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On the image of p-adic regulators

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1 Introduction

Let V denote a complete discrete valuation ring with a fraction field K of characteristic 0 and a perfect residue field k of positive characteristic p, let $V_0 = W(k)$ denote the ring of Witt vectors with coefficients in k, and K_0 its fraction field. Set $G_K = \mathrm{Gal}(\overline{K}/K)$. Let X be a smooth and projective scheme over V.

For $i \geqq 0$, $2i - n - 1 \geqq 0$, consider the Galois representation $L = H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i))$. In the work of Bloch and Kato on the special values of L-functions of motives [2], the question of describing the images of the Soulé's p-adic Chern character

$$r_p^{\text{ét}} : K_{2i-n-1}(X) \otimes \mathbf{Q} \to H^1(G_K, L), \quad \text{for } 2i - n - 1 \geqq 1 ,$$

and of the étale cycle class map $\mathrm{cl}_X : (CH^i(X) \otimes \mathbf{Q})_{\mathrm{hom} \sim 0} \to H^1(G_K, L)$ arises. We will show here that, as Bloch and Kato predicted, they are contained in the subgroups $H^1_f(G_K, L)$ of crystalline extensions. The factorization through H^1_f of the cycle class map was shown earlier by Nekovář [11] via the de Rham conjecture.

The problem can be reduced to a question of existence and compatibility between various constructions in the theory of p-adic periods. Briefly, our argument goes as follows. There is a Chern character and a cycle class map into a version of syntomic cohomology $H^*_f(X, \mathscr{K}(*))$ (introduced by us in [14]) based on convergent crystalline cohomology. The relevant theory of Chern classes in the convergent crystalline cohomology is described in the appendix to this paper.

The map $h : H^n_f(X, \mathscr{K}(i)) \to H^n_{\text{ét}}(X_K, \mathbf{Q}_p(i))$, which we constructed in [14], commutes with the K-theory characteristic classes, hence with the regulators and cycle classes. This is shown by mapping both the syntomic and the étale cohomology to the cohomology of the topos of 'rigid smooth sheaves' introduced by Faltings in [5]. We show that this last cohomology also carries K-theory characteristic classes. Now, the standard methods allow to reduce the question to the comparison of Chern classes of line bundles on projective spaces, which can be done explicitely.

Finally, since we have proved earlier [14] that the map h is compatible with the spectral sequences

$$H^i_f(G_K, H^j_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(n))) \Rightarrow H^{i+j}_f(X, \mathscr{K}(n)),$$
$$H^i(G_K, H^j_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(n))) \Rightarrow H^{i+j}_{\text{ét}}(X_K, \mathbf{Q}_p(n))),$$

we are done.

I would like to thank Spencer Bloch for very helpful conversations related to the subject of this paper.

Throughout the paper, for a scheme X, $\mathscr{X}$ will denote the associated formal scheme.

2 *p*-adic cohomologies

2.1 *f*-cohomology

Choose a uniformizer π of V. Let Ł be the category of smooth and quasi projective schemes over V. We will define a cohomology theory H_f on Ł generalizing the groups H^1_f of Bloch and Kato [2].

For $X \in$ Ł, let $(X_k/V)_{\text{conv}}$ be the convergent topos of Ogus [15] and let $\mathscr{K}_{X_k/V}$ be its structure isocrystal. Let $\Omega^{\bullet}_X$ be the natural resolution of $\mathscr{K}_{X_k/V}$ on $(X_k/V)_{\text{conv}}$ by sheaves acyclic for the projection $u_{X_k/V} : (X_k/V)_{\text{conv}} \to X_{k,\text{Zar}}$ to the Zariski topos of X_k. We have $u_{X_k/V*}\Omega^{\bullet}_X \simeq \mathscr{O}_{\mathscr{X}}[1/p] \otimes \Omega^{\bullet}_{X/V}$, and we set $F^n_X := \Omega^{\geqq n}_X$. There are natural embeddings $\mathscr{K}_{X_k/V} \hookrightarrow \Omega^{\bullet}_X$ and $F^n_X \hookrightarrow \Omega^{\bullet}_X$, and a map $\phi_n : \phi^*\mathscr{K}_{X_k/V_0} \to \mathscr{K}_{X_k/V_0}$, $\phi_n = p^{-n}$ Id, where ϕ is the Frobenius on $(X_k/V_0)_{\text{conv}}$.

The functors H_f are hypercohomologies of certain complexes of sheaves of differential forms in the Zariski topology. For a small affine Spec$(R) \in$ Ł (small means that there is an étale map $V[T_1^{\pm 1}, \ldots, T_d^{\pm 1}] \to R$), let $C(R)$ denote a set of data consisting of closed embeddings of Spec(R) into an affine space $\mathbb{A}^n_{V_0} = \text{Spec}(R')$ and into an affine space $\mathbb{A}^m_V = \text{Spec}(R'')$, a V_0-morphism $\alpha : R' \to R''$ over the identity on Spec(R), and a Frobenius lift ϕ on $\widehat{R'}$. Note that, for a small affine Spec$(R) \in$ Ł, the collection of closed embeddings $(R, C(R))$ (with maps given by linear maps on the affine spaces inducing the identity on Spec(R)) forms a cofiltered family (compute the coincidence schemes in the category of flat schemes).

Having a pair $(R, C(R))$, we will denote by T', T'' the widenings

$$T'' = (R''^{\wedge}, I''^{\wedge}, R/\pi R \xrightarrow{\text{Id}} R/\pi R), \qquad T' = (R'^{\wedge}, I'^{\wedge}, R/\pi R \xrightarrow{\text{Id}} R/\pi R),$$

where I'' is the ideal of $R/\pi R$ in R'' and $R''^{\wedge}$ denotes the I''-adic completion of R''. Similarly for T'.

Denote by $\mathscr{S}^{\bullet}(n)$ the complex of sheaves on the large Zariski site of Ł associated to the complex of presheaves of abelian groups sending small affine $\mathrm{Spec}(R)$ to $\mathrm{inj\,lim}_{(R,C(R))}\Omega^{\bullet}_{(R,C(R))}(\mathscr{K}(n))$, where we set

$$\Omega^{\bullet}_{(R,C(R))}(\mathscr{K}(n)) := \mathrm{Cone}(\mathscr{K}_{T'}\otimes\Omega^{\bullet}_{R'/V_0}\oplus F^n\Omega^{\bullet}_{R''/V}\xrightarrow{\beta}\mathscr{K}_{T'}\otimes\Omega^{\bullet}_{R'/V_0}\oplus\ \mathscr{K}_{T''}\otimes\Omega^{\bullet}_{R''/V})[-1]\,,$$

with

$$F^n\Omega^{\bullet}_{R''/V}=J^{(n)}\otimes\Omega^0_{R''/V}\to J^{(n-1)}\otimes\Omega^1_{R''/V}\to\cdots\,,$$

where $J^{(i)}$ is the filtration of $\mathscr{K}_{T''}$ defined in [14], and $\beta(x,y)=(x-\phi_n(x), x-y)$. Here, to define the restriction map for a morphism $\mathrm{Spec}(S)\to\mathrm{Spec}(R)$, we choose a presentation of S, $S\simeq R[X_1,\dots,X_k]/I$, and take the induced from $\mathrm{Spec}(R)$ embeddings into affine spaces. This definition is easily seen to be independent of the presentation chosen. Since $\mathscr{K}_{T'}\otimes\Omega^{\bullet}_{R'/V_0}\simeq i_*Ru_{(R/\pi)/V_0*}\mathscr{K}_{(R/\pi)/V_0}$ and $F^n\Omega^{\bullet}_{R''/V}\simeq i_*Ru_{(R/\pi)/V*}F^n_R$, for any $X\in$ Ł, there is a distinguished triangle

(1) $$\mathscr{S}^{\bullet}(n)_X\xrightarrow{\varepsilon_{V_0}\oplus\varepsilon_V}i_*Ru_{X_k/V_0*}\mathscr{K}_{X_k/V_0}\oplus i_*Ru_{X_k/V*}F^n_X$$

(2) $$\xrightarrow{\beta}i_*Ru_{X_k/V_0*}\mathscr{K}_{X_k/V_0}\oplus i_*Ru_{X_k/V*}\mathscr{K}_{X_k/V}\to\,,$$

where $i: X_{k,\mathrm{Zar}}\hookrightarrow X_{\mathrm{Zar}}$. There is also an anticommutative and associative multiplication on $\mathscr{S}^{\bullet}(n)$ induced (as in [14]) by the de Rham product on $\Omega^{\bullet}_{R'/V_0}$ and $\Omega^{\bullet}_{R''/V}$. More precisely, it comes from a homotopic family of maps of complexes

$$\cup_{\alpha}: \Omega^{\bullet}_{(R,C(R))}(\mathscr{K}(n))\otimes\Omega^{\bullet}_{(R,C(R))}(\mathscr{K}(m))\to\Omega^{\bullet}_{(R,C(R))}(\mathscr{K}(n+m)),\quad \alpha\in\mathbf{Z}_p\,,$$

$$\begin{aligned}&(x_1,x_2,x_3,x_4)\cup_{\alpha}(y_1,y_2,y_3,y_4)\\ &\quad=(x_1\cup y_1,\ x_2\cup y_2,\\ &\qquad x_3\cup(\alpha y_1+(1-\alpha)\phi_m(y_1))+(-1)^{\deg x_1}((1-\alpha)x_1+\alpha\phi_n(x_1))\cup y_3\,,\\ &\qquad x_4\cup(\alpha y_1+(1-\alpha)y_2)+(-1)^{\deg x_1}((1-\alpha)x_1+\alpha x_2)\cup y_4)\,.\end{aligned}$$

Let $X\in$ Ł. Consider the maps

$$\varepsilon: \mathscr{S}^{\bullet}(n)_X\xrightarrow{\varepsilon_{V_0}}i_*Ru_{X_k/V_0*}\mathscr{K}_{X_k/V_0}\xrightarrow{(1,1)}i_*Ru_{X_k/V_0*}\mathscr{K}_{X_k/V_0}\oplus i_*Ru_{X_k/V*}\mathscr{K}_{X_k/V}\,,$$

$$\begin{aligned}\varepsilon': \mathscr{S}^{\bullet}(n)_X&\xrightarrow{\varepsilon_{V_0}\oplus\varepsilon_V}i_*Ru_{X_k/V_0*}\mathscr{K}_{X_k/V_0}\oplus i_*Ru_{X_k/V*}F^n_X\\ &\xrightarrow{(\phi_n,1)}i_*Ru_{X_k/V_0*}\mathscr{K}_{X_k/V_0}\oplus i_*Ru_{X_k/V*}\mathscr{K}_{X_k/V}\end{aligned}$$

($\varepsilon=\varepsilon'$ in the derived category). Here, for two maps $f: A\to B$, $g: A\to C$, $(f,g): A\to B\oplus C$ denotes the map sending $a\in A$ to $(f(a),f(b))\in B\oplus C$, and 1 stands for the identity. They induce products

$$\cup_{V_0} : \mathscr{S}^\bullet(n)_X \otimes^{\mathbf{L}} i_* Ru_{X_k/V_0 *}\mathscr{K}_{X_k/V_0} \xrightarrow{\cup(\varepsilon_{V_0}\otimes 1)} i_* Ru_{X_k/V_0 *}\mathscr{K}_{X_k/V_0}\ ,$$

$$\cup_V : \mathscr{S}^\bullet(n)_X \otimes^{\mathbf{L}} i_* Ru_{X_k/V *}F_X^m \xrightarrow{\cup(\varepsilon_V\otimes 1)} i_* Ru_{X_k/V *}F_X^{n+m}\ ,$$

$$\cup' : \mathscr{S}^\bullet(n)_X \otimes^{\mathbf{L}} (i_* Ru_{X_k/V_0 *}\mathscr{K}_{X_k/V_0} \oplus i_* Ru_{X_k/V *}\mathscr{K}_{X_k/V})$$
$$\xrightarrow{(\cup\oplus\cup)(\varepsilon\otimes 1)} (i_* Ru_{X_k/V_0 *}\mathscr{K}_{X_k/V_0} \oplus i_* Ru_{X_k/V *}\mathscr{K}_{X_k/V})\ ,$$

which, as we can easily check, together with $\cup_1$ define operations of $\mathscr{S}^\bullet(n)_X$ on the distinguished triangle 1, which are compatible with the morphisms.

Set $H_f(X, \mathscr{K}(n)) := R\Gamma(X, \mathscr{S}^\bullet(n))$. This definition is easily checked to be independent of all the choices made.

Lemma 2.1. *For any $X \in Ł$, there is a canonical quasi isomorphism*

$$H_f(X, \mathscr{K}(n)) \simeq H_f(X, \mathscr{K}_{X_k/V}\{-n\})\ ,$$

where $H_f(X, \mathscr{K}_{X_k/V}\{-n\})$ is the cohomology defined in [14].

