p-adic motivic cohomology in arithmetic

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Abstract. We will discuss recent progress by many people in the program of representing motivic cohomology with torsion coefficients of arithmetic schemes by various arithmetic p-adic cohomologies: étale, logarithmic de Rham-Witt, and syntomic. To illustrate possible applications in arithmetic geometry we will sketch proofs of the absolute purity conjecture in étale cohomology and comparison theorems of p-adic Hodge theory.

Mathematics Subject Classification (2000). Primary 14F42, 14F20, 11G25.

Keywords. motivic cohomology, K-theory, étale cohomology, crystalline cohomology, p-adic Hodge theory.

1. Introduction

In this report, we will survey results about the relationship between motivic cohomology and p-adic cohomologies of arithmetic schemes. We will concentrate on the stable ranges where the motivic cohomology with torsion coefficients tends to be represented by various cohomologies of arithmetic type. Much has happened in that subject in the last few years and a detailed survey can be found in [10]. For us motivic cohomology will mean Bloch higher Chow groups and its approximation – the gamma gradings of algebraic K-theory. They are defined using algebraic cycles and vector bundles, respectively, and are connected via an Atiyah-Hirzebruch spectral sequence. By inverting the Bott element or in some stable ranges motivic cohomology becomes isomorphic to its (étale) topological version. That is the content of the Beilinson-Lichtenbaum and Quillen-Lichtenbaum conjectures both of which follow (over fields) from the Bloch-Kato conjecture (now proved at least for mod 2 coefficients). Torsion étale motivic cohomology has a direct relationship with various arithmetic cohomologies: logarithmic de Rham-Witt (an arithmetic version of crystalline cohomology in positive characteristic), syntomic cohomology (an arithmetic version of crystalline cohomology in mixed characteristic), and arithmetic étale cohomology. As a result we can represent p-adic cohomology classes as algebraic cycle classes in a way reminiscent of the classical situation. This turns out to be – both conceptually and technically – a powerful tool. We present two applications: a proof of the absolute purity conjecture in étale cohomology and proofs of comparison theorems in p-adic Hodge theory.

2. K-theory

2.1. Milnor K-theory. For a field k, the Milnor K-groups $K_*^M(k)$ are defined to be the quotient of the tensor algebra of the multiplicative group k^* of k by the ideal generated by the Steinberg relation $x \otimes (1-x)$ for $x \neq 0, 1$. This rather simple construction turns out to be a fundamental object in the subject.

If m is relatively prime to the characteristic of k, Kummer theory gives that $K_1^M(k)/m \simeq k^*/k^{*m} \xrightarrow{\sim} H^1(k_{\text{ét}}, \mu_m)$. Using cup product on Galois cohomology we get the Galois symbol map

$$K_n^M(k)/m \to H^n(k_{\text{ét}}, \mu_m^{\otimes n}).$$

Conjecture 2.1. (Bloch-Kato) *The above symbol map is an isomorphism.*

Voevodsky proved [40] the Bloch-Kato conjecture for m a power of 2 (Milnor conjecture) and has recently announced a proof for general m [41].

If $p > 0$ is equal to the characteristic of k, the Bloch-Kato conjecture could be interpreted as the theorem of Bloch-Gabber-Kato [3] giving an isomorphism

$$\mathrm{dlog} : K_n^M(k)/p^r \xrightarrow{\sim} H^0(k_{\text{ét}}, \nu_r^n),$$

where $\nu_r^n = W_r\Omega^n_{X,\log}$ is the logarithmic de Rham-Witt sheaf. It is a subsheaf of the de Rham-Witt sheaf $W_r\Omega^n_X$ generated locally for the étale topology by $\mathrm{dlog}\,\overline{x}_1 \wedge \ldots \wedge \mathrm{dlog}\,\overline{x}_n$, where $\overline{x} \in W_r\mathcal{O}_X$ are the Teichmüller lifts of units. It fits into the short exact sequence of pro-sheaves (F is the Frobenius)

$$0 \to \nu_\cdot^n \to W_\cdot\Omega^n_X \xrightarrow{F-1} W_\cdot\Omega^n_X \to 0.$$

2.2. Algebraic K-theory. For a noetherian scheme X, let $\mathcal{M}(X)$ and $\mathcal{P}(X)$ denote the categories of coherent and locally free sheaves on X, respectively. The higher algebraic K and K' groups of X are defined as homotopy groups of certain simplicial spaces associated to the above categories: $K_i(X) = \pi_i(\mathcal{K}(X))$, $K'_i(X) = \pi_i(\mathcal{K}'(X))$. For $i = 0$, $K_0(X)$ and $K'_0(X)$ are the Grothendieck groups of vector bundles and coherent sheaves, respectively. For a field k, the product structure on K-theory gives a natural homomorphism $K_n^M(k) \to K_n(k)$ that is an isomorphism for $n \leq 2$.

K-groups mod m, $K_i(X, \mathbf{Z}/m)$, are defined by taking homotopy groups with $\mathbf{Z}/m$ coefficients of the above spaces. They are related to $K_*(X)$ by a universal coefficient sequence. Exterior powers of vector bundles induce a descending filtration $F_\gamma^* K_*(X, \mathbf{Z}/m)$ (γ-filtration) and the graded pieces $\mathrm{gr}_\gamma^* K_*(X, \mathbf{Z}/m)$ can be considered an approximation to motivic cohomology groups.

