$ are injective and have the same image (equal to principal homogeneous spaces coming from $\mathrm{Spec}(W)$). So the projections p and q are isomorphisms. Since $H^1(W, N)$ is right exact, this yields that $F^{\bullet}_N$ defines a δ-functor. □

The thus-obtained δ-isomorphism $H^*_f(G_K, N_K(\bar{K})) \tilde{\to} H^*(\mathrm{Spec}(W), N)$ is given in degree 0 by the inverse of the map $N(W) \to N_K(K)$ and, thus, is equal to the isomorphism defined at the beginning of the proof.

We turn now to the cohomology of the complex $\mathrm{Cone}(B^{\bullet}_{N_K(\bar{K})} \xrightarrow{t} A^{\bullet G_K}_{N_K(\bar{K})})[-1]$. In diagram (4) all squares but the third one commute. The third one commutes up to canonical homotopy $h\colon F_N^k \to R^{\cdot}(K, \mathscr{A}^{\cdot}_{N_K(\bar{K})})^{k-1}$, $h(x, y, z) = -z$. Indeed,

$$\begin{aligned}(hd + dh)(x, y, z) &= h(dx, dy, a(y) - b(x) - dz) - dz \\ &= -a(y) + b(x) + dz - dz \\ &= b(x) - a(y) \\ &= (b \cdot p - a \cdot q)(x, y, z).\end{aligned}$$

This homotopy commutativity is sufficient for the induced canonical maps of cones of the horizontal maps to give for short exact sequences of group schemes maps between corresponding distinguished triangles. Lemma 8.1 gives that all the constructed maps between cones are quasi isomorphisms. Thus, since the natural map

$$\Gamma_x(\mathrm{Spec}(W), G^{\cdot}(\mathrm{Spec}(W)_{fl}, N)) \to \mathrm{Cone}(R^{\cdot}(W, N) \to R^{\cdot}(K, j^*N))[-1],$$

$$(4)\qquad \begin{array}{ccc}
B^{\cdot}_{N_K(\bar{K})} & \xrightarrow{\ t\ } & A^{\cdot G_K}_{N_K(\bar{K})} \\
\Vert & & \downarrow{\scriptstyle a_1} \\
B^{\cdot}_{N_K(\bar{K})} & \xrightarrow{\ a\ } & R^{\cdot}(K, \mathscr{A}^{\cdot}_{N_K(\bar{K})}) \\
\uparrow{\scriptstyle q} & & \Vert \\
F^{\cdot}_N & \xrightarrow{\ a\cdot q\ } & R^{\cdot}(K, \mathscr{A}^{\cdot}_{N_K(\bar{K})}) \\
\downarrow{\scriptstyle p} & & \Vert \\
R^{\cdot}(W, N) & \xrightarrow{\ b\ } & R^{\cdot}(K, \mathscr{A}^{\cdot}_{N_K(\bar{K})}) \\
\Vert & & \uparrow{\scriptstyle b_2} \\
R^{\cdot}(W, N) & \xrightarrow{\ b_1\ } & R^{\cdot}(K, N) \\
\Vert & & \uparrow \\
R^{\cdot}(W, N) & \longrightarrow & R^{\cdot}(K, j^*N)
\end{array}$$

is a quasi isomorphism, we have obtained the needed δ-isomorphism

$$\alpha_x\colon {}_xH^*_{f,-1,0}(G_K, N_K(\bar{K})) \to H^*_x(\mathrm{Spec}(W), N).$$

The existence of the diagram from the statement of the theorem now follows readily from the definition of α and α_x, and the fact that the right vertical composition in diagram (4) induces ω.

Finally, since the trace map $H_x^3(\mathrm{Spec}(W), \mu_{p^n}) \stackrel{\sim}{\to} \mathbf{Z}/p^n$ is defined [17, III.1, II.1] as the composition

$$H_x^3(\mathrm{Spec}(W), \mu_{p^n}) \stackrel{\sim}{\leftarrow} H^2(\mathrm{Spec}(K), \mu_{p^n}) \underset{\sim}{\stackrel{\omega}{\leftarrow}} H^2(G_K, \mu_{p^n}(\overline{K})) \underset{\sim}{\stackrel{\mathrm{inv}}{\to}} \mathbf{Z}/p^n$$

and ω commutes with α_x, we get the compatibility of α_x with traces. □

Remark 5. Let a, b be integers, $a \leqslant -1, b \geqslant 0, b - a \leqslant p - 2$. The above theorem remains valid (and can be proved in the same way) for the cohomology groups with weight between a and b. Moreover, the isomorphisms α, α_x for various choices of the range of the weight are compatible.

THEOREM 8.2. *Assume $p \geqslant 5$. Let N, M, P belong to $\mathscr{G}_W$ and $N \times M \to P$ be a pairing of associated flat sheaves on* Spec(W). *Let $N_K(\overline{K}) \times M_K(\overline{K}) \to P_K(\overline{K})$ be the induced pairing of G_K-representations. Then the diagram*

$$\begin{array}{ccc} H^i_{f,-1,0}(G_K, N_K(\overline{K})) \otimes {}_xH^j_{f,-1,0}(G_K, M_K(\overline{K})) & \xrightarrow{\cup} & {}_xH^{i+j}_{f,-2,0}(G_K, P_K(\overline{K})) \\ \downarrow{\scriptstyle \alpha\otimes\alpha_x}\wr & & \downarrow{\scriptstyle \alpha_x}\wr \\ H^i(\mathrm{Spec}(W), N) \otimes H^j_x(\mathrm{Spec}(W), M) & \xrightarrow{\cup} & H^{i+j}_x(\mathrm{Spec}(W), P) \end{array}$$

commutes. The same holds for cohomology without support.

Proof. We will follow step by step the construction of the isomorphisms α, α_x (diagram (4)). We will show (in a lengthy but straightforward computation), that, due to Proposition 3.1, complexes appearing at particular steps can be equipped with compatible products. We do not need here the full strength of Proposition 3.1 but only the existence of the map of complexes $\cup_0$ and the induced map $\tilde{\cup}_0$ (denoted below simply by $\cup$), which would follow from the compatibility of the partial maps $\cup$ with the maps f_i, $i = 1, 2, 3$. Recall that

$$a_1 \cup (a_2, c_2) = (a_1 \cup a_2, (-1)^{\deg a_1} f_1(a_1) \cup c_2).$$

Set $X = N_K(\overline{K})$, $Y = M_K(\overline{K})$, $Z = P_K(\overline{K})$.

(a) Take

$$\cup\colon B^{\bullet}_X \otimes \mathrm{Cone}(B^{\bullet}_Y \xrightarrow{\iota} A^{\bullet G_K}_Y)[-1] \to \mathrm{Cone}(B^{\bullet}_Z \xrightarrow{\iota} A^{\bullet G_K}_Z)[-1]$$

equal to $\cup$ defined in the proof of Proposition 4.2.

(b) To define

$$\cup\colon B^{\bullet}_X \otimes \mathrm{Cone}(B^{\bullet}_Y \xrightarrow{a} R^{\bullet}(K, \mathscr{A}^{\bullet}_Y))[-1] \to \mathrm{Cone}(B^{\bullet}_Z \xrightarrow{a} R^{\bullet}(K, \mathscr{A}^{\bullet}_Z))[-1],$$

induce at first, from the Galois product $A_X^\bullet \otimes A_Y^\bullet \to A_Z^\bullet$, the product $\mathscr{A}_X^\bullet \otimes \mathscr{A}_Y^\bullet \to \mathscr{A}_Z^\bullet$. It commutes with the pairing $\pi^*\mathscr{X} \otimes \pi^*\mathscr{Y} \to \pi^*\mathscr{Z}$, where π: $\mathrm{Spec}(K)_{fl} \to \mathrm{Spec}(K)_{ét}$, and for a G_K-discrete module W, $\mathscr{W}$ denotes the associated étale sheaf. Set $\cup$ equal to $\tilde{\cup}_0$ from Proposition 3.1. It commutes with the product in (a).

