K-THEORY OF LOG-SCHEMES I

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Abstract. We set down some basic facts about the algebraic and topological K-theory of log-schemes. In particular, we show that the l-adic topological log-étale K-theory of log-regular schemes computes the l-adic étale K-theory of the largest open sets where the log-structure is trivial.

Contents

1.	Introduction	1
2.	Topologies on log-schemes	2
2.1.	The Kummer log-flat and the Kummer log-syntomic topology	2
2.2.	The valuative topologies	8
3.	Coherent and locally free sheaves on log-schemes	10
3.1.	Invertible sheaves	11
3.2.	Locally free sheaves of higher rank	16
4.	Algebraic K-theory of log-schemes	20
4.1.	Basic properties	20
4.2.	Calculations	22
5.	Topological K-theory of log-schemes	24
5.1.	Homotopy theory of simplicial presheaves and sheaves	24
5.2.	Topological K-theory	27
5.3.	Topological log-étale K/n-theory	28
	References	33

1. Introduction

The purpose of this paper is to set down some basic facts about the algebraic and topological K-theory of log-schemes. Log-schemes come equipped with several natural topologies. The main two are the Kummer log-étale topology, well suited to study l-adic phenomena, and the Kummer log-flat topology (together with its derivative – the Kummer log-syntomic topology) reasonably well-suited to study p-adic phenomena. These topologies are often enhanced by adding log-blow-ups as coverings, a procedure that yields better behaved topoi.

The investigation of coherent and locally free sheaves in these topologies as well as of the related descent questions was initiated by Kato in [24]. In particular, Kato was able to compute the Picard groups of strictly local rings. A foundational study of the algebraic K-theory of the Kummer log-étale topos (i.e. the Quillen K-theory of locally free sheaves in that topos) was done by Hagihara in [14]. He has shown that over a separably closed field Kummer log-étale K-theory satisfies devissage, localization as well as Poincaré duality for log-regular regular schemes. Using these facts and an equivariant K-theory computation of the Kummer log-étale K'-theory of log-points (fields equipped with log-structure) he obtained a structure theorem (see Theorem 4.13 below) for Kummer log-étale K-theory of a certain class of log-schemes including those coming from a smooth variety with a divisor with strict normal crossings.

This paper builts on the results of Kato and Hagihara. In section 2 we focus on some basic properties of the topologies we will use. In section 3 we study coherent and locally free sheaves in these topologies. Since

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Kato's paper remains unfinished and unpublished, for the convenience of the reader (and the author), this section contains some of Kato's proofs as well as supplies proofs of the results only announced in [24]. In section 4 we study algebraic K-theory. We generalize Hagihara's work to schemes over fields with Kummer log-étale topology and to arbitrary schemes with Kummer log-flat topology. This is rather straightforward and is done by studying equivariant K-theory of finite flat group schemes instead of just finite groups as in Hagihara. The following structure theorem follows. Let X be a regular, log-regular scheme with the log-structure associated to a divisor D with strict normal crossing. Let $\{D_i | i \in I\}$ be the set of the irreducible (regular) components of D. For an index set $J \subset I$ denote by D_J the intersection of irreducible components indexed by J and by $\Lambda_{|J|}$ (resp. $\Lambda'_{|J|}$) the free abelian groups generated by the set $\{(a_1, \ldots, a_{|J|}) | a_i \in \mathbf{Q}/\mathbf{Z} \setminus \{0\}\}$ (resp. the set $\{(a_1, \ldots, a_{|J|}) | a_i \in (\mathbf{Q}/\mathbf{Z})' \setminus \{0\}\}$).

Theorem 1.1. *For any $q \geq 0$ we have the canonical isomorphism*

$$K_q(X_{\text{kfl}}) \simeq \bigoplus_{J \subset I} K_q(D_J) \otimes \Lambda_{|J|}$$

Moreover, if D is equicharacteristic then canonically

$$K_q(X_{\text{két}}) \simeq \bigoplus_{J \subset I} K_q(D_J) \otimes \Lambda'_{|J|}$$

Section 5 is devoted to topological K-theory. By definition this is K-cohomology of the various sites considered in this paper. The main theorem (Theorem 5.14 and Corollary 5.17) states that l-adic log-étale K-theory of a log-regular scheme computes the étale K-theory of the largest open set on which the log-structure is trivial.

Theorem 1.2. *Let X be a log-regular scheme satisfying condition (*) from section 5. Let n be a natural number invertible on X. Then the open immersion $j : U \hookrightarrow X$, where $U = X_{\text{tr}}$ is the maximal open set of X on which the log-structure is trivial, induces an isomorphism*

$$j^* : K_m^{\text{vét}}(X, \mathbf{Z}/n) \xrightarrow{\sim} K_m^{\text{ét}}(U, \mathbf{Z}/n), \quad m \geq 0.$$

This follows from the fact that we can resolve singularities of log-regular schemes by log-blow-ups and that the étale sheaves of nearby cycles can be killed by coverings that are étale where the log-structure is trivial and tamely ramified at infinity.

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For a log-scheme X, M_X will always denote the log-structure of X. Unless otherwise stated all the log-structures on schemes are fine and saturated (in short: fs) and come from the étale topology, and all the operations on monoids are performed in the fine and saturated category.

2. Topologies on log-schemes

In this section we collect some very basic facts about topologies on log-schemes.

2.1. **The Kummer log-flat and the Kummer log-syntomic topology.**

2.1.1. *The log-étale, log-syntomic, and log-flat morphisms.* The notion of the log-étale and the log-flat morphism recalled below is the one of Kato [23, 3.1.2]. The notion of log-syntomic morphism we introduce is modeled on that. Our main reason for introducing it is the local lifting property it satisfies (see Lemma 2.9).

def **Definition 2.1.** Let $f : Y \to X$ be a morphism of log-schemes. We say that f is log-étale (resp. log-flat, resp. log-syntomic) if locally on X and Y for the (classical) étale (resp. fppf, resp. syntomic) topology, there exists a chart $(P \to M_X, Q \to M_Y, P \to Q)$ of f such that the induced morphisms of schemes

- $Y \to X \times_{\mathrm{Spec}(\mathbf{Z}[P])} \mathrm{Spec}(\mathbf{Z}[Q])$,
- $\mathrm{Spec}(\mathcal{O}_Y[Q^{gp}]) \to \mathrm{Spec}(\mathcal{O}_Y[P^{gp}])$

are classically étale (resp. flat, resp. syntomic).

Recall the definition of (classical) syntomic morphism.

Definition 2.2. Let $f : Y \to X$ be a morphism of schemes. We say that f is syntomic if locally on X and Y for the classical étale topology f can be written as $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, with $B = A[X_1, \ldots, X_r]/(f_1, \ldots, f_s)$, where the sequence $(f_1, \ldots, f_s)$ is regular in $A[X_1, \ldots, X_r]$ and the algebras $A[X_1, \ldots, X_r]/(f_1, \ldots, f_i)$ are flat over A, for all i.

Syntomic morphisms are stable under composition and base change.

Remark 2.3. We should mention that, a priori, in the Definition 2.1 we have used (after Kato [23, 3.1]) the following meaning of property being local on X and Y: there exist coverings $(X_i \to X)_i$ and $(Y_{ij} \to X_i \times_X Y)_j$, for each i, for the corresponding topology such that each morphism $Y_{ij} \to X_i$ has the required property. By Lemma 2.8 below, this is equivalent for log-étale, log-flat, and log-syntomic morphisms, to the more usual meaning: for every point $y \in Y$ and its image $x \in X$, there exist neighbourhoods U and V of y and x respectively (for the corresponding topology) such that U maps to V and the morphism $U \to V$ has the required property. In particular, in the Definition 2.1 we may use the second meaning of "locally" and change the second condition to "$\mathrm{Spec}(\mathcal{O}_X[Q^{gp}]) \to \mathrm{Spec}(\mathcal{O}_X[P^{gp}])$ is classically étale (resp. flat, resp. syntomic)."

Remark 2.4. The notion of log-syntomic morphism presented here is not the same as the one used by Kato [21, 2.5]. Recall that Kato defines an integral morphism $f : Y \to Z$ of fine log-schemes to be log-syntomic if étale locally Y (over Z) embeds into a log-smooth Z-scheme via an exact classically regular embedding over Z. In particular, Kato's log-syntomic morphisms are classically flat while ours are not necessarily so.

cond **Lemma 2.5.** *Let S be a nonempty scheme and let $h : G \to H$ be a homomorphism of finitely generated abelian groups. Then the morphism $\mathcal{O}_S[G] \to \mathcal{O}_S[H]$ is étale (resp. flat or syntomic) if and only if the kernel and the cokernel of h are finite groups whose orders are invertible on S (resp. if the kernel of h is a finite group whose order is invertible on S).*

Proof. The étale case follow from [22, 3.4]. The "if" part of the flat case follows from [22, 4.1]. We will now show that if the induced morphism $f : k[G] \to k[H]$, where k is a field is flat, then the kernel N of h is torsion of order invertible in k. Take an element g from N. It is easy to see that the kernel of the multiplication by $g - 1$ on $k[G]$ is generated, as an ideal, by elements $1 + g + \ldots + g^{n-1}$, such that $g^n = 1$. By the flatness of f, the images of these elements in $k[H]$ generate as an ideal the whole of $k[H]$. In particular, the element g has to be of finite order d and the ideal of the multiplication by $g - 1$ on $k[G]$ is generated by the element $1 + g + \ldots + g^{d-1}$. But $f(1 + g + \ldots + g^{d-1}) = d$. Hence d is invertible in k, as wanted. The syntomic case follows from Lemma 2.6 below. □

synflat **Lemma 2.6.** *With the notation as in the above lemma, the morphism $\mathcal{O}_S[G] \to \mathcal{O}_S[H]$ is flat if and only if it is syntomic.*

Proof. Since syntomic morphism is flat, we have to show that if the morphism $\mathcal{O}_S[G] \to \mathcal{O}_S[H]$ is flat it is already syntomic. Let $N = \ker(G \to H)$. Our morphism $\mathcal{O}_S[G] \to \mathcal{O}_S[H]$ factors as $\mathcal{O}_S[G] \to \mathcal{O}_S[G/N] \to \mathcal{O}_S[H]$. Since the morphism $\mathcal{O}_S[G] \to \mathcal{O}_S[H]$ is flat, the group N is torsion of order invertible on S (see the previous lemma). This yields that the first morphism in our factorization is étale hence syntomic. This allows us to reduce the question to proving that if the morphism $h : G \to H$ is injective then the induced morphism $\mathbf{Z}[G] \to \mathbf{Z}[H]$ is syntomic.

Write $H = H_1/G_1$, for $H_1 = G \oplus \mathbf{Z}r_1 \oplus \ldots \oplus \mathbf{Z}r_n$ and a subgroup G_1 of H_1. Since $H_{1,tor} = G_{tor}$ and the map $G \to H$ is injective, the group G_1 is finitely generated and torsion-free. Write $G_1 = \mathbf{Z}a_1 \oplus \ldots \oplus \mathbf{Z}a_k$. We claim that $\mathbf{Z}[H] \simeq \mathbf{Z}[H_1]/(a_1 - 1, \ldots, a_k - 1)$ and the sequence $\{a_1 - 1, \ldots, a_k - 1\}$ is regular. Set $H_{1,l} := H_1/\mathbf{Z}a_1 \oplus \ldots \oplus \mathbf{Z}a_l$. Note that, since the group $\mathbf{Z}a_1 \oplus \ldots \oplus \mathbf{Z}a_l \oplus \mathbf{Z}a_{l+1}$ is torsion free, the element a_{l+1} is not torsion in $H_{1,l}$. This easily implies (cf. [5, 2.1.6]) that $a_{l+1} - 1$ is not a zero-divisor in $\mathbf{Z}[H_{1,l}]$. To finish, it suffices to check that the natural map $\mathbf{Z}[H_{1,l}]/(a_{l+1} - 1) \to \mathbf{Z}[H_{1,l}/\mathbf{Z}a_{l+1}]$ is an isomorphism. But this is clear since we have the inverse induced by $\overline{x} \mapsto x$, for $x \in H_{1,l}$. □

proper **Lemma 2.7.** (1) *Log-étale, log-flat, log-syntomic morphisms are stable under compositions and under base changes.*

(2) *Let $f : Y \to X$ be a strict morphism of log-schemes, i.e., a morphism such that $f^*M_X \overset{\simeq}{\to} M_Y$. Then f is log-étale (resp. log-flat, resp. log- syntomic) if and only if the underlying morphism of schemes is (classically) étale (resp. flat, resp. syntomic).*

(3) *Let S be a scheme and let $P \to Q$ be a morphism of monoids. Then the induced morphism of log-schemes* $\mathrm{Spec}(\mathcal{O}_S[Q]) \to \mathrm{Spec}(\mathcal{O}_S[P])$ *is log-étale (resp. log-flat, resp. log-syntomic) if and only if the morphism of schemes* $\mathrm{Spec}(\mathcal{O}_S[Q^{gp}]) \to \mathrm{Spec}(\mathcal{O}_S[P^{gp}])$ *is (classically) étale (resp. flat, resp. syntomic).*

Proof. The only nonobvious statement is the one concerning compositions, which follow easily from Lemma 2.8 below. □

comp1 **Lemma 2.8.** *Let $f : Y \to X$ be a morphism of log-schemes and let $\beta : P \to M_X$ be a chart. Assume that f is log-étale (resp. log-flat, resp. log-syntomic). Then, étale (resp. flat, resp. syntomic) locally on X and on Y in the classical sense, there exists a chart $(P \to M_X, Q \to M_Y, P \to Q)$ including β satisfying the conditions in Definition 2.1. We can require further $P^{gp} \to Q^{gp}$ to be injective.*

Proof. For the log-étale and the log-flat topology this is Lemma 3.1.6 from [23]. We will argue in a similar fashion for the log-syntomic case taking into account that (unlike in [23]) our monoids are always saturated.