Similarly, we get groups $K'_i(X, \mathbf{Z}/m)$. The natural homomorphism

$$K_i(X, \mathbf{Z}/m) \to K'_i(X, \mathbf{Z}/m)$$

is an isomorphism if X is regular ("Poincaré duality"), because every coherent sheaf has a finite resolution by locally free sheaves. For Z a closed subscheme of X with open complement U, there is a localization sequence in K'-theory

$$\to K'_{i+1}(U, \mathbf{Z}/m) \to K'_i(Z, \mathbf{Z}/m) \to K'_i(X, \mathbf{Z}/m) \to K'_i(U, \mathbf{Z}/m) \to$$

It is relatively easy to derive in this generality and (in arithmetic applications) it gives K-theory great technical advantage over other motivic cohomologies. The other important technical advantage is the ease with which one can define higher Chern classes into various cohomologies. To be able to follow Gillet's construction [17] all one really needs is that the cohomology of the classifying simplicial scheme BGL (and some of its variants) is the expected one.

Varying X, one can view $\mathcal{K}(X)$ as a presheaf of simplicial spaces. Let $\mathcal{K}/m$ denote the presheaf of corresponding spaces mod m. Assume that X is regular. Then the Mayer-Vietoris property of K-theory gives that $K_*(X, \mathbf{Z}/m) \simeq H^{-*}(X_{\mathrm{Zar}}, \mathcal{K}/m)$.

If m is invertible on X, under some additional technical assumptions on X, the étale K-theory of Dwyer-Friedlander [5] can be computed using presheaves $\mathcal{K}/m$: $K_j^{\text{ét}}(X, \mathbf{Z}/m) \simeq H^{-j}(X_{\text{ét}}, \mathcal{K}/m)$, for $j \geq 0$.

Conjecture 2.2. (Quillen-Lichtenbaum) *The change of topology map*

$$\rho_j : K_j(X, \mathbf{Z}/m) \to K_j^{\text{ét}}(X, \mathbf{Z}/m)$$

is an isomorphism for $j \geq \operatorname{cd}_m X_{\text{ét}}$ (the étale cohomological dimension of X).

Here and below we will review the current status of the Quillen-Lichtenbaum conjecture. Recall that Thomason [37] proved that the map ρ_j induces an isomorphism

$$\tilde{\rho}_j : K_j(X, \mathbf{Z}/m)[\beta_m^{-1}] \xrightarrow{\sim} K_j^{\text{ét}}(X, \mathbf{Z}/m),$$

where $K_j(X, \mathbf{Z}/m)[\beta_m^{-1}]$ denotes the j'th graded piece of the ring obtained by inverting the action of the Bott element β_m on $K_*(X, \mathbf{Z}/m)$. If $\mu_m \subset \Gamma(X, \mathcal{O}_X)$, then β_m is defined as an element in $K_2(X, \mathbf{Z}/m)$ canonically lifting a chosen primitive m'th root of unity in $K_1(X)$. A refined version of the proof of this theorem [38] allowed him to show that for a variety of schemes (not necessarily over an algebraically closed fields) the map ρ_j is surjective for j larger than (roughly) $N = (\dim X)^3$ and its kernel is annihilated by β_m^N. Over an algebraically closed field and for quasi-projective X we can do better: Walker shows [42] that in that case ρ_j is split surjective for $j \geq 2d$ and its kernel is annihilated by β_m^d, $d = \dim X$ (see also [12]).

Étale K-theory has a direct relationship to étale cohomology. Namely Gabber's rigidity and Suslin's computation of K-theory of algebraically closed fields imply that the sheaves of fundamental groups $\tilde{\pi}_i(\mathcal{K}/m)$ are isomorphic to $\mu_m^{\otimes i/2}$ for i even and are 0 for i odd. Then the local to global spectral sequence becomes

$$E_2^{p,q} = \begin{cases} H^p(X_{\text{ét}}, \mu_m^{q/2}) & \text{for } q \text{ even} \\ 0 & \text{for } q \text{ odd} \end{cases} \Rightarrow K_{q-p}^{\text{ét}}(X, \mathbf{Z}/m).$$

Action of Adams operations shows that this spectral sequence degenerates at E_2 modulo torsion of a bounded order depending only on $\operatorname{cd}_m X_{\text{ét}}$. In particular, the Chern classes

$$c_{i,j}^{\text{ét}} : \operatorname{gr}_\gamma^i K_j^{\text{ét}}(X, \mathbf{Z}/m) \to H^{2i-j}(X_{\text{ét}}, \mu_m^{\otimes i})$$

are isomorphisms modulo small torsion.

If X is smooth over a perfect field of characteristic $p > 0$, Geisser-Levine [15] using motivic cohomology (see below) prove the isomorphism $\tilde{\pi}_n(\mathcal{K}/p^r) \simeq \nu_r^n$. Since ν_r^n vanishes for $n > \dim X$, the local to global spectral sequence

$$H^k(X, \tilde{\pi}_n(\mathcal{K}/p^r)) \Rightarrow K_{n-k}(X, \mathbf{Z}/p^r)$$

gives the important vanishing result: $K_n(X, \mathbf{Z}/p^r) = 0$ for $n > \dim X$.