Proof. Let $\operatorname{Spec}(R_i)$, $i \in I$, be a covering of X by small affine and fix embeddings $C(R_i)$, $i \in I$. Then, by filtered cohomological descent,

$$\begin{aligned} R\Gamma(X, \mathscr{S}^\bullet(n)) &\simeq \operatorname{Cone}(H((X_k/V_0)_{\text{conv}}, \mathscr{K}_{X_k/V_0}) \oplus H((X_k/V)_{\text{conv}}, F_X^n) \\ &\quad \xrightarrow{\beta} H((X_k/V_0)_{\text{conv}}, \mathscr{K}_{X_k/V_0}) \oplus H((X_k/V)_{\text{conv}}, \mathscr{K}_{X_k/V}))[-1] \\ &\simeq \operatorname{Cone}(\mathscr{K}_{T'_\bullet} \otimes \Omega^\bullet_{R'_\bullet/V_0} \oplus F^n\Omega^\bullet_{R''_\bullet/V} \\ &\quad \xrightarrow{\beta} \mathscr{K}_{T'_\bullet} \otimes \Omega^\bullet_{R'_\bullet/V_0} \oplus \mathscr{K}_{T''_\bullet} \otimes \Omega^\bullet_{R''_\bullet/V})[-1] \\ &\simeq H_f(X, \mathscr{K}_{X_k/V}\{-n\}) \end{aligned}$$

□

There exist morphisms $c_0^f : \mathbf{Q}_p \to \mathscr{S}^\bullet(0)$, $c_1^f : \mathscr{O}^*[-1] \to \mathscr{S}^\bullet(1)$ in $\mathbf{D}^+(Ł_{\text{ZAR}})$. The morphism c_0^f is induced by the canonical map $\mathbf{Q}_p \to \mathscr{K}_{T'}$. It is the unit for our multiplication. Concerning c_1^f, it suffices to define a compatible family of maps $R^*[-1] \to \Omega^\bullet_{(R,C(R))}(\mathscr{K}(1))$. For a pair $(R, C(R))$, define the map

$$\begin{aligned} R^*[-1] \to \operatorname{Cone}(&\mathscr{K}_{T'} \otimes \Omega^\bullet_{R'/V_0} \oplus F^1\Omega^\bullet_{R''/V} \\ &\to \mathscr{K}_{T'} \otimes \Omega^\bullet_{R'/V_0} \oplus \mathscr{K}_{T''} \otimes \Omega^\bullet_{R''/V})[-1] \end{aligned}$$

as the composition

$$\begin{aligned} R^*[-1] &\to \operatorname{Cone}((R/\pi)^* \oplus R^* \xrightarrow{-1+1} (R/\pi)^*)[-2] \\ &\xleftarrow{\sim} \operatorname{Cone}(\operatorname{Cone}((1+I'^\wedge) \xrightarrow{-1} (R'^\wedge)^*) \oplus \operatorname{Cone}((1+J''^\wedge) \xrightarrow{-1} \widetilde{R}^*) \\ &\quad \xrightarrow{-1+1} \operatorname{Cone}((1+I''^\wedge) \xrightarrow{-1} R''^{\wedge *}))[-2] \\ &\xrightarrow{\omega} \operatorname{Cone}(\mathscr{K}_{T'} \otimes \Omega^\bullet_{R'/V_0} \oplus F^1\Omega^\bullet_{R''/V} \\ &\quad \to \mathscr{K}_{T'} \otimes \Omega^\bullet_{R'/V_0} \oplus \mathscr{K}_{T''} \otimes \Omega^\bullet_{R''/V})[-1]\ , \end{aligned}$$

where J'' is the ideal of R in R'' and $\widetilde{R}$ is the J''-adic completion of R''. Here the map ω is defined in degree 0 by

$$\log \oplus \log : (1 + I'^{\wedge}) \oplus (1 + J''^{\wedge}) \to \mathscr{K}_{T'} \otimes \Omega^0_{R'/V_0} \oplus J^{(1)} \otimes \Omega^0_{R''/V}, \quad \text{by}$$

$$R'^{\wedge *} \oplus \widetilde{R}^* \oplus (1 + I''^{\wedge}) \to \mathscr{K}_{T'} \otimes \Omega^1_{R'/V_0} \oplus \mathscr{K}_{T''} \otimes \Omega^1_{R''/V} \oplus \mathscr{K}_{T'} \otimes \Omega^0_{R'/V_0} \oplus \mathscr{K}_{T''} \otimes \Omega^0_{R''/V},$$

$$(x, y, z) \to (\operatorname{dlog} x, \operatorname{dlog} y, p^{-1}\log(\phi(x)/x^p), \log z)$$

in degree 1, and by $\operatorname{dlog} : R''^{\wedge *} \to \mathscr{K}_{T''} \otimes \Omega^1_{R''/V}$ in degree 2.

Lemma 2.2. *Let $\mathscr{L}$ be an invertible sheaf on X and $c_1^f(\mathscr{L})$ its Chern class in $H^2_f(X, \mathscr{K}(1))$. Then $\varepsilon_{V_0}(c_1^f(\mathscr{L})) \in H^2((X_k/V_0)_{\mathrm{conv}}, \mathscr{K}_{X_k/V_0})$ and $\varepsilon_V(c_1^f(\mathscr{L})) \in H^2((X_k/V)_{\mathrm{conv}}, F^1_X)$ are the crystalline Chern classes of $i^*\mathscr{L}$, $i : X_k \hookrightarrow X$, and $\mathscr{L}$ respectively.*

Proof. Concerning ε_{V_0}, it suffices to show that it is compatible with c_1^f and the convergent topos map c_1 (see the appendix). Locally, let $\operatorname{Spec}(R)$ be an affine open of X. Then we need to show that, for every embedding $C(R)$, the following diagram commutes

$$\begin{array}{ccc} (R/\pi)^*[-1] & \xleftarrow{\sim} & \operatorname{Cone}(1 + I'^{\wedge} \xrightarrow{-1} (R'^{\wedge})^*)[-1] \\ \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle (\log\, \operatorname{dlog})} \\ Ru_{(R/\pi)/V_0 *}\mathscr{K}_{(R/\pi)/V_0} & \xrightarrow{\sim} & \mathscr{K}_{T'} \otimes \Omega^{\bullet}_{R'/V_0}, \end{array}$$

what follows from Lemma A.1. The statement about ε_V follows similarly from Lemma A.2. □

Proposition 2.1. *Let $n > 0$, $\xi = c_1^f(\mathscr{O}(1)) \in H^2_f(\mathbf{P}^n_V, \mathscr{K}(1))$. For $X \in Ł$, take $\pi_{\mathbf{P}^n_V}$, π_X to be the projections of $\mathbf{P}^n_X$ onto $\mathbf{P}^n_V$ and X, then the natural map*

$$\oplus_i \pi^*_{\mathbf{P}^n_V}(\xi)^i \cup \pi^*_X : \bigoplus_{i=0}^{n} \mathscr{S}^{\bullet}(j-i)_X[-2i] \to R\pi_{X*}\mathscr{S}^{\bullet}(j)_{\mathbf{P}^n_X}$$

is an isomorphism for all integers j.

Proof. It follows easily from the distinguished triangle 1, compatibility of our products with the maps in this triangle, compatibility of the f-cohomology c_1's with the crystalline c_1's (Lemma 2.2), and the projective space theorems (Propositions A.1, A.2) in the (filtered) convergent crystalline cohomology. □

Like in [8, 2.2], we can now construct universal classes $C^f_{i,n} \in H^{2i}_f(B.GL_n/V, \mathscr{K}(i))$ (classes of the universal locally free sheaf on $B.GL_n/V$). This yields

Proposition 2.2. *There exists a unique theory of Chern classes, which to every simplicial smooth quasi projective scheme X over V and every locally free sheaf of finite type $\mathscr{E}$ on X associates an element $c^f(\mathscr{E}) = \prod_i c_i^f(\mathscr{E}) \in \prod_i H_f^{2i}(X, \mathscr{K}(i))$ such that c_0^f and c_1^f for invertible sheaves are induced by the above c_0^f and c_1^f, and the usual functoriality and additivity properties hold.*

Proof. The only nontrivial point is the additivity property. By the argument of [8, 2.10] it suffices to check it in the universal case, i.e., over $X = B.GL(n,m)/V$.

First, we claim that the odd degree cohomology groups $H^{2*+1}((X_k/V_0)_{\mathrm{conv}}, \mathscr{K}_{X_k/V_0}) = H^{2*+1}((X_k/V)_{\mathrm{conv}}, \mathscr{K}_{X_k/V})$ are trivial. Indeed, the simplicial scheme X being smooth and affine, it suffices to show that $H_{dR}^{2*+1}(B.GL(n,m)/W_l(k)) = 0$, where the subscript l denotes the reduction mod p^l. Since we have a good theory of Chern classes in the de Rham cohomology, arguing like in the proof of Proposition 3.4, we get that, by Dold–Thom isomorphism, $H_{dR}^*(B.GL(n,m)/W_l(k)) \hookrightarrow H_{dR}^*(B.B_{n+m}/W_l(k))$. On the other hand, the same computation as in the proof of the Proposition 3.4 yields that $H_{dR}^*(B.GL_{n+m}/W_l(k)) \hookrightarrow H_{dR}^*(B.B_{n+m}/W_l(k))$ as well. In fact, iterating the Dold–Thom isomorphism, we get that $H_{dR}^*(B.B_{n+m}/W_l(k))$ is a $H_{dR}^*(B.GL_{n+m}/W_l(k))$-algebra generated by Chern classes of some vector bundles over $B.B_{n+m}/W_l(k)$. Since we know that

$$H_{dR}^*(B.GL_{n+m}/W_l(k)) \simeq W_l(k)[x_1, \ldots, x_{n+m}], \quad x_i \in H_{dR}^{2i}(B.GL_{n+m}/W_l(k)),$$

that proves our claim.

Next, the long exact sequence

$$\begin{aligned} \to H_f^i(X, \mathscr{K}(j)) &\to H^i((X_k/V_0)_{\mathrm{conv}}, \mathscr{K}_{X_k/V_0}) \oplus H^i((X_k/V)_{\mathrm{conv}}, F_X^j) \\ &\to H^i((X_k/V_0)_{\mathrm{conv}}, \mathscr{K}_{X_k/V_0}) \oplus H^i((X_k/V)_{\mathrm{conv}}, \mathscr{K}_{X_k/V}) \\ &\to H_f^{i+1}(X, \mathscr{K}(j)) \to \end{aligned}$$

implies that

$$\begin{aligned} H_f^{2i}(B.GL(n,m)/V, \mathscr{K}(i)) &\hookrightarrow H^{2i}((X_k/V_0)_{\mathrm{conv}}, \mathscr{K}_{X_k/V_0}) \\ &\oplus H^{2i}((X_k/V)_{\mathrm{conv}}, \mathscr{K}_{X_k/V}). \end{aligned}$$

Since this inclusion is compatible with the theory of Chern classes for vector bundles on $B.GL(n,m)/V$ (Lemma 2.2) and we know additivity in the (filtered) convergent cohomology (Proposition A.2), we are done. □

Theorem 2.1. *For every $X \in Ł$ and integer $j \geqq 0$, there exist functorial maps*

$$c_j^f = \prod c_{ij}^f : K_j(X) \to \prod H_f^{2i-j}(X, \mathscr{K}(i)),$$

such that c_0^f coincides with the above Chern classes and c_j^f, for $j > 0$, is *a group homomorphism. Moreover, the induced Chern character*

$$r_p^f = \mathrm{ch} : K_j(X) \to \prod_{i \geqq 0} H_f^{2i-j}(X, \mathscr{K}(i))$$

is a ring homomorphism.

Proof. As in [8]. □

2.2 Étale cohomology

Recall [5], that étale cohomology can be computed via the topos of "sheaves of local systems". We will need here its large version. Let X be a smooth, separated scheme of finite type over V.

Let $\widetilde{X}_{\mathrm{ZAR}}$ be the following topos. An object of $\widetilde{X}_{\mathrm{ZAR}}$ is a collection $L = ((L_Y), (r_{Y_1 Y_2}))$ of locally constant étale sheaves L_Y on Y_K, for every smooth, separated scheme Y over X and, for every pair $f : Y_2 \to Y_1$, a morphism $r_{Y_1 Y_2} : f^*_{Y_1 Y_2} L_{Y_1} \to L_{Y_2}$ such that $r_{Y_2 Y_3} r_{Y_1 Y_2} = r_{Y_1 Y_3}$ and $r_{YY} = \mathrm{Id}$. One also requires that for every trancuated Zariski hypercovering $U_1 \rightrightarrows U_0 \to U$, L_U is the maximal locally constant subsheaf of $\ker(j_{0*} L_{U_0} \rightrightarrows j_{1*} L_{U_1})$, where $j_i : (U_i)_K \to U_K$. Morphism $f : L \to M$ in $\widetilde{X}_{\mathrm{ZAR}}$ is a collection of morphisms of locally constant sheaves $f_Y : L_Y \to M_Y$ compatible with $r_{Y_1 Y_2}$'s. Here, locally constant étale sheaf on Y means an element of the Ind-category of finite étale covers of Y.

We will also denote by $\widetilde{X}_{\mathrm{ZAR}}$ the equivalent topos, where all Y's are assumed to be affine. In particular, every locally constant sheaf L on X_K defines a sheaf in $\widetilde{X}_{\mathrm{ZAR}}$, and one proves [5, III.c] that the étale cohomology $H_{\text{ét}}(X_K, L)$ is isomorphic to $H(\widetilde{X}_{\mathrm{ZAR}}, L)$.

There is a left exact functor ψ_{ZAR} from $\widetilde{X}_{\mathrm{ZAR}}$ to the large Zariski topos of X, sending L to the sheaf $Y \mapsto L_Y(Y_K)$. We have $H(\widetilde{X}_{\mathrm{ZAR}}, L) = H(X_{\mathrm{ZAR}}, R\psi L)$. Let $\mathscr{E}_k^\bullet(n)$ and $\mathscr{E}^\bullet(n)$ be the following complexes of sheaves in $\mathbf{D}^+(\text{Ł}_{\mathrm{ZAR}})$

$$\mathscr{E}_k^\bullet(n) := R\psi_{\mathrm{ZAR}} \mu_k(n), \qquad \mathscr{E}^\bullet(n) := \mathbf{Q}_p \otimes^{\mathbf{L}} R \varprojlim R\psi_{\mathrm{ZAR}}(\mu_k(n))_k \ ,$$

where $\mu_k(n)$ is the n'th twist of p^k-roots of unity on $\widetilde{\text{Ł}}_K$. There is a compatible family of multiplications on $\mathscr{E}_k^\bullet(n)$ and $\mathscr{E}^\bullet(n)$, which is easily seen to be associative and anticommutative. Set $H(\widetilde{X}, \mathbf{Q}_p(n)) := H(X, \mathscr{E}^\bullet(n))$.