Let $(P' \to M_X, Q' \to M_Y, P' \to Q')$ be a chart satisfying the conditions in Definition 2.1. Fix $y \in Y, x = f(y) \in X$. By replacing P' with the inverse image P_1 (which is always saturated) of $M_{X,x}$ under the map

$$P^{gp} \oplus (P')^{gp} \to M^{gp}_{X,x}; \qquad (a, b) \mapsto ab,$$

and by replacing Q' with the pushout $P_1 \leftarrow P' \to Q'$, we may assume that $\beta : P \to M_X$ factors as $P \to P' \to M_X$. By (Zariski) localization we may also assume that $P'/(P')^* \simeq M_{X,x}/\mathcal{O}^*_{X,x}$ and $Q'/(Q')^* \simeq M_{Y,y}/\mathcal{O}^*_{Y,y}$.

Assume for the moment that the morphism $(P')^{gp} \to (Q')^{gp}$ is injective. Consider the pushout diagrams with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P^{gp} \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (P')^{gp} \arrow[r] \arrow[d] & (Q')^{gp} \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[r] & T \arrow[r] & W \arrow[r] & 0,
\end{tikzcd}$$

where the group G is the cokernel of the map $P^{gp} \to (P')^{gp}$. We want to construct the group H. For that, it suffices to show that the map $T \to W$ has a section. Consider a direct summond $\mathbf{Z}/n\mathbf{Z}$ of W. Let $t \in T$ be a preimage of a generator of $\mathbf{Z}/n\mathbf{Z}$. Then $t^n = b, b \in G$. Take $b' \in (P')^*$ in the preimage of b. Since $P'/(P')^* \simeq M_{X,x}/\mathcal{O}^*_{X,x}$ and P^{gp} maps onto $M^{gp}_{X,x}/\mathcal{O}^*_{X,x}$ such a b' exists. Define the group G_1 by adjoining the n'th root of b' to $(P')^*$. By localizing in the classical syntomic topology, we can now change P' and Q' into the pushouts $P' \leftarrow (P')^* \to G_1$ and $Q' \leftarrow (P')^* \to G_1$. Note that we can do that since the morphism $(P')^* \to G_1$ is injective with finite cokernel, hence the induced morphism $\mathrm{Spec}(\mathbf{Z}[G_1]) \to \mathrm{Spec}(\mathbf{Z}[(P')^*])$ is syntomic (Lemma 2.5) and surjective. Moreover, the above pushouts taken in the category of monoids are already fine and saturated. Now, $b = a^n$ for some $a \in G$. Changing t to t/a gives us an element in the preimage of our generator of $\mathbf{Z}/n\mathbf{Z}$ whose n'th power is one, hence the section we wanted.

Let now Q be the inverse image of $M_{Y,y}$ under the map $H \to M^{gp}_{Y,y}$ (it is saturated). Since $P'/(P')^* \simeq M_{X,x}/\mathcal{O}^*_{X,x}$ and $Q'/(Q')^* \simeq M_{Y,y}/\mathcal{O}^*_{Y,y}$ this gives a local chart at y. We claim that the natural morphism $P \to Q$ gives us the chart we wanted. The map $P^{gp} \to Q^{gp}$ is clearly injective. Let Q_1 be the pushout $P' \leftarrow P \to Q$. There is a natural morphism $Q_1 \to Q'$. By Zariski localizing on Y, we may assume that

$Q_1/Q_1^* \xrightarrow{\sim} Q'/(Q')^*$. Since the map $Q_1^{gp} \to (Q')^{gp}$ is an isomorphism, this yields that $Q_1 \xrightarrow{\sim} Q'$. Hence the morphism $P \to Q$ is indeed the chart we wanted.

Let now $(P \to M_X, Q \to M_Y, P \to Q)$ be a chart satisfying the conditions in Definition 2.1. It remains to show that we may assume $P^{gp} \to Q^{gp}$ to be injective. Indeed, let N be the kernel of $P^{gp} \to Q^{gp}$. Consider the pushout diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P^{gp} & \longrightarrow & H & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P^{gp}/N & \longrightarrow & Q^{gp} & \longrightarrow & W & \longrightarrow & 0.
\end{array}$$

It is easy to construct the group H. Let now Q' be the inverse image of $M_{Y,y}$ under the map $H \to M^{gp}_{Y,y}$ (it is saturated). Q' gives a local chart at y. Since N is the kernel of the map $(Q')^{gp} \to Q^{gp}$ and N is a finite group of order invertible on Y, there exists an open set $U \subset X \times_{\mathrm{Spec}(\mathbf{Z}[P])} \mathrm{Spec}(\mathbf{Z}[Q'])$ such that the order of N is invertible on U. Then the morphism $U \to X \times_{\mathrm{Spec}(\mathbf{Z}[P])} \mathrm{Spec}(\mathbf{Z}[Q'])$ is log-étale. On the other hand this morphism is (perhaps after Zariski localization) strict. Hence it is étale. This shows that $(P \to M_X, Q' \to M_Y, P \to Q')$ is a chart satisfying the conditions in Definition 2.1 such that the map $P^{gp} \to (Q')^{gp}$ is injective. $\square$

lifting **Lemma 2.9.** *Let $Y \hookrightarrow X$ be a exact closed immersion defined by a nilideal. Étale locally log-syntomic morphisms over Y can be lifted to log-syntomic morphisms over X.*

Proof. Immediate from Lemma 2.8 (note that it suffices to localize in the étale topology on Y) and the well-known lifting property for classical syntomic morphisms that we recall below. $\square$

Lemma 2.10. *Let A be a commutative ring, $B \to A$ a closed immersion defined by a nilideal, and $C = A[X_1, \ldots, X_n]/(G_1, \ldots, G_r)$ an A-algebra such that the sequence $(G_1, \ldots, G_r)$ is regular and each $A[X1, \ldots, X_n]/(G_1, \ldots, G_i)$, $i \leq r$, is flat over A. Let $(\check{G}_1, \ldots, \check{G}_r)$ be liftings of $(G_1, \ldots, G_r)$ to $B[X_1, \ldots, X_n]$. Then the sequence $(\check{G}_1, \ldots, \check{G}_r)$ is regular and each $B[X_1, \ldots, X_n]/(\check{G}_1, \ldots, \check{G}_i)$, $i \leq r$, is flat over B.*

2.1.2. *Kummer topologies.* Recall first the definition of Kummer morphisms.

Definition 2.11. (1) A homomorphism of monoids $h : P \to Q$ is said to be of Kummer type if it is injective and, for any $a \in Q$, there exists $n \geq 1$ such that $a^n \in h(P)$.
(2) A morphism $f : X \to Y$ of log-schemes is of Kummer type if for any $x \in X$, the induced homomorphism of monoids $(M/\mathcal{O}^*)_{Y,\overline{f(x)}} \to (M/\mathcal{O}^*)_{X,\overline{x}}$ is of Kummer type in the sense of (1).

One checks [27, 2.1.2] that Kummer morphisms are stable under base changes and compositions.

Remark 2.12. Note that if the morphism $P \to Q$ is Kummer, then by Lemma 2.6 the associated morphism $\mathrm{Spec}(\mathbf{Z}[Q]) \to \mathrm{Spec}(\mathbf{Z}[P])$ is both log-flat and log-syntomic.

Definition 2.13. Let X be a log-scheme. A morphism $Y \to X$ is called Kummer log-étale (resp. log-flat, log-syntomic) if it is log-étale (resp. log-flat, log-syntomic) and of Kummer type and the underlying morphism of schemes is locally of finite presentation. The log-étale (resp. log-flat, log-syntomic) topology on the category of Kummer log-étale (resp. log-flat, log-syntomic) morphisms over X is defined by taking as coverings families of morphisms $\{f_i : U_i \to T\}_i$ such that each f_i is log-étale (resp. log-flat, log-syntomic) and $T = \bigcup_i f_i(U_i)$ (set theoretically).

This defines a Grothendieck topology by the following result of Nakayama [27, 2.2.2].

Na **Lemma 2.14.** *Let $f : Y \to X$ be a morphism of log-schemes that is Kummer and surjective. Then, for any log-scheme $X' \to X$, the morphism $Y \times_X X' \to X'$ is surjective. In fact, for any $y \in Y$ and $x \in X'$ having the same image in X, there exists $z \in Y \times_X X'$ mapping to x and to y.*

The following proposition describes a very useful cofinal system of coverings for the Kummer log- étale, log-flat and log-syntomic sites.

factor **Proposition 2.15.** *Let $f : Y \to T$ be a Kummer log-étale (resp. log-flat, log-syntomic) morphism. Let $y \in Y, t = f(y)$, and $P \to M_T$ be a chart such that $P^* \simeq \{1\}$. Then there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "g"'] & T_n \arrow[d] \\ Y \arrow[r, "f"] & T, \end{tikzcd}$$

where y is in the image of g, h is classically étale (resp. flat, syntomic), g is Kummer log-étale (resp. log-flat, log-syntomic), and n is invertible on X (resp. any, any).

Proof. We will argue the case of Kummer log-flat topology. The other cases are similar. By Lemma 2.8 localizing on Y (but keeping $y \in Y$) for the flat topology, we get a chart $(P \to M_T, Q \to M_Y, P \to Q)$ as in Definition 2.1 such that $P^{gp} \to Q^{gp}$ is injective. Note that localizing on T is not necessary. Arguing further as in the proof of Proposition A.2 in [28] we may assume that Q is torsion free. Hence $P^{gp} \simeq Q^{gp}$ as abelian groups. Write $n : P \to Q \to P^{1/n}$ for some n, where $P^{1/n}$ is a P-monoid such that $P \to P^{1/n}$ is isomorphic to $n : P \to P$. Set $X = Y \times_{T_Q} T_n$, where $T_n = T \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P^{1/n}]$, $T_Q = T \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$. By definition the map $h : X \to T_n$ is classically flat and, since $Q^{gp} \hookrightarrow P^{1/n,gp}$, the induced map $g : X \to Y$ is surjective and Kummer log-flat. $\square$

factor1 **Corollary 2.16.** *Let $f : Y \to T$ be a Kummer log-flat (resp. log-syntomic) morphism. Let $P \to M_T$ be a chart such that $P^* \simeq \{1\}$. Then there exists a Kummer log-flat (resp. log-syntomic) covering $V \to Y$ such that for some n the map $V \times_T T_n \to T_n$ is classically flat (resp.syntomic).*

Proof. We will treat the flat case. The syntomic case is similar. By Proposition 2.15, there exists n such that for a flat covering $V \to Y$ the induced map $V \to T$ factors as $V \to T_n \to T$, where the map $V \to T_n$ is classically flat. We have the following cartesian diagram

$$\begin{tikzcd} V_n = V \times_{T_n} T_n \times_T T_n \arrow[r] \arrow[d] & T_n \times_T T_n \arrow[r, "p_2"] \arrow[d, "p_1"'] & T_n \arrow[d] \\ V \arrow[r] & T_n \arrow[r] & T, \end{tikzcd}$$

Since p_1, p_2 are classicaly flat, so is the map $V_n \to T_n$, as wanted. $\square$

Similarly one proves the following

factor2 **Corollary 2.17.** *Let $f : Y \to T$ be a Kummer log-étale covering. Let $P \to M_T$ be a chart such that $P^* \simeq \{1\}$. Then, Zariski locally on T, there exists a Kummer log-étale covering $V \to T$ refining f such that, for some n invertible on T, the map $V \times_T T_n \to T_n$ is classically étale.*

For a log-scheme X, we will denote by $X_{\text{két}}$ (resp. X_{kfl}, X_{ksyn}) the site defined above. In what follows, I will denote sites and the associated topoi in the same way. I hope that this does not lead to a confusion.

We will need to know that certain presheaves are sheaves for the Kummer topologies.

sheaf **Proposition 2.18.** *Let X be a log-scheme. Then the presheaf $(Y \to X) \mapsto \Gamma(Y, \mathcal{O}_Y)$ is a sheaf on all Kummer sites.*

Proof. It is clearly enough to show this for the Kummer log-flat site. In that case it follows from a Kummer descent argument (see Lemma 3.28 below). $\square$

More generally

sheaf1 **Proposition 2.19.** *Let X be a log-scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X_{Zar}. Then the presheaf*

$$(f : T \to X) \mapsto \Gamma(T, f^*\mathcal{F})$$

is a sheaf on all Kummer sites.

Proof. It is clearly enough to show this for the Kummer log-flat site. In that case it follows from the proof of the Kummer descent argument below via exhibiting an explicite contracting homotopy (see Lemma 3.28). □

And in a different direction, we have the following theorem. The proof presented here is that of Kato [24, 3.1].

Theorem 2.20. *Let X be a log-scheme, and let Y be a log-scheme over X. Then the functor*

$$\mathrm{Mor}_X(\ ,Y) : T \mapsto \mathrm{Mor}_X(T,Y)$$

on (fs/X) is a sheaf for all the Kummer topologies.