2.3. Application: the absolute purity conjecture. The relationship between algebraic K-theory and étale cohomology just described was used by Thomason [36] and Gabber [13] to prove the absolute purity conjecture in étale cohomology. Thomason derived it (up to small torsion) from absolute purity in K-theory and Gabber – after some reductions – from vanishing results in K-theory.

Conjecture 2.3. *Let $i : Y \hookrightarrow X$ be a closed immersion of noetherian, regular schemes of pure codimension d. Let n be an integer invertible on X. Then*

$$\mathcal{H}^q_Y(X_{\text{ét}}, \mathbf{Z}/n) \simeq \begin{cases} 0 & \textit{for } q \neq 2d \\ \mathbf{Z}/n(-d) & \textit{for } q = 2d \end{cases}$$

Proof. Thomason's proof works (for example) for schemes of finite type over $\mathbf{Z}$ and m all of whose prime divisors are at least $\dim X+1$. Localization sequence immediately gives absolute purity in K-theory: one defines K-theory with support $K_Y(X)$ to be the homotopy fiber of the restriction $K(X) \to K(X \setminus Y)$ and by localization and Poincaré duality we get the isomorphism $K_{Y,*}(X) \simeq K_*(Y)$. Inverting the Bott element yields absolute cohomological purity in étale K-theory: $K^{\text{ét}}_{Y,*}(X) \simeq K^{\text{ét}}_*(Y)$. This can be now easily transferred to étale cohomology via the Atiyah-Hirzebruch spectral sequence. Namely, we have the sheafification of the Atiyah-Hirzebruch spectral sequence with support

$$E_2^{p,q} = \begin{cases} \mathcal{H}^p_Y(X_{\text{ét}}, \mu_m^{\otimes i}) & \text{for } q = 2i \\ 0 & \text{for } q \neq 2i \end{cases}$$

strongly converging to the sheaf associated to $K^{\text{ét}}_{Y,q-p}(-, \mathbf{Z}/m) \simeq K^{\text{ét}}_{q-p}(-_Y, \mathbf{Z}/m)$. Evoking once more the Atiyah-Hirzebruch spectral sequence (this time on Y) one computes that this sheaf is periodic (of period two) and $\mathbf{Z}/m$ for $q = p$ and trivial for $q - p = 1$. Action of Adams operations shows that the above spectral sequence degenerates modulo a constant (depending on étale cohomological dimension of X) and that the same constant kills $E_\infty^{2j,2j}$ for $j \neq q$.

To prove absolute purity in general Gabber appeals to the Atiyah-Hirzebruch spectral sequence only in the situation where it does in fact degenerate. First, he defines a well-behaved global cycle class $c(Y) \in H^{2d}_Y(X_{\text{ét}}, \mu_m^{\otimes d})$ that allows him to reduce absolute purity to a punctual one: for a regular strict local ring $\mathcal{O}$ of dimension d with closed point $i_x : x \to \operatorname{Spec} \mathcal{O}$ the cycle class gives an isomorphism $cl(x) : \mu_{m,x} \simeq i_x^! \mu_m^{\otimes d}[2d]$. Induction now reduces it to a vanishing result: $H^p(\mathcal{O}[f^{-1}]_{\text{ét}}, \mu_m) = 0$ for $p \neq 0, 1$, where $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ for the maximal ideal $\mathfrak{m} \subset \mathcal{O}$. Here he can assume $\mathcal{O}$ to be of arithmetic type.

Next, he considers the following Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = \begin{cases} H^p(\mathcal{O}[f^{-1}]_{\text{ét}}, \mu_m^{q/2}) & \text{for } q \text{ even} \\ 0 & \text{for } q \text{ odd} \end{cases} \Rightarrow K_{q-p}^{\text{ét}}(\mathcal{O}[f^{-1}], \mathbf{Z}/m)$$

where the étale K-groups $K_{q-p}^{\text{ét}}(\mathcal{O}[f^{-1}], \mathbf{Z}/m)$ are equal to $K_0(\mathcal{O}[f^{-1}], \mathbf{Z}/m)(q/2)$ for q even and $K_1(\mathcal{O}[f^{-1}], \mathbf{Z}/m)((q-1)/2)$ for q odd. Inductively, using local affine Lefschetz and duality he gets vanishing of $H^p(\mathcal{O}[f^{-1}]_{\text{ét}}, \mu_m) = 0$ for $p \neq 0, 1, d-1, d$. That kills some colums in the above spectral sequence and the degeneration at E_2 follows. Now, vanishing of level 2 of the filtration on K-groups implies that $E_2^{p,q} = E_\infty^{p,q} = 0$ for $p \geq 2$ yielding $H^p(\mathcal{O}[f^{-1}]_{\text{ét}}, \mu_m) = 0$ for $p \neq 0, 1$, as wanted. □

3. Motivic cohomology

3.1. Motivic cohomology over a field. For a separated scheme X over a field, Bloch higher Chow groups [1] are the cohomology groups of a certain complex of abelian groups. To define it, denote by $\triangle^n$ the algebraic n-simplex $\operatorname{Spec} \mathbf{Z}[t_0, \ldots, t_i]/(\sum t_i - 1)$. Let $z^r(X, i)$ denote the free abelian group generated by irreducible codimension r subvarieties of $X \times \triangle^i$ meeting all faces properly. Let $z^r(X, *)$ be the chain complex thus defined with boundaries given by pullbacks of cycles along face maps. Denote by $H^i(X, \mathbf{Z}(r))$ the cohomology of the complex $\mathbf{Z}(r) := z^r(X, 2r - *)$ in degree i. The motivic cohomology with coefficients $\mathbf{Z}/m$ is the cohomology of the complex $\mathbf{Z}/m(r) = \mathbf{Z}(r) \otimes \mathbf{Z}/m$. It fits into the usual universal coefficient sequence.