We will need an analogous topos $\widetilde{X}_{\mathrm{Et}}$, where we put the sheaf condition via the étale instead of the Zariski topology. We have the commutative diagram

$$\begin{array}{ccc} \widetilde{X}_{\mathrm{Et}} & \xrightarrow{\ \widetilde{\varepsilon}_*\ } & \widetilde{X}_{\mathrm{ZAR}} \\ {\scriptstyle \psi_{\mathrm{Et}}} \downarrow & & \downarrow {\scriptstyle \psi_{\mathrm{ZAR}}} \\ X_{\mathrm{Et}} & \xrightarrow{\ \varepsilon_*\ } & X_{\mathrm{ZAR}} \end{array}$$

Lemma 2.3. *For any locally constant p-torsion sheaf L on $X_{K,\mathrm{Et}}$ and the map $j : X_{K,\mathrm{Et}} \to X_{\mathrm{Et}}$, there are canonical quasi isomorphisms*

$$\text{(i)} \quad Rj_*L \simeq R\psi_{\mathrm{Et}}L, \qquad \text{(ii)} \quad R\psi_{\mathrm{ZAR}}R\,\widetilde{\varepsilon}_*L \overset{\sim}{\leftarrow} R\psi_{\mathrm{ZAR}}L.$$

Proof. (i) First, we check that the natural map $R\psi_{\mathrm{Et}}L \to Rj_*j^*R\psi_{\mathrm{Et}}L$ is an isomorphism. Take a geometric point $\bar{x}$ over X. We have to show that the natural map $\operatorname{inj\,lim} H^q(\widetilde{U}_{\text{ét}}, L) \overset{j^*}{\to} \operatorname{inj\,lim} H^q(\widetilde{U_{K,\text{ét}}}, L)$, where the limit is over étale neighbourhoods of $\bar{x}$ in X, is an quasi isomorphism. Take such a U. There is a canonical map ρ from the étale topos of U_K to $\widetilde{U}_{\text{ét}}$ factoring through $\widetilde{U_{K,\text{ét}}}$ [5],[14]. We have a commutative diagram

$$\begin{array}{ccc} H^q(\widetilde{U}_{\text{ét}}, L) & \overset{j^*}{\to} & H^q(\widetilde{U_{K,\text{ét}}}, L) \\ \searrow \rho^* & & \swarrow \rho^* \\ & H^q_{\text{ét}}(U_K, L) & \end{array}$$

The ρ^*'s being isomorphisms, we are done.

Next, since $L = j^*\psi_{\mathrm{Et}}L$, we have a natural map $Rj_*L \to Rj_*j^*R\psi_{\mathrm{Et}}L$, which we claim is a quasi isomorphism. Again, looking at stalks, we get a map

$$\operatorname{inj\,lim} H^q_{\text{ét}}(U_K, L) \to \operatorname{inj\,lim} H^q(\widetilde{U_{K,\text{ét}}}, L)$$

(the limit being over étale neighbourhoods of a geometric point $\bar{x}$ in X). The map $H^q_{\text{ét}}(U_K, L) \to H^q(\widetilde{U_{K,\text{ét}}}, L)$ being an isomorphism [5], we are done.
(ii) Looking at stalks of both sides at $\bar{x}$ on X, we get a map

$$\operatorname{inj\,lim} H^q(\widetilde{U}_{\mathrm{Zar}}, L) \to \operatorname{inj\,lim} H^q(\widetilde{U}_{\text{ét}}, L)\,.$$

Since $H^q(\widetilde{U}_{\mathrm{Zar}}, L) \to H^q(\widetilde{U}_{\text{ét}}, L)$ is an isomorphism [5, IIIc], we are done. □

There exist morphisms $\tilde{c}_0 : \mathbf{Q}_p \to \mathscr{E}^\bullet(0)$, $\tilde{c}_1 : \mathcal{O}^*[-1] \to \mathscr{E}^\bullet(1)$ in $\mathbf{D}^+(Ł_{\mathrm{ZAR}})$. For $\tilde{c}_0$, we use the compatible family of natural maps $\mathbf{Z}/p^k \to R\psi_{\mathrm{ZAR}}\mathbf{Z}/p^k$. So defined $\tilde{c}_0$ is clearly a unit for our multiplication. We set $\tilde{c}_1$ to be equal to the following compatible family of maps

$$\begin{aligned} \mathcal{O}^*[-1] &\to Rj_*\mathcal{O}^*[-1] \to Rj_*R\varepsilon_{K*}\mathcal{O}^*[-1] \overset{(1)}{\to} Rj_*R\varepsilon_{K*}\mu_k \\ &\simeq R\varepsilon_*Rj_*\mu_k \underset{\sim}{\overset{(2)}{\to}} R\varepsilon_*R\psi_{\mathrm{Et}}\mu_k \simeq R\psi_{\mathrm{ZAR}}R\widetilde{\varepsilon}_*\mu_k \underset{\sim}{\overset{(3)}{\leftarrow}} R\psi_{\mathrm{ZAR}}\mu_k\,, \end{aligned}$$

where the morphism (1) comes from the Kummer exact sequence on $Ł_{K,\mathrm{Et}}$ ($Ł_{K,\mathrm{Et}}$ being the large étale topos of smooth quasi projective schemes over K), and the quasi isomorphisms $(2), (3)$ follow from Lemma 2.3.

Before we proceed, let us mention how all of the above relates to the constructions in the étale cohomology leading to the Soulé's regulator. Define in $\mathbf{D}^+(Ł_{K,\mathrm{ZAR}})$

$$\tau_k^\bullet(n) := R\varepsilon_{K*}\mu_k(n) \quad \text{and} \quad \tau^\bullet(n) := \mathbf{Q}_p \otimes^{\mathbf{L}} R\varprojlim R\varepsilon_{K*}(\mu_k(n))_k\,.$$

There is a compatible family of multiplications on $\tau_k^{\bullet}(n)$ and on $\tau^{\bullet}(n)$, which is easily seen to be associative and anticommutative.

The map $c_0^{\text{ét}}: \mathbf{Q}_p \to \tau^{\bullet}(0)$ is induced by the natural maps $\mathbf{Z}/p^k \to R\varepsilon_{K*}\mathbf{Z}/p^k$, and the map $c_1^{\text{ét}}: \mathscr{O}^*[-1] \to \tau^{\bullet}(1)$ is induced by the compositions

$$\mathscr{O}^*[-1] \to R\varepsilon_{K*}\mathscr{O}^*[-1] \to R\varepsilon_{K*}\mu_k = \tau_k^{\bullet}(1)\,,$$

where the last map comes from the Kummer exact sequence. The maps $c_0^{\text{ét}}, c_1^{\text{ét}}$ induce corresponding maps into $Rj_*\tau^{\bullet}(i)$, $i = 0, 1$, which we will also denote by $c_0^{\text{ét}}, c_1^{\text{ét}}$, if there is no confusion.

Clearly, for $Y \in Ł_K$, $H_{\text{ét}}(Y, \mu_k(n)) = H(Y, \tau_k^{\bullet}(n))$. There are quasi isomorphisms

$$\omega_k : Rj_*\tau_k^{\bullet}(n) \simeq \mathscr{E}_k^{\bullet}(n) \quad \text{and } \omega : Rj_*\tau^{\bullet}(n) \simeq \mathscr{E}^{\bullet}(n)\,.$$

Indeed, by Lemma 2.3,

$$\begin{aligned}\mathscr{E}_k^{\bullet}(n) = R\psi_{\text{ZAR}}\mu_k(n) &\simeq R\psi_{\text{ZAR}}R\widetilde{\varepsilon}_*\mu_k(n) \simeq R\varepsilon_*R\psi_{\text{Et}}\mu_k(n)\\ &\simeq R\varepsilon_*Rj_*\mu_k(n) \simeq Rj_*\tau_k^{\bullet}(n)\,.\end{aligned}$$

The maps $\tilde{c}_0$, $c_0^{\text{ét}}$ are clearly compatible with ω. Since $\omega c_1^{\text{ét}} = \tilde{c}_1$, so are the c_1's. Also, using Godement resolutions, we conclude that ω is compatible with the products induced on $H^*(X, \mathscr{E}^{\bullet}(n))$ and on $H^*(X, Rj_*\tau^{\bullet}(n)) = H^*(X_K, \tau^{\bullet}(n)) = H^*_{\text{ét}}(X_K, (\mathbf{Q})_p(n))$.

Proposition 2.3. *Let* $n > 0$, $\xi = \tilde{c}_1(\mathscr{O}(1)) \in H^2(\widetilde{\mathbf{P}}^n_V, \mathbf{Q}_p(1))$. *For* $X \in Ł$ *and all integers* j, *there is a natural isomorphism*

$$\oplus_i \pi^*_{\mathbf{P}^n_V}(\xi)^i \cup \pi^*_X : \bigoplus_{i=0}^{n} \mathscr{E}^{\bullet}(j-i)_X[-2i] \to R\pi_{X*}\mathscr{E}^{\bullet}(j)_{\mathbf{P}^n_X}\,.$$

Proof. We easily reduce to the statement that, for every k, l and j, the morphism

$$\oplus_i \pi^*_{\mathbf{P}^n_V}(\xi)^i \cup \pi^*_X : \bigoplus_{i=0}^{n} H^{l-2i}(X, \mathscr{E}_k^{\bullet}(j-i)) \to H^l(\mathbf{P}^n_X, \mathscr{E}_k^{\bullet}(j))$$

is an isomorphism. Now, applying the isomorphism ω_k and using its compatibility with the multiplications and the c_1's, we reduce to the Dold–Thom isomorphism in étale cohomology, which we know is true. □

All of the above yields a theory of Chern classes in $H^{2i}(\widetilde{X}, \mathbf{Q}_p(i))$ for simplicial smooth quasi projective schemes X over V and locally free sheaves of finite type on X. Via the morphism ω, it is compatible with the theory of étale Chern classes for X_K. Since the map ω is an isomorphism, we conclude that the Chern class theory in $H^{2i}(\widetilde{X}, \mathbf{Q}_p(i))$ has the expected uniqueness, normalization, functoriality, and additivity properties. Using the Gillet's machinery [8], we can extend these Chern classes to characteristic classes for K-theory. We will postpone the question of comparison with the "étale" K-theory classes to Sect. 3.2.

3 Comparison

3.1 Auxiliary topos

We will relate the étale regulators and the f-regulators via regulators into cohomology of an auxiliary topos: the topos of 'sheaves of rigid local systems' from [5].

For a smooth, separated scheme X of finite type over V, we have a topos $\widetilde{\mathscr{X}}$. An object of $\widetilde{\mathscr{X}}$ is a collection $L = ((L_Y), (r_{Y_1Y_2}))$ of locally constant étale sheaves L_Y on $\mathrm{Spec}(\mathscr{A}_K)$, for every formally smooth p-adic formal scheme $Y = \mathrm{Spf}(\mathscr{A})$ over $\mathscr{X}$; for every pair $Y_2 \to Y_1 = \mathrm{Spf}(\mathscr{A}_2) \to \mathrm{Spf}(\mathscr{A}_1)$, a morphism $r_{Y_1Y_2}\colon f^*_{Y_1Y_2}L_{Y_1} \to L_{Y_2}$, where $f^*_{Y_1Y_2}$ is the induced map $\mathrm{Spec}(\mathscr{A}_{2,K}) \to \mathrm{Spec}(\mathscr{A}_{1,K})$, satisfying the usual compatibilities. Further the definition is analogous to that of $\widetilde{X}$.

There is a map $\bar{\imath}\colon \widetilde{\mathscr{X}} \to \widetilde{X}_{\mathrm{ZAR}}$. Define the complexes of sheaves $\mathscr{F}_k^{\bullet}(n)$ and $\mathscr{F}^{\bullet}(n)$ in $\mathbf{D}^+(\text{Ł}_{\mathrm{ZAR}})$

$$\mathscr{F}_k^{\bullet}(n) := R\psi_{\mathrm{ZAR}} R\bar{\imath}_* \mu_k(n), \qquad \mathscr{F}^{\bullet}(n) := \mathbf{Q}_p \otimes^{\mathbf{L}} R \varprojlim \mathscr{F}_k^{\bullet}(n)\,,$$

where $\mu_k(n)$ is the n'th twist of p^k-roots of unity on $\widetilde{\text{Ł}}$. There is an associative and anticommutative product on $\mathscr{F}_k^{\bullet}(n)$ and on $\mathscr{F}^{\bullet}(n)$. Clearly $H(X, \mathscr{F}_k^{\bullet}(n)) = H(\widetilde{\mathscr{X}}, \mu_k(n))$. Set $H(\widetilde{\mathscr{X}}, \mathbf{Q}_p(n)) := H(X, \mathscr{F}^{\bullet}(n))$.

There exist morphisms $\tilde{c}_0\widehat{\ }\colon \mathbf{Q}_p \to \mathscr{F}^{\bullet}(0)$, $\tilde{c}_1\widehat{\ }\colon \mathscr{O}^*[-1] \to \mathscr{F}^{\bullet}(1)$ in $\mathbf{D}^+(\text{Ł}_{\mathrm{ZAR}})$. We define both of them by composing $\tilde{c}_i$'s with the natural maps $\mu_k(n) \to R\bar{\imath}_*\mu_k(n)$. $\tilde{c}_0\widehat{\ }$ is clearly a unit for our product.