(c) To define a map of complexes $\cup$: $C_N^\bullet \otimes C_M^\bullet \to C_P^\bullet$, use the Godement products and the products defined in (b). To be able to evoke Proposition 3.1, we need only to know that the maps $R^\bullet(W, N) \xrightarrow{b} R^\bullet(K, \mathscr{A}_X^\bullet)$ commute with products. But since b is equal to the composition

$$R^\bullet(W, N) \to R^\bullet(K, N_K) \to R^\bullet(K, \pi^*N_K) \to R^\bullet(K, \pi^*\mathscr{X}) \to R^\bullet(K, \mathscr{A}_X^\bullet),$$

where j: $\mathrm{Spec}(W) \to \mathrm{Spec}\,(K)$, this follows from the fact that the maps $j^*N \to N_K$, $\pi^*N_K \to N_K$, and $N_K \to \mathscr{X}$ commute with the pairings. Induce the map

$$\cup\colon \mathrm{cocycles}(F_N^\bullet) \otimes \mathrm{cocycles}(F_M^\bullet) \to \mathrm{cocycles}(F_P^\bullet)$$

in the obvious way. This map passes to cohomology groups. Define now a map (only on cocycle $\otimes$ cocycle)

$$\cup\colon F_N^\bullet \otimes \mathrm{Cone}(F_M^\bullet \xrightarrow{a\cdot q} R^\bullet(K, \mathscr{A}_Y^\bullet))[-1] \to \mathrm{Cone}(F_P^\bullet \xrightarrow{a\cdot q} R^\bullet(K, \mathscr{A}_Z^\bullet))[-1]$$

by $a_N \cup (a_M, b_M) = (a_N \cup a_M, (-1)^{\deg a_N} a \cdot q(a_N) \cup b_M)$. We claim that $a_N \cup (a_M, b_M)$ is a cocycle. Indeed,

$$\begin{aligned} &d(a_N \cup a_M, (-1)^{\deg a_N} a \cdot q(a_N) \cup b_M) \\ &\quad = (d(a_N \cup a_M),\ -a \cdot q(a_N \cup a_M) - (-1)^{\deg a_N} d[a \cdot q(a_N) \cup b_M]) \\ &\quad = (d(a_N \cup a_M),\ -a \cdot q(a_N) \cup a \cdot q(a_M) - a \cdot q(a_N) \cup db_M) \\ &\quad = 0 \end{aligned}$$

because $da_N = da_M = -db_M - a \cdot q(a_M) = 0$ by assumption. This map commutes with the product from (b).

(d) The compatibilities listed above are now sufficient to define the product

$$\begin{aligned} \cup\colon R^\bullet(W, N) \otimes \mathrm{Cone}(R^\bullet(W, M) \xrightarrow{b} R^\bullet(K, \mathscr{A}_Y^\bullet))[-1] \\ \to \mathrm{Cone}(R^\bullet(W, P) \xrightarrow{b} R^\bullet(K, \mathscr{A}_Z^\bullet))[-1]. \end{aligned}$$

On cocycle $\otimes$ cocycle, it commutes with the map constructed in (c) modulo a coboundary. To see this, note that since $H_x^i(\mathrm{Spec}(W), M) = 0$ unless $i = 2, 3$, it is enough to consider elements of the form $a_N \otimes (0, b_M)$. We have then

$$\begin{aligned} a_N \cup (0, b_M) - p(a_N) \cup (0, b_M) &= (0, (-1)^{\deg a_N}(a \cdot q(a_N) - b \cdot p(a_N)) \cup b_M) \\ &= (0, (-1)^{\deg a_N + 1}(hd + dh)(a_N) \cup b_M) \\ &= (-1)^{\deg a_N} d(0, h(a_N) \cup b_M) \end{aligned}$$

since $da_N = db_M = 0$.

(e) It is now easy to construct compatible products

$$\begin{aligned} &R^{\cdot}(W, N) \otimes \operatorname{Cone}(R^{\cdot}(W, M) \xrightarrow{b_1} R^{\cdot}(K, M_K))[-1] \\ &\qquad \xrightarrow{\cup} \operatorname{Cone}(R^{\cdot}(W, P) \xrightarrow{b_1} R^{\cdot}(K, P_K))[-1], \\ &R^{\cdot}(W, N) \otimes \operatorname{Cone}(R^{\cdot}(W, M) \to R^{\cdot}(K, j^*M))[-1] \\ &\qquad \xrightarrow{\cup} \operatorname{Cone}(R^{\cdot}(W, P) \to R^{\cdot}(K, j^*P))[-1], \end{aligned}$$

compatible with the product from (d) and the Godement product

$$R^{\cdot}(W, N) \otimes \Gamma_x(\operatorname{Spec}(W), G^{\cdot}(\operatorname{Spec}(W)_{fl}, M)) \to \Gamma_x(\operatorname{Spec}(W), G^{\cdot}(\operatorname{Spec}(W)_{fl}, P)).$$

Since $H^2(\operatorname{Spec}(W), P) = 0$, the last assertion of the theorem is straightforward. □

We find the following well-known [17, III.1.3] result.

COROLLARY 8.1. *Assume $p \geqslant 5$. For any $N \in \mathcal{G}_W(p^n)$,*

$$H^i(\operatorname{Spec}(W), N) \times H_x^{3-i}(\operatorname{Spec}(W), N^D) \to H_x^3(\operatorname{Spec}(W), \mu_{p^n}) \xrightarrow{\sim} \mathbf{Z}/p^n$$

is a perfect pairing. □

Let K be a number field absolutely unramified above p and $\mathcal{O}_K$ its ring of integers. Let S be any finite set of primes of K containing all primes above p and ∞, $S' = S_{fin} - \{v \mid p\}$, $U = \operatorname{Spec}(\mathcal{O}_K) - S_{fin}$, and $X = \operatorname{Spec}(\mathcal{O}_K) - S'$. Then the category $\mathcal{G}_X$ is abelian and, by the functor $L\colon G \to G(\overline{K})$, equivalent to $\operatorname{Rep}_{cr, S, -1, 0}$; cf. [1, 1.5]. Let $G = \operatorname{Gal}(\overline{K}/K)$ and $G_S = \operatorname{Gal}(K_S/K)$, where K_S is the maximal extension of K unramified outside of S. For every prime v, let K_v be the completion of K at v, $\mathcal{O}_v$ its ring of integers, and $D_v = \operatorname{Gal}(\overline{K}_v/K_v)$. Fix also an embedding $\overline{K} \hookrightarrow \overline{K_v}$. This induces the maps $D_v \to G \to G_S$.

For $v \mid p$, we have the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(K_v) & \xrightarrow{j_v} & \operatorname{Spec}(\mathcal{O}_v) \\ \downarrow{\scriptstyle p_v} & \searrow^{k_v} & \downarrow{\scriptstyle \pi_v} \\ U & \xrightarrow[u]{} & X, \end{array}$$

in which the left triangle makes sense and commutes for all $v \in S$.

Let $H_c^*(Y, \cdot)$, for Y an open subscheme of the spectrum of the ring of integers in a number field, denote the flat cohomology groups with compact support from [17, III.0]. Recall that, for a flat sheaf $\mathscr{F}$ on Y, $H_c^*(Y, \mathscr{F})$ is defined as the cohomology of the complex

$$\operatorname{Cone}(R^{\cdot}(Y, \mathscr{F}) \to \oplus_{v \in I} R^{\cdot}(\operatorname{Spec}(K_v), k_v^* \mathscr{F}))[-1],$$

where I is the set of finite primes of K not corresponding to a point of Y.