Remark 3.1. There is another commonly used construction of motivic cohomology due to Suslin-Voevodsky [9]. It works well in characteristic zero but it is not well suited for studying mod p-phenomena in positive characteristic p.

Motivic cohomology groups are trivial for $i > 2n$ and $i > n + \dim X$ for dimension reasons. Beilinson-Soulé conjecture (still open) postulates that they vanish for $i < 0$. We have $H^{2n}(X, \mathbf{Z}(n)) \simeq CH_n(X)$, the classical Chow group. For a field k, $H^i(k, \mathbf{Z}(n))$ are trivial for $i > n$ and agree with the Milnor group $K_n^M(k)$ for $i = n$. In particular, $H^n(k, \mathbf{Z}/m(n)) \simeq K_n^M(k)/m$.

For Z a closed subcheme of X of codimension c with open complement U, there is a localization sequence

$$\to H^{i-2c}(Z, \mathbf{Z}(n-c)) \to H^i(X, \mathbf{Z}(n)) \to H^i(U, \mathbf{Z}(n)) \to H^{i+1-2c}(Z, \mathbf{Z}(n-c)) \to$$

It is a difficult theorem to prove. It is also rather difficult in general to construct higher cycle classes $c_{i,n} : H^i(X, \mathbf{Z}(n)) \to H^i(X, n)$ into various bigraded cohomology theories relevant to arithmetic. The original method of Bloch [2] requires weak purity as well as homotopy property both of which fail for some commonly used p-adic cohomologies. In particular, we are still missing a definition of cycle classes into syntomic cohomology independent of the theory of p-adic periods.

It is even more difficult to show that the higher Chow groups and K'-theory are related by an Atiyah-Hirzebruch spectral sequence

$$E_2^{s,t} = H^{s-t}(X, \mathbf{Z}(-t)) \Rightarrow K'_{-s-t}(X). \tag{3.1}$$

This sequence was first constructed for fields by Bloch-Lichtenbaum [4], then generalized to quasi-projective varieties by Friedlander-Suslin [11], and finally to schemes of finite type by Levine [25],[23]. By different methods, it was also constructed by Grayson-Suslin [18], [34] and Levine [26]. If X is regular, the action of Adams operations shows that this sequence degenerates modulo small torsion and the resulting filtration differs from the γ-filtration by a small torsion. In particular, we get that

$$\operatorname{gr}^i_\gamma K_j(X) \otimes \mathbf{Q} \simeq H^{2i-j}(X, \mathbf{Q}(i)).$$

Varying X, one gets a sheaf $\mathbf{Z}(n) := z^n(-, 2n - *)$ in the étale topology. We have $\mathbf{Z}(0) \simeq \mathbf{Z}$ on a normal scheme and $\mathbf{Z}(1) \simeq \mathbf{G}_m[-1]$ on a smooth scheme. For a separated, noetherian scheme X of finite Krull dimension, the Mayer-Vietoris property yields the isomorphism $H^i(X, \mathbf{Z}(n)) \simeq H^i(X_{\mathrm{Zar}}, \mathbf{Z}(n))$. For X smooth, filtering $z^*(X, *)$ by codimension, we get the very useful Gersten resolution

$$0 \to \mathcal{H}^p(\mathbf{Z}(n)) \to \bigoplus_{x \in X^{(0)}} (i_x)_* H^p(k(x), \mathbf{Z}(n)) \to \bigoplus_{x \in X^{(1)}} (i_x)_* H^{p-1}(k(x), \mathbf{Z}(n-1)) \to$$

Here $X^{(s)}$ denotes the set of points in X of codimension s.

For X smooth and m invertible on X, rigidity in higher Chow groups and étale cohomology and the vanishing of $H^i(k_{\text{ét}}, \mathbf{Z}/m(n))$ for an algebraically closed field k [33] imply that $\mathbf{Z}/m(n)_{\text{ét}} \xrightarrow{\sim} \mu_m^{\otimes n}$. We get the isomorphism

$$c^{\text{ét}}_{i,n} : H^i(X_{\text{ét}}, \mathbf{Z}/m(n)) \xrightarrow{\sim} H^i(X_{\text{ét}}, \mu_m^{\otimes n}).$$

Conjecture 3.2. (Beilinson-Lichtenbaum) *The canonical map*

$$\rho_{i,n} : H^i(X_{\mathrm{Zar}}, \mathbf{Z}/m(n)) \to H^i(X_{\text{ét}}, \mathbf{Z}/m(n))$$

is an isomorphism for $i \leq n$.

It is clear that the Bloch-Kato conjecture is a special case of the above conjecture. What is not obvious is that it also implies it.