The natural maps $\mu_k(n) \to R\bar{\imath}_*\mu_k(n)$ induce compatible morphisms $t_k\colon \mathscr{E}_k^{\bullet}(n) \to \mathscr{F}_k^{\bullet}(n)$, hence a morphism $t\colon \mathscr{E}^{\bullet}(n) \to \mathscr{F}^{\bullet}(n)$. t is clearly compatible with the maps c_0, c_1 and products. Passing to cohomology, we get a map

$$t\colon H^*(\widetilde{X}, \mathbf{Q}_p(n)) \to H^*(\widetilde{\mathscr{X}}, \mathbf{Q}_p(n))\,,$$

which is an isomorphism for X proper (Proposition 6.1 of [13]).

Proposition 3.1. *Let $n > 0$, $\xi = \tilde{c}_1\widehat{\ }(\mathscr{O}(1)) \in H^2(\mathbf{P}^n_V, \mathscr{F}^{\bullet}(1))$. For $X \in$ Ł and all integers j, there is a natural isomorphism*

$$\oplus_i \pi^*_{\mathbf{P}^n_V}(\xi)^i \cup \pi^*_X\colon \bigoplus_{i=0}^{n} \mathscr{F}^{\bullet}(j-i)_X[-2i] \to R\pi_{X*}\mathscr{F}^{\bullet}(j)_{\mathbf{P}^n_X}\,.$$

Proof. We easily reduce to the statement that, for every j, k, l and for every geometric point $\bar{x}$ of the special fiber of X, the morphism

$$\begin{aligned}\oplus_i \operatorname{inj\,lim}\,(\pi^*_{\mathbf{P}^n_V}(\xi)^i \cup \pi^*_U)\colon &\bigoplus_{i=0}^{n} \operatorname{inj\,lim} H^{l-2i}(U, \mathscr{F}_k^{\bullet}(j-i))\\ &\to \operatorname{inj\,lim} H^l(\mathbf{P}^n_U, \mathscr{F}_k^{\bullet}(j))\end{aligned}$$

is an isomorphism, where the limit over the affine étale (sic!) neighbourhoods of $\bar{x}$ in X (cf., [14, Proposition 6.2]). Equivalently, we want the morphism

$$\oplus_i \varinjlim(\pi^*_{\mathbf{P}^n_V}(\xi)^i \cup \pi^*_U) : \bigoplus_{i=0}^{n} \varinjlim H^{l-2i}(\widetilde{\mathscr{U}}_{\text{ét}}, \mu_k(j-i)) \to \varinjlim H^l((\mathscr{U} \times \mathbf{P}^{n\wedge}_V)^{\sim}_{\text{ét}}, \mu_k(j))$$

to be an isomorphism. We claim that the natural morphism

$$\operatorname{inj}\lim H^l((\mathscr{U} \times \mathbf{P}^{n\wedge}_V)^{\sim}_{\text{ét}}, \mu_k(j)) \to H^l((\operatorname{Spf}(\mathscr{O}_{\widehat{X,\bar{x}}}) \times \mathbf{P}^{n\wedge}_V)^{\sim}_{\text{ét}}, \mu_k(j)) \, ,$$

where $\mathscr{O}_{X,\bar{x}}$ denotes the strict henselization at $\bar{x}$, is an isomorphism. First, we will prove that the above morphism is an isomorphism for any locally constant p-torsion sheaf L on $X \times T$, where $T = \operatorname{Spec}(A)$ is any smooth separated V-scheme. By $K(\pi, 1)$-Lemma [6] and Gabber's base change, we get

$$\begin{aligned} \varinjlim H^l((\mathscr{U} \times \mathscr{T})^{\sim}_{\text{ét}}, L) &\xleftarrow{\sim} \varinjlim H^l(((U \times A)^h)^{\sim}_{\text{ét}}, L) \simeq \varinjlim H^l((U \times A)^h[1/p]_{\text{ét}}, L) \\ &\xrightarrow{\sim} H^l((\mathscr{O}_{X,\bar{x}} \times A)^h[1/p]_{\text{ét}}, L) \simeq H^l(((\mathscr{O}_{X,\bar{x}} \times A)^h)^{\sim}_{\text{ét}}, L) \\ &\xrightarrow{\sim} H^l((\operatorname{Spf}(\mathscr{O}_{\widehat{X,\bar{x}}}) \times \mathscr{T})^{\sim}_{\text{ét}}, L) \end{aligned}$$

Here h denotes the henselization at p. We finish by Mayer–Vietoris.

Since

$$\operatorname{inj}\lim H^{l-2i}(\widetilde{\mathscr{U}}_{\text{ét}}, \mu_k(j-i)) \xrightarrow{\sim} H^{l-2i}(\operatorname{Spf}(\mathscr{O}_{X,\bar{x}})^{\sim}_{\text{ét}}, \mu_k(j-i))$$

by [14, Proposition 6.1], by the above isomorphism we reduce to showing that the morphism

$$\oplus_i \pi^*_{\mathbf{P}^n_V}(\xi)^i \cup \pi^*_{\mathscr{O}_{\widehat{X,\bar{x}}}} : \bigoplus_{i=0}^{n} H^{l-2i}(\operatorname{Spf}(\mathscr{O}_{\widehat{X,\bar{x}}})^{\sim}_{\text{ét}}, \mu_k(j-i)) \to H^l((\operatorname{Spf}(\mathscr{O}_{\widehat{X,\bar{x}}}) \times \mathbf{P}^{n\wedge}_V)^{\sim}_{\text{ét}}, \mu_k(j))$$

is an isomorphism. Consider the morphisms

$$t_k^{\text{ét}} : H^l(\operatorname{Spec}(\mathscr{O}_{X,\bar{x}})^{\sim}_{\text{ét}}, \mu_k(j)) \to H^l(\operatorname{Spf}(\mathscr{O}_{\widehat{X,\bar{x}}})^{\sim}_{\text{ét}}, \mu_k(j)) \, ,$$

$$t_k^{\text{ét}} : H^l(\operatorname{Spec}((\mathscr{O}_{X,\bar{x}}) \times \mathbf{P}^n_V)^{\sim}_{\text{ét}}, \mu_k(j)) \to H^l((\operatorname{Spf}(\mathscr{O}_{\widehat{X,\bar{x}}}) \times \mathbf{P}^{n\wedge}_V)^{\sim}_{\text{ét}}, \mu_k(j)) \, .$$

The first one is an isomorphism because both sides are cohomologies of the fundamental groups of $\mathscr{O}_{X,\bar{x}}[1/p]$ and $\mathscr{O}_{\widehat{X,\bar{x}}}[1/p]$ respectively, and we know these groups to be isomorphic [4]. Concerning the second one, project everything via the functor ψ onto the étale topos of $\operatorname{Spec}(\mathscr{O}_{X,\bar{x}}) \times \mathbf{P}^n_V$. The proper base change theorem being true for this scheme (cf., [7]) we can proof that $t_k^{\text{ét}}$ is also an isomorphism paraphrasing the arguments from the proof of Proposition 6.1 of [14].

We have now reduced everything to the Dold–Thom isomorphism in étale cohomology, which we know is true. □

All of the above yields a theory of Chern classes in $H^{2i}(\widetilde{\mathscr{X}},\mathbf{Q}_p(i))$ for simplicial smooth quasi projective schemes X over V and locally free sheaves of finite type on X. Via the morphism $t: H^{2i}(\widetilde{X},\mathbf{Q}_p(i)) \to H^{2i}(\widetilde{\mathscr{X}},\mathbf{Q}_p(i))$, it is compatible with the theory of Chern classes in $H^{2i}(\widetilde{X},\mathbf{Q}_p(i))$. Since the last theory behaves as expected, so does the former one, i.e., we have uniqueness, normalization, functoriality and additivity properties. Now, the Gillet's method [8] extends these Chern classes to characteristic classes for K-theory.

3.2 Comparison between the étale cohomology and the auxiliary topos cohomology

Proposition 3.2. *For any $X \in$ Ł, the following diagrams commute*

$$
\begin{array}{ccc}
K_j(X_K) & \xrightarrow{c^{\text{ét}}_{ij}} & H^{2i-j}_{\text{ét}}(X_K,\mathbf{Q}_p(i)) \\
\uparrow & & w \uparrow l \\
K_j(X) & \xrightarrow{\tilde{c}_{ij}} & H^{2i-j}(\widetilde{X},\mathbf{Q}_p(i))
\end{array}
\qquad
\begin{array}{ccc}
K_j(X) & \xrightarrow{\tilde{c}_{ij}} & H^{2i-j}(\widetilde{X},\mathbf{Q}_p(i)) \\
\downarrow & & \downarrow t \\
K_j(X) & \xrightarrow{\widehat{\tilde{c}_{ij}}} & H^{2i-j}(\widetilde{\mathscr{X}},\mathbf{Q}_p(i))\,.
\end{array}
$$

Proof. All the K-theory classes can be defined following [8] (cf., the proof of Proposition 2.2). Using functoriality for $\widetilde{X}$-classes, we reduce to checking that the following diagram commutes (in the homotopy category)

$$
\begin{array}{ccc}
& B.GL(\mathscr{O}_{X_K}) & \\
\widetilde{C}_i \swarrow & & \searrow C^{\text{ét}}_i \\
\mathscr{K}(2i,\mathscr{E}^{\bullet}_K(i)') & \xrightarrow{\omega} & \mathscr{K}(2i,\tau^{\bullet}(i)')\,,
\end{array}
$$

where $\mathscr{E}^{\bullet}_K(i)'$ and $\tau^{\bullet}(i)'$ are injective resolutions of $\mathscr{E}^{\bullet}_K(i)$ and $\tau^{\bullet}(i)$ on X_K, $\mathscr{K}$ is the Dold–Puppe functor of the corresponding truncations, and the maps of pointed simplicial sets $\widetilde{C}_i$, $C^{\text{ét}}_i$ come from the universal Chern classes $\widetilde{C}^K_{i,n} \in H^{2i}(B.GL_n/K,\mathscr{E}^{\bullet}(i))$ and $C^{\text{ét}}_{i,n} \in H^{2i}(B.GL_n/K,\tau^{\bullet}(i))$. Hence, it suffices to check that $\widetilde{C}^K_{i,n}$ and $C^{\text{ét}}_{i,n}$ are compatible with $\omega: H^{2i}(B.GL_n/K,\mathscr{E}^{\bullet}(i)) \simeq H^{2i}(B.GL_n/K,\tau^{\bullet}(i))$. That, in turn, follows easily from the compatibility of ω with c_0, c_1 and products.

For the other diagram we argue in the same way. □

3.3 Comparison between the f-cohomology and the auxiliary topos cohomology

The following proposition will allow us to reduce computations to affine schemes.

Proposition 3.3. *Any torsor* $\pi\colon W \to X$ *for a locally free sheaf of finite type over a quasi projective smooth* V*-scheme* X *induces a cohomology isomorphism*

$$H^*(\widetilde{\mathscr{X}}, L) \xrightarrow{\sim} H^*(\widetilde{\mathscr{W}}, \pi^* L)$$

for a sheaf L *on* $\mathrm{Spec}(K)_{\text{ét}}$.

Proof. Project everything via the functor ψ to the étale topos of X. It suffices to show that the natural morphism

$$\text{inj}\lim H^*(\widetilde{\mathscr{U}}, L) \to \text{inj}\lim H^*(\widetilde{\mathscr{W}_U}, \pi^* L)\ ,$$

where the limit is over affine étale neighbourhoods U of a geometric point $\bar{x}$ on the special fiber of X, is an isomorphism. Here, we can assume that $W_U = U \times \mathbb{A}^n_V$.