Remark 6. One can also define the groups $H_c^*(Y, \mathscr{F})$ using instead of the field K_v, the fraction field of the henselization of $\mathcal{O}_K$ at v (as in [17, III.0]). In the case of finite flat group schemes, these definitions yield isomorphic cohomology theories.

THEOREM 8.3. *There are isomorphisms of δ-functors*

$$H^*(X, \cdot) \overset{\alpha}{\underset{\sim}{\leftarrow}} H^*_{f,-1,0}(G_S, L(\cdot)), \qquad H_c^*(X, \cdot) \overset{\alpha_c}{\underset{\sim}{\leftarrow}} {}_!H^*_{f,-1,0}(G_S, L(\cdot)),$$

which make the diagrams with exact rows

$$\begin{array}{ccccccccc}
\longrightarrow & {}_!H^i_f(G_S, N(\overline{K})) & \longrightarrow & H^i_f(G_S, N(\overline{K})) & \longrightarrow & \oplus_{v \in S'} H^i(D_v, N(\overline{K})) & \longrightarrow \\
& \wr\downarrow \alpha_c & & \wr\downarrow \alpha & & \wr\downarrow \oplus\omega_1 & \\
\longrightarrow & H^i_c(X, N) & \longrightarrow & H^i(X, N) & \longrightarrow & \oplus_{v \in S'} H^i(\operatorname{Spec}(K_v), N) & \longrightarrow,
\end{array}$$

$$\begin{array}{ccccccccc}
\longrightarrow & \oplus_{v|p}\, {}_vH^i_f(D_v, N(\overline{K})) & \longrightarrow & H^i_f(G_S, N(\overline{K})) & \longrightarrow & H^i(G_S, N(\overline{K})) & \longrightarrow \\
& \wr\downarrow \oplus\alpha_v & & \wr\downarrow \alpha & & \wr\downarrow \omega_2 & \\
\longrightarrow & \oplus_{v|p} H^i_v(\operatorname{Spec}(\mathcal{O})_v, N) & \longrightarrow & H^i(X, N) & \longrightarrow & H^i(U, N) & \longrightarrow,
\end{array}$$

$$\begin{array}{ccccccccc}
\longrightarrow & H^i_c(G_S, N(\overline{K})) & \longrightarrow & {}_!H^i_f(G_S, N(\overline{K})) & \longrightarrow & \oplus_{v|p} H^i_f(D_v, N(\overline{K})) & \longrightarrow \\
& \wr\downarrow \omega_{2,c} & & \wr\downarrow \alpha_c & & \wr\downarrow \oplus\alpha & \\
\longrightarrow & H^i_c(U, N) & \longrightarrow & H^i_c(X, N) & \longrightarrow & \oplus_{v|p} H^i(\operatorname{Spec}(\mathcal{O})_v, N) & \longrightarrow
\end{array}$$

commute. Here ω_1, ω_2 are the canonical maps, and $\omega_{2,c}$ is defined so as to make the diagram with exact rows

$$\begin{array}{ccccccccc}
\longrightarrow & H^i_c(G_S, N(\overline{K})) & \longrightarrow & H^i(G_S, N(\overline{K})) & \longrightarrow & \oplus_{v \in S_{fin}} H^i(D_v, N(\overline{K})) & \longrightarrow \\
& \wr\downarrow \omega_{2,c} & & \wr\downarrow \omega_2 & & \wr\downarrow \oplus\omega_1 & \\
\longrightarrow & H^i_c(U, N) & \longrightarrow & H^i(U, N) & \longrightarrow & \oplus_{v \in S_{fin}} H^i(\operatorname{Spec}(K_v), N) & \longrightarrow
\end{array}$$

commute. In particular, the isomorphism α_c commutes with the trace maps

$$H_c^3(X, \mu_{p^n}) \to \mathbf{Z}/p^n, \qquad {}_!H^3_{f,-1,0}(G_K, \mu_{p^n}(\overline{K})) \to \mathbf{Z}/p^n.$$

Proof. Recall that, for crystalline representation T, ${}_!H_f^*(G_S, T)$ is defined by the complex

$${}_!D_T^\cdot = \mathrm{Cone}\left(S_T^{\cdot G_S} \bigoplus \oplus_{v|p} B_{T,v}^\cdot \xrightarrow{r+\mathrm{res}-t} \oplus_{v|p} A_{T,v}^{\cdot D_v} \bigoplus \oplus_{v\in S'} S_{T,v}^{\cdot D_v}\right)[-1],$$

where $S_T^\cdot = C^\cdot(G_S, T)$, $S_{T,v}^\cdot = C^\cdot(D_v, T)$, and $A_{T,v}^\cdot = C^\cdot(D_v, E_{T,v}^\cdot)$.

Let $N \in \mathscr{G}_X$ and $T = N(\overline{K})$. We will now construct a quasi isomorphism between the complex ${}_!D_T^\cdot$, and the complex

$${}_!C^\cdot(X, N) := \mathrm{Cone}(R^\cdot(U, u^*N) \bigoplus \oplus_{v|p} R^\cdot(\mathcal{O}_v, \pi_v^* N) \to \oplus_{v\in S_{fin}} R^\cdot(K_v, k_v^* N))[-1].$$

First, note that the D_v-equivariant isomorphisms $N(\overline{K}) \xrightarrow{\sim} N_{K_v}(\overline{K_v})$ induce an isomorphism of ${}_!D_T^\cdot$ with the complex

$$\mathrm{Cone}\left(S_T^{\cdot G_S} \bigoplus \oplus_{v|p} B_{T_v}^\cdot \xrightarrow{r+\mathrm{res}-t} \oplus_{v|p} A_{T_v}^{\cdot D_v} \bigoplus \oplus_{v\in S'} S_{T_v}^{\cdot D_v}\right)[-1],$$

where $T_v = N_{K_v}(\overline{K_v})$, $A_{T_v}^\cdot = C^\cdot(D_v, E_{T_v}^\cdot)$, $S_{T_v}^\cdot = C^\cdot(D_v, T_v)$. Next, introduce the following notation. Let $\mathscr{S}_{T_v}^\cdot$, $\mathscr{A}_{T_v}^\cdot$, $\mathscr{S}_T^\cdot$ denote the complexes of flat sheaves associated to the Galois complexes $S_{T_v}^\cdot$, $A_{T_v}^\cdot$, $S_T^\cdot$ respectively. We have thus

$$\Gamma(U, \mathscr{S}_T^\cdot) \xleftarrow{\sim} S_T^\cdot, \qquad \Gamma(\mathrm{Spec}(K_v), \mathscr{S}_{T_v}^\cdot) \xleftarrow{\sim} S_v^{\cdot D_v}, \qquad \Gamma(\mathrm{Spec}(K_v), \mathscr{A}_{T_v}^\cdot) \xleftarrow{\sim} A_{T_v}^{\cdot D_v},$$

and since N_U is étale, $\mathscr{S}_T^\cdot$ resolves N_U on U and $\mathscr{S}_{T_v}^\cdot$, and $\mathscr{A}_{T_v}^\cdot$ resolve N_{K_v} on $\mathrm{Spec}(K_v)$. These resolutions are acyclic (since they are such already for Galois cohomology and we work over fields of characteristic 0 or open sets of integers in number fields). For every $v|p$, consider the maps

$$a_v\colon B_{T_v}^\cdot \xrightarrow{t_v} A_{T_v}^{\cdot D_v} \to R^\cdot(K_v, \mathscr{A}_{T_v}^\cdot),$$

$$b_v\colon R^\cdot(\mathcal{O}_v, N) \xrightarrow{b_{v,1}} R^\cdot(K_v, N) \xrightarrow{b_{v,2}} R^\cdot(K_v, \mathscr{A}_{T_v}^\cdot),$$

and the complexes $C_{N_v}^\cdot$, $F_{N_v}^\cdot$ introduced in the proof of Theorem 8.1. Recall (Lemma