Theorem 3.3. *(Suslin-Voevodsky [35], Geisser-Levine [16]) The Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture.*

To prove this via Gersten resolution one passes to the following statement for fields: the Bloch-Kato isomorphism $H^n(F, \mathbf{Z}/m(n)) \simeq K_n^M(F)/n \xrightarrow{\sim} H^n(F_{\text{ét}}, \mu_m^{\otimes n})$ for all fields F that are finitely generated over the base field implies that $\rho_{i,n}$: $H^i(F, \mathbf{Z}/m(n)) \to H^i(F_{\text{ét}}, \mathbf{Z}/m(n))$ is an isomorphism for all such F and $i \leq n$. This is proved by descending induction on the degree of cohomology by "bootstrapping" the Bloch-Kato isomorphism into relative cohomology of cubical complexes.

Unconditionally, we have two important results

Theorem 3.4. *(Levine [24]) If $\mu_m \in \Gamma(X, \mathcal{O}_X)$, inverting the Bott element $\beta_m \in H^0(X_{\mathrm{Zar}}, \mathbf{Z}/m(1))$ gives an isomorphism*

$$\tilde{\rho}_{i,n} : H^i(X_{\mathrm{Zar}}, \mathbf{Z}/m(n))[\beta_m^{-1}] \xrightarrow{\sim} H^i(X_{\text{ét}}, \mathbf{Z}/m(n)).$$

Theorem 3.5. *(Suslin [33]) The map $\rho_{i,n}$ is an isomorphism for X smooth over an algebraically closed field and $n \geq \dim X$.*

We can now use the Atiyah-Hirzebruch spectral sequence (3.1) and its étale analogue

$$E_2^{s,t} = H^{s-t}(X_{\text{ét}}, \mathbf{Z}/m(-t)) \Rightarrow K^{\text{ét}}_{-s-t}(X, \mathbf{Z}/m)$$

constructed by Levine to pass from motivic cohomology to K-theory and to conclude that

Theorem 3.6. *(Levine [23]) The Beilinson-Lichtenbaum conjecture implies the Quillen-Lichtenbaum conjecture.*

For X smooth over a perfect field of characteristic $p > 0$, Geisser-Levine [15] have shown that there is a quasi-isomorphism (in the Zariski and étale topology) $\mathbf{Z}/p^r(n) \xrightarrow{\sim} \nu_r^n[-n]$. They derive it from the fact that for any field k of characteristic p, $H^i(k, \mathbf{Z}/p^r(n)) = 0$ for $i \neq n$, which in turn they induce from the Bloch-Kato isomorphism $H^n(k, \mathbf{Z}/p^r(n)) \xleftarrow{\sim} K_n^M(k)/p^r \xrightarrow{\sim} \nu_r^n(k)$. As a result we get

$$H^{i+n}(X, \mathbf{Z}/p^r(n)) \simeq H^i(X_{\mathrm{Zar}}, \nu_r^n), \qquad H^{i+n}(X_{\text{ét}}, \mathbf{Z}/p^r(n)) \simeq H^i(X_{\text{ét}}, \nu_r^n).$$

The above implies that $\tilde{\pi}_n(\mathcal{K}/p^r) \simeq \nu_r^n$: via the Bloch-Lichtenbaum spectral sequence $H^{s-t}(k, \mathbf{Z}/p^r(-t)) \Rightarrow K_{-s-t}(k, \mathbf{Z}/p^r)$, the computation of $H^i(k, \mathbf{Z}/p^r(n))$ yields the isomorphism $K_n^M(k)/p^r \to K_n(k, \mathbf{Z}/p^r)$; having that it suffices now to evoke Gersten resolution.

3.2. Motivic cohomology over Dedekind domains.

The construction of Bloch higher Chow groups and some of its basic properties (most notably the localization exact sequence and the Atiyah-Hirzebruch spectral sequence) as well as some computations of motivic sheaves can be extended to schemes of finite type over a Dedekind scheme ([25], [23], [14]). Here is an example. Let X be a smooth scheme over a complete discrete valuation ring V of mixed characteristic $(0, p)$ with a perfect residue field k. Denote by $i : Y \hookrightarrow X$ and $j : U \hookrightarrow X$ the special and generic fibers, respectively. We will sketch how assuming the Bloch-Kato conjecture mod p we get a quasi-isomorphism [14]

$$i^*\mathbf{Z}/p^n(r)_{\text{ét}} \to S_n(r) \quad \text{for } r < p-1.$$

Here $S_n(r)$ is the syntomic complex of Fontaine-Messing [8] (philosophically) defined as the mapping cone of the map $Ru_*J^{[r]}_{X_n/W_n(k)} \xrightarrow{1-\phi^r} Ru_*\mathcal{O}_{X_n/W_n(k)}$, where $W(k)$ is the ring of Witt vectors of k, $\mathcal{O}_{X_n/W_n(k)}$ is the crystalline structure sheaf, $J_{X_n/W_n(k)} = \ker(\mathcal{O}_{X_n/W_n(k)}) \to \mathcal{O}_{X_n})$, $u : X_n/W_n(k)_{\mathrm{cr,\acute{e}t}} \to X_{n,\text{ét}}$ is the natural projection, and $\phi^r = \phi/p^r$ is the divided Frobenius. We always get the long exact sequence

$$\to H^i(X_n, S_n(r)) \to H^i_{\mathrm{cr}}(X_n/W_n(k), J^{[r]}) \stackrel{1-\phi^r}{\to} H^i_{\mathrm{cr}}(X_n/W_n(k)) \to$$