Consider the commutative diagram

$$\begin{array}{ccc}
\text{inj}\lim H^*(\widetilde{\mathscr{U}_{\text{ét}}}, L) & \longrightarrow & \text{inj}\lim H^*((\mathscr{U} \times \widehat{\mathbb{A}_V}^n)^{\sim}_{\text{ét}}, L) \\
\wr\big\downarrow & & \gamma\big\downarrow \\
H^*(\mathrm{Spf}(\mathscr{O}_{\widehat{X,\bar{x}}})^{\sim}_{\text{ét}}, L) & \longrightarrow & H^*((\mathrm{Spf}(\mathscr{O}_{\widehat{X,\bar{x}}}) \times \widehat{\mathbb{A}_V}^n)^{\sim}_{\text{ét}}, L) \\
\wr\big\uparrow & & \wr\big\uparrow \\
H^*(\mathrm{Spec}(\mathscr{O}_{X,\bar{x}})^{\sim}_{\text{ét}}, L) & \longrightarrow & H^*(((\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V)^h)^{\sim}_{\text{ét}}, L) \\
\Big\| & & \Big\| \\
H^*_{\text{ét}}(\mathrm{Spec}(\mathscr{O}_{X,\bar{x}})[1/p], L) & \xrightarrow{\ \ell\ } & H^*_{\text{ét}}(((\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V)^h)[1/p], L)\ .
\end{array}$$

The morphism γ is an isomorphism by the following commutative diagram

$$\begin{array}{ccc}
\varinjlim H^*((\mathscr{U} \times \widehat{\mathbb{A}_V}^n)^{\sim}_{\text{ét}}, L) & \xleftarrow{\sim} & \varinjlim H^*(((U \times \mathbb{A}^n_V)^h)^{\sim}_{\text{ét}}, L) \simeq \varinjlim H^*_{\text{ét}}((U \times \mathbb{A}^n_V)^h_K, L) \\
\gamma\big\downarrow & & \big\downarrow \\
H^*((\mathscr{O}_{\widehat{X,\bar{x}}} \times \widehat{\mathbb{A}_V}^n)^{\sim}_{\text{ét}}, L) & \xleftarrow{\sim} & H^*(((\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V)^h)^{\sim}_{\text{ét}}, L) \simeq H^*_{\text{ét}}((\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V)^h_K, L)\ .
\end{array}$$

It suffices to prove that the map ℓ is an isomorphism. Since both sides define cohomological functors it suffices to check their behaviour in degree 0 and for injectives on $\mathrm{Spec}(K)_{\text{fl}}$. We clearly have the vanishing of both cohomologies in degrees higher than 0 for injectives. In degree 0, assume that L is constructable. Then $L = \coprod \mathrm{Spec}(K_i)$, where K_i is a finite field extension of K. Let V_i be the corresponding ring of integers. Take any K_i. Then $V_i = \bigoplus_{k=0}^{l} x_i^k V$, for some $x_i \in V_i$. If $f \in H^0_{\text{ét}}((\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V)^h[1/p], L)$, then $f = f_i$, $f_i \in \mathrm{Mor}_K(\mathrm{Spec}((\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V)^h[1/p]), \mathrm{Spec}(K_i))$. If $f_i \neq 0$, then, since $(\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V)^h$ is normal and integral, f_i comes from a V-morphism $f_i : V_i \to (\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V)^h$ and V_i is necessarily étale over V. Now, $f_i(x_i) \in (\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V)^h$ is a root of a polynomial with all the roots in the maximal unramified extension V^{unr} of V. Since $V^{\text{unr}} \subset \mathscr{O}_{X,\bar{x}}$ and $(\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V)^h$ is an integral domain, $f_i(x_i) \in V^{\text{unr}} \subset \mathscr{O}_{X,\bar{x}}$, and ℓ is surjective. Since $\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V$ is an integral domain, $\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V \hookrightarrow (\mathscr{O}_{X,\bar{x}} \times \mathbb{A}^n_V)^h$, and ℓ is injective. □

We will need the following description of Gillet's K-theory characteristic classes [1], [16]. Let $X \in$ Ł be an affine scheme. Consider the evaluation maps

$$\mathrm{ev}_k : X \times B_{\bullet} GL_k(X) \to B_{\bullet} GL_k/V$$

(in the category of simplicial schemes). The universal class $C^f_{i,k} \in H^{2i}(B_{\bullet} GL_k/V, \mathscr{S}^{\bullet}(i))$ gives a class $\mathrm{ev}_k^*(C^f_{i,k}) \in H^{2i}(X \times B_{\bullet} GL_k(X), \mathscr{S}^{\bullet}(i))$. Varying k, we get a class in $H^{2i}(X \times B_{\bullet} GL(X), \mathscr{S}^{\bullet}(i))$, where $B_{\bullet} GL/V = \mathrm{inj}\lim B_{\bullet} GL_k/V$. Since

$$\begin{aligned} H^{2i}(X \times B_{\bullet} GL(X), \mathscr{S}^{\bullet}(i)) &\simeq \mathrm{Hom}_{\mathscr{H}o^*}(B_{\bullet} GL(X), R\Gamma(X, \mathscr{K}(2i, \mathscr{S}^{\bullet}(i)'_X)) \\ &= \mathrm{Hom}_{\mathscr{H}o^*}(\mathbf{Z}_{\infty} B_{\bullet} GL(X), R\Gamma(X, \mathscr{K}(2i, \mathscr{S}^{\bullet}(i)'_X)) , \end{aligned}$$

where $\mathscr{H}o^*$ is the homotopy category of pointed simplicial sets and $\mathscr{S}^{\bullet}(i)'_X$ is an injective resolution of $\mathscr{S}^{\bullet}(i)_X$, we get a map (in the homotopy category)

$$C^f_i : \mathbf{Z}_{\infty} B_{\bullet} GL(X) \to R\Gamma(X, \mathscr{K}(2i, \mathscr{S}^{\bullet}(i)'_X)) ,$$

where $\mathbf{Z}_{\infty}$ is the sheafified version of the integral completion functor of Bousfield–Kan. That yields our characteristic classes for $j > 0$

$$c^f_{ij} : K_j(X) = \pi_j \mathbf{Z}_{\infty} B_{\bullet} GL(X) \xrightarrow{C^f_i} \pi_j R\Gamma(X, \mathscr{K}(2i, \mathscr{S}^{\bullet}(i)'_X)) \simeq H^{2i-j}(X, \mathscr{S}^{\bullet}(i)) .$$

Same description is valid for all the other cohomologies we use here.

Recall [5], [14] that, for a well-pointed $X \in$ Ł, there is a map $l : \mathscr{S}^{\bullet}(n)_X \to \mathscr{F}^{\bullet}(n)_X$ in $\mathbf{D}^+(X)$, functorial with respect to well-pointed morphisms $f : X \to Y$, $X, Y \in$ Ł. In particular, the map l extends to well-pointed simplicial schemes. Here, well-pointed scheme is a scheme with a particular choice of base-points, namely, a choice of a geometric generic formal point for every connected formal component. Well-pointed morphism is a morphism of well-pointed schemes equipped with a choice of path between the corresponding base points. Every morphism can be well-pointed. Well-pointed simplicial scheme has a well-pointed scheme in every degree and the base-points are compatible with the face operators.

The construction of the above map l uses the fact that, for a well-pointed $X \in$ Ł, the complex $\mathscr{F}^{\bullet}(n)_X = \mathbf{Q}_p \otimes^{\mathbf{L}} R\lim_{\leftarrow} R\psi_{\mathrm{ZAR}} R\bar{i}_* \mu_k(n)$ is quasi isomorphic to the complex of sheaves associated to the complex of presheaves sending an open affine U of X to the standard nonhomogenous continuous cochain complex $C^\bullet(\pi_1(\mathscr{U}[1/p]),\mathbf{Q}_p(n))$ of $\mathbf{Q}_p(n)$.

Proposition 3.4. *For any $X \in Ł$, the following diagram commutes*

$$\begin{array}{ccc} & K_j(X) & \\ c_{ij}^f \swarrow & & \searrow \widehat{\tilde{c}_{ij}} \\ H^{2i-j}(X,\mathscr{S}^\bullet(i)) & \xrightarrow{l} & H^{2i-j}(X,\mathscr{F}^\bullet(i))\,. \end{array}$$

Proof. First, let $j=0$. Since l is compatible with products and we have projective space theorems, it suffices to check that l is compatible with Chern classes of invertible sheaves. Let $\mathscr{L}$ be an invertible sheaf on X. Concerning c_0, note that $l\colon H^0(X,\mathscr{S}^\bullet(0)) \to H^0(X,\mathscr{F}^\bullet(0))$ is an isomorphism [14]. Hence $l(c_0^f(\mathscr{L}))$ is a unit in $H^0(X,\mathscr{F}^\bullet(0))$ and consequently is equal to $\widehat{\tilde{c}_0}(\mathscr{L})$.

For c_1, choose an open small $\mathscr{L}$ is represented by a cocycle $\{f_{ij} \in \Gamma(U_{ij},\mathscr{O}_X^*)\}$. Assume also that there are good embeddings $C(R_i)$ of R_i. Let us describe the class

$$\begin{aligned} c_1^f(\mathscr{L}) &\in H_f^2(X,\mathscr{S}^\bullet(1)) \xleftarrow{\sim} H_f^2(N_\bullet U,\mathscr{S}^\bullet(1)) \\ &= H^2(\mathrm{Cone}(\mathscr{K}_{T'_\bullet} \otimes \Omega^\bullet_{R'_\bullet/V_0} \oplus F^1\Omega^\bullet_{R''_\bullet/V} \xrightarrow{\beta} \mathscr{K}_{T'_\bullet} \otimes \Omega^\bullet_{R'_\bullet/V_0} \\ &\quad \oplus \mathscr{K}_{T''_\bullet} \otimes \Omega^\bullet_{R''_\bullet/V})[-1])\,. \end{aligned}$$

It is represented (cf., Sect. 2.1) by a 2-cocycle in

$$\mathrm{Cone}(\mathscr{K}_{T'_\bullet} \otimes \Omega^\bullet_{R'_\bullet/V_0} \oplus F^1\Omega^\bullet_{R''_\bullet/V} \xrightarrow{\beta} \mathscr{K}_{T'_\bullet} \otimes \Omega^\bullet_{R'_\bullet/V_0} \oplus \mathscr{K}_{T''_\bullet} \otimes \Omega^\bullet_{R''_\bullet/V})[-1]$$

given on R_{ij} and on R_{ijk} respectively by

$$\begin{aligned} &(\mathrm{dlog}\, f'_{ij}, \mathrm{dlog}\, \tilde{f}_{ij}, p^{-1}\log(\phi(f'_{ij})/f'^{p}_{ij})) \\ &\qquad \in \mathscr{K}_{T'_{ij}} \otimes \Omega^1_{R'_{ij}/V_0} \oplus \mathscr{K}_{T''_{ij}} \otimes \Omega^1_{R''_{ij}/V} \oplus \mathscr{K}_{T'_{ij}} \otimes \Omega^0_{R'_{ij}/V_0}, \\ &(\log(f'_{jk}f'^{-1}_{ik}f'_{ij}), \log(\tilde{f}_{jk}\tilde{f}'^{-1}_{ik}\tilde{f}_{ij})) \in \mathscr{K}_{T'_{ijk}} \otimes \Omega^0_{R'_{ijk}/V_0} \oplus J^{(1)} \otimes \Omega^0_{R''_{ijk}/V}\,, \end{aligned}$$

where f'_{ij} and $\tilde{f}_{ij}$ lift f_{ij} to R'_{ij} and R''_{ij} respectively, and $\tilde{f}_{ij}$ is the image of f'_{ij} under the map $R'_{ij} \to R''_{ij}$.

Now, let's see what happens to this cocycle under the map l. The computation is rather tedious – we advice the reader to consult [14] for details. First, recall the definition of this map. We have sequences of morphisms between complexes of sheaves of $\pi_{\widehat{ij}} = \pi_1(\widehat{R}_{ij}[1/p])$-modules

$$\mathscr{K}_{T'_{ij}} \otimes \Omega^\bullet_{R'_{ij}/V_0} \to \Omega^\bullet_{B(\widehat{R}_{ij})} \to S^\bullet(\Omega^\bullet_{B(\widehat{R}_{ij})}) \xleftarrow{(1)} S^\bullet(\mathscr{K}_{B(\widehat{R}_{ij})}) \xrightarrow{\sim} S^\bullet(B(\widehat{R}_{ij})(1))\,;$$

$$\mathscr{K}_{T''_{ij}} \otimes \Omega^\bullet_{R''_{ij}/V} \to \Omega^\bullet_{B_{dR}(\widehat{R}_{ij})} \to S^\bullet(\Omega^\bullet_{B_{dR}(\widehat{R}_{ij})}) \xleftarrow{(2)} S^\bullet(\mathscr{K}_{B_{dR}(\widehat{R}_{ij})}) \xrightarrow{\sim} S^\bullet(B_{dR}(\widehat{R}_{ij})(1))\,,$$

where $S^\bullet$ are the continuous homogeneous cochains of $\pi_{\widehat{ij}}$ and (1) and (2) are quasi isomorphisms. Here, $\Omega^\bullet_{B(\widehat{R_{ij}})}$ and $\Omega^\bullet_{B_{dR}(\widehat{R_{ij}})}$ are certain resolutions of the structure crystals evaluated on $B(\widehat{R_{ij}})$ and $B_{dR}(\widehat{R_{ij}})$ respectively (and twisted by -1). These sequences yield a morphism

$$\begin{aligned}\Omega^\bullet_{(R_{ij},C(R_{ij}))}(\mathscr{K}(1)) \quad &\to \mathrm{Cone}(C^\bullet(\pi_{\widehat{ij}}, B(\widehat{R}_{ij})(1)) \oplus C^\bullet(\pi_{\widehat{ij}}, F^0B_{dR}(\widehat{R_{ij}})(1)) \\ &\xrightarrow{\beta} C^\bullet(\pi_{\widehat{ij}}, B(\widehat{R}_{ij})(1)) \oplus C^\bullet(\pi_{\widehat{ij}}, B_{dR}(\widehat{R_{ij}})(1)))[-1] ,\end{aligned}$$

where $\beta(x,y) = (x - p^{-1}\phi_{ij}(x), x-y)$. Now, the fundamental exact sequence

$$0 \to \mathbf{Q}_p(1)_{\widehat{R}_{ij}} \to B(\widehat{R}_{ij})(1) \oplus F^0B_{dR}(\widehat{R_{ij}})(1) \xrightarrow{\beta} B(\widehat{R}_{ij})(1) \oplus B_{dR}(\widehat{R_{ij}})(1) \to 0$$

yields a quasi isomorphism

$$\begin{aligned}C^\bullet(\pi_{\widehat{ij}}, \mathbf{Q}_p(1)_{\widehat{R}_{ij}}) &\to \mathrm{Cone}(C^\bullet(\pi_{\widehat{ij}}, B(\widehat{R}_{ij})(1)) \oplus C^\bullet(\pi_{\widehat{ij}}, F^0B_{dR}(\widehat{R_{ij}})(1)) \\ &\xrightarrow{\beta} C^\bullet(\pi_{\widehat{ij}}, B(\widehat{R}_{ij})(1)) \oplus C^\bullet(\pi_{\widehat{ij}}, B_{dR}(\widehat{R_{ij}})(1)))[-1] ,\end{aligned}$$