Proof. We claim that it suffices to show that the functors

w1 (2.1)
$$T \mapsto \Gamma(T,\mathcal{O}_T), \quad T \mapsto \Gamma(T,M_T).$$

are sheaves for the Kummer log-flat topology. To see that assume that $X = \mathrm{Spec}(\mathbf{Z})$ with the trivial log-structure, Y is an affine scheme with a chart $P \to \Gamma(Y,M_Y)$. Let F,G,H be the following functors from $(fs)/\mathrm{Spec}(\mathbf{Z})$ to $(Sets)$

$$\begin{aligned} F(T) &= \{\text{ring homomorphisms } \Gamma(Y,\mathcal{O}_Y) \to \Gamma(T,\mathcal{O}_T)\},\\ G(T) &= \{\text{monoid homomorphisms } P \to \Gamma(T,M_T)\},\\ H(T) &= \{\text{monoid homomorphisms } P \to \Gamma(T,\mathcal{O}_T)\}. \end{aligned}$$

The functor $\mathrm{Mor}_X(\ ,Y) : T \mapsto \mathrm{Mor}_X(T,Y)$ is the fiber product $F \to H \leftarrow G$, where the first arrow is induced by $P \to \Gamma(Y,\mathcal{O}_Y)$ and the second one by $\Gamma(T,M_T) \to \Gamma(T,\mathcal{O}_T)$. It follows that it suffices to show that the functors F,G,H are sheaves.

Take now a presentation

$$\Gamma(Y,\mathcal{O}_Y) = \mathbf{Z}[T_i; i\in I]/(f_j; j \in J),$$
$$\mathbf{N}^r \rightrightarrows \mathbf{N}^s \to P.$$

We get that $F(T)$ is the kernel of $\Gamma(T,\mathcal{O}_T)^I \to \Gamma(T,\mathcal{O}_T)^J$ and $G(T)$ and $H(T)$ are the equalizers of $\Gamma(T,M_T)^s \rightrightarrows \Gamma(T,M_T)^t$ and $\Gamma(T,\mathcal{O}_T)^s \rightrightarrows \Gamma(T,\mathcal{O}_T)^t$, respectively. Thus it suffices to show that the functors in (2.1) are sheaves.

For the functor $T \mapsto \Gamma(T,\mathcal{O}_T)$ this follows from Lemma 3.28. For the functor $T \mapsto \Gamma(T,M_T)$ we first show that it is a sheaf for the classical flat topology. If $T' \to T$ is a fppf covering, then we know that the sequence

$$\Gamma(T,\mathcal{O}_T^*) \to \Gamma(T',\mathcal{O}_{T'}^*) \rightrightarrows \Gamma(T'',\mathcal{O}_{T''}^*)$$

where $T' = T' \times_T T'$, is exact. Since $M_{T'}/\mathcal{O}_{T'}^*$ and $M_{T''}/\mathcal{O}_{T''}^*$ are pulbacks of $M_T/\mathcal{O}_T^*$, the sequence

$$\Gamma(T,M_T/\mathcal{O}_T^*) \to \Gamma(T',M_{T'}/\mathcal{O}_{T'}^*) \rightrightarrows \Gamma(T'',M_{T''}/\mathcal{O}_{T''}^*)$$

is exact as well. Next we treat Kummer coverings

Lemma 2.21. *Take $T = \mathrm{Spec}(A)$ for a local ring A equipped with a chart $P \to \Gamma(T,M_T)$, $P \simeq (M_T/\mathcal{O}_T^*)_t$, where t is the closed point of T. Let Q be a monoid with no torsion. Let $P \to Q$ be a homomorphism of Kummer type. Let $T' = T \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ endowed with the log-structure associated to Q. Let $T'' = T' \times_T T'$. Then*

$$\Gamma(T,M_T) \to \Gamma(T',M_{T'}) \rightrightarrows \Gamma(T'',M_{T''})$$

is exact.

Proof. Set $A' = \Gamma(T',\mathcal{O}_{T'}) = A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$, $A'' = \Gamma(T'',\mathcal{O}_{T''}) = A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q \oplus (Q^{gp}/P^{gp})]$. By Lemma 3.28 the sequence $A \to A' \rightrightarrows A''$ is exact. Let I, I', I'' be the ideals of A, A', A'', respectively, generated by the images of $P \setminus \{1\}, Q\setminus\{1\}, Q\setminus\{1\}$, respectively. Let V, V', V'' be the subgroups of $A^*, (A')^*, (A'')^*$, respectively, consisting of elements that are congruent to 1 modulo I, I', I'', respectively. Since $A/I \simeq A'/I'$ the sequence $V \to V' \rightrightarrows V''$ is exact. It remains to show that the sequence

$$\Gamma(T,M_T)/V \to \Gamma(T',M_{T'})/V' \rightrightarrows \Gamma(T'',M_{T''})/V''$$

is exact. This sequence is isomorphic to

$$P \oplus (A/I)^* \to Q \oplus (A/I)^* \rightrightarrows Q \oplus \{(A/I)[Q^{gp}/P^{gp}]\}^*,$$

where the two arrows in the middle are $\beta_1 : (q,u) \mapsto (q,u)$, $\beta_2 : (q,u) \mapsto (q,qu)$. The exactness of the last sequence follows from the exactness of $P \to Q \to Q \oplus (Q^{gp}/P^{gp})$. □

□

Denote by $\mathbf{G}_m^\times$ the functor $T \mapsto \Gamma(T, M_T^{\mathrm{gp}})$ on (fs/X). The above theorem yields

Corollary 2.22. ([24, 3.6]) *The functor $\mathbf{G}_m^\times$ is a sheaf for all the Kummer topologies.*

Proof. This argument is also due to Kato [24, 3.6]. It suffices to show that $\mathbf{G}_m^\times$ is a sheaf for the Kummer log-flat topology. Let $T' \to T$ be a Kummer log-flat covering equipped with a chart $P \to \Gamma(T, M_T)$. Set $T'' = T' \times_T T'$. We have $\Gamma(T, M_T^{gp}) = \operatorname{inj\,lim}_a \Gamma(T, a^{-1}M_T)$, where a ranges over all elements of P. Since both T' and T'' are of Kummer type over T, we also have $\Gamma(T', M_{T'}^{gp}) = \operatorname{inj\,lim}_a \Gamma(T', a^{-1}M_{T'})$ and $\Gamma(T'', M_{T''}^{gp}) = \operatorname{inj\,lim}_a \Gamma(T'', a^{-1}M_{T''})$. It follows that the exactness of the sequence

$$\Gamma(T, M_T^{gp}) \to \Gamma(T', M_{T'}^{gp}) \rightrightarrows \Gamma(T'', M_{T''}^{gp})$$

is reduced to the exactness of the sequence

$$\Gamma(T, M_T) \to \Gamma(T', M_{T'}) \rightrightarrows \Gamma(T'', M_{T''})$$

that was proved above. □

valuative

2.2. **The valuative topologies.**

The valuative topologies refine Kummer topologies with log-blow-up coverings. That makes them slightly pathological (blow-ups do not change the global sections of sheaves) but also allows for better functorial properties [15].

Definition 2.23. Let X be a log-scheme. A morphism $Y \to X$ is called Zariski (resp. étale, log-étale, log-flat, log-syntomic) valuative if it is a composition of Zariski open (resp. étale, Kummer log-étale, Kummer log-flat, Kummer log-syntomic) morphisms and log-blow-ups. The Zariski (resp. étale, log-étale, log-flat, log-syntomic) valuative topology on this category of morphisms over X is defined by taking as coverings families of morphisms $\{f_i : U_i \to T\}_i$ such that each f_i is Zariski (resp. étale, log-étale, log-flat, log-syntomic) valuative and $T = \bigcup_i f_i(U_i)$ universally (i.e., this equality is valid after any base change by a map $S \to T$ of log-schemes). We will denote the corresponding site by X_{val} (resp. $X_{\mathrm{v\acute{e}t}}$, $X_{\mathrm{vk\acute{e}t}}$, X_{vkfl}, X_{vksyn}).

Note that, since any base change of a log-blow-up is a log-blow-up [30, Cor.4.8], the above definition makes sense. We have the following commutative diagram of continuous maps of sites

$$\begin{array}{ccccccccc}
X_{\mathrm{vkfl}} & \longrightarrow & X_{\mathrm{vksyn}} & \longrightarrow & X_{\mathrm{vk\acute{e}t}} & \longrightarrow & X_{\mathrm{v\acute{e}t}} & \longrightarrow & X_{\mathrm{val}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_{\mathrm{kfl}} & \longrightarrow & X_{\mathrm{ksyn}} & \longrightarrow & X_{\mathrm{k\acute{e}t}} & \longrightarrow & X_{\mathrm{\acute{e}t}} & \longrightarrow & X_{\mathrm{Zar}}.
\end{array}$$

Remark 2.24. Note that the site $X_{\mathrm{vk\acute{e}t}}$ is the same as the full log-étale site $X_{\mathrm{\acute{e}t}}^{\log}$ [15].

Denote by $\mathcal{O}_{X^*}$ (or by $\mathcal{O}_X$ if there is no risk of confusion) the structure sheaf of the topos on X induced by one of the above topologies, i.e., the sheaf associated to the presheaf $(Y \to X) \mapsto \Gamma(Y, \mathcal{O}_Y)$.

We will now describe points of the topoi associated to some of the above sites. Recall [27, 2.4] that a log-geometric point is a scheme $\operatorname{Spec}(k)$, for a separably closed field k, equipped with a saturated monoid M such that the map $a \mapsto a^n$ on $P = M/k^*$ is bijective for any integer n prime to the characteristic of k. Log-geometric points form a conservative system for the Kummer log-étale topos [27, 2.5]. We get enough points of the full log-étale topos by taking (valuative) log-geometric points, i.e., log-geometric points with M/k^* valuative (recall that a saturated monoid P is called valuative if for any $a \in P^{gp}$, either a or a^{-1} is in P). There is an alternative way of describing a conservative family of points for the log-étale topos. For $x \in X$, choose a chart $x \in U$, $U \to \operatorname{Spec} \mathbf{Z}[P]$. For each finitely generated and nonempty ideal $J \subset P$, let U_J be the log-blow-up of U along J. These U_J's form an inverse system indexed by the set of finitely generated and nonempty ideals J partially ordered by divisibility. Take now a compatible system of log-geometric points of the $U_J's$ lying above x.

A conservative family of points of X_{val} (resp. $X_{\mathrm{v\acute{e}t}}$) can be described in a similar fashion by taking compatible systems of Zariski (resp. geometric) points. Recall [23, 1.3.5] that in the case of X_{val} and a chart $X \to \operatorname{Spec} \mathbf{Z}[P]$ there is a canonical bijection between this set of points and all pairs $(V, \mathfrak{p})$ such that V is a valuative submonoid of P^{gp} containing P and $\mathfrak{p}$ is a point of $X_V = X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[V]$ satisfying the following condition: If $a \in V$ and the image of a in $\mathcal{O}_{X_V,\mathfrak{p}}$ is invertible, then $a \in V^*$. We have then the following description of stalks of the structure sheaf: $\mathcal{O}_{X_{\mathrm{val}},(V,\mathfrak{p})} \simeq \mathcal{O}_{X_V,\mathfrak{p}}$.

fact **Lemma 2.25.** *Let $Y \to X$ be a log-flat valuative morphism. Then there is a log-blow-up $Y' \to Y$ (hence necessarily a covering) such that the morphism $Y' \to X$ can be written as a composition $Y' \to T \to X$, where $Y' \to T$ is Kummer log-flat and $T \to X$ is a log-blow-up.*

Proof. Since composition of log-blow-ups is a log-blow-up [30, Cor.4.11], it is enough to show this for a composition $Y \to Z \to X$ of a log-blow-up $Y \to Z$ with a Kummer log-flat morphism $Z \to X$. Recall that by [20, 3.13] we can find a log-blow-up $B \to X$ such that the base change $Y' := Y \times_X B \to B$ is exact. Here a morphism of log-schemes $f : T \to S$ is called exact if, for every $t \in T$, the morphism $f : M_{S,\overline{s}}/\mathcal{O}^*_{S,\overline{s}} \to M_{T,\overline{t}}/\mathcal{O}^*_{T,\overline{t}}$, $s = f(t)$, is exact, i.e., $(f^{gp})^{-1}(M_{T,\overline{t}}/\mathcal{O}^*_{T,\overline{t}}) = M_{S,\overline{s}}/\mathcal{O}^*_{S,\overline{s}}$. Consider now the following commutative diagram

$$\begin{tikzcd} Y \arrow[r] & Z \arrow[r] & X \\ Y' \arrow[u] \arrow[r, "\sim"] & Z \times_X B \arrow[u] \arrow[r] & B. \arrow[u] \end{tikzcd}$$

Since base change of a log-blow-up is a log-blow-up [30, Cor.4.8] the morphisms $Y' \to Y$, $Z \times_X B \to Z$ and $Y' \to Z \times_X B$ are log-blow-ups. But because the composition $Y' \to B$ is exact, the morphism $Z \times_X B \to B$ is Kummer, and the log-schemes are saturated, the morphism $Y' \to Z \times_X B$ is actually an isomorphism. Hence $Y' \to B$ is Kummer log-flat as wanted. □

For a general scheme X, the presheaf $(Y \to X) \mapsto \Gamma(Y, \mathcal{O}_Y)$ on X_{val} is not always a sheaf (see [12, 2.5]). Let, for example, $X = \operatorname{Spec}(k[T_1, T_2]/(T_1^2, T_2^2))$ with the log-structure $\mathbf{N}^2 \to \mathcal{O}_X$; $e_i \mapsto T_i$, and let $Y \to X$ be the log-blow-up of the ideal generated by e_1 and e_2. Then the map $\Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y)$ is not injective. On the other hand, since Y covers X and $Y \times_X Y \simeq Y$, the map $\Gamma(X, \mathcal{O}_{X_{\mathrm{val}}}) \to \Gamma(Y, \mathcal{O}_{Y_{\mathrm{val}}})$ is necessarily an isomorphism. We have however the proposition below. But first we need to recall the notion of a log-regular scheme.