By the theory of p-adic periods [21] we have the distingushed triangle

$$\to S_n(r) \to \tau_{\leq n} i^* Rj_* \mu_{p^n}^{\otimes r} \to \nu_n^{r-1}[-r] \to$$

This triangle can be seen as a realization of the "localization" sequence for the étale motivic sheaves: we apply the above computations of motivic sheaves over fields and a purity result $\mathbf{Z}(r-1)_{\text{ét}}[-2] \stackrel{\sim}{\to} \tau_{\leq r+1} Ri^! \mathbf{Z}(r)_{\text{ét}}$ (contingent on the Beilinson-Lichtenbaum conjecture mod p) to the localization sequence and get the distinguished triangle

$$\to i^* \mathbf{Z}/p^n(r)_{\text{ét}} \to \tau_{\leq r} i^* Rj_* \mu_{p^n}^{\otimes r} \to \nu_n^{r-1}[-r] \to .$$

Comparing the above two triangles, we get that the cycle class map $i^*\mathbf{Z}/p^n(r)_{\text{ét}} \to S_n(r)$ is a quasi-isomorphism inducing a cycle map (an isomorphism for X proper)

$$c^{\mathrm{syn}}_{i,r} : H^i(X_{\text{ét}}, \mathbf{Z}/p^n(r)) \to H^i(X, S_n(r)), \quad r < p-1.$$

4. Application: p-adic Hodge theory

In p-adic Hodge theory we attempt to understand p-adic Galois representations coming from the étale cohomology of varieties over p-adic fields via the de Rham cohomology of these varieties. The maps relating étale and de Rham cohomology groups are called p-adic period morphisms. Just as in the classical case, we would like to see them as integration of differential forms. Motivic cohomology allows us to do that [30], [31]. We will sketch briefly how.

Remark 4.1. The main comparison theorems of p-adic Hodge theory were proved earlier by two different methods: by Fontaine-Messing-Kato [8], [20], Kato [19], and Tsuji [39] via a study of p-adic nearby cycles and by Faltings [6], [7] using the theory of almost étale extensions.

4.1. The good reduction case.

Let k be a perfect field of positive characteristic p, $W(k)$ the corresponding ring of Witt vectors and K its field of fractions. Let $\overline{K}$ be an algebraic closure of K and let $\mathrm{Gal}(\overline{K}/K)$ denote its Galois group. Let X be a smooth proper scheme over V of relative dimension d. We have a functor which carries the crystalline cohomology groups of X with all their structures into representations of $\mathrm{Gal}(\overline{K}/K)$. For $p-2 \geq r \geq i$, set

$$\mathbf{L}(H^i_{\mathrm{cr}}(X_n/V_n)\{-r\}) := (F^0(H^i_{\mathrm{cr}}(X_n/V_n)\{-r\} \otimes B^+_{\mathrm{cr},n}))^{1=\phi^0}.$$

Here $B^+_{\mathrm{cr},n} = H^*_{\mathrm{cr}}(\mathrm{Spec}(\overline{V}_n)/W_n(k))$ is one of Fontaine's rings of periods. It is equipped with a decreasing filtration $F^i B^+_{\mathrm{cr},n}$, Frobenius, and an action of the group $\mathrm{Gal}(\overline{K}/K)$. The crystalline cohomology groups $H^i_{\mathrm{cr}}(X_n/V_n) \simeq H^i_{dR}(X_n/V_n)$ have a natural Hodge filtration and ϕ^0 comes from the tensor of divided Frobeniuses $\phi^j = \phi/p^j$. The twist $\{-r\}$ refers to twisting the Hodge filtration and the Frobenius.

Conjecture 4.2. (Crystalline conjecture) *For p large enough, there exists a canonical Galois equivariant period isomorphism*

$$\alpha_{\mathrm{cr}} : H^i(X_{\overline{K}}, \mu_n^{\otimes r}) \xrightarrow{\sim} \mathbf{L}(H^i_{\mathrm{cr}}(X_n/V_n)\{-r\}).$$

The proof using K-theory we sketch here works for $r \geq 2d$, $p-2 \geq 2r+d$ (or rationally with no restriction on p).

Since $B^+_{\mathrm{cr},n} \simeq H^*_{\mathrm{cr}}(\overline{V}_n/V_n)$, by the Künneth formula $H^*_{\mathrm{cr}}(X_n/V_n) \otimes B^+_{\mathrm{cr}} \simeq H^*_{\mathrm{cr}}(X_{\overline{V},n}/V_n)$, where $\overline{V}$ is the integral closure of V in $\overline{K}$. The defining property of syntomic cohomology yields a natural map (in fact an isomorphism)

$$H^i(X_{\overline{V}}, S_n(r)) \to \mathbf{L}(H^i_{\mathrm{cr}}(X_n/V_n)\{-r\}).$$