and finally our map $l_{ij} : \Omega^\bullet_{(R_{ij},C(R_{ij}))}(\mathscr{K}(1)) \to C^\bullet(\pi_{\widehat{ij}}, \mathbf{Q}_p(1)_{\widehat{R}_{ij}})$ from which we construct l by gluing. Recall that

$$\Omega^\bullet_{B(\widehat{R_{ij}})} := \mathscr{K}_{X_k/V_0}\{-1\}_{B(\widehat{R_{ij}})} \otimes_{B_t(\widehat{R_{ij}})[1/p]} (\mathrm{proj}\lim \mathscr{B}(\widehat{R_{ij}})_n[1/p]) \otimes_{R'_J} \Omega^\bullet_{R'_J/V_0} ,$$

where $B_t(\widehat{R_{ij}})$ is the V_0-enlargement (with the ideal $(\xi,[(\pi)])$) of R_{ij}/π equal to the p-adic completion of $W(S(\widehat{R}_{ij}))[\xi^p/p]$, and $\mathscr{B}(\widehat{R_{ij}})_n$ is the algebra of the n'th enlargement associated to the widening given by the algebra $B_t(\widehat{R_{ij}})\hat{\otimes}_{V_0}\widehat{R}'_{ij}$ completed along $B_t(\widehat{R_{ij}})/(\xi,[(\pi)])$. Here, $S(\widehat{R}_{ij}) = \mathrm{proj}\lim \overline{\widehat{R}}_{ij}/p\overline{\widehat{R}}_{ij}$, where the maps in the projective system are the p-th power maps and $\overline{\widehat{R}}_{ij}$ is the normalization of $\widehat{R}_{ij}$ in the maximal étale extension of $\widehat{R}_{ij}[1/p]$, and $\xi = [(p)] + p[(-1)]$, where $(p),(-1) \in S(\widehat{R}_{ij})$ are the reductions mod p of sequences of p-roots of p and -1 respectively. $\mathscr{K}_{X_k/V_0}\{-1\}_{B(\widehat{R_{ij}})}$ is the evaluation of the structure crystal $\mathscr{K}_{X_k/V_0}\{-1\}$ (twisted by -1) on $B(\widehat{R_{ij}})$.

Consider the element $h'_{ij} = [(1/f_{ij})] \otimes f'_{ij} \in B_t(\widehat{R_{ij}})\hat{\otimes}_{V_0}\widehat{R}'_{ij}$, where $(f_{ij}) \in S(\widehat{R}_{ij})$ is a sequence of p-power roots of f_{ij}. It is easy to check that both $[(f_{ij})] \otimes 1$ and $1 \otimes f'_{ij}$ have the same image in $B_t(\widehat{R_{ij}})/(\xi,[(\pi)])$, thus we have a well defined $\log h'_{ij}$ in every $\mathscr{B}(\widehat{R_{ij}})_n$. Recall that

$$\Omega^\bullet_{B_{dR}(\widehat{R_{ij}})} := \mathscr{K}_{X_k/V}\{-1\}_{\mathscr{B}_{dR}(\widehat{R_{ij}})} \otimes_{R''_{ij}} \Omega^\bullet_{R''_{ij}/V} ,$$

where $\mathscr{B}_{dR}(\widehat{R_{ij}}) = \mathscr{B}^+_{dR}(\widehat{R_{ij}})[t^{-1}]$ with $\mathscr{B}^+_{dR}(\widehat{R_{ij}}) = \mathrm{proj}\lim E^+(\widehat{R_{ij}})/t^nE^+(\widehat{R_{ij}})$. Here $E^+(\widehat{R_{ij}}) = \bigcup_{n \geqq 0} t^{-n}F^nE(\widehat{R_{ij}})$ as a $B^+_{dR}(\widehat{R_{ij}})$-subalgebra of $E(\widehat{R_{ij}})[t^{-1}]$, where $E(\widehat{R_{ij}}) = \mathrm{proj}\lim K \otimes (W_V(S(\widehat{R_{ij}}))\hat{\otimes}_V\widehat{R''_{ij}})/K \otimes I^n$, and I is the ideal of $\overline{\widehat{R_{ij}}}^\wedge$ in $W_V(S(\widehat{R_{ij}}))\hat{\otimes}_V\widehat{R''_{ij}}$.

Consider the element $h''_{ij} = [(1/f_{ij})] \otimes \tilde{f}_{ij} \in W_V(S(\widehat{R_{ij}}))\hat{\otimes}_V \widehat{R''_{ij}}$. Since $[(f_{ij})] \otimes 1$ and $1 \otimes \tilde{f}_{ij}$ have the same image in $\overline{\widehat{R_{ij}}}^{\wedge}$, $\log h''_{ij}$ is well defined already in $E(\widehat{R_{ij}})$. Moreover, since $\log h''_{ij}$ lies in the closure of the ideal $K \otimes I$, $\log h''_{ij} \in F^0 \Omega^0_{B_{dR}(\widehat{R_{ij}})}$.

Consider the element $(\log h'_{ij}, \log h''_{ij}) \in C^0(\pi_{\widehat{ij}}, \Omega^0_{B(\widehat{R_{ij}})}) \oplus C^0(\pi_{\widehat{ij}}, F^0\Omega^0_{B_{dR}(\widehat{R_{ij}})})$ as a 1-class in the complex

$$\begin{aligned}\mathrm{Cone}(C^{\bullet}(\pi_{\widehat{\bullet}}, \Omega^{\bullet}_{B(\widehat{R_{\bullet}})}) \oplus C^{\bullet}(\pi_{\widehat{\bullet}}, F^0\Omega^{\bullet}_{B_{dR}(\widehat{R_{\bullet}})}) &\xrightarrow{\beta} C^{\bullet}(\pi_{\widehat{\bullet}}, \Omega^{\bullet}_{B(\widehat{R_{\bullet}})}) \\ &\oplus C^{\bullet}(\pi_{\widehat{\bullet}}, \Omega^{\bullet}_{B_{dR}(\widehat{R_{\bullet}})}))[-1]\,.\end{aligned}$$

Let's compute its differential. In the Čech direction, we get on R_{ijk} the element

$$\begin{aligned}&(\log([(f_{jk}^{-1} f_{ik} f_{ij}^{-1})] \otimes f'_{jk} f'^{-1}_{ik} f'_{ij}), \log([(f_{jk}^{-1} f_{ik} f_{ij}^{-1})] \otimes \tilde{f}_{jk} \tilde{f}_{ik}^{-1} \tilde{f}_{ij}))\\ &\quad = \log([(f_{jk}^{-1} f_{ik} f_{ij}^{-1})]) + \log(f'_{jk} f'^{-1}_{ik} f'_{ij}), \log([(f_{jk}^{-1} f_{ik} f_{ij}^{-1})])\\ &\qquad + \log(\tilde{f}_{jk} \tilde{f}_{ik}^{-1} \tilde{f}_{ij})\,.\end{aligned}$$

The equality above follows from the fact that both $[(f_{jk}^{-1} f_{ik} f_{ij}^{-1})]$ and $f'_{jk} f'^{-1}_{ik} f'_{ij}$ (respectively, $[(f_{jk}^{-1} f_{ik} f_{ij}^{-1})]$ and $\tilde{f}_{jk} \tilde{f}_{ik}^{-1} \tilde{f}_{ij}$) have the image in $B_t(\widehat{R}_{ij})\hat{\otimes}_{V_0}\widehat{R}'_{ij}$ (respectively, in $\overline{\widehat{R}}_{\widehat{ij}}$) equal to 1. In the f-direction, we get $(\mathrm{dlog}\, f'_{ij}, \mathrm{dlog}\, \tilde{f}_{ij}, p^{-1}\log(1 \otimes (\phi_{ij}(f'_{ij})/f'^{p}_{ij}), 0)$. Finally, to compute the differential in the Galois direction, note that $\pi_{\widehat{ij}}$ acts on both $\log h'_{ij}$ and $\log h''_{ij}$ via its action on $[(1/f_{ij})]$. The last action is given by a character $\pi_{\widehat{ij}} \xrightarrow{\chi} \mathbf{Z}_p(1)^* \xrightarrow{\alpha} W(S(\widehat{R}_{ij}))^*$. Hence the differential carries $(\log h'_{ij}, \log h''_{ij})$ into a cocycle in $C^1(\pi_{\widehat{ij}}, \Omega^0_{B(\widehat{R_{ij}})}) \oplus C^1(\pi_{\widehat{ij}}, F^0\Omega^0_{B_{dR}(\widehat{R_{ij}})})$ sending $g \in \pi_{\widehat{ij}}$ to $(\log \alpha\chi(g), \log \alpha\chi(g))$.

Let us now describe the class $\tilde{c}_{\widehat{1}}(\mathscr{L}) \in H^2(\widetilde{\mathscr{X}}, \mathbf{Q}_p(1))$. It can be represented by a 2-cocycle in $C^{\bullet}(\pi_{\widehat{\bullet}}, \mathbf{Q}_p(1)_{\widehat{R}_{\bullet}})$ given on R_{ij} by

$$\pi_{\widehat{ij}} \xrightarrow{\chi^{-1}} \mathbf{Z}_p(1)^* \xrightarrow{\alpha} (\mathrm{proj}\lim \overline{\widehat{R}}_{ij})^* \in C^1(\pi_{\widehat{ij}}, \mathbf{Q}_p(1)_{\widehat{R}_{ij}}), \quad g \mapsto g(f_{ij})(f_{ij}^{-1})\,,$$

and on R_{ijk} by $(f_{jk} f_{ik}^{-1} f_{ij}) \in C^0(\pi_{\widehat{ikj}}, \mathbf{Q}_p(1)_{\widehat{R}_{ijk}})$. Since the isomorphisms

$$\mathbf{Q}_p(1)_{\widehat{R}_{ij}} \otimes B(\widehat{R}_{ij}) \xrightarrow{\sim} \mathscr{K}_{B(\widehat{R}_{ij})} \qquad \mathbf{Q}_p(1)_{\widehat{R}_{ij}} \otimes B_{dR}(\widehat{R}_{ij}) \xrightarrow{\sim} \mathscr{K}_{B_{dR}(\widehat{R}_{ij})}$$

are induced by the map $\log \alpha$, the above computation shows that the differential of $(\log h'_{ij}, \log h''_{ij})$ is equal to the difference of the images of the 2-cocycles defining $c_1^f(\mathscr{L})$ and $\tilde{c}_{\widehat{1}}(\mathscr{L})$, as wanted.

Assume now that $j > 0$. By Jouanolou's trick [9], we can assume X to be affine. It is clear from the above description of the K-theory characteristic classes and the fact that the simplicial scheme $X \times B_{\bullet} GL_k(X)$ is well-pointed, that it suffices to show that the class $\mathrm{ev}_k^*(C^f_{i,k})$ is mapped via l to $\mathrm{ev}_k^*(\widetilde{C}_{\widehat{i,k}})$.

Although $B_{\bullet} GL_k/V$ is not well-pointed in the sense described above, there is a choice of a base point, namely the zero section, compatible with all the maps in $B_{\bullet} GL_k/V$. One can also (Zariski) cover $B_{\bullet} GL_k/V$ with a semisimplicial scheme build from small open affines containing the zero section. This suffices [14] to define a map $l : \mathscr{S}^{\bullet}(i) \to \mathscr{F}^{\bullet}(i)$ in the derived category over the nerve of that scheme, hence a map $l : H^*(B_{\bullet} GL_k/V, \mathscr{S}^{\bullet}(i)) \to H^*(B_{\bullet} GL_k/V, \mathscr{F}^{\bullet}(i))$. Consider the following diagram

$$\begin{array}{ccc} H^*(B_{\bullet} GL_k/V, \mathscr{S}^{\bullet}(i)) & \xrightarrow{\mathrm{ev}_k^*} & H^*(X \times B_{\bullet} GL_k(X), \mathscr{S}^{\bullet}(i)) \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l} \\ H^*(B_{\bullet} GL_k/V, \mathscr{F}^{\bullet}(i)) & \xrightarrow{\mathrm{ev}_k^*} & H^*(X \times B_{\bullet} GL_k(X), \mathscr{F}^{\bullet}(i)) . \end{array}$$

We want to prove that this diagram commutes and that the left map l is compatible with the universal classes. Concerning commutativity, had $B_{\bullet} GL_k/V$ been well-pointed, it would have followed from functoriality (with respect to well-pointed morphisms) of the map l. Since that is not the case, set $g_1 = \mathrm{ev}_k l$ and $g_2 = l \mathrm{ev}_k$, and consider the induced maps of spectral sequences

$$\begin{array}{ccc} E_1^{s,t} = H^t(B_s GL_k/V, \mathscr{S}^{\bullet}(i)) & \Rightarrow & H^{s+t}(B_{\bullet} GL_k/V, \mathscr{S}^{\bullet}(i)) \\ \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle g_i} \\ E_1^{\prime s,t} = H^t(X \times B_s GL_k(X), \mathscr{F}^{\bullet}(i)) & \Rightarrow & H^{s+t}(X \times B_{\bullet} GL_k(X), \mathscr{F}^{\bullet}(i)) \end{array}$$

Since now the morphism $\mathrm{ev}_k : X \times B_s GL_k(X) \to B_s GL_k/V$ is well-pointed, we see that the maps g_1, g_2 induce the same map on the terms of the spectral sequence, hence on the limit cohomology groups as well.