Definition 2.26. A log-scheme X is called log-regular at $x \in X$ if $\mathcal{O}_{X,\overline{x}}/I_{\overline{x}}\mathcal{O}_{X,\overline{x}}$ is regular and $\dim(\mathcal{O}_{X,\overline{x}}) = \dim(\mathcal{O}_{X,\overline{x}}/I_{\overline{x}}\mathcal{O}_{X,\overline{x}}) + \operatorname{rank}_{\mathbf{Z}}((M_X^{gp}/\mathcal{O}_X^*)_{\overline{x}})$, where $I_{\overline{x}} = M_{X,\overline{x}} \setminus \mathcal{O}^*_{X,\overline{x}}$. We say that X is log-regular if X is log-regular at every point $x \in X$.

Proposition 2.27. *Let X be a log-regular log-scheme. Then the presheaf $(Y \to X) \mapsto \Gamma(Y, \mathcal{O}_Y)$ is a sheaf on all valuative sites.*

Proof. It is clearly enough to show this for the log-flat valuative site. Since this presheaf is a sheaf on the Kummer log-flat site, by Lemma 2.25, it suffices to show that if $\pi : B \to T$ is a log-blow-up of a log-scheme $T \to X$, log-flat valuative over X, then $\Gamma(T, \mathcal{O}_T) \to \Gamma(B, \mathcal{O}_B)$ is an isomorphism. We will show that $\mathcal{O}_T \xrightarrow{\sim} R\pi_*\mathcal{O}_B$. Assume for the moment that T is log-regular. Then T behaves like a toric variety, and this is a well-known result. As the argument in [30] shows the key-point is that (flat) locally there is a chart $P \to \mathcal{O}_T$, with a torsion free monoid P, such that

w3 (2.2) $\quad$ for every injective morphism $P \to Q$, $\quad \operatorname{Tor}_i^{\mathbf{Z}[P]}(\mathcal{O}_T, \mathbf{Z}[Q]) = 0, \quad i \geq 1.$

We will show that this is also the case for our (general now) T. By induction, assume that a log-scheme $Z \to X$, log-flat valuative over X satisfies the condition (2.2). We have to show that any log-scheme $T \to Z$, Kummer log-flat or log-blow-up over Z, also satisfies this condition. We will show the argument in the case when $T \to Z$ is Kummer log-flat. The argument for log-blow-up is similar but simpler.

Consider a “good” chart

$$\begin{array}{ccc} T & \longrightarrow & \operatorname{Spec}(\mathbf{Z}[Q]) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \operatorname{Spec}(\mathbf{Z}[P]), \end{array}$$

where the monoid P has no torsion, the morphism $P \to Q$ is injective, and the morphism $T \to Y_1$, $Y_1 := Y \times_{\operatorname{Spec}(\mathbf{Z}[P])} \operatorname{Spec}(\mathbf{Z}[Q])$ is flat. A slight modification of an argument of Nakayama in [28, A.2.], yields that, modulo a flat localization, we may assume Q to be torsion free as well. Since the morphism $T \to Y_1$ is flat, we just need to show that $\operatorname{Tor}_i^{\mathbf{Z}[Q]}(\mathcal{O}_{Y_1}, \mathbf{Z}[Q_1]) = 0$, $i \geq 1$, for any injection $Q \to Q_1$. But this follows from the fact that $\operatorname{Tor}_i^{\mathbf{Z}[P]}(\mathcal{O}_Y, \mathbf{Z}[P_1]) = 0$, $i \geq 1$, for any injection $P \to P_1$. □

3. Coherent and locally free sheaves on log-schemes

Let us first collect some basic facts about coherent and locally free sheaves in the various topologies on log-schemes discussed above. Let $\mathcal{F}(X)_*$ be the category of $\mathcal{O}_X$-modules, where $*$ stands for one of the considered here topologies. It is an abelian category. Let $\mathcal{P}(X)_*$ denote the category of $\mathcal{O}_X$-modules that are locally a direct factor of a free module of finite type. By [3, I.2.15.1.ii] this is the same as the category of locally free sheaves of finite type. Let $\mathcal{M}(X)_*$ denote the category of coherent $\mathcal{O}_X$-modules, i.e., $\mathcal{O}_X$-modules that are of finite type and precoherent. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is called precoherent [3, I.3.1] if for every object $Y \to X$ in X_* and for every map $\mathcal{E} \xrightarrow{f} \mathcal{F}|Y_*$ from a locally free finite type $\mathcal{O}_Y$-module $\mathcal{E}$, the kernel of f is of finite type.

Lemma 3.1. (1) *The category $\mathcal{M}(X)_*$ is abelian and closed under extensions.*
(2) *The category $\mathcal{P}(X)_*$ is additive and when embedded in $\mathcal{F}(X)_*$ with the induced notion of a short exact sequence, it is exact.*

Proof. The first statement follows from [3, I.3.3]. For the second one it suffices to check that $\mathcal{P}(X)_*$ is closed under extensions in $\mathcal{F}(X)_*$. That follows from the fact that $\mathcal{O}_{X_*}$-modules of finite type are closed under extensions [3, I.3.3] and that all epimorphisms $\mathcal{M}_1 \to \mathcal{M}_2$, $\mathcal{M}_2 \in \mathcal{P}(X)_*$, locally admit a section [3, I.1.3.1]. □

The simplest coherent sheaves come from the Zariski topology. Let X_* denote one of the Kummer topologies and let $\varepsilon_X : X_* \to X_{\mathrm{Zar}}$ be the natural projection. We have

Lemma 3.2. *The pullback functor $\varepsilon_X^* : \mathrm{Q}\mathcal{M}(X_{\mathrm{Zar}}) \to \mathcal{F}(X_*)$ (from the category of quasicoherent Zariski sheaves) is fully faithful.*

Proof. Immediate from Proposition 2.19. □

Proposition 3.3. *Let X_* satisfy the following property*

w4 (3.1) $$\varepsilon^* \text{ is exact for a cofinal system of coverings in } X_*$$

Then the structure sheaf $\mathcal{O}_{X_*}$ is coherent on all Kummer sites.

Proof. We need to check that for any object $Y \to X$ in the Kummer site X_* the kernel of any morphism $f : \mathcal{O}_{Y_*}^m \to \mathcal{O}_{Y_*}$ is of finite type. But f comes from a Zariski morphism $f' : \mathcal{O}_{Y_{\mathrm{Zar}}}^m \to \mathcal{O}_{Y_{\mathrm{Zar}}}$ and by exactness $\varepsilon_Y^* \ker f' = \ker f$. Since $\ker f'$ is of finite type so is $\ker f$. □

Corollary 3.4. *If X has property (3.1) then the $\mathcal{F}$ is coherent if and only if there exists a covering $X_i \to X$ of X such that $\mathcal{F}|X_i$ is isomorphic to $\varepsilon_{X_i}^* \mathcal{F}_i'$ for some coherent sheaf $\mathcal{F}_i'$ on $X_{i,\mathrm{Zar}}$.*

Example 3.5. A log-scheme X such that $(M_X/\mathcal{O}_X^*)_{\overline{x}} \simeq \mathbf{N}^{r(x)}$ has property (3.1). In particular, X can be a strict closed subscheme of a regular, log-regular scheme.

Definition 3.6. The coherent sheaves or locally free sheaves in the (essential) image of the functor ε^* are called *classical.*

equiv **Lemma 3.7.** *Let X be a log-regular log-scheme. Let $Y \to X$ be a log-blow-up. Then the restrictions*

$$r : \mathcal{F}(X)_* \to \mathcal{F}(Y)_*, \quad r : \mathcal{M}(X)_* \to \mathcal{M}(Y)_*, \quad r : \mathcal{P}(X)_*^{\mathrm{val}} \to \mathcal{P}(Y)_*$$

are equivalences of categories for $*$ any of the valuative topologies.

Proof. Let $\mathcal{M} \in \mathcal{F}(Y)_*$. Consider the functor $\pi : \mathcal{F}(X)_* \to \mathcal{F}(Y)_*$ given by $\pi(\mathcal{M}) : (T \to X) \mapsto \Gamma(T \times_X Y, \mathcal{M})$. Since $Y \times_X Y \simeq Y$, the compositions $r\pi$ and πr are naturally equivalent to the identity. Hence the restriction induces an equivalence of categories $\mathcal{F}$. The remaining equivalences follow since the map $Y \to X$ is covering. □

vect **Lemma 3.8.** *Let X be a log-regular quasi-compact log-scheme. Let $\mathcal{F} \in \mathcal{P}(X)_{\mathrm{vkfl}}$ be a locally free sheaf of rank n. Then, for some log-blow-up $T \to X$, $\mathcal{F}|T_{\mathrm{vkfl}}$ is isomorphic to a pullback of a locally free sheaf of rank n from T_{kfl}.*

Proof. By Lemma 2.25 we can restrict our attention to trivializing coverings of the form $Y \to T \to X$, where $Y \to T$ is a Kummer log-flat covering and $T \to X$ is a log-blow-up. Since the isomorphism classes of locally free sheaves of rank n are classified by the first Čech cohomology groups of the sheaf $\mathbf{GL}_n$, the statement of the lemma follows now easily from the following commutative diagram

$$\begin{array}{ccccc}
\mathbf{GL}_n(Y) & \longrightarrow & \mathbf{GL}_n(Y \times_X Y) & \longrightarrow & \mathbf{GL}_n(Y \times_X Y \times_X Y) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\mathbf{GL}_n(Y) & \longrightarrow & \mathbf{GL}_n(Y \times_T Y) & \longrightarrow & \mathbf{GL}_n(Y \times_T Y \times_T Y),
\end{array}$$

where the equalities hold already on the level of schemes (since $T \times_X T \simeq T$). □

inj **Corollary 3.9.** *Let X be a log-regular quasi-compact log-scheme. Then the pullback functor*

$$\operatorname{inj}\lim_{Y} \mathcal{P}(Y_{\mathrm{kfl}}) \to \mathcal{P}(X_{\mathrm{vkfl}})$$

is an equivalence of categories, where the limit is over log-blow-ups $Y \to X$.

synfl **Lemma 3.10.** *Let X be a log-regular log-scheme. The the pullback functor $\mathcal{P}(X_{\mathrm{vksyn}}) \to \mathcal{P}(X_{\mathrm{vkfl}})$ is an equivalence of categories.*

Proof. Let $\mathcal{E}$ be a locally free sheaf on X_{vkfl}. Denote by $\mathcal{E}'$ its restriction to X_{vksyn}. It is a sheaf. We claim that $\mathcal{E}'$ is actually a locally free sheaf and that $\varepsilon^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}$, where $\varepsilon : X_{\mathrm{vkfl}} \to X_{\mathrm{vksyn}}$ is the natural map. By Corollary 2.16 and Lemma 2.25, $\mathcal{E}$ can be trivialized by a covering of the form $U \to T \to Y \to X$, where $U \to T$ is a (classical) flat covering, $T \to X$ is a Kummer log-syntomic covering, and $Y \to X$ is a log-blow-up. The restriction of $\mathcal{E}$ to T, $\mathcal{E}|T$, comes from flat topology hence by faithfully flat descent from a Zariski locally free sheaf. This allows us to show that $(\varepsilon^*\mathcal{E}')|T \xrightarrow{\sim} \mathcal{E}|T$, as wanted. □

Basically the same argument gives the following

syn=fl **Lemma 3.11.** *For any Noetherian log-scheme X, the pullback functors*

$$\mathcal{P}(X_{\mathrm{ksyn}}) \to \mathcal{P}(X_{\mathrm{kfl}}), \qquad \mathcal{M}(X_{\mathrm{ksyn}}) \to \mathcal{M}(X_{\mathrm{kfl}})$$

are equivalences of categories.

3.1. **Invertible sheaves.** We will compute now the groups $H^1(X_*, \mathbf{G}_m)$ of isomorphism classes of invertible sheaves for X local and equipped with one of the Kummer topologies. The main ideas here are due to Kato [24]. Let X be a log-scheme. We have the following Kummer exact sequences on X_{kfl}, respectively $X_{\mathrm{két}}$,

$$0 \to \mathbf{Z}/n(1) \to \mathbf{G}_m^{\times} \xrightarrow{n} \mathbf{G}_m^{\times} \to 0,$$
$$0 \to \mathbf{Z}/n(1) \to \mathbf{G}_m^{\times} \xrightarrow{n} \mathbf{G}_m^{\times} \to 0,$$

for any nonzero integer n, respectively for any integer n which is invertible on X. Here $\mathbf{Z}/n(1)$ is by definition the kernel of the multiplication by n on the multiplicative group $\mathbf{G}_m$.