8.1) that $F^\cdot_{N_v}$ defines a δ-functor and the projections

$$p_v\colon F^\cdot_{N_v} \to R^\cdot(\mathcal{O}_v, N), \qquad q_v\colon F^\cdot_{N_v} \to B^\cdot_{T_v},$$

$$(5)\qquad \begin{array}{ccc}
S_T^{\cdot G_S} \oplus B^\cdot_{T_\wp} & \xrightarrow{r+\mathrm{res}-t} & A^{\cdot D_\wp}_{T_\wp} \oplus S^{\cdot D_v}_{T_v} \\
\downarrow & & \downarrow \\
R^\cdot(U, \mathcal{S}^\cdot_T) \oplus B^\cdot_{T_\wp} & \xrightarrow{s+\mathrm{res}'-a} & R^\cdot(K_\wp, \mathcal{A}^\cdot_{T_\wp}) \oplus R^\cdot(K_v, \mathcal{S}^\cdot_{T_v}) \\
\uparrow{\scriptstyle 1\oplus q} & & \| \\
R^\cdot(U, \mathcal{S}^\cdot_T) \oplus F^\cdot_{N_\wp} & \xrightarrow{s+\mathrm{res}'-a\cdot q} & R^\cdot(K_\wp, \mathcal{A}^\cdot_{T_\wp}) \oplus R^\cdot(K_v, \mathcal{S}^\cdot_{T_v}) \\
\downarrow{\scriptstyle 1\oplus p} & & \| \\
R^\cdot(U, \mathcal{S}^\cdot_T) \oplus R^\cdot(\mathcal{O}_\wp, N) & \xrightarrow{s+\mathrm{res}'-b} & R^\cdot(K_\wp, \mathcal{A}^\cdot_{T_\wp}) \oplus R^\cdot(K_v, \mathcal{S}^\cdot_{T_v}) \\
\uparrow & & \uparrow{\scriptstyle b_2\oplus 1} \\
R^\cdot(U, N) \oplus R^\cdot(\mathcal{O}_\wp, N) & \xrightarrow{f-b_1} & R^\cdot(K_\wp, N) \oplus R^\cdot(K_v, N) \\
\uparrow & & \uparrow \\
R^\cdot(U, u^*N) \oplus R^\cdot(\mathcal{O}_\wp, \pi^*_\wp N) & \longrightarrow & R^\cdot(K_\wp, k^*_\wp N) \oplus R^\cdot(K_v, k^*_v N),
\end{array}$$

define isomorphisms of δ-functors. Set also

$$s_v\colon R^\cdot(U, \mathcal{S}^\cdot_T) \xrightarrow{\mathrm{res}'_v} R^\cdot(K_v, \mathcal{S}^\cdot_{T_v}) \to R^\cdot(K_v, \mathcal{A}^\cdot_{T_v}) \qquad \text{for } v|p;$$

$$\mathrm{res}'_v\colon R^\cdot(U, \mathcal{S}^\cdot_T) \xrightarrow{p^*_v} R^\cdot(K_v, p^*_v\mathcal{S}^\cdot_T) \to R^\cdot(K_v, \mathcal{S}^\cdot_{T_v}),$$

$$f_v\colon R^\cdot(U, N) \to R^\cdot(K_v, N), \qquad \text{for } v \in S,$$

where the map $p^*_v\mathcal{S}^\cdot_T \to \mathcal{S}^\cdot_{T_v}$ is induced by the D_v-equivariant restriction map res_v: $S^\cdot_T \to S^\cdot_{T_v}$.

Our quasi isomorphism can now be constructed from diagram (5), in which all squares but the third one commute. The third one commutes up to canonical homotopy (see the proof of Theorem 8.1). Here the prime $\wp$ lies above p, and the prime $v \in S$ does not.

As for the groups $H^*_f(G_S, N(\overline{K}))$, set in the above construction $S' = \emptyset$ to obtain a quasi isomorphism between the complex

$$D^\cdot_T = \mathrm{Cone}\left(S_T^{\cdot G_S} \bigoplus \oplus_{v|p} B^\cdot_{T_v} \xrightarrow{r-t} \oplus_{v|p} A^{\cdot D_v}_{T,v}\right)[-1]$$

computing $H_f^*(G_S, N(\overline{K}))$ and the complex

$$C^\cdot(X, N) := \mathrm{Cone}\left(R^\cdot(U, u^*N) \bigoplus \oplus_{v|p} R^\cdot(\mathcal{O}_v, \pi_v^*N) \to \oplus_{v|p} R^\cdot(K_v, k_v^*N)\right)[-1].$$

To finish, consider the canonical maps

$$\theta\colon R^\cdot(X, N) \to C^\cdot(X, N),$$

$$\theta_!\colon \mathrm{Cone}(R^\cdot(X, N) \to \oplus_{v\in S'} R^\cdot(K_v, k_v^*N))[-1] \to {}_!C^\cdot(X, N).$$

We claim that they are quasi isomorphisms. Indeed, note first that it is enough to prove this fact for modified complexes $C^\cdot(X, N)$, ${}_!C^\cdot(X, N)$, in which, for every $v \in S_{fin}$, the ring $\mathcal{O}_v$ and the field K_v are replaced by the henselization $\mathcal{O}_v^h$ of $\mathcal{O}_K$ at v and its field of fractions K_v^h. The maps θ and $\theta_!$ have to be modified correspondingly. Next, extend the above definitions of the cones and the maps to any flat sheaf $\mathscr{F}$ on X. Finally, note that higher cohomology groups of $C^\cdot(X, \mathscr{F})$ vanish for injectives, and since

$$H^0(X, \mathscr{F})$$
$$= \ker\left(H^0(U, u^*\mathscr{F}) \bigoplus \oplus_{v|p} H^0(\mathrm{Spec}(\mathcal{O}_v^h), \pi_v^*\mathscr{F}) \to \oplus_{v|p} H^0(\mathrm{Spec}(K_v^h), k_v^*\mathscr{F})\right),$$

θ induces an isomorphism on cohomology groups in degree 0. That $\theta_!$ is a quasi isomorphism is now clear.

The construction of δ-isomorphisms α, and α_c is thus complete.

The existence of the diagrams in the statement of the theorem follows easily from the fact that the maps ω_1, ω_2, $\omega_{2,c}$ can be induced by the vertical compositions in diagram (5).

It remains to compare the traces. Recall that the trace map $H_c^3(X, \mu_{p^n}) \xrightarrow{\sim} \mathbf{Z}/p^n$ [17, III.3, II.3] is defined via the maps

$$\oplus_{v\in S_{fin}} H^2(\mathrm{Spec}(K_v), \mu_{p^n}) \to H_c^3(U, \mu_{p^n}) \xrightarrow{\sim} H_c^3(X, \mu_{p^n}),$$

$$H^2(\mathrm{Spec}(K_v), \mu_{p^n}) \xrightarrow{\sim} H^2(D_v, \mu_{p^n}(\overline{K}_v)) \underset{\mathrm{inv}}{\xrightarrow{\sim}} \mathbf{Z}/p^n,$$

where the first arrow is a surjection. Thus since α_c commutes with $\omega_{2,c}$, which, in turn, commutes with ω_1, the commutativity of traces follows. □

Remark 7. Let a, b be integers, $a \leqslant -1, b \geqslant 0, b - a \leqslant p - 2$. Again, the above theorem is valid for the cohomology groups with weight between a and b. The isomorphisms α, α_c for different choices of the range of the weight are compatible.