For the universal classes, we will use the splitting principle (cf., [3], [16]). Let $T_1/V = \mathbf{G}_m/V$, $T_k/V = (\mathbf{G}_m)^k/V$. Consider the maps

$$B_{\bullet}\ T_k/V \xrightarrow{h_1} B_{\bullet} B_k/V \xrightarrow{h_2} B_{\bullet} GL_k/V ,$$

where B_k/V is the standard Borel subgroup of GL_k/V. We claim that the map

$$(h_2 h_1)^* : H^*(B_{\bullet} GL_k/V, \mathscr{F}^{\bullet}(i)) \to H^*(B_{\bullet} T_k/V, \mathscr{F}^{\bullet}(i))$$

is injective. Indeed, by the homotopy property (3.3), h_1^* is an isomorphism. Concerning h_2^*, we have the following commutative diagram

$$\begin{array}{ccc}
B_{\bullet}B_k/V = B(*, B_k/V, *) & \longrightarrow & B(*, GL_k/V, *) = B_{\bullet}GL_k/V \\
\Big\uparrow{\scriptstyle \varepsilon_1} & & \Big\uparrow{\scriptstyle \varepsilon_2} \\
B([B_k/V\backslash GL_k/V]_{\bullet}, GL_k/V, *) & \xrightarrow{\gamma_1} & B([GL_k/V\backslash GL_k/V]_{\bullet}, GL_k/V, *) \\
\Big\downarrow{\scriptstyle \omega_1} & & \Big\downarrow{\scriptstyle \omega_2} \\
B(\mathrm{Fl}_k/V, GL_k/V, *) & \xrightarrow{\gamma_2} & B(*, GL_k/V, *)\ ,
\end{array}$$

where Fl_k/V is the full flag scheme corresponding to $B_k/V\backslash GL_k/V$, $B(X, G/V, Y)$ is the double bar construction [10]: $n \mapsto X \times G^n/V \times Y$. This construction can be extended in an obvious way to the case when X or Y itself is a simplicial scheme. $B(X, G/V, Y)$ is then a bisimplicial scheme. Also,

$$[B_k/V\backslash GL_k/V]_{\bullet} = B(*, B_k/V, GL_k/V) \xrightarrow{\sim} \mathrm{cosk}(GL_k/V \to \mathrm{Fl}_k/V) \xrightarrow{\omega_1} \mathrm{Fl}_k/V$$

$$[GL_k/V\backslash GL_k/V]_{\bullet} = B(*, GL_k/V, GL_k/V) \xrightarrow{\sim} \mathrm{cosk}(GL_k/V \to V) \xrightarrow{\omega_2} V$$

are the simplicial models of the corresponding homogeneous spaces.

The maps ω_i induce isomorphisms on our cohomology as easily follows from the existence of local sections of the morphisms $GL_k/V \to \mathrm{Fl}_k/V$ and $GL_k/V \to V$. Since γ_2 can be represented as a tower of morphisms from the projective bundle associated to a vector bundle over a simplicial scheme to that scheme, by the Dold–Thom isomorphism (Proposition 3.1), γ_2^* is injective, hence so is γ_1^*. To finish, we claim that the maps ε_i^* are isomorphisms. Indeed, $B([B_k/V\backslash GL_k/V]_{\bullet}, GL_k/V, *) = B(*, B_k/V, [GL_k/V\backslash GL_k/V]_{\bullet})$ and ε_1 is induced by the canonical map $[GL_k/V\backslash GL_k/V]_{\bullet} \to V$. Since the last one is a cohomology isomorphism, we are done. Similarly for ε_2^*.

Arguing as above, we get that $(h_2h_1)^*$ commutes with the map l. Our problem reduces thus to checking that $l((h_2h_1)^*(C^f_{i,k})) = (h_2h_1)^*(\widetilde{C}_{i,\widehat{k}})$. But by additivity of Chern classes $(h_2h_1)^*(C^f_{i,k}) = \sigma_i(a^f_1, \dots, a^f_k)$, where a^f_j, $1 \leqq j \leqq k$ is the pullback via the j'th projection of the first Chern class $C^f_{1,1}$ of the universal locally free sheaf on $B_{\bullet}T_1/V = B_{\bullet}GL_1/V$. Similarly, $(h_2h_1)^*(\widetilde{C}_{i,\widehat{k}}) = \sigma_i(\widehat{\tilde{a}_1}, \dots, \widehat{\tilde{a}_k})$. Here, σ_i is the i'th elementary symmetric function. Since the map l is compatible with cup products and every projection $\mathrm{pr}_j : B_{\bullet}T_k/V \to B_{\bullet}T_1/V$ is compatible with our base point (the zero section), it suffices to show that $l(C^f_{1,1}) = \widetilde{C}_{\widehat{1,1}}$.

Here, we can reduce the computations to schemes. Consider the projective space $\mathbf{P}^m_V$ with its usual affine covering $\{U_i\}$, $i = 0, \dots, m$, and the associated Čech nerve $N_{\bullet}U$. The standard trivialization of the canonical invertible sheaf $\mathcal{O}(1)$ on $\mathbf{P}^m_V$ over $\{U_i\}$ defines a map $\psi_m : N.U \to B.GL_1/V$. We have a commutative diagram

$$\begin{array}{ccccc}
H^2(\mathbf{P}^m_V, \mathcal{S}^\bullet(1)) & \xrightarrow{\sim} & H^2(N.U, \mathcal{S}^\bullet(1)) & \xleftarrow{\psi_m^*} & H^2(B.GL_1/V, \mathcal{S}^\bullet(1)) \\
\downarrow l & & \downarrow l & & \downarrow l \\
H^2(\mathbf{P}^m_V, \mathcal{F}^\bullet(1)) & \xrightarrow{\sim} & H^2(N.U, \mathcal{F}^\bullet(1)) & \xleftarrow{\psi_m^*} & H^2(B.GL_1/V, \mathcal{F}^\bullet(1))\,.
\end{array}$$

For m large enough, the homotopy property yields that the bottom ψ_m^* is an isomorphism [16, 2.3]. Since in both theories $\psi_m^*(a_1) = c_1(\mathcal{O}(1))$, it suffices to check that $l(c_1^f(\mathcal{O}(1))) = \tilde{c}\,\widehat{}_1(\mathcal{O}(1))$, which we have already done above. □

Let now $X \in \text{Ł}$ be projective. Consider the morphism

$$l : H^*_f(X, \mathcal{K}(*)) \to H^*_{\text{ét}}(X_K, \mathbf{Q}_p(*)), \quad h = \omega t^{-1} l\,.$$

Corollary 3.1. *For* $i \geqq 0$, $2i - n - 1 \geqq 0$, *the following diagram commutes*

$$\begin{array}{ccc}
 & K_{2i-n-1}(X) \otimes \mathbf{Q} & \\
 & r_p^{\text{ét}} \swarrow \quad \searrow r_p^f & \\
H^{n+1}_{\text{ét}}(X_K, \mathbf{Q}_p(i)) & \xleftarrow{h} & H^{n+1}_f(X, \mathcal{K}(i))\,.
\end{array}$$

Proof. By Propositions 3.2 and 3.4. □

Theorem 3.1. *Let* X *be any smooth projective scheme over* V *and* $2i - n - 1 \geqq 1$. *Then the étale regulator induces a map*

$$r_p^{\text{ét}} : K_{2i-n-1}(X) \otimes \mathbf{Q} \to H^1(G_K, H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i)))\,,$$

which factors through the subgroup $H^1_f(G_K, H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i)))$.

Proof. It follows from Corollary 3.1, and Corollary 5.1 from [14], which shows that the following diagram commutes.

$$\begin{array}{ccc}
H^{n+1}_f(X, \mathcal{K}(i))_0 & \xrightarrow{l} & H^{n+1}_{\text{ét}}(X_K, \mathbf{Q}_p(i))_0 \\
\uparrow \wr & & \downarrow \rho \\
H^1_f(G_K, H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i))) & \xrightarrow{l'} & H^1(G_K, H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i)))\,,
\end{array}$$

where l' is the canonical embedding of crystalline extensions into the unrestricted extensions and ρ comes from the Hochschild–Serre spectral sequence. Here

$$\begin{aligned} H_f^{n+1}(X,\mathscr{K}(i))_0 &= \ker(H_f^{n+1}(X,\mathscr{K}(i)) \to H_f^0(G_K, H_{\text{ét}}^{n+1}(X_{\overline{K}}, \mathbf{Q}_p(i)))), \quad \text{and} \\ H_{\text{ét}}^{n+1}(X_K,\mathbf{Q}_p(i))_0 &= \ker(H_{\text{ét}}^{n+1}(X_K,\mathbf{Q}_p(i)) \to H_{\text{ét}}^{n+1}(X_{\overline{K}},\mathbf{Q}_p(i))) . \end{aligned}$$

□

Theorem 3.2. *The image of the cycle class map*

$$(CH^i(X)\otimes\mathbf{Q})_{\hom\sim 0} \xrightarrow{\mathrm{cl}_X} H^1(G_K, H_{\text{ét}}^{2i-1}(X_{\overline{K}},\mathbf{Q}_p(i)))$$

is contained in the subgroup $H^1_f(G_K, H_{\text{ét}}^{2i-1}(X_{\overline{K}},\mathbf{Q}_p(i)))$.

Proof. Propositions 3.2, 3.4, and 4.1 from [14] yield a commutative diagram

$$\begin{array}{ccc} (CH^i(X)\otimes\mathbf{Q})_{\hom\sim 0} & \xrightarrow{\mathrm{cl}_X^f} & H_f^{2i}(X,\mathscr{K}(i))_0 \\ \Big\| & & \Big\downarrow l \\ (CH^i(X)\otimes\mathbf{Q})_{\hom\sim 0} & \xrightarrow{\mathrm{cl}_X} & H_{\text{ét}}^{2i}(X_K,\mathbf{Q}_p(i))_0 \end{array}$$

where, for an irreducible subscheme Y of codimension i in X, $\mathrm{cl}_X^f([Y]) = (-1)^{i-1}/(i-1)!c_i^f(\mathscr{O}_Y)$ (since the map l is injective [14], it is a good definition). We finish as in Theorem 3.1. □

Appendix A. Convergent crystalline Chern classes

Let X be a scheme over k. There is an exact sequence of sheaves on $(X/V)_{\mathrm{conv}}$:

$$0 \to \mathscr{I}_{X/V} \to \mathscr{O}_{X/V} \to \mathscr{O}_X \to 0 ,$$

where the value of $\mathscr{I}_{X/V}$ on the enlargement $Z \hookrightarrow \mathscr{S}$ is the ideal $I_Z(\mathscr{S})$ of Z in $\mathscr{S}$ and $\mathscr{O}_{X,\mathscr{S}} = \mathscr{O}_Z$. Since $I_Z^n(\mathscr{S}) \subset \pi\mathscr{O}_{\mathscr{S}}$ for some n, we have a multiplicative analogue

$$0 \to 1 + \mathscr{I}_{X/V} \to \mathscr{O}_{X/V}^* \to \mathscr{O}_X^* \to 0 .$$

Since $\mathscr{O}_{\mathscr{S}}$ is p-adically complete, the series

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}x^n/n, \quad x \in I_Z(\mathscr{S}) ,$$

converges and defines a homomorphism of abelian sheaves $\log\colon 1+\mathscr{I}_{X/V} \to \mathscr{K}_{X/V}$. We define the canonical (in X) map $c_1 : \mathscr{O}_X^*[-1] \to Ru_{X/V*}\mathscr{K}_{X/V}$ in $\mathbf{D}^+(X_{\mathrm{Zar}})$ as the composition

$$\mathscr{O}_X^*[-1] \to Ru_{X/V*}\mathscr{O}_X^*[-1] \to Ru_{X/V*}(1+\mathscr{I}_{X/V}) \xrightarrow{\log} Ru_{X/V*}\mathscr{K}_{X/V} .$$

Assume now that X is embedded in a p-adic formal scheme $\mathscr{Y}$, formally smooth over V.

Lemma A.1. *The following diagram commutes in* $\mathbf{D}^+(\mathbf{X}_{\mathrm{Zar}})$

$$
\begin{array}{ccccc}
c_1 : \mathscr{O}_X^*[-1] & \longrightarrow & Ru_{X/V*}\mathscr{O}_X^*[-1] & \longrightarrow & Ru_{X/V*}\mathscr{K}_{X/V} \\
\wr \Big\uparrow & & & & \Big\downarrow \\
\mathrm{Cone}(1+J & \xrightarrow{-1} & \mathscr{O}^*_{\mathscr{Y}_{/X}})[-1] & \xrightarrow{(\log,\,\mathrm{dlog})} & \mathscr{K}_T \otimes \Omega^{\bullet}_{\mathscr{Y}/V} \simeq u_{X/V*}\Omega^{\bullet}_{\mathscr{Y}}\ ,
\end{array}
$$

where T is the widening corresponding to $\mathscr{Y}$ and X, J is the ideal of X in $\mathscr{Y}_{/X}$, and $\Omega^{\bullet}_{\mathscr{Y}}$ is the acyclic for $u_{X/V*}$ resolution of $\mathscr{K}_{X/V}$ associated to $\mathscr{Y}$.