The following theorem was basically proved by Kato in [24, Theorem 4.1]. We supplied the missing arguments.

kaa **Theorem 3.12.** *Let X be a log-scheme and assume X to be locally Noetherian. Let $\varepsilon : X_{\mathrm{kfl}} \to X_{\mathrm{fl}}$ be the canonical map. Let G be a commutative group scheme over the underlying scheme of X satisfying one of the following two conditions*

1. *G is finite flat over the underlying scheme of X;*
2. *G is smooth and affine over the underlying scheme of X.*

We endow G with the inverse image of the log-structure of X. Then we have a canonical isomorphism

$$R^1\varepsilon_*G \simeq \operatorname*{inj\,lim}_{n\neq 0} \mathcal{H}\mathrm{om}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} (\mathbf{G}_m^{\times}/\mathbf{G}_m).$$

Proof. Let X be a log-scheme and let G be a sheaf of abelian groups on X_{kfl}. Define a canonical homomorphism of sheaves on X_{kfl}

$$\mu : \qquad \operatorname*{inj\,lim}_{n\neq 0} \mathcal{H}\mathrm{om}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} (\mathbf{G}_m^{\times}/\mathbf{G}_m) \to R^1\varepsilon_*G$$

as follows. Let h be a local section of $\mathcal{H}\mathrm{om}(\mathbf{Z}/n(1), G)$. The Kummer exact sequence on X_{kfl}

$$0 \to \mathbf{Z}/n(1) \to \mathbf{G}_m^{\times} \xrightarrow{n} \mathbf{G}_m^{\times} \to 0$$

yields the composition

$$\mathbf{G}_m^{\times} = \varepsilon_*\mathbf{G}_m^{\times} \xrightarrow{\partial} R^1\varepsilon_*(\mathbf{Z}/n(1)) \xrightarrow{h} R^1\varepsilon_*G,$$

where ∂ is the connecting morphism. Since multiplication by n on $\mathbf{G}_m$ on the site X_{fl} is surjective, the map ∂ kills $\mathbf{G}_m$. That gives us the definition of the map μ.

It is easy to see now that the first case of the theorem follows from the second. Indeed, if G is a finite flat commutative group scheme on X we can take its (see [26, A.5]) smooth resolution, i.e., an exact sequence of sheaves on X_{fl}

$$0 \to G \to L \to L' \to 0,$$

where both L and L' are smooth and affine group schemes over the underlying scheme of X. We endow both L and L' with the inverse image log-structure. By applying the pushforward ε_* to the above exact sequence and using the fact that $L = \varepsilon_*L \to L' = \varepsilon_*L'$ is surjective on X_{fl} we get an exact sequence

$$0 \to R^1\varepsilon_*G \to R^1\varepsilon_*L \to R^1\varepsilon_*L'.$$

Hence bijectivity of the map μ for G is reduced to the bijectivity of this map for L and L'.

It suffices now to prove the following proposition

invertible **Proposition 3.13.** *Assume $X = \mathrm{Spec}(A)$, where A is strictly local, and assume that G is represented by a smooth commutative group scheme over X endowed with the induced log-structure. Assume that $P \xrightarrow{\sim} (M_X/\mathcal{O}_X^*)_x$, where x is the closed point of X. Then the map*

$$\operatorname*{inj\,lim}_{n} \mathrm{Hom}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} P^{\mathrm{gp}} \xrightarrow{\mu} H^1(X_{\mathrm{kfl}}, G),$$

is an isomorphism.

Proof. For $n \geq 1$, consider $X_n = X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P^{1/n}]$, with the induced log-structure. Here $P^{1/n}$ is a P-monoid such that $P \to P^{1/n}$ is isomorphic to $n : P \to P$. The map $X_n \to X$ is a covering in X_{kfl}. Denote by $X_{n,i}$ the fiber product of $i+1$ copies of X_n over X. For any sheaf of abelian groups G on X_{kfl}, we have a Čech complex

$$C^{\cdot}_{G,n} : \qquad \Gamma(X_{n,0}, G) \to \Gamma(X_{n,1}, G) \to \Gamma(X_{n,2}, G) \to \dots$$

Assume that A is Noetherian and complete. Then our proposition is proved in two steps via the following two lemmas

39 **Lemma 3.14.** *Assume $X = \mathrm{Spec}(A)$, where A is strictly local, and assume that G is represented by a smooth commutative group scheme over X endowed with the induced log-structure. Then*

$$\operatorname*{inj\,lim}_n H^1(C^{\cdot}_{G,n}) \xrightarrow{\sim} H^1(X_{\mathrm{kfl}}, G).$$

Proof. From Čech cohomology we know that the map $\operatorname{inj\,lim}_n H^1(C_{G,n}) \to H^1(X_{\mathrm{kfl}}, G)$ is injective and its cokernel injects into $\operatorname{inj\,lim}_n H^1((X_n)_{\mathrm{kfl}}, G)$. Hence it suffices to show that $\operatorname{inj\,lim}_n H^1((X_n)_{\mathrm{kfl}}, G) = 0$. Take an element α of $H^1((X_n)_{\mathrm{kfl}}, G)$. Let $T \to X_n$ be a log-flat Kummer covering such that α dies in $H^1(T_{\mathrm{kfl}}, G)$. By Corollary 2.16, we may assume that for some m, we have a factorization $T \to X_{mn} \to X_n$, where $T \to X_{mn}$ is a classically flat covering. It follows that the class α on X_{mn} is trivialised by a classically flat cover. Thus the class of α in $H^1((X_{mn})_{\mathrm{kfl}}, G)$ comes from $H^1((X_{mn})_{\mathrm{fl}}, G)$. But the group scheme G being smooth, $H^1((X_{mn})_{\mathrm{fl}}, G) \simeq H^1((X_{mn})_{\text{ét}}, G)$. Finally, since X_{mn} is a disjoint union of a finite number of Spec of strictly local rings, we have $H^1((X_{mn})_{\text{ét}}, G) = 0$, as wanted. □

Before stating the second lemma, we would like to show that the composition of the structure map $P^{\mathrm{gp}} \to H^0(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times})$ with the map $\mu_h : H^0(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times}) \to H^1(X_{\mathrm{kfl}}, G)$ induced by a section $h \in \mathrm{Hom}(\mathbf{Z}/n(1), G)$ factors through $H^1(C^{\cdot}_{G,n})$. For that, consider the classical commutative group scheme $H_n = \mathrm{Spec}(\mathbf{Z}[P^{\mathrm{gp}}/(P^{\mathrm{gp}})^n])$ over $\mathrm{Spec}(\mathbf{Z})$. It defines the sheaf $T \mapsto \mathrm{Hom}(P^{\mathrm{gp}}/(P^{\mathrm{gp}})^n, \Gamma(T, \mathbf{Z}/n(1)))$ on X_{kfl}. The group scheme H_n acts on X_n over X, and we have $H_n \times_{\mathbf{Z}} X_n \simeq X_n \times_X X_n$. Hence, $X_{n,i} \simeq (H_n)^{\times i} \times_{\mathbf{Z}} X_n$. For a sheaf of abelian groups G on X_{kfl}, let G_n be the sheaf of abelian groups on X_{kfl} defined by $G_n(T) = \Gamma(T \times_X X_n, G)$. The sheaf H_n acts on G_n. The Čech complex $C^{\cdot}_{G,n}$ can now be written as

$$C^{\cdot}_{G,n}: \quad \mathrm{Mor}(1, G_n) \xrightarrow{\partial_0} \mathrm{Mor}(H_n, G_n) \xrightarrow{\partial_1} \mathrm{Mor}(H_n^{\times 2}, G_n) \xrightarrow{\partial_2} \ldots,$$

where Mor refers to morphisms of sheaves of sets, and

$$\partial_0(x) = (\sigma \mapsto \sigma x - x), \quad \partial_1(x) = ((\sigma, \tau) \mapsto \sigma x(\tau) - x(\sigma\tau) + x(\sigma)), \ldots$$

Note that the above complex is the standard complex that computes the cohomology of the H_n-module G_n (see [6, II.3]).

Consider now G with the trivial action of H_n. Note that

$$H^1(H_n, G) = \mathrm{Hom}(H_n, G) = \mathrm{Hom}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} P^{\mathrm{gp}}.$$

It can be easily checked that the map

$$\mathrm{Hom}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} P^{\mathrm{gp}} \simeq H^1(H_n, G) \to H^1(H_n, G_n) \simeq H^1(C^{\cdot}_{G,n}) \to H^1(X_{\mathrm{kfl}}, G)$$

maps $h \otimes a$ to the image of a under the above composition. We can now state the second lemma.

l2 **Lemma 3.15.** *Assume $X = \mathrm{Spec}(A)$, where A is a Noetherian complete local ring with separably closed residue field, and assume that G is represented by a smooth commutative group scheme over X endowed with the induced log-structure. Assume that $P \xrightarrow{\sim} (M_X/\mathcal{O}_X^*)_x$, where x is the closed point of X. Then, for any $n \neq 0$,*

$$\mu: \quad \mathrm{Hom}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} P^{\mathrm{gp}} \xrightarrow{\sim} H^1(C^{\cdot}_{G,n}).$$

Proof. Let's treat first the case when A is Artinian. Let I (resp. J) be the ideal of A (resp. $\mathcal{O}_{X_n}$) generated by the image of $P \setminus \{1\}$ (resp. $P^{1/n} \setminus \{1\}$). Then I (resp. J) is a nilpotent ideal. Define a descending filtration G^i on the H_n-module G and G^i_n on the H_n-module G_n by

$$G^i(T) = \ker(G(T) \to G(T \times_X \mathrm{Spec}(\mathcal{O}_X/I^i))); \qquad G^i_n(T) = \ker(G_n(T) \to G(T \times_X \mathrm{Spec}(\mathcal{O}_{X_n}/J^i))).$$

Since I and J are nilpotent, we have that $G^i(T) = G^i_n(T) = 0$ for a large enough i. Since the group scheme G is smooth, for $i \geq 1$ we get

$$\mathrm{gr}^i(G)(T) \simeq \mathit{Lie}(G) \otimes_A \Gamma(T, I^i\mathcal{O}_T/I^{i+1}\mathcal{O}_T), \quad \mathrm{gr}^i(G_n)(T) \simeq \mathit{Lie}(G) \otimes_A \Gamma(T, J^i\mathcal{O}_T/J^{i+1}\mathcal{O}_T).$$

Also, since $\mathcal{O}_X/I \xrightarrow{\sim} \mathcal{O}_{X_n}/J$, we have that $\mathrm{gr}^0(G)(T) \xrightarrow{\sim} \mathrm{gr}^0(G_n)(T)$. We will prove now the following lemma

vanish **Lemma 3.16.** *For any $i \geq 1$ and any $m \geq 1$, the groups $H^m(H_n, \mathrm{gr}^i(G))$ and $H^m(H_n, \mathrm{gr}^i(G_n))$ are zero.*

Proof. Let $i \geq 1$ and consider the standard complex $C^{\cdot}(H_n, \mathrm{gr}^i(G))$ that computes the cohomology of the H_n-module $\mathrm{gr}^i(G)$. Then for $m \geq 0$, since H_n is flat over $\mathbf{Z}$ and G is smooth over X, for a certain number k we have

$$\begin{aligned} C^m(H_n, \mathrm{gr}^i(G)) &= \mathrm{Mor}(H_n^{\times m}, \mathrm{gr}^i(G)) = \mathrm{gr}^i(G)(H_n^{\times m} \times_{\mathbf{Z}} X) = \mathit{Lie}(G) \otimes_A \Gamma(H_n^{\times m} \times_{\mathbf{Z}} X, I^i\mathcal{O}/I^{i+1}\mathcal{O}) \\ &= \mathit{Lie}(G) \otimes_A \mathcal{O}_{H_n^{\times m} \times_{\mathbf{Z}} X} \otimes_A I^i/I^{i+1} = \mathbf{G}_a^k(H_n^{\times m} \times_{\mathbf{Z}} X) \otimes_A I^i/I^{i+1} \\ &= \mathrm{Mor}(H_n^{\times m}, \mathbf{G}_a^k) \otimes_A I^i/I^{i+1} = C^m(H_n, \mathbf{G}_a^k) \otimes_A I^i/I^{i+1}. \end{aligned}$$

Similarly, for the standard complex $C^{\cdot}(H_n, \mathrm{gr}^i(G_n))$ that computes the cohomology of the H_n-module $\mathrm{gr}^i(G_n)$, we get

$$C^m(H_n, \mathrm{gr}^i(G_n)) = \mathrm{Mor}(H_n^{\times m}, \mathrm{gr}^i(G_n)) = \mathrm{Mor}(H_n^{\times m}, \mathbf{G}_a^k) \otimes_A J^i/J^{i+1} = C^m(H_n, \mathbf{G}_a^k) \otimes_A J^i/J^{i+1},$$

Since H_n is diagonalizable and it acts trivially on $\mathbf{G}_a^k$, we know that $H^m(H_n, \mathbf{G}_a^k) = 0$ for $m \geq 1$ [33, Exp.I, Theorem 5.3.3]. Moreover, $\mathbf{G}_a^k$ embeds into $\mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k)$ with an H_n-equivariant section [33, Exp.I, Prop. 4.7.4]. Hence

$$\mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k) \simeq \mathbf{G}_a^k \oplus \mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k)/\mathbf{G}_a^k$$

as H_n-modules. That gives us that

$$C^{\cdot}(H_n, \mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k)) \simeq C^{\cdot}(H_n, \mathbf{G}_a^k) \oplus C^{\cdot}(H_n, \mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k)/\mathbf{G}_a^k).$$

Now, $C^{\cdot}(H_n, \mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k))$ has an A-linear contracting homotopy [33, Exp.I, Lemma 5.2.]. It follows that $C^{\cdot}(H_n, \mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k)) \otimes_A I^i/I^{i+1}$ also has a contracting homotopy. Hence $H^m(C^{\cdot}(H_n, \mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k)) \otimes_A I^i/I^{i+1}) = 0$, for $m \geq 1$, and by the above splitting $H^m(C^{\cdot}(H_n, \mathbf{G}_a^k) \otimes_A I^i/I^{i+1}) = 0$, as wanted.