It follows that to prove the conjecture, by a standard argument, it suffices to construct a Galois equivariant map

$$\alpha_{i,r} : H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes r}) \to H^i(X_{\overline{V}}, S_n(r))$$

compatible with Poincaré duality and some cycle classes. To construct this map as an integration we will use the following diagram

$$\begin{array}{ccc}
F^r_\gamma/F^{r+1}_\gamma K_{2r-i}(X_{\overline{V}}, \mathbf{Z}/p^n) & \xrightarrow[j^*]{\sim} & F^r_\gamma/F^{r+1}_\gamma K_{2r-i}(X_{\overline{K}}, \mathbf{Z}/p^n) \\
\downarrow{\scriptstyle \rho_{2r-i}} & & \downarrow{\scriptstyle \rho_{2r-i}} \\
F^r_\gamma/F^{r+1}_\gamma K^{\text{ét}}_{2r-i}(X_{\overline{V}}, \mathbf{Z}/p^n) & \xrightarrow[j^*]{} & F^r_\gamma/F^{r+1}_\gamma K^{\text{ét}}_{2r-i}(X_{\overline{K}}, \mathbf{Z}/p^n) \\
\downarrow{\scriptstyle c^{\mathrm{syn}}_{r,2r-i}} & & \downarrow{\scriptstyle c^{\text{ét}}_{r,2r-i}} \\
H^i(X_{\overline{V}}, S_n(r)) & \xleftarrow{\alpha_{i,r}} & H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes r}).
\end{array}$$

The right-hand side allows us to represent étale classes by higher algebraic cycle classes on $X_{\overline{K}}$. Those can be lifted (via j^*) to the integral model $X_{\overline{V}}$ and we can integrate differential forms along them to get the period map $\alpha_{i,r}$. Specifically, by Quillen-Lichtenbaum conjecture or by Suslin the map ρ_{2r-i} is an isomorphism for $2r-i \geq \mathrm{cd}_p X_{\text{ét}} = 2d$. The degeneration of the étale Atiyah-Hirzebruch spectral sequence gives that $c^{\text{ét}}_{r,2r-i}$ is an isomorphism modulo small torsion. Also the restriction j^* is an isomorphism: since the scheme $X_{\overline{V}}$ is smooth we can pass to K'-theory; by localization, the kernel and cokernel of j^* is controlled by mod p^n K'-groups of special fibers and those can be killed by totally ramified extensions of V of degree p^n. For p and r as above, we define the map $\alpha_{i,r}$ to make this diagram commute.

Corollary 4.3. *For $r \geq d + i/2$, $p-2 \geq r + d/2$, there exists a unique period map $\alpha_{i,r} : H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes r}) \to H^i(X_{\overline{V}}, S_n(r))$ compatible with the étale and syntomic higher Chern classes from K-theory mod p^n of $X_{\overline{K}}$ and $X_{\overline{V}}$.*

Based on [29], [28] we expect all the existing constructions of the period maps to be compatible with higher Chern classes hence equal.

Assume that we are able to define syntomic higher cycle maps without using p-adic periods. Then a construction of the period map $\alpha_{i,r}$ as an integral can be done in a more precise way by the following diagram.

$$\begin{tikzcd} H^i(X_{\overline{V}}, \mathbf{Z}/p^n(r)) \arrow[r, "j^*", "\sim"'] \arrow[d, "\rho_{i,r}"] & H^i(X_{\overline{K}}, \mathbf{Z}/p^n(r)) \arrow[d, "\rho_{i,r}"] \\ H^i(X_{\overline{V},\text{ét}}, \mathbf{Z}/p^n(r)) \arrow[r, "j^*"] \arrow[d, "c^{\text{syn}}_{i,r}"] & H^i(X_{\overline{K},\text{ét}}, \mathbf{Z}/p^n(r)) \arrow[d, "c^{\text{ét}}_{i,r}", "\wr"'] \\ H^i(X_{\overline{V}}, S_n(r)) & H^i(X_{\overline{K}}, \mu_{p^n}^{\otimes r}) \arrow[l, "\alpha_{i,r}"'] \end{tikzcd} \tag{4.1}$$

Arguing as above, we see that the restriction map j^* is an isomorphism. The map $\rho_{i,r}$ is an isomorphism for $r \geq i$ by the Beilinson-Lichtenbaum conjecture or for $r \geq d$ by Suslin. That gives the definition of $\alpha_{i,r}$ in these two cases and a proof of the Crystalline conjecture for all i and $2d \leq r \leq p-2$. Notice that then all the maps in the above diagram are isomorphisms.

Remark 4.4. Our period map $\alpha_{i,r}$ goes in the opposite direction than the period maps constructed by other methods. This implies that one can simply use Poincaré duality to prove that the map is an isomorphism. Rationally that works well but integrally it doubles the lower bound on p.

4.2. The semistable reduction case.

Let now K be a complete discrete valuation field of mixed characteristic $(0,p)$ with ring of integers V and a perfect residue field k. Let $X^\times$ be a fine log-smooth proper $V^\times$-scheme, where V is equipped with the log-structure associated to the closed point, such that the generic fiber X_K is smooth over K and the special fiber $X_0^\times$ is of Cartier type. A standard example would be a scheme X with simple semistable reduction.