THEOREM 8.4. *Assume $p \geqslant 5$. Let N, M, P belong to $\mathcal{G}_X$ and $N \times M \to P$ be a pairing of associated flat sheaves over X. Let $N(\overline{K}) \times M(\overline{K}) \to P(\overline{K})$ be the induced pairing of G_S-representations. Then the diagram*

$$\begin{array}{ccc} H^i_{f,-1,0}(G_S, N(\overline{K})) \oplus {}_!H^{3-i}_{f,-1,0}(G_S, M(\overline{K})) & \xrightarrow{\cup} & {}_!H^3_{f,-2,0}(G_S, P(\overline{K})) \\ \wr\downarrow \alpha\otimes\alpha_c & & \wr\downarrow \alpha_c \\ H^i(X, N) \otimes H^{3-i}_c(X, M) & \xrightarrow{\cup} & H^3_c(X, P) \end{array}$$

commutes.

Proof. The method is the same as in the proof of Theorem 8.2: we will follow step by step the construction of the isomorphisms α, α_c (diagram (5)). Denote by ${}_!C^\cdot_i(N)$ the cones (shifted by -1) of the ith horizontal map (counted from the top) in diagram (5). Let $C^\cdot_i(N)$ be its version without support (obtained by setting $S' = \emptyset$ in ${}_!C^\cdot_i(N)$). Let $X = N(\overline{K})$, $Y = M(\overline{K})$, and $Z = P(\overline{K})$ and assume for notational simplicity that there is only one prime $\wp$ above p and at most one prime $v \in S'$.

(a) Take $\cup$: $D^\cdot_X \otimes {}_!D^\cdot_Y \to {}_!D^\cdot_Z$ equal to $\cup$ defined in Theorem 5.2.

(b) To define the product $\cup$: $C^\cdot_1(N) \otimes {}_!C^\cdot_1(M) \to {}_!C^\cdot_1(P)$ as $\tilde{\cup}_0$ from Proposition 3.1, it suffices that for every $v \in S_{fin}$ the maps $S^\cdot_{T,v} \to S^\cdot_{T_v}$, $T = X, Y, Z$, commute with the standard products. This follows from the commutativity of the maps $G(\overline{K}) \to G_{K_v}(\overline{K_v})$, $G = N, M, P$, $v \in S_{fin}$, with pairings. Proposition 3.1 also gives us then compatibility of this product with the one defined in (a).

(c) In order to construct the product $\cup$: $C^\cdot_2(N) \otimes {}_!C^\cdot_2(M) \to {}_!C^\cdot_2(P)$ induce from Galois products the products

$$\mathscr{A}^\cdot_{X_\wp} \otimes \mathscr{A}^\cdot_{Y_\wp} \to \mathscr{A}^\cdot_{Z_\wp}, \qquad \mathscr{S}^\cdot_X \otimes \mathscr{S}^\cdot_Y \to \mathscr{S}^\cdot_Z, \qquad \mathscr{S}^\cdot_{X_v} \otimes \mathscr{S}^\cdot_{Y_v} \to \mathscr{S}^\cdot_{Z_v}, \; v \in S.$$

The last two commute with the pairings

$$f^*\mathscr{X} \otimes f^*\mathscr{Y} \to f^*\mathscr{Z}, \qquad f_v^*\mathscr{X}_v \otimes f_v^*\mathscr{Y}_v \to f_v^*\mathscr{Z}_v$$

respectively. Here f: $U_{fl} \to U_{ét}$, f_v: $\mathrm{Spec}(K_v)_{fl} \to \mathrm{Spec}(K_v)_{ét}$, and $\mathscr{T}$, $\mathscr{T}_v$ are étale sheaves corresponding to the G_S-module T and D_{K_v}-module T_v, $T = X, Y, Z$, respectively. Moreover, we find easily that the maps $\mathscr{S}^\cdot_{T_\wp} \to \mathscr{A}^\cdot_{T_\wp}$ and $p_v^*\mathscr{S}^\cdot_T \to \mathscr{S}^\cdot_{T_v}$, $T = X, Y, Z$, commute with the constructed products. Now Proposition 3.1 gives us the desired map of complexes $\cup$ as well as its compatibility with the product in (b).

(d) Define the map (only on cocycle $\otimes$ cocycle) $\cup$: $C^\cdot_3(N) \otimes {}_!C^\cdot_3(M) \to {}_!C^\cdot_3(P)$ as

$$(a_N, b_N, c_N) \cup (a_M, b_M, c_M, d_M)$$
$$= (a_N \cup a_M, b_N \cup b_M, c_N \cup \tilde{b}_M + (-1)^{\deg a_N}\tilde{a}_N \cup c_M, (-1)^{\deg a_N}\tilde{a}_N \cup d_M),$$

where $b_N \cup b_M$ was defined (and is a cocycle) in Theorem 8.2. This map commutes with the product from (c). We claim that $(a_N, b_N, c_N) \cup (a_M, b_M, c_M, d_M)$ is a cocycle. Indeed,

$$\begin{aligned}
&d((a_N, b_N, c_N) \cup (a_M, b_M, c_M, d_M)) \\
&\quad = (d(a_N \cup a_M), d(b_N \cup b_M), \\
&\qquad a \cdot q(b_N \cup b_M) - s(a_N \cup a_M) - d(c_N \cup \tilde{b}_M) - (-1)^{\deg a_N} d(\tilde{a}_N \cup c_M), \\
&\qquad -\mathrm{res}'(a_N \cup a_M) - (-1)^{\deg a_N} d(\tilde{a}_N \cup d_M)).
\end{aligned}$$

But from the cocycle condition, we have

$$\mathrm{res}'(a_N \cup a_M) + (-1)^{\deg a_N} d(\tilde{a}_N \cup d_M) = \tilde{a}_N \cup \tilde{a}_M + \tilde{a}_N \cup dd_M = 0$$

and

$$\begin{aligned}
&a \cdot q(b_N \cup b_M) - s(a_N \cup a_M) - d(c_N \cup \tilde{b}_M) - (-1)^{\deg a_N} d(\tilde{a}_N \cup c_M) \\
&\quad = \tilde{b}_N \cup \tilde{b}_M - \tilde{a}_N \cup \tilde{a}_M - dc_N \cup \tilde{b}_M - \tilde{a}_N \cup dc_M \\
&\quad = (\tilde{b}_N - dc_N) \cup \tilde{b}_M - \tilde{a}_N \cup (\tilde{a}_M + dc_M) \\
&\quad = \tilde{a}_N \cup \tilde{b}_M - \tilde{a}_N \cup \tilde{b}_M \\
&\quad = 0.
\end{aligned}$$

(e) Due to the compatibilities listed in (c) and in the proof of Theorem 8.2, we can use Proposition 3.1 to find a map of complexes $\cup$: $C_4^{\cdot}(N) \otimes {}_!C_4^{\cdot}(M) \to {}_!C_4^{\cdot}(P)$. In the degrees we are interested in, where we can assume $b_N = 0$ or $b_M = 0$, it commutes with the map from (d) modulo a coboundary: we find

$$\begin{aligned}
&(a_N, b_N, c_N) \cup_d (a_M, b_M, c_M, d_M) - (a_N, p(b_N), hb_N + c_N) \\
&\qquad \cup_e (a_M, p(b_M), hb_M + c_M, d_M) \\
&\quad = (0, 0, c_N \cup a \cdot q(b_M) + (-1)^{\deg a_N} \tilde{a}_N \cup c_M - (hb_N + c_N) \\
&\qquad \cup b \cdot p(b_M) - (-1)^{\deg a_N} \tilde{a}_N \cup (hb_M + c_M), 0),
\end{aligned}$$

where h is the homotopy between $b \cdot p$ and $a \cdot q$ defined in the proof of Theorem 8.1. It remains to show that $c_N \cup a \cdot q(b_M) - (hb_N + c_N) \cup b \cdot p(b_M) - (-1)^{\deg a_N} \tilde{a}_N \cup hb_M$

is a coboundary. It is clear for $b_M = 0$; if $b_N = 0$, it is equal to

$$-c_N \cup dhb_M + (-1)^{\deg a_N} dc_N \cup hb_M = (-1)^{\deg a_N} d(c_N \cup hb_M),$$

as desired.