Similarly, $H^m(H_n, \mathrm{gr}^i(G_n))) = H^m(C^{\cdot}(H_n, \mathbf{G}_a^k) \otimes_A J^i/J^{i+1}) = 0$, for $m \geq 1$. □

Using the above lemma, we get

$$\mathrm{Hom}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} P^{\mathrm{gp}} = H^1(H_n, G) \xrightarrow{\sim} H^1(H_n, \mathrm{gr}^0(G)) \xrightarrow{\sim} H^1(H_n, \mathrm{gr}^0(G_n)) \xleftarrow{\sim} H^1(H_n, G_n) = H^1(C^{\cdot}_{G,n}),$$

as wanted.

Let's turn now to the general case of A complete. We will basically "go to the limit over the argument for A Artinian". Denote the maximal ideal of A by m_A. Note that $G(A) \xrightarrow{\sim} \mathrm{proj\,lim}_i\, G(A/\mathrm{m}_A^i)$ and $G(X_{n,k}) \xrightarrow{\sim} \mathrm{proj\,lim}_i\, G(X_{n,k} \otimes_A A/\mathrm{m}_A^i)$. Moreover, since G is smooth, we have that the maps

$$G(A/\mathrm{m}_A^{i+1}) \to G(A/\mathrm{m}_A^i), \qquad G(X_{n,k} \otimes_A A/\mathrm{m}_A^{i+1}) \to G(X_{n,k} \otimes_A A/\mathrm{m}_A^i)$$

are surjective. Hence we get the following exact sequences

$$\begin{aligned} 0 \to & G(A) \to G(X_{n,0}) \to D \to 0, \\ 0 \to & E \to G(X_{n,1}) \to G(X_{n,2}), \end{aligned}$$

where $E = \mathrm{proj\,lim}_i\, E_i$ and $D = \mathrm{proj\,lim}_i\, D_i$, and E_i and D_i are defined by the following exact sequences

$$\begin{aligned} 0 \to & G(A/\mathrm{m}_A^i) \to G(X_{n,0} \otimes_A A/\mathrm{m}_A^i) \to D_i \to 0, \\ 0 \to & E_i \to G(X_{n,1} \otimes_A A/\mathrm{m}_A^i) \to G(X_{n,2} \otimes_A A/\mathrm{m}_A^i). \end{aligned}$$

We have $D_i \subset E_i$ and E_i/D_i is H^1 of the complex $C^{\cdot}_{G,n}$ for $\mathrm{Spec}(A/\mathrm{m}_A^i)$. Also $E/D \simeq H^1(C^{\cdot}_{G,n})$ and, since the maps $D_{i+1} \to D_i$ are surjective, $E/D \simeq \mathrm{proj\,lim}_i(E_i/D_i)$. On the other hand, let $\mathrm{Hom}(\mathbf{Z}/n(1), G)_i$ denote the group $\mathrm{Hom}(\mathbf{Z}/n(1), G)$ over $\mathrm{Spec}(A/\mathrm{m}_A^i)$. Since $\mathcal{H}\mathrm{om}(\mathbf{Z}/n(1), G)$ is representable by an étale scheme [1, Exp.XI, Prop. 3.12], [2, Exp.XV, Prop.16], we have $\mathrm{Hom}(\mathbf{Z}/n(1), G) \simeq \mathrm{Hom}(\mathbf{Z}/n(1), G)_i$, $i \geq 1$. The proof of our lemma for A Artinian gives that

$$\mathrm{Hom}(\mathbf{Z}/n(1), G)_i \otimes_{\mathbf{Z}} P^{\mathrm{gp}} \xrightarrow{\sim} E_i/D_i.$$

Hence taking limits

$$\mathrm{Hom}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} P^{\mathrm{gp}} \xrightarrow{\sim} E/D \simeq H^1(C^{\cdot}_{G,n}),$$

as wanted. □

In the general case we have to argue a little bit more. Let $\widehat{X} = \mathrm{Spec}(\widehat{A})$, where $\widehat{A}$ is the completion of A. Endow $\widehat{X}$ with the inverse image log-structure. Since $\mathcal{H}\mathrm{om}(\mathbf{Z}/n(1), G)$ is represented by an étale scheme and the morphism $A \to \widehat{A}$ is a covering for the fpqc topology, $\mathrm{Hom}(\mathbf{Z}/n(1), G)$ does not change when we pass to the completion. It suffice thus to show that $H^1(X_{\mathrm{kfl}}, G) \to H^1(\widehat{X}_{\mathrm{kfl}}, G)$ is injective. Let $\alpha \in H^1(X_{\mathrm{kfl}}, G)$ be a class that dies in $H^1(\widehat{X}_{\mathrm{kfl}}, G)$. By fpqc descent, α is a class of a representable smooth affine G-torsor Y over X (equipped with the inverse image log-structure). Since X is strictly local, Y has an X-rational point. Hence $\alpha = 0$. □

□

picard **Corollary 3.17.** *Let $X = \mathrm{Spec}(A)$ be a log-scheme such that A is Noetherian and strictly local. We have the following isomorphisms*

$$H^1(X_{\mathrm{kfl}}, \mathbf{G}_m) \simeq (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z}), \quad H^1(X_{\mathrm{k\acute{e}t}}, \mathbf{G}_m) \simeq (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})',$$

where x denotes the closed point of X and $(\mathbf{Q}/\mathbf{Z})' = \oplus_{l \neq char(x)} \mathbf{Q}_l/\mathbf{Z}_l$.

Proof. The case of X_{kfl} follows from Proposition 3.13. Inspecting its proof we see that together with Corollary 2.17 it actually proves the statement for $H^1(X_{\mathrm{k\acute{e}t}}, \mathbf{G}_m)$ as well. □

picardZar **Corollary 3.18.** *Let $X = \mathrm{Spec}(A)$ be a log-scheme equipped with a Zariski log-structure such that A is Noetherian and local. We have the following isomorphisms*

$$H^1(X_{\mathrm{kfl}}, \mathbf{G}_m) \simeq (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z}), \quad H^1(X_{\mathrm{k\acute{e}t}}, \mathbf{G}_m) \simeq (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})'.$$

Proof. The proof of Proposition 3.13 goes through with few small changes. In Lemma 3.14 we have to use the fact that X_{mn} is a product of a finite number of Spec of local rings and we have $H^1((X_{mn})_{\mathrm{\acute{e}t}}, \mathbf{G}_m) = H^1((X_{mn})_{\mathrm{Zar}}, \mathbf{G}_m) = 0$. Similarly, at the very end of the proof of the proposition we get that, since X is local, and Y is a $\mathbf{G}_m$-torsor, it has a rational point. □

example **Example 3.19.** We can obtain invertible sheaves on the Kummer log-flat site in the following way. Take a log-scheme X with a chart $P \to M_X$. Consider the covering $Y = X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ associated to a Kummer map $P \to Q$. The $\mathcal{O}_X$-module $f_*\mathcal{O}_Y$ on X_{kfl}, $f : Y \to X$, has an action of the group scheme $H = \mathrm{Spec}(\mathbf{Z}[Q^{\mathrm{gp}}/P^{\mathrm{gp}}])$. It decomposes under this action into a direct sum of invertible sheaves $f_*\mathcal{O}_Y \simeq \oplus_a \mathcal{O}_X(a)$, $a \in Q^{\mathrm{gp}}/P^{\mathrm{gp}}$. Here $\mathcal{O}_X(a)$ is the part of $f_*\mathcal{O}_Y$ on which H acts via the character $H \to \mathbf{G}_m$ corresponding to a. More specifically,

$$f_*\mathcal{O}_Y(Y) \simeq \mathcal{O}_{Y\times_X Y} \simeq \mathcal{O}_X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q \oplus Q^{\mathrm{gp}}/P^{\mathrm{gp}}] = \oplus_{a \in Q^{\mathrm{gp}}/P^{\mathrm{gp}}} a\mathcal{O}_Y$$

and $\mathcal{O}_X(a)|Y_{\mathrm{kfl}} = \varepsilon^* a\mathcal{O}_Y$. The element of $H^1(X_{\mathrm{kfl}}, \mathbf{G}_m)$ corresponding to the invertible sheaf $\mathcal{O}_X(a)$ is given by the image of a^m under

$$H^0(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times}) \to H^1(X_{\mathrm{kfl}}, \mathbf{Z}/m(1)) \to H^1(X_{\mathrm{kfl}}, \mathbf{G}_m),$$

where the first arrow is the connecting map of the Kummer sequence

$$0 \to \mathbf{Z}/m(1) \to \mathbf{G}_m^{\times} \xrightarrow{m} \mathbf{G}_m^{\times} \to 0$$

Here m is a number such that $a^m \in P^{\mathrm{gp}}$ and the above image is independent of m chosen. If X and x are as in the above corollary then this element corresponds to $a \otimes m^{-1}$ of $(M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})$.

To get nontrivial Kummer log-flat coherent sheaves note that, for $a \in Q$ and for the natural map $\alpha : Q \to M_Y$, the element $a \otimes \alpha(a) \in f_*\mathcal{O}_Y(Y)$ is a global section of $\mathcal{O}_X(a)$. Define $\mathcal{O}_X\{a\}$ to be the image of the map $\alpha(a) : \mathcal{O}_X \to \mathcal{O}_X(a)$.

As the next corollary we get the following log-version of Hilbert 90.

Theorem 3.20. *(Hilbert 90) Let X be a log-scheme whose underlying scheme is locally Noetherian. Then the canonical maps*

$$H^1(X_{\mathrm{fl}}, \mathbf{G}_m^{\times}) \xrightarrow{\sim} H^1(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times}), \qquad H^1(X_{\mathrm{\acute{e}t}}, \mathbf{G}_m^{\times}) \xrightarrow{\sim} H^1(X_{\mathrm{k\acute{e}t}}, \mathbf{G}_m^{\times})$$

are isomorphisms.

Proof. We have the short exact sequence

$$0 \to \mathbf{G}_m \to \mathbf{G}_m^{\times} \to \mathbf{G}_m^{\times}/\mathbf{G}_m \to 0$$

of sheaves on X_{fl}. For $X = \mathrm{Spec}(A)$, where A is a Noetherian strictly local ring, this yields $H^1(X_{\mathrm{fl}}, \mathbf{G}_m^{\times}) = 0$. Indeed, we have $H^1(X_{\mathrm{fl}}, \mathbf{G}_m) = 0$. And, since $\mathbf{G}_m^{\times}/\mathbf{G}_m = \varepsilon^*(\mathbf{G}_m^{\times}/\mathbf{G}_m)$, where $\varepsilon : X_{\mathrm{fl}} \to X_{\text{ét}}$ is the natural map, $H^1(X_{\mathrm{fl}}, \mathbf{G}_m^{\times}/\mathbf{G}_m) = H^1(X_{\text{ét}}, \mathbf{G}_m^{\times}/\mathbf{G}_m) = 0$ (cf., [25, II.3]).

The above implies that this theorem is equivalent to the following local form. □

cor **Corollary 3.21.** *Let X be a log-scheme whose underlying scheme is* Spec *of a Noetherian strictly local ring. Then the groups $H^1(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times})$, $H^1(X_{\text{két}}, \mathbf{G}_m^{\times})$ and $H^1(X_{\text{ét}}, \mathbf{G}_m^{\times})$ are zero.*

Proof. Let $X = \mathrm{Spec}(A)$, where A is a Noetherian strictly local ring. Assume that $P \stackrel{\sim}{\to} (M_X/\mathcal{O}_X^*)_x$, where x is the closed point of X. We will show that $H^1(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times}) = 0$ (the proof for the Kummer log-étale site is almost the same and the case of the étale site is obvious). From Čech cohomology we know that the map $\mathrm{inj\,lim}_n \check{H}^1(X_n/X, \mathbf{G}_m^{\times}) \to H^1(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times})$ is injective and its cokernel injects into $\mathrm{inj\,lim}_n H^1((X_n)_{\mathrm{kfl}}, \mathbf{G}_m^{\times})$. Hence it suffices to show that $\mathrm{inj\,lim}_n H^1((X_n)_{\mathrm{kfl}}, \mathbf{G}_m^{\times}) = 0$ and $\check{H}^1(X_n/X, \mathbf{G}_m^{\times}) = 0$. Here the covering $X_n = X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$, $Q = P^{1/n}$.

First, let's show that $\mathrm{inj\,lim}_n H^1((X_n)_{\mathrm{kfl}}, \mathbf{G}_m^{\times}) = 0$. Take an element α of $H^1((X_n)_{\mathrm{kfl}}, \mathbf{G}_m^{\times})$. Let $T \to X_n$ be a log-flat Kummer covering such that α comes from $\check{H}^1(T/X_n, \mathbf{G}_m^{\times})$. By Corollary 2.16, for some m, we may assume that we have a factorization $T \to X_{mn} \to X_n$, where $T \to X_{mn}$ is classically flat and surjective. It follows that the class α on X_{mn} comes from $\check{H}^1(T \times_{X_n} X_{mn}/X_{mn}, \mathbf{G}_m^{\times})$. Thus the class of α in $H^1((X_{mn})_{\mathrm{kfl}}, \mathbf{G}_m^{\times})$ comes from $H^1((X_{mn})_{\mathrm{fl}}, \mathbf{G}_m^{\times})$. Since X_{mn} is a disjoint union of a finite number of Spec of strictly local rings, the last group is trivial as we have shown above.