Proof. Recall [15, 5.4] that there exists a natural complex $\Omega^{\bullet}_{\mathscr{Y}}(\mathscr{O}_{X/V})$ and a map $\mathscr{O}_{X/V} \to \Omega^0_{\mathscr{Y}}(\mathscr{O}_{X/V})$ of complexes of sheaves on $(X/V)_{\mathrm{conv}}$ such that $\Omega^i_{\mathscr{Y}}(\mathscr{O}_{X/V})$ is naturally a $\Omega^0_{\mathscr{Y}}(\mathscr{O}_{X/V})$-module, $u_{X/V*}\Omega^i_{\mathscr{Y}}(\mathscr{O}_{X/V}) \simeq \mathscr{O}_T \otimes \Omega^i_{\mathscr{Y}/V}$, and $R^k u_{X/V*}\Omega^i_{\mathscr{Y}}(\mathscr{O}_{X/V}) = 0$, for $k > 0$. We claim that there is a surjective homomorphism $\Omega^0_{\mathscr{Y}}(\mathscr{O}_{X/V}) \to \mathscr{O}_X$. Indeed, it suffices to argue locally, so assume that $X, \mathscr{Y}$ are affine and we have an enlargement $T' = (A', I', z : \mathrm{Spec}(A'/I') \to X)$ of X/V. Then

$$\Omega^0_{\mathscr{Y}}(\mathscr{O}_{X/V})(T') = \mathscr{O}_{X/V}(T' \times T) \simeq \operatorname{proj\,lim} \mathscr{O}_{(T' \times T)_n}\ ,$$

where $(T' \times T)_n$ is the level n enlargement associated to the widening $T' \times T$. Since the natural morphism $\mathscr{O}_{T' \times T} \to \operatorname{proj\,lim} \mathscr{O}_{(T' \times T)_n}$ is a topological isomorphism (cf., [15, 3,3]), we have a natural surjection $\operatorname{proj\,lim} \mathscr{O}_{(T' \times T)_n} \to A'/I'$.

Let now $\mathscr{Z}$ be the kernel of the map $\Omega^0_{\mathscr{Y}}(\mathscr{O}_{X/V})^* \to \mathscr{O}^*_X$. The last isomorphism gives that there is a well defined logarithm map $\log : 1 + \mathscr{Z} \to \Omega^0_{\mathscr{Y}}$. Denote by $\mathscr{N}^{\times}$ the complex

$$1 + \mathscr{Z} \xrightarrow{\mathrm{dlog}} \Omega^1_{\mathscr{Y}} \to \Omega^2_{\mathscr{Y}} \to \cdots$$

We have a natural map $\mathscr{N}^{\times} \xrightarrow{\log^{\bullet}} \Omega^{\bullet}_{\mathscr{Y}}$ defined in degree 0 by $\log : 1 + \mathscr{Z} \to \Omega^0_{\mathscr{Y}}$ and in higher degrees by the identity. The commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{N}^{\times} & \longrightarrow & \Omega^{\bullet\times}_{\mathscr{Y}} & \longrightarrow & \mathscr{O}^*_X & \longrightarrow & 0 \\
 & & \Big\uparrow & & \Big\uparrow & & \Big\| & & \\
0 & \longrightarrow & 1 + \mathscr{I}_{X/V} & \longrightarrow & \mathscr{O}^*_{X/V} & \longrightarrow & \mathscr{O}^*_X & \longrightarrow & 0\ ,
\end{array}
$$

where $\Omega^{\bullet\times}_{\mathscr{Y}}$ is the complex

$$\Omega^0_{\mathscr{Y}}(\mathscr{O}_{X/V})^* \xrightarrow{\mathrm{dlog}} \Omega^1_{\mathscr{Y}} \to \Omega^2_{\mathscr{Y}} \to \cdots\ ,$$

yields a commutative diagram

$$\begin{array}{ccccc} Ru_{X/V*}\mathcal{O}_X^*[-1] & \longrightarrow & Ru_{X/V*}(1+\mathcal{I}_{X/V}) & \xrightarrow{\log} & Ru_{X/V*}\mathcal{K}_{X/V} \\ \| & & \downarrow & & \downarrow \\ Ru_{X/V*}\mathcal{O}_X^*[-1] & \longrightarrow & Ru_{X/V*}\mathcal{N}^\times & \xrightarrow{\log^\bullet} & Ru_{X/V*}\Omega^\bullet_{\mathcal{Y}} \end{array}$$

Note also that the map $\mathrm{Cone}(\mathcal{N}^\times \xrightarrow{-1} \Omega^{\bullet\times}_{\mathcal{Y}})[-1] \to \mathcal{N}^\times \xrightarrow{\log^\bullet} \Omega^\bullet_{\mathcal{Y}}$ is equal in $\mathbf{D}^+((X/V)_{\mathrm{conv}})$ to the map

$$\mathrm{Cone}(\mathcal{N}^\times \xrightarrow{-1} \Omega^{\bullet\times}_{\mathcal{Y}})[-1] \xrightarrow{\sim} \mathrm{Cone}(1+\mathcal{Z} \xrightarrow{-1} \Omega^0_{\mathcal{Y}}(\mathcal{O}_{X/V})^*)[-1] \xrightarrow{(\log,\mathrm{dlog})} \Omega^\bullet_{\mathcal{Y}}$$

(the complex $\mathrm{Cone}(1+\mathcal{Z} \xrightarrow{-1} \Omega^0_{\mathcal{Y}}(\mathcal{O}_{X/V})^*)[-1]$ maps quasi isomorphically to $\mathrm{Cone}(\mathcal{N}^\times \xrightarrow{-1} \Omega^{\bullet\times}_{\mathcal{Y}})[-1]$ via the identity in degree 0 and $(\mathrm{dlog}, 1)$ in degree 1). To finish, we claim that $u_{X/V*}\mathcal{Z} \simeq J$ and $R^k u_{X/V*}\mathcal{Z} = 0$, for $k > 0$. Indeed, since $u_{X/V*}\Omega^0_{\mathcal{Y}}(\mathcal{O}_{X/V}) \simeq \mathcal{O}_T \simeq \mathcal{O}_{\mathcal{Y}_{/X}}$ and $R^k u_{X/V*}\Omega^0_{\mathcal{Y}}(\mathcal{O}_{X/V}) = 0$, for $k > 0$, it suffices to show that $u_{X/V*}\mathcal{O}_X = \mathcal{O}_X$ and $R^k u_{X/V*}\mathcal{O}_X = 0$, for $k > 0$. The last fact follows from [15, 4.4]. Concerning $u_{X/V*}\mathcal{O}_X$, we easily compute [15, 4] that it is equal to $\operatorname{proj\,lim} \mathcal{O}_{X,T_n}$, which by [15, 3.3] is isomorpic to $\mathcal{O}_X$. This yields the commutative diagram of exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & u_{X/V*}(1+\mathcal{Z}) & \longrightarrow & u_{X/V*}\Omega^0_{\mathcal{Y}}(\mathcal{O}_{X/V})^* & \longrightarrow & u_{X/V*}\mathcal{O}_X^* & \longrightarrow & 0 \\ & & \wr\uparrow & & \wr\uparrow & & \wr\uparrow & & \\ 0 & \longrightarrow & 1+J & \longrightarrow & \mathcal{O}^*_{\mathcal{Y}_{/X}} & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & 0, \end{array}$$

which finishes our proof. □

Assume now that a p-adic formal scheme $\mathcal{X}$, formally smooth over V, lifts X and is itself embedded in a p-adic formal scheme $\mathcal{Y}$, formally smooth over V.

Lemma A.2. *The following diagram commutes in* $\mathbf{D}^+(X_{\mathrm{Zar}})$

$$\begin{array}{ccccc} \mathcal{O}_X^*[-1] & \xrightarrow{c_1} & Ru_{X/V*}\mathcal{K}_{X/V} & \longrightarrow & u_{X/V*}\Omega^\bullet_{\mathcal{Y}} \\ \uparrow & & & & \uparrow \\ \mathcal{O}_{\mathcal{X}}^*[-1] & \xleftarrow{\sim} & \mathrm{Cone}(1+J_{\mathcal{X}} \xrightarrow{-1} \mathcal{O}^*_{\mathcal{Y}_{/\mathcal{X}}})[-1] & \xrightarrow{(\log,\mathrm{dlog})} & F^1\Omega^\bullet_{\mathcal{Y}/V} \end{array}$$

where $J_{\mathcal{X}}$ *is the ideal of* $\mathcal{X}$ *in* $\mathcal{Y}_{/\mathcal{X}}$.

Proof. By Lemma A.1, it suffices to show that the diagram

$$\begin{array}{ccccc}
\mathcal{O}_X^*[-1] & \xleftarrow{\sim} & \mathrm{Cone}(1+J \xrightarrow{-1} \mathcal{O}^*_{\mathcal{Y}_{/X}})[-1] & \xrightarrow{(\log, \mathrm{dlog})} & \mathcal{K}_T \otimes \Omega^{\bullet}_{\mathcal{Y}/V} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{O}_{\mathcal{X}}^*[-1] & \xrightarrow{\sim} & \mathrm{Cone}(1+J_{\mathcal{X}} \xrightarrow{-1} \mathcal{O}^*_{\mathcal{Y}_{/\mathcal{X}}})[-1] & \xrightarrow{(\log, \mathrm{dlog})} & F^1\Omega^{\bullet}_{\mathcal{Y}/V}\ ,
\end{array}$$

where T is the widening corresponding to $\mathcal{Y}$ and X and J is the ideal of X in $\mathcal{Y}_{/X}$, commutes, which is obvious. □

In particular, when $\mathcal{Y} = \mathcal{X}$, the crystalline c_1 corresponds to the de Rham $c_1 : \mathcal{O}^*_{\mathcal{X}}[-1] \to \Omega^{\geqq 1}_{\mathcal{X}/V}$ via the quasi isomorphism $\mathcal{K}_{X/V} \xrightarrow{\sim} \Omega^{\bullet}_{\mathcal{X}}$.

Proposition A.1. *Let* $n > 0$, $\xi = c_1(\mathcal{O}(1)) \in H^2((\mathbf{P}^n_k/V)_{\mathrm{conv}}, \mathcal{K}_{\mathbf{P}^n_k/V})$. *For a smooth scheme* X *over* k, *the natural map*

$$\oplus_i \pi^*_{\mathbf{P}^n_k/V}(\xi)^i \cup \pi_X^* : \bigoplus_{i=0}^{n} Ru_{X/V*}\mathcal{K}_{X/V}[-2i] \to Ru_{X/V*}R\pi_{X*}\mathcal{K}_{\mathbf{P}^n_X/V}$$

is an isomorphism.

Proof. It suffices to check it locally on X, so assume that X lifts to a p-adic formal scheme $\mathcal{X}$, formally smooth over V. Since $Ru_{X/V*}\mathcal{K}_{X/V} \simeq Ru_{X/V*}\Omega^{\bullet}_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{X}}[1/p] \otimes \Omega^{\bullet}_{\mathcal{X}/V}$, $Ru_{\mathbf{P}^n_X/V*}\mathcal{K}_{\mathbf{P}^n_X/V} \simeq Ru_{\mathbf{P}^n_X/V*}\Omega^{\bullet}_{\mathbf{P}^n_{\mathcal{X}}} \simeq \mathcal{O}_{\mathbf{P}^n_{\mathcal{X}}}[1/p] \otimes \Omega^{\bullet}_{\mathbf{P}^n_{\mathcal{X}}/V}$, and via this isomorphism ξ corresponds to the de Rham class (Lemma A.2), we are reduced to a similar statement in the de Rham cohomology, which is known to be true. □

Similarly in the lifted situation we have the following

Proposition A.2. *Let* $n > 0$, $\xi = c_1(\mathcal{O}(1)) \in H^2((\mathbf{P}^n_k/V)_{\mathrm{conv}}, F^1_{\mathbf{P}^n_V})$. *For a* p*-adic formal scheme* $\mathcal{X}$, *formally smooth over* V, *lifting a scheme* X, *the natural map*

$$\begin{aligned}\oplus_i \pi^*_{\mathbf{P}^n_k/V}(\xi)^i \cup \pi_X^* &: \bigoplus_{i=0}^{n} Ru_{X/V*}F^{j-i}_{\mathcal{X}}[-2i] \\ &\to Ru_{X/V*}R\pi_{X*}F^j_{\mathbf{P}^n_{\mathcal{X}}}\end{aligned}$$

is an isomorphism for all integers j.

Proposition A.3. *There exists a unique theory of Chern classes, which to every simplicial smooth scheme* X *over* k *and every locally free sheaf of finite type* $\mathcal{E}$ *on* X *associates an element* $c(\mathcal{E}) = \prod_i c_i(\mathcal{E}) \in \prod_i H^{2i}((X/V)_{\mathrm{conv}}, \mathcal{K}_{X/V})$ *such that*

- (*normalization*) *if* $\mathscr{E}$ *is invertible*, *then* $c_0(\mathscr{E}) = 1$ *and* $c_1(\mathscr{E})$ *is induced by the above constructed class*;
- (*functoriality*) *if* $f : X \to Y$ *and* $\mathscr{E}$ *is a locally free sheaf of finite type on* Y, *then* $c(f^*\mathscr{E}) = f^*(c(\mathscr{E}))$;
- (*additivity*) *if* $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ *is an exact sequence of locally free sheaves of finite type*, *then* $c(\mathscr{E}) = c(\mathscr{E}')c(\mathscr{E}'')$.

Proof. We define Chern classes via the above Dold–Thom isomorphism. Normalization and functoriality follow. To check additivity, note that by [8, 2.10], it suffices to do it in the universal case, i.e., over $B_{\bullet}GL(n,m)_k$. The canonical isomorphism

$$H^*_{dR}(\widehat{B}.GL(n,m)/V)[1/p] \xrightarrow{\sim} H^*((B.GL(n,m)_k/V)_{\mathrm{conv}}, \mathscr{K}_{B.GL(n,m)_k/V})$$

carries, by Lemma A.2, de Rham Chern classes into our convergent Chern classes. Since additivity for de Rham Chern classes is well known, we are done. Uniqueness follows from the splitting principle. □

Similarly, we have a theory of Chern classes in the filtered cohomology $c_i(\mathscr{E}) \in H^{2i}((X_k/V)_{\mathrm{conv}}, F^i_X)$, for a simplicial smooth V-scheme X and a locally free finite type sheaf $\mathscr{E}$ on X, unique in an appropriate sense (additivity is proved as above by reduction to de Rham cohomology).

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