(f) The product $\cup: C_5^\cdot(N) \otimes {}_!C_5^\cdot(M) \to {}_!C_5^\cdot(P)$ can be defined by Proposition 3.1 since the maps

$$R^\cdot(\mathcal{O}_{\wp}, N) \to R^\cdot(K_{\wp}, N), \qquad R^\cdot(U, N) \to R^\cdot(K_v, N), v \in S_{fin},$$

commute with products. The product $\cup$ commutes with the product from (e): to apply Proposition 3.1 we need only to show that the maps $N \to \mathscr{S}_X^\cdot$ and $N_{K_v} \to \mathscr{S}_{X,v}^\cdot$, $v \in S_{fin}$, commute with products. But since the map $N \to \mathscr{S}_X^\cdot$ is equal to the composition

$$N \to f^*N \to f^*\mathscr{X}_U \to f^*\mathscr{X} \to \mathscr{S}_X^\cdot,$$

where $X_U = N_U(\overline{K})$, its commutativity with products follows from the fact that the map $N \to \mathscr{X}_U$ commutes with products. The map $N_{K_v} \to \mathscr{S}_{X,v}^\cdot$ can be dealt with in a fashion similar to the map $N_{K_v} \to \mathscr{A}_{X,v}^\cdot$ in the proof of Theorem 8.2 (point (c)).

(g) The fact that the maps $j_v^*\pi_v^* \to k_v^*$ and $p_v^*u^* \to k_v^*$ commute with the pairing $N \times M \to P$ allows us to use Proposition 3.1 to construct a product $\cup: C_6^\cdot(N) \otimes {}_!C_6^\cdot(M) \to {}_!C_6^\cdot(P)$ compatible with the one defined in (f).

(h) Using the Godement products, we construct by Proposition 3.1 the product

$$\cup: R^\cdot(X, N) \otimes \mathrm{Cone}(R^\cdot(X, M) \to \oplus_{v \in S'} R^\cdot(K_v, k_v^* M))[-1]$$
$$\to \mathrm{Cone}(R^\cdot(X, P) \to \oplus_{v \in S'} R^\cdot(K_v, k_v^* P))[-1]$$

compatible with the product from (g). It is easy to see that it gives us the flat product $\cup: H^i(X, N) \otimes H_c^j(X, M) \to H_c^{i+j}(X, P)$ from [17, III.0.4]. □

Again, we obtain the following well-known result [17, III.3.3].

COROLLARY 8.2. *Assume $p \geqslant 5$. Let $N \in \mathscr{G}_X(p^n)$. Then*

$$H^i(X, N) \times H_c^{3-i}(X, N^D) \to H_c^3(X, \mu_{p^n}) \tilde{\to} \mathbf{Z}/p^n$$

is a nondegenerate pairing. □

9. Examples. To get acquainted with our cohomology groups, we will try to evaluate them for Tate twists. For twists bigger than 1, they turn out to be isomorphic to Galois cohomology groups. Assume throughout that p is odd.

9.1. Local cohomology of $\mathbf{Q}_p(n)$. We quote the following computations from [3].

$$\mathbf{D}_{dR}(\mathbf{Q}_p(n)) = K\{-n\},$$

$$\chi_{cr}(K, \mathbf{Q}_p(n)) = \begin{cases} 0 & \text{if } n \leqslant 0 \\ -[K:\mathbf{Q}_p] & \text{if } n > 0. \end{cases}$$

Thus

$$\dim_{\mathbf{Q}_p} H_f^1(G_K, \mathbf{Q}_p(n)) = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ [K:\mathbf{Q}_p] & \text{if } n > 0, \end{cases}$$

$$H_f^i(G_K, \mathbf{Q}_p(n)) \xrightarrow{\sim} H^i(G_K, \mathbf{Q}_p(n)) \qquad \text{for } n \neq 1.$$

Moreover, $H_f^1(G_K, \mathbf{Q}_p) \hookrightarrow \mathrm{Hom}_{cont}(G_K, \mathbf{Q}_p)$ corresponds to unramified homomorphisms, $H_f^1(G_K, \mathbf{Q}_p(1)) \xleftarrow{\sim} \text{proj lim } \mathcal{O}_K^*/(\mathcal{O}_K^*)^{p^n} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, and ${}_xH_f^2(G_K, \mathbf{Q}_p(1)) \overset{\text{ord}}{\underset{\sim}{\to}} \mathbf{Q}_p$.

9.2. Global cohomology of $\mathbf{Q}_p(n)$. Let r_1, r_2 be the number of real and complex places of K respectively. Then since

$$\chi_{cr}(G_S, \mathbf{Q}_p(n)) = \sum_{v|\infty} \dim_{\mathbf{Q}_p}(\mathbf{Q}_p(n)^{D_v}) + \sum_{v|p} \chi_{cr}(K_v, \mathbf{Q}_p(n)),$$

where K_v is the completion of K at v and $D_v = \mathrm{Gal}(\overline{K}_v/K_v)$,

$$\chi_{cr}(G_S, \mathbf{Q}_p(n)) = \begin{cases} r_1 + r_2 & \text{if } n \text{ even} \leqslant 0 \\ r_2 & \text{if } n \text{ odd} \leqslant 0 \\ -r_2 & \text{if } n \text{ even} > 0 \\ -r_1 - r_2 & \text{if } n \text{ odd} > 0. \end{cases}$$

Let $S' = S_{fin} - \{v|p\}$, $V = \mathrm{Spec}(\mathcal{O}_{S'})$, $U = \mathrm{Spec}(\mathcal{O}_{S_{fin}})$, $X = \mathrm{Spec}(\mathcal{O}_K)$, and $j\colon U \hookrightarrow X$. Table 1 lists the dimensions of the cohomology groups of $\mathbf{Q}_p(n)$. To get the dimensions of $H_f^3(G_S, \mathbf{Q}_p(n))$, we have used the following simple lemma:

LEMMA 9.1. *Assume* $p > 2$.
(i) *If* $S' \neq \varnothing$ *and* V *is a crystalline representation (integral or rational), then*

$$H_f^3(G_S, V) = 0.$$

TABLE 1

	$H_f^0(G_S, \mathbf{Q}_p(n))$	$H_f^1(G_S, \mathbf{Q}_p(n))$	$H_f^2(G_S, \mathbf{Q}_p(n))$	$H_f^3(G_S, \mathbf{Q}_p(n))$
$n < 0$	0	0	r_2 if n odd $r_1 + r_2$ if n even	0
$n = 0$	1	0	$r_1 + r_2 - 1$	0
$n = 1$	0	$r_1 + r_2 - 1 + \#S'$	$\#S' - 1$	1 if $S' = \varnothing$ 0 if $S' \neq \varnothing$
$n > 1$	0	$r_1 + r_2$ if n odd r_2 if n even	0	0

(ii) *If $S' = \varnothing$ and V is a torsion or rational crystalline representation, then*

$$H_f^3(G_S, V) \cong H^0(G_S, V^D)^*.$$

Proof. It comes immediately from the exact sequence

$$H^2(G_S, V) \to \oplus_{v|p} H^2(D_v, V) \to H_f^3(G_S, V) \to 0,$$

global Tate duality, and [17, I.4.16]. □

As for the rest, recall that any extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(n)$, $n \neq 1$, in the category of G_S-representations is in fact unramified at every $v \in S'$ [15, 3.5]. Therefore the groups $H_f^*(G_S, \mathbf{Q}_p(n))$, $n \neq 1$, do not depend on S. For $n > 1$, local computations yield the isomorphisms

$$H_f^i(G_S, \mathbf{Q}_p(n)) \xrightarrow{\sim} H^i(G_S, \mathbf{Q}_p(n)) \cong H_{\acute{e}t}^i(U, \mathbf{Q}_p(n)).$$

If we now assume that $S' = \varnothing$, then the Euler characteristic and the vanishing of $H^2(G_S, \mathbf{Q}_p(n))$ [18] give the values in the table. For $n < 0$, we assume $S' = \varnothing$ and evoke the duality.