Now, let's show that $\check{H}^1(X_n/X, \mathbf{G}_m^{\times}) = 0$. Consider the exact sequence of presheaves (!) on X_{kfl}

$$0 \to \mathbf{G}_m \to \mathbf{G}_m^{\times} \to \mathbf{G}_m^{\times}/\mathbf{G}_m \to 0.$$

It gives us the exact sequence of Čech cohomology groups

$$\to \check{H}^0(X_n/X, \mathbf{G}_m^{\times}/\mathbf{G}_m) \stackrel{\partial}{\to} \check{H}^1(X_n/X, \mathbf{G}_m) \to \check{H}^1(X_n/X, \mathbf{G}_m^{\times}) \to \check{H}^1(X_n/X, \mathbf{G}_m^{\times}/\mathbf{G}_m) \to$$

By Proposition 3.13 the connecting morphism ∂ is surjective. Indeed, consider an element $a \otimes n^{-1} \in \check{H}^1(X_n/X, \mathbf{G}_m) \simeq P^{\mathrm{gp}} \otimes \mathbf{Z}/n$, $a \in P^{\mathrm{gp}}$. Choose an element $b \in Q^{gp}$ such that $b^n = a$. It belongs to $\check{H}^0(X_n/X, \mathbf{G}_m^{\times}/\mathbf{G}_m)$. To see that recall that the exact sequence of the covering X_n/X

$$0 \to \Gamma(X, \mathcal{O}_X) \to \Gamma(X_n, \mathcal{O}_{X_n}) \to \Gamma(X_n \times_X X_n, \mathcal{O}_{X_n \times_X X_n})$$

is isomorphic to

$$0 \to A \to A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q] \stackrel{\beta_1 - \beta_2}{\longrightarrow} A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q \oplus Q^{\mathrm{gp}}/P^{\mathrm{gp}}],$$

where $\beta_1(x) = 1 \otimes x$, $x \in Q$, $\beta_2(x) = 1 \otimes (x, x \mod P^{\mathrm{gp}})$. Hence

$$(\beta_1^* - \beta_2^*)(b) = 1 \otimes b - 1 \otimes (b, b \mod P^{\mathrm{gp}}) = 1 \otimes b - 1 \otimes b = 0$$

and $b \in \check{H}^0(X_n/X, \mathbf{G}_m^{\times}/\mathbf{G}_m)$, as wanted. One easily now checks that $\partial(b) = a \otimes n^{-1}$.

It remains to show that $\check{H}^1(X_n/X, \mathbf{G}_m^{\times}/\mathbf{G}_m) = 0$. Or that the sequence

$$\mathbf{G}_m^{\times}/\mathbf{G}_m(X_n) \stackrel{d_0}{\to} \mathbf{G}_m^{\times}/\mathbf{G}_m(X_n \times_X X_n) \stackrel{d_1}{\to} \mathbf{G}_m^{\times}/\mathbf{G}_m(X_n \times_X X_n \times_X X_n)$$

is exact. By Lemma 3.28 this sequence is isomorphic to

$$Q^{gp} \stackrel{d_0}{\to} Q^{gp} \stackrel{d_1}{\to} Q^{gp},$$

where $d_0 = 0$ and $d_1 = 1$. Hence it is exact, as wanted. □

3.2. **Locally free sheaves of higher rank.** For isomorphism classes of locally free sheaves of arbitrary rank we have the following theorem stated already by Kato [24, Cor. 6.4].

gl **Theorem 3.22.** *Let $X = \operatorname{Spec}(A)$ be a log-scheme such that A is Noetherian and strictly local. The map*

$$\prod^n \check{H}^1(X_{\mathrm{kfl}}, \mathbf{G}_m) \to \check{H}^1(X_{\mathrm{kfl}}, \mathbf{GL}_n)$$

given by the diagonal embedding $\prod^n \mathbf{G}_m \hookrightarrow \mathbf{GL}_n$ induces an isomorphism

$$\check{H}^1(X_{\mathrm{kfl}}, \mathbf{GL}_n) \simeq S_n \backslash (\prod^n (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})),$$

where $S_n\backslash$ denotes the quotient by the action of the symmetric group of degree n on the product of n copies. Similarly, we have an isomorphism

$$\check{H}^1(X_{\mathrm{k\acute{e}t}}, \mathbf{GL}_n) \simeq S_n \backslash (\prod^n (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})').$$

Proof. Assume that $P \xrightarrow{\sim} (M_X/\mathcal{O}_X^*)_x$. For $m \geq 1$, let $X_m = X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P^{1/m}]$, with the induced log-structure.

40 **Lemma 3.23.** *We have*

$$\operatorname{inj}\lim_m \check{H}^1(X_m/X, \mathbf{GL}_n) \xrightarrow{\sim} \check{H}^1(X_{\mathrm{kfl}}, \mathbf{GL}_n).$$

Proof. The injectivity is obvious. For the surjectivity, consider a class $\alpha \in \check{H}^1(X_{\mathrm{kfl}}, \mathbf{GL}_n)$. Let $T \to X$ be a log-flat Kummer covering such that $\alpha \in \check{H}^1(T/X, \mathbf{GL}_n)$. By Corollary 2.16, we may assume that for some m, we have a factorization $T \to X_m \to X$, where $T \to X_m$ is classically flat and surjective. Since X_m is a disjoint union of a finite number of Spec of strictly local rings we have $\check{H}^1(X_{m,\mathrm{fl}}, \mathbf{GL}_n) = 0$. It follows that α is trivialised on X_m hence $\alpha \in \check{H}^1(X_m/X, \mathbf{GL}_n)$, as wanted. □

Lemma 3.24. *Assume $X = \operatorname{Spec}(A)$, where A is a Noetherian complete local ring with separably closed residue field. Then, for any $n \neq 0$,*

$$(\operatorname{Hom}(H_m, \mathbf{GL}_n)/\equiv) \xrightarrow{\sim} \check{H}^1(X_m/X, \mathbf{GL}_n),$$

where H_m is the group scheme $\operatorname{Spec}(\mathbf{Z}[P^{\mathrm{gp}}/(P^{\mathrm{gp}})^m])$ and $/\equiv$ means the quotient set by the inner conjugation by elements of $\mathbf{GL}_n(A)$.

Proof. We proceed as in the proof of Lemma 3.15 and keep its notation. Note that

$$(\operatorname{Hom}(H_m, \mathbf{GL}_n)/\equiv) = H^1(H_m, \mathbf{GL}_n)$$

Let's treat first the case when A is Artinian. Consider the corresponding filtrations $\mathbf{GL}_n^i$, $\mathbf{GL}_{n,m}^i$ of $\mathbf{GL}_n$ and $\mathbf{GL}_{n,m}$. The computation of the graded pieces goes through and, since $Lie(\mathbf{GL}_n) \simeq \mathbf{G}_a^{n^2}$, so does the proof of Lemma 3.16. Hence

$$H^k(H_n, \mathrm{gr}^i(\mathbf{GL}_n)) = H^k(H_n, \mathrm{gr}^i(\mathbf{GL}_{n,m})) = 0, \quad i \geq 1, k \geq 1.$$

Using now the exact sequences

$$0 \to \mathbf{GL}_n^{i-1}/\mathbf{GL}_n^i \to \mathbf{GL}_n/\mathbf{GL}_n^i \to \mathbf{GL}_n/\mathbf{GL}_n^{i-1} \to 0$$

(starting from i such that $\mathbf{GL}_n^i = 0$) we get that $H^1(H_n, \mathbf{GL}_n) \xrightarrow{\sim} H^1(H_n, \mathrm{gr}^0(\mathbf{GL}_n))$. Similarly, $H^1(H_n, \mathbf{GL}_{n,m}) \xrightarrow{\sim} H^1(H_n, \mathrm{gr}^0(\mathbf{GL}_{n,m}))$. Since $\mathrm{gr}^0(\mathbf{GL}_n) \xrightarrow{\sim} \mathrm{gr}^0(\mathbf{GL}_{n,m})$, we are done.

Let's turn now to the general case of A complete. We compute

$$\begin{aligned}(\operatorname{Hom}(H_m, \mathbf{GL}_n)/\equiv) &= S_n \backslash \operatorname{Hom}(H_m, \prod^n \mathbf{G}_m) = S_n \backslash \prod^n \operatorname{Hom}(H_m, \mathbf{G}_m) \\ &= S_n \backslash \prod^n \operatorname{Hom}(\mathbf{Z}/m(1), \mathbf{G}_m) \otimes P^{gp} = S_n \backslash \prod^n \mathbf{Z}/m \otimes P^{\mathrm{gp}}.\end{aligned}$$

The same computation works over each A/m_A^i. Passing now to the limit over i it suffices to show that the natural map

$$\check{H}^1(X_m, \mathbf{GL}_n) \to \operatorname{proj}\lim_i \check{H}^1(X_{m,i}, \mathbf{GL}_n),$$

where $X_{m,i}$ is the base change of X_m to A/m_A^i, is injective. By straightforward computation this follows from the fact that $\mathbf{GL}_n$ defines a sheaf for the Kummer log-flat topology. □

In the general case we have to argue a little bit more. Let $\widehat{X} = \operatorname{Spec}(\widehat{A})$, where $\widehat{A}$ is the completion of A. Endow $\widehat{X}$ with the inverse image log-structure. Since $\operatorname{Hom}(H_m, \mathbf{GL}_n)/\equiv$ does not change when we pass to the completion (see above), it suffice to show that $\check{H}^1(X_{\mathrm{kfl}}, \mathbf{GL}_n) \to \check{H}^1(\widehat{X}_{\mathrm{kfl}}, \mathbf{GL}_n)$ is injective. This is proved exactly like the corresponding fact in the proof of Lemma 3.15.

The proof for $X_{\text{két}}$ is analogous (using Corollary 2.17). □

glZar **Corollary 3.25.** *In the above theorem we may take $X = \operatorname{Spec}(A)$ to be a log-scheme equipped with a Zariski log-structure such that A is Noetherian and local.*

Proof. The proof of Theorem 3.22 goes through with few small changes. In Lemma 3.23 we have to use the fact that X_m is a product of a finite number of Spec of local rings and we have $\check{H}^1((X_m)_{\mathrm{fl}}, \mathbf{GL}_n) = 0$. Similarly, at the very end of the proof of the theorem we get that, since X is local, and Y is a $\mathbf{GL}_n$-torsor, it has a rational point. □

split1 **Corollary 3.26.** *Let $X = \operatorname{Spec}(A)$ be a log-scheme such that A is Noetherian and strictly local. Let $\mathcal{F}$ be a locally free finite type $\mathcal{O}_X$-module on X_{kfl} (resp. $X_{\text{két}}$). Then $\mathcal{F}$ is a direct sum of invertible sheaves on X_{kfl} (resp. $X_{\text{két}}$). Similarly for A local and equipped with a Zariski log-structure.*

The following proposition will be useful in computing K-theory groups. It was originally stated by Kato [24, Prop. 6.5].

cohom **Proposition 3.27.** *Let X be an affine log-scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X_{kfl} such that for some Kummer log-flat covering $Y \to X$ the restriction $\mathcal{F}|Y$ is isomorphic to the inverse image of a quasi-coherent sheaf on the small Zariski site of Y. Then $H^n(X_{\mathrm{kfl}}, \mathcal{F}) = 0$ for any $n \geq 1$. Similar statement holds for $X_{\text{két}}$.*

Proof. Consider the case of X_{kfl}. Assume first that $\mathcal{F}$ is isomorphic to the inverse image of a quasi-coherent sheaf on the small Zariski site of X. Since $H^n(X_{\mathrm{Zar}}, \mathcal{F}) = 0$, $n \geq 1$, we may assume that X is equipped with a chart $P \to M_X$, $P^* = \{1\}$. We may work on the small site of the Kummer log-flat site built from affine maps. It suffices now to show that our sheaf $\mathcal{F}$ is flasque. We will show that for every covering $Y \to X$ from some cofinal system of coverings the Čech cohomology groups $\check{H}^n(Y/X, \mathcal{F}) = H^n(C^{\cdot}(Y/X))$, $n \geq 1$, are trivial. Since our coverings are log-flat and of Kummer type, by Corollary 2.16 we may assume that there exists a factorization of $Y \to X$ into $f : Y \to Y_1$ and $g : Y_1 \to X$, where f is affine, strictly flat and a covering and $Y_1 = Y \times_{\operatorname{Spec}(\mathbf{Z}[P])} \operatorname{Spec}(\mathbf{Z}[Q])$, for a Kummer morphism $u : P \to Q$.

We will show now that the complex $C^{\cdot}(Y/X)$ has trivial cohomology in degrees higher than 0. Assume first that the augmentation $\Gamma(X, \mathcal{F}) \xrightarrow{g^*} C^{\cdot}(Y_1/X)$ is a quasi-isomorphism. We will check that this implies that the augmentation $\Gamma(X, \mathcal{F}) \xrightarrow{(gf)^*} C^{\cdot}(Y/X)$ is a quasi–isomorphism as well. The reader will note that because the schemes Y, Y_1, and X are assumed to be affine, all the schemes appearing in the argument below are affine as well. Consider the double complex

$$C^{\cdot}(Y, Y_1, X) : (i,j) \mapsto \Gamma(Y^{(i+1)} \times_X Y_1^{(j+1)}, \mathcal{F}),$$

where, for any $n \geq 1$, $Y^n = (Y/X)^{\times n}$, and $Y_1^n = (Y_1/X)^{\times n}$. Consider the natural maps $C^{\cdot}(Y/X) \xrightarrow{f_1^*} C^{\cdot}(Y, Y_1, X)$ and $C^{\cdot}(Y_1/X) \xrightarrow{g_1^*} C^{\cdot}(Y, Y_1, X)$. First, we claim that f_1^* is a quasi-isomorphism. For that, it suffices to show that, for any $n \geq 1$, the map $\Gamma(Y^n, \mathcal{F}) \xrightarrow{f_1^*} C^{\cdot}(Y^n \times_X Y_1/Y^n)$ is a quasi-isomorphism. Since the projection $Y^n \times_X Y_1 \to Y^n$ admits a section $Y^n \xrightarrow{s_n} Y^n \times_X Y_1$, this is clear. Next, we will show that g_1^* is a quasi-isomorphism. It suffices to show that the augmentation $\Gamma(Y_1^n,\mathcal{F})\xrightarrow{g_1^*} C^\cdot(Y_1^n\times_X Y/Y_1^n)$ is a quasi-isomorphism. Consider the composition $Y^n \stackrel{f^{n-1}\times s_1}{\to} Y_1^n\times_X Y\to Y_1^n$. It is equal to the map f^n, which is faithfully flat. By faithfully flat descent, since the base-change of the augmentation g_1^* by f^n is a quasi-isomorphism $\Gamma(Y^n,\mathcal{F})\to C^\cdot(Y^{(n+1)}/Y^n)$ (the morphism $Y^{(n+1)}=Y^n\times_{Y_1^n} Y_1^n\times_X Y\to Y^n$ admitting a section), so is the augmentation g_1^*.