Conjecture 4.5. (Semistable conjecture) *There exists a natural period isomorphism*

$$\alpha_{st} : H^*(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\text{st}} \simeq H^*_{\text{cr}}(X_0^\times / W(k)^0) \otimes_{W(k)} B_{\text{st}}$$

preservings Galois action, monodromy, filtration and Frobenius.

Here the period ring B_{st} is equipped with Galois action, Frobenius and monodromy operators. It maps naturally into another ring of periods B_{dR}, which is equipped with a decreasing filtration. The log-crystalline cohomology groups $H^*_{\text{cr}}(X^\times/W(k)^0)[1/p]$ (analogues of limit Hodge structures) are also equipped with Frobenius and monodromy operators. There is also a canonical isomorphism of $K \otimes_{W(k)} H^*_{\text{cr}}(X_0^\times/W(k)^0) \simeq H^*_{dR}(X_K/K)$ via which Hodge filtration induces a descending filtration on these groups. The period isomorphism and its base change to B_{dR} should preserve all these structures. As a corollary, one gets that the étale cohomology as a Galois representation can be recovered from the log-crystalline cohomology

$$H^*(X_{\overline{K}},\mathbf{Q}_p)\simeq (H^*_{\mathrm{cr}}(X_0^\times/W(k)^0)\otimes_{W(k)}B_{\mathrm{st}})^{N=0,\phi=1}\cap F^0(B_{dR}\otimes_K H^*_{dR}(X_K/K)).$$

In the above formula the kernel of the monodromy was computed by Kato to be $\mathbf{Q}\otimes \operatorname{proj}\lim_n H^*_{\mathrm{cr}}(X^\times_{\overline{V},n}/W_n(k))$. If we now take into account both Frobenius and the filtration, we can pass to log-syntomic cohomology and we see that to prove the conjecture it suffices to construct a Galois equivariant family of maps

$$\alpha^n_{i,r}: H^i(X_{\overline{K}},\mu_{p^n}^{\otimes r})\to H^i_{\mathrm{cr}}(X^\times_{\overline{V}},S_n(r)),$$

at least for r large enough, that is compatible with Poincaré duality and the trace map.

The main problem with trying to carry over our motivic proof of the Crystalline conjecture to this setting is that the integral model $X_{\overline{V}}$ is in general singular. It becomes then very difficult to control the kernel and cokernel of the restriction map j^*. However the singularities are rather mild (they are of toric type) and we find [32] that every model $X^\times_{V'}$, for a finite extension V'/V, can be desingularized by a log-blow-up $Y^\times\to X^\times_{V'}$. Since we are blowing up only strata this desingularization does not change the log-syntomic cohomology. Obviously it does not change the étale cohomology either, so to define the maps $\alpha^n_{i,r}$ we can work with the regular models $Y^\times$. We have the usual "integration" diagram

$$\begin{array}{ccc}
F^r_\gamma/F^{r+1}_\gamma K_{2r-i}(Y,\mathbf{Z}/p^n) & \xrightarrow[j^*]{} & F^r_\gamma/F^{r+1}_\gamma K_{2r-i}(Y_K,\mathbf{Z}/p^n)\\
\Big\downarrow{\scriptstyle c^{\mathrm{syn}}_{r,2r-i}\rho_{2r-i}} & & \Big\downarrow{\scriptstyle c^{\text{ét}}_{r,2r-i}\rho_{2r-i}}\\
H^i(Y^\times,S_n(r)) & \xleftarrow{\alpha^n_{i,r}} & H^i(Y_K,\mu_{p^n}(r)).
\end{array}$$

The right-hand side of the diagram behaves like before. The restriction j^* is an isomorphism for $2r-i>\dim X_K+1$ because by the localization sequence its kernel and cokernel are controlled by $K'_j(X_k,\mathbf{Z}/p^n)$, which vanishes for $j>\dim X_K$ by Geisser-Levine. Hence we can integrate differential forms against higher cycles (on the integral model Y) to get the period maps $\alpha^n_{i,r}$. Again as a corollary we get a uniqueness statement for semistable period maps.

Remark 4.6. Notice that the above vanishing result of Geisser-Levine and the resulting bijectivity of the restriction map j^* are entirely p-adic phenomena. The analogous statements mod l are false. This is in contrast with the good reduction case where j^* is an isomorphism mod l as well.

Question 4.7. Is it possible to define log-motivic complexes and cohomology that would specialize to log-syntomic cohomology ? More precisely, one would like to have a log-analogue of the motivic diagram (4.1) for a semistable scheme $X^\times$ (with logs everywhere in the left column). For that we need a good definition of log-motivic complexes $\mathbf{Z}/p^n(r)^\times$ and log-syntomic cycle classes

$$c^{\mathrm{syn}}_{i,r}: H^i(X^\times,\mathbf{Z}/p^n(r)^\times)\to H^i(X^\times,S_n(r))$$

(isomorphisms for X proper and $i \leq r < p-1$). We would expect the restriction map

$$j^* : H^i(X^\times, \mathbf{Z}/p^n(r)^\times) \to H^i(X_K, \mathbf{Z}/p^n(r))$$

to be an isomorphism.

This question is closely related to that of the existence of limit motivic cohomology (see the recent work of Marc Levine [27] on that subject in the case of schemes over a field).

Acknowledgments. I would like to thank Thomas Geisser and Marc Levine for helpful comments.

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