It remains to compute $H_f^*(G_S, \mathbf{Q}_p)$ and $H_f^*(G_S, \mathbf{Q}_p(1))$. The exact sequence

$$0 \to H_f^1(G_S, \mathbf{Q}_p(1)) \to H^1(G_S, \mathbf{Q}_p(1)) \xrightarrow{\oplus \mathrm{ord}_v} \oplus_{v|p} \mathbf{Q}_p \to 0$$

yields the isomorphism

$$H_f^1(G_S, \mathbf{Q}_p(1)) \xleftarrow{\sim} \mathcal{O}_{S'}^* \otimes \mathbf{Q}_p.$$

By local computations $H_f^1(G_S, \mathbf{Q}_p) \hookrightarrow \mathrm{Hom}_{cont}(\mathrm{Gal}(\overline{K}/K), \mathbf{Q}_p)$ is the group of (everywhere) unramified homomorphisms and, thus, trivial. $H_f^2(G_S, \mathbf{Q}_p)$ is independent of S, and for $S' = \varnothing$ duality gives the isomorphism $H_f^2(G_S, \mathbf{Q}_p) \cong H_f^1(G_S, \mathbf{Q}_p(1))^*$. If $S' = \varnothing$, $H_f^2(G_S, \mathbf{Q}_p(1))$ vanishes by duality; otherwise, since ${}_!H_f^2(G_S, \mathbf{Q}_p(1)) \cong$

$H^0_f(G_S, \mathbf{Q}_p)^* = 0$ and ${}_!H^3_f(G_S, \mathbf{Q}_p(1)) \overset{\text{tr}}{\underset{\sim}{\to}} \mathbf{Q}_p$, from the definition of compact support cohomology, we get the short exact sequence

$$0 \to H^2_f(G_S, \mathbf{Q}_p(1)) \to \oplus_{v \in S'} H^2(D_v, \mathbf{Q}_p(1)) \xrightarrow{\oplus \mathrm{inv}_v} \mathbf{Q}_p \to 0.$$

Combined with the exact sequence [17, II.2.1]

$$(6) \qquad 0 \to H^2_{fl}(V, \mathbf{G}_m) \to \oplus_{v \in T} \mathrm{Br}(K_v) \xrightarrow{\oplus \mathrm{inv}_v} \mathbf{Q}/\mathbf{Z} \to 0,$$

where $T = S - \{v|p\}$ and K_v is the completion of K at v, it yields the isomorphism

$$H^2_f(G_S, \mathbf{Q}_p(1)) \tilde{\to} V_p(\mathrm{Br}(\mathcal{O}_{S'})),$$

where $V_p(\mathrm{Br}(\mathcal{O}_{S'}))$ is the p-Tate module of the Brauer group of $\mathcal{O}_{S'}$.

9.3. *Local cohomology of* $\mathbf{Z}/p^k(n)$. Assume that K is unramified and $|n| \leqslant p-2$. We have

$$\mathbf{D}_{cr}(\mathbf{Z}/p^k(n)) = V/p^k\{-n\}.$$

From the exact sequence

$$0 \to H^0_f(\mathrm{G}_K, \mathbf{Z}/p^k(n)) \to F^0(V/p^k\{-n\}) \xrightarrow{1-\phi^0} V/p^k \to H^1_f(G_K, \mathbf{Z}/p^k(n)) \to 0$$

and the fact that, if $n \leqslant 0$, then $\phi^0 = p^{-n}\sigma$, where σ is the Frobenius on V, we derive that

$$\chi_{cr}(K, \mathbf{Z}/p^k(n)) = \begin{cases} 0 & \text{if } n \leqslant 0 \\ -[K:\mathbf{Q}_p]k & \text{if } n > 0, \end{cases}$$

$$H^0_f(G_K, \mathbf{Z}/p^k(n)) = \begin{cases} 0 & \text{if } n \neq 0 \\ \mathbf{Z}/p^k & \text{if } n = 0, \end{cases}$$

$$H^1_f(G_K, \mathbf{Z}/p^k(n)) = \begin{cases} 0 & \text{if } n < 0 \\ \mathbf{Z}/p^k & \text{if } n = 0 \\ V/p^k & \text{if } n > 0. \end{cases}$$

Moreover, $H^1_f(G_K, \mathbf{Z}/p^k) \hookrightarrow \mathrm{Hom}_{cont}(G_K, \mathbf{Z}/p^k)$ corresponds to unramified homomorphisms, $H^1_f(G_K, \mathbf{Z}/p^k(1)) \cong H^1_{fl}(G_K, \mu_{p^k})$ and

$$H^i_f(G_K, \mathbf{Z}/p^k(n)) \tilde{\to} H^i(G_K, \mathbf{Z}/p^k(n)), \qquad \text{for } n > 1.$$

9.4. Global cohomology of $\mathbf{Z}/p^k(n)$. Assume that the number field K is absolutely unramified above p and $|n| \leqslant p-2$. Since

$$\chi_{cr}(G_S, \mathbf{Z}/p^k(n)) = \sum_{v|\infty} \lg(\mathbf{Z}/p^k(n)^{D_v}) + \sum_{v|p} \chi_{cr}(K_v, \mathbf{Z}/p^k(n)),$$

we have

$$\chi_{cr}(G_S, \mathbf{Z}/p^k(n)) = \begin{cases} (r_1+r_2)k & \text{if } n \text{ even} \leqslant 0 \\ r_2 k & \text{if } n \text{ odd} \leqslant 0 \\ -r_2 k & \text{if } n \text{ even} > 0 \\ -(r_1+r_2)k & \text{if } n \text{ odd} > 0. \end{cases}$$

Table 2 shows what we know about the cohomology groups of $\mathbf{Z}/p^k(n)$. Here, the vanishing of $H^0_f(G_S, \mathbf{Z}/p^k(n))$, $n \neq 1$, follows from local computations. Lemma 9.1 gives the groups $H^3_f(G_S, \mathbf{Z}/p^k(n))$. The free parts of $H^*_f(G_S, \mathbf{Z}/p^k(n))$ are given by the rational case.

To compute $H^*_f(G_S, \mu_{p^k})$ recall that $H^*_f(G_S, \mu_{p^k}) \cong H^*_{fl}(V, \mu_{p^k})$. The Kummer exact sequence on V yields the exact sequences

$$0 \to \mathcal{O}^*_{S'}/\mathcal{O}^{*p^k}_{S'} \to H^1_{fl}(V, \mu_{p^k}) \to {}_{p^k}H^1_{fl}(V, \mathbf{G}_m) \to 0,$$

$$0 \to \mathrm{Pic}(V)/p^k \to H^2_{fl}(V, \mu_{p^k}) \to {}_{p^k}H^2_{fl}(V, \mathbf{G}_m) \to 0.$$

They are split since the exact sequence (6) shows that

$${}_{p^k}H^2_{fl}(V, \mathbf{G}_m) = \begin{cases} 0 & \text{if } S' = \varnothing \\ (\mathbf{Z}/p^k)^{\#S'-1} & \text{if } S' \neq \varnothing. \end{cases}$$