Finally, we have that

$$f_1^*(gf)^*=g_1^*g^*:\Gamma(X,\mathcal{F})\to C^\cdot(Y,Y_1,X).$$

Since f_1^*, g_1^*, and g^* are quasi-isomorphisms, so is $(gf)^*$.

descent **Lemma 3.28.** *Let $A=\Gamma(X,\mathcal{O}_X)$. Then the augmentation*

des (3.2) $$A\xrightarrow{e^*} C^\cdot(Y_1/X)$$

is a quasi-isomorphism.

Proof. The essential point is that the morphism of monoids $u:P\to Q$ is exact, i.e., $P=(u^{gp})^{-1}(Q)$ in P^{gp}, where $u^{gp}:P^{gp}\to Q^{gp}$. Set $G=Q^{gp}/P^{gp}$. The augmentation e^* is isomorphic to

$$A\xrightarrow{e^*} A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q]\xrightarrow{d_0} A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus G]\xrightarrow{d_1} A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus G^{\oplus 2}]\xrightarrow{d_2}\dots$$

Here the A-linear morphism $d_n: A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus G^{\oplus n}]\to A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus G^{\oplus n+1}]$ is equal to the alternating sum of maps $\beta_1,\beta_2,\dots,\beta_{n+2}$, where

$$\beta_k(b_1,b_2,\dots,b_{n+1})=\begin{cases}(b_1,b_1b_2^{-1}\cdots b_{n+1}^{-1},b_2,\dots,b_{n+1}) & \text{if } k=1\\ (b_1,b_2,\dots,b_{k-1},1,b_{k+1},\dots,b_{n+1}) & \text{if } k\neq 1,\end{cases}$$

for $b_1\in Q$, $b_2,\dots,b_{n+1}\in G$. Consider now the following A-module homomorphisms $h_{n+1}: A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus G^{\oplus n}]\to A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus G^{\oplus n-1}]$ for $n\geq 1$,

$$h_{n+1}(b_1,b_2,\dots,b_{n+1})=\begin{cases}(-1)^n(b_1,b_2,\dots,b_n) & \text{if } b_{n+1}=1\\ 0 & \text{if } b_{n+1}\neq 1\end{cases}$$

We claim that h_n's together with the morphism $h_1: A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q]\to A$ sending $1\otimes b$ to b if $b\in P$ and to 0 if $b\notin P$ (h_1 is well-defined since u is exact), form a contracting homotopy, i.e., that $h_1e^*=\mathrm{Id}$, $h_2d_0+e^*h_1=\mathrm{Id}$, and $h_{n+2}d_n+d_{n-1}h_{n+1}=\mathrm{Id}$, $n\geq 1$. We compute that

$$\begin{aligned}(h_2d_0+e^*h_1)(b_1) &= \begin{cases}h_2((b_1,1)-(b_1,1))+1\otimes b_1 & \text{if } b_1\in P\\ h_2((b_1,b_1)-(b_1,1)) & \text{if } b_1\notin P\end{cases}\\ &= 1\otimes b_1\end{aligned}$$

(use that u is exact), and that, for $n\geq 1$,

$$\begin{aligned}h_{n+2}d_n(b_1,b_2,\dots,b_{n+1}) &= h_{n+2}[(b_1,b_1b_2^{-1}\cdots b_{n+1}^{-1},b_2,\dots,b_{n+1})-(b_1,1,b_3,\dots,b_{n+1})+\dots\\ &\quad +(-1)^{n+1}(b_1,b_2,b_3,\dots,b_{n+1},1)]\\ &= \begin{cases}(-1)^{n+1}(b_1,b_1b_2^{-1}\cdots b_{n+1}^{-1},b_2,\dots,b_n)-(-1)^{n+1}(b_1,1,b_3,\dots,b_n)+\dots\\ \quad -(b_1,b_2,b_3,\dots,b_n,1)+(b_1,b_2,b_3,\dots,b_{n+1}) & \text{if } b_{n+1}=1\\ (b_1,b_2,b_3,\dots,b_{n+1}) & \text{if } b_{n+1}\neq 1\end{cases}\end{aligned}$$

$$\begin{aligned}d_{n-1}h_{n+1}(b_1,b_2,\dots,b_{n+1}) &= \begin{cases}(-1)^n d_{n-1}(b_1,b_2,\dots,b_n) & \text{if } b_{n+1}=1\\ 0 & \text{if } b_{n+1}\neq 1\end{cases}\\ &= \begin{cases}(-1)^n(b_1,b_1b_2^{-1}\cdots b_n^{-1},b_2,\dots,b_n)-(-1)^n(b_1,1,b_3,\dots,b_n)+\dots\\ \quad +(b_1,b_2,b_3,\dots,b_n,1) & \text{if } b_{n+1}=1\\ 0 & \text{if } b_{n+1}\neq 1\end{cases}\end{aligned}$$

Hence we get that

$$(h_{n+2}d_n + d_{n-1}h_{n+1})(b_1, b_2, \dots, b_{n+1}) = \begin{cases} (b_1, b_2, \dots, b_n, 1) & \text{if } b_{n+1} = 1 \\ (b_1, b_2, b_3, \dots, b_{n+1}) & \text{if } b_{n+1} \neq 1, \end{cases}$$

as wanted. □

This proves the vanishing of cohomology for $\mathcal{F} = \mathcal{O}_X$. For general $\mathcal{F}$, the complex $\Gamma(X,\mathcal{F}) \xrightarrow{e^*} C^{\cdot}(Y_1/X)$ is isomorphic to the tensor product (over A) of the complex (3.2) with $\Gamma(X,\mathcal{F})$. Since the contracting homotopy we have constructed above is A-linear, this complex is clearly exact.

Let us turn now to the case of general Y. By Corollary 2.16 and faithfully flat descent we may assume that $Y = \mathrm{Spec}(A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P^{1/m}])$ for some m. Then (see the proof of Proposition 3.13)

$$\check{H}^n(Y/X, \mathcal{F}) = H^n(H_m, f_*\mathcal{F})$$

where H_m is the group scheme $\mathrm{Spec}(\mathbf{Z}[P^{gp}/(P^{gp})^m])$. Since H_m is diagonalizable, we know that $H^n(H_m, f_*\mathcal{F}) = 0$ for $n \geq 1$ [33, Exp. I, Thm. 5.3.3]. This finishes our proof for the Kummer log-flat topology. The proof for the Kummer log-étale topology is analogous (replace Corollary 2.16 with Corollary 2.17). □

The above proposition implies the following

split **Proposition 3.29.** *Let X be a log-scheme and let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence of locally free finite rank $\mathcal{O}_X$-sheaves on X_{kfl} or $X_{\mathrm{két}}$. Then

1. *if X is affine, this exact sequence splits;*
2. *$\mathcal{F}$ is classical if and only if so is $\mathcal{F}'$ and $\mathcal{F}''$.*

Proof. Consider the following exact sequence of sheaves on X_{kfl}

$$0 \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{F}') \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{F}) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{F}'') \to 0.$$

Since, by Proposition 3.27, $H^1(X_{\mathrm{kfl}}, \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{F})) = 0$, (1) follows. To prove (2) reduce to the case of X affine and use (1). Treat the case of $X_{\mathrm{két}}$ similarly. □

4. Algebraic K-theory of log-schemes

We present in this section basic properties and some examples of calculations of algebraic (Quillen) K-theory of log-schemes for the topologies discussed earlier. Hagihara [14] was the first one to study algebraic K-theory of Kummer log-étale topos. Most of his results hold for log-schemes over (separably) closed fields. Working with equivariant K-theory for finite flat group schemes instead of finite groups and using some of the results from earlier sections we show that they hold in greater generality. In particular, for the Kummer log-flat site.

Let X be a Noetherian log-scheme. Let $K(X_*) = K(\mathcal{P}(X)_*)$ denote the higher K-theory groups of the exact category $\mathcal{P}(X)_*$ as defined by Quillen [32]. Similarly, let $K'(X_*) = K(\mathcal{M}(X)_*)$ be the Quillen's K-theory of the abelian category $\mathcal{M}(X)_*$. Denote by $\mathbf{K}(X_*), \mathbf{K}'(X_*)$ the Waldhausen spectra [35, 1.5.3] corresponding respectively to the categories $\mathcal{P}(X)_*, \mathcal{M}(X)_*$. Recall that they are functorial with respect to exact functors. We have

$$\pi_i(\mathbf{K}(X_*)) = K_i(X_*), \quad \pi_i(\mathbf{K}'(X_*)) = K'_i(X_*)$$

Let $\mathbf{K}/n(X_*), \mathbf{K}'/n(X_*)$ be the associated mod-n spectra. Set

$$K_i(X_*, \mathbf{Z}/n) = \pi_i(\mathbf{K}/n(X_*)), \quad K'_i(X_*, \mathbf{Z}/n) = \pi_i(\mathbf{K}'/n(X_*)).$$

4.1. **Basic properties.** We easily check that we have the following morphisms

- $K(X_*) \to K'(X_*)$ if $\mathcal{O}_{X_*}$ is a coherent sheaf;
- $f^* : K(X_*) \to K(Y_*)$, for any morphism $f : Y \to X$;
- $f^* : K'(X_*) \to K'(Y_*)$ for any object $f : Y \to X$ in X_* or f classically flat and $*$ any Kummer site.

Less obvious is the existence of pushforward for exact closed immersions.

Lemma 4.1. *The pushforward functor $i_* : K'(Y_*) \to K'(X_*)$ exists for an exact closed immersion $i : Y \hookrightarrow X$, X such that $(M_X/\mathcal{O}_X^*)_{\overline{x}} \simeq \mathbf{N}^{r(x)}$ for every point $x \in X$, and $*$ any Kummer topology.*

Proof. This follows easily for the Kummer étale topology from the exactness of i_* on all abelian sheaves (check on stalks at log-geometric points of X). We present here the argument for the Kummer log-flat topology (the log-syntomic case is analogous). In that case it can be reduced to the exactness of i_* for the Zariski topology. Let $f : \mathcal{F}_1 \to \mathcal{F}_2$ be a surjective morphism of Kummer log-flat coherent sheaves on Y. Cover X with étale open sets U that are affine and equipped with charts $P \to M_U$, $P \simeq \mathbf{N}^r$. For each U, by Corollary 2.16 and faithfully flat descent, there exists an n such that the map $f|U$ comes from a Zariski map $f_{\mathrm{Zar}} : \mathcal{F}_{1,\mathrm{Zar}} \to \mathcal{F}_{2,\mathrm{Zar}}$ on $U_{Y,n} = Y_U \times_U U_n$. Since $\varepsilon^* : U_{Y,n,\mathrm{kfl}} \to U_{Y,n,\mathrm{Zar}}$ is exact and faithful, the map f_{Zar} is surjective as well. It follows that the pushforward $i_* f_{\mathrm{Zar}} : i_*\mathcal{F}_{1,\mathrm{Zar}} \to i_*\mathcal{F}_{2,\mathrm{Zar}}$ is a surjection on X_n. Since $\varepsilon^* i_* \simeq i_* \varepsilon^*$ (easy to check), we are done. □

The following two propositions follow from Corollary 3.9, Lemma 3.10, and Lemma 3.11.

Proposition 4.2. *Let X be a log-regular quasi-compact log-scheme. Then*

1. $\operatorname{inj\,lim}_Y K_*(Y_{\mathrm{kfl}}) \xrightarrow{\sim} K_*(X_{\mathrm{vkfl}})$, *where the limit is over log-blow-ups $Y \to X$;*
2. $K_*(X_{\mathrm{vksyn}}) \xrightarrow{\sim} K_*(X_{\mathrm{vkfl}})$.

Proposition 4.3. *For any Noetherian log-scheme X, the pullback functors induce isomorphisms*

$$K_*(X_{\mathrm{ksyn}}) \xrightarrow{\sim} K_*(X_{\mathrm{kfl}}), \quad K'_*(X_{\mathrm{ksyn}}) \xrightarrow{\sim} K'_*(X_{\mathrm{kfl}}).$$

The following two propositions are proved in a similar way to their classical versions.

Proposition 4.4. *Let X be a Noetherian, log-scheme satisfying property (3.1). Then the natural immersion $i : X_{\mathrm{red}} \hookrightarrow X$ induces an isomorphism $i_* : K'_q(X_{\mathrm{red},*}) \xrightarrow{\sim} K'_q(X_*)$, for any Kummer topology.*

Proposition 4.5. *Let $\{X_i\}$ be a filtered system of Noetherian log-schemes. Assume that all the schemes X_i satisfy property (3.1) and the transition maps $\alpha_{ij} : X_j \to X_i$ are affine and classically flat. Then, for any Kummer site $*$,*

$$\operatorname*{inj\,lim}_