TABLE 2

	H^0_f	$H^1_f(G_S, \mathbf{Z}/p^k(n))$	$H^2_f(G_S, \mathbf{Z}/p^k(n))$	$H^3_f(G_S, \mathbf{Z}/p^k(n))$
$n<0$	0	${}_{p^k}A^n_S$	$(\mathbf{Z}/p^k)^{r_2}$ if n odd $(\mathbf{Z}/p^k)^{r_1+r_2}$ if n even $\oplus A^n_S/p^k$	0
$n=0$	$\mathbf{Z}/p^k$	${}_{p^k}B_S$	$(\mathbf{Z}/p^k)^{r_1+r_2-1}$ $\oplus B_S/p^k$	0
$n=1$	0	$(\mathbf{Z}/p^k)^{r_1+r_2-1+\#S'}$ $\oplus_{p^k}\mathrm{Pic}(V)$	$(\mathbf{Z}/p^k)^{\#S'-1}$ $\oplus \mathrm{Pic}(V)/p^k$	0 if $S' \neq \varnothing$ $\mathbf{Z}/p^k$ if $S' = \varnothing$
$n>1$	0	$(\mathbf{Z}/p^k)^{r_1+r_2}$ if n odd $(\mathbf{Z}/p^k)^{r_2}$ if n even $\oplus_{p^k}C^n_S$	C^n_S/p^k	0

Next note that multiplication by p^k on $\mathbf{Z}_p(n)$ yields the split exact sequence

$$0 \to H^1_f(G_S, \mathbf{Z}_p(n))/p^k \to H^1_f(G_S, \mathbf{Z}/p^k(n)) \to {}_{p^k}H^2_f(G_S, \mathbf{Z}_p(n)) \to 0.$$

(There is no torsion in $H^1_f(G_S, \mathbf{Z}_p(n))$ because it would have to come from the cokernel of $H^0(G_S, \mathbf{Q}_p(n)) \to H^0(G_S, \mu_{p^\infty}(n)))$ which is clearly trivial for $n = 0$, and for $n \neq 0$ we have $H^0(G_S, \mu_{p^\infty}(n)) \hookrightarrow H^0(D_v, \mu_{p^\infty}(n)) = 0$ $(v|p)$ by local computations.) Thus, if we set $A^n_S = H^2_f(G_S, \mathbf{Z}_p(n))(p)$, for $n < 0$, $B_S = H^2_f(G_S, \mathbf{Z})(p)$ and $C^n_S = H^2_f(G_S, \mathbf{Z}(n))(p)$, for $n > 1$, and note that $H^2_f(G_S, \mathbf{Z}_p(n))/p^k \tilde{\to} H^2_f(G_S, \mathbf{Z}/p^k(n))$, for $n \neq 1$, we get the groups in Table 2.

About the groups A^n_S, B^n_S, and C^n_S, we can say the following. From the exact sequences (2) and (3) (Section 5) we derive that

$$H^*_f(G_S, \mathbf{Z}/p^k(n)) \cong H^*(G_S, \mathbf{Z}/p^k(n)) \cong H^*_{\acute{e}t}(U, \mathbf{Z}/p^k(n)), \qquad \text{for } n > 1,$$

$$_!H^*_f(G_S, \mathbf{Z}/p^k(n)) \cong H^*_c(G_S, \mathbf{Z}/p^k(n)) \cong H^*_{\acute{e}t}(X, j_!\mathbf{Z}/p^k(n)), \qquad \text{for } n < 0.$$

Also, since $H^*_f(G_S, \mathbf{Z}/p^k) \cong H^*_{\acute{e}t}(V, \mathbf{Z}/p^k)$,

$$_{p^k}B_S = H^1_f(G_S, \mathbf{Z}/p^k) \hookrightarrow \mathrm{Hom}_{cont}(\mathrm{Gal}(\overline{K}/K), \mathbf{Z}/p^k)$$

corresponds to homomorphisms unramified outside of S'. If $S' = \varnothing$ for k big enough we find

$$B_S = H^1_f(G_S, \mathbf{Z}/p^k) \cong H^1_{\acute{e}t}(X, \mathbf{Z}/p^k) \cong H^2_{fl}(X, \mathbf{Z}/p^k(1))^* \cong \mathrm{Pic}(X)(p)^*$$

by flat duality, and if $-n \leqslant p - 3$

$$A^n_S = H^1_f(G_S, \mathbf{Z}/p^k(n)) \cong {}_!H^1_f(G_S, \mathbf{Z}/p^k(n)) \cong H^1_{\acute{e}t}(X, j_!\mathbf{Z}/p^k(n))$$

$$\cong H^2_{\acute{e}t}(U, \mathbf{Z}/p^k(-n+1))^* \cong (C_S^{-n+1})^*$$

by Artin-Verdier duality.

To finish, we would like to point out the following. In the totally real case, the main conjecture of Iwasawa theory as proved in [20] gives us that, for n odd and negative,

$$|\zeta_K(n)|_p = \frac{\# H^2_{\acute{e}t}(U, \mathbf{Z}_p(-n+1))}{\# H^1_{\acute{e}t}(U, \mathbf{Z}_p(-n-1))},$$

where $|\cdot|_p$ is the p-primary part of a rational number. Thus, by computations done earlier,

$$|\zeta_K(n)|_p = \# H^2_f(G_S, \mathbf{Z}_p(n)) \qquad \text{for } -n \leqslant p - 3.$$

9.5. Relation to K-theory. If $n > 1$, by the work of Borel and Soulé [19] we have the isomorphism

$$K_{2n-k}(\mathcal{O}_K) \otimes \mathbf{Q}_p \xrightarrow{\sim} H^k_{\acute{e}t}\left(\operatorname{Spec}\left(\mathcal{O}_K\left[\frac{1}{p}\right]\right), \mathbf{Q}_p(n)\right), \qquad \text{when } k = 1, 2.$$

If we now identify the right side with $H^k_f(G_S, \mathbf{Q}_p(n))$, where $S = \{v \mid p \cdot \infty\}$, and use the fact that both sides do not depend on S, we find the isomorphism

$$K_{2n-k}(\mathcal{O}_{S'}) \otimes \mathbf{Q}_p \xrightarrow{\sim} H^k_f(G_S, \mathbf{Q}_p(n)) \qquad \text{for any } S \text{ and } k = 1, 2.$$

In particular,

$$H^2_f(G_S, \mathbf{Q}_p(n)) = 0 \qquad \text{for any } S.$$

In the integral case, still assuming $n > 1$, there exist, by Soulé [18, IV.1, IV.3.3], the surjections

$$K_{2n-2}(\mathcal{O}_{S'}; \mathbf{Z}/p^k) \to H^2_f(G_S, \mathbf{Z}/p^k(n)), \qquad K_{2n-1}(\mathcal{O}_{S'}; \mathbf{Z}/p) \to H^1_f(G_S, \mathbf{Z}/p(n)),$$

and, for K totally real and n even, the surjection

$$K_{2n-2}(\mathcal{O}_{S'}) \otimes \mathbf{Z}_p \to H^2_f(G_S, \mathbf{Z}_p(n)) \cong C^n_S.$$

Earlier, we found the isomorphisms

$$K_1(\mathcal{O}_{S'}) \otimes \mathbf{Z}_p \xrightarrow{\sim} H^1_f(G_S, \mathbf{Z}_p(1)), \qquad K_1(\mathcal{O}_{S'}) \otimes \mathbf{Q}_p \xrightarrow{\sim} H^1_f(G_S, \mathbf{Q}_p(1)).$$

Also the exact sequence

$$0 \to K_1(\mathcal{O}_{S'})/p^k \to K_1(\mathcal{O}_{S'}; \mathbf{Z}/p^k) \to {}_{p^k}K_0(\mathcal{O}_{S'}) \to 0$$

yields the isomorphism

$$K_1(\mathcal{O}_{S'}; \mathbf{Z}/p^k) \xrightarrow{\sim} H^1_f(G_S, \mathbf{Z}/p^k(1)).$$

Assuming $S' = \varnothing$, we also find the isomorphism

$$(K_0(\mathcal{O}_K) \otimes \mathbf{Z}_p)(p) \xrightarrow{\sim} H^2_f(G_S, \mathbf{Z}_p(1))$$

and the exact sequence

$$0 \to \mathbf{Z}/p^k \to K_0(\mathcal{O}_K; \mathbf{Z}/p^k) \to H^2_f(G_S, \mathbf{Z}/p^k(1)) \to 0.$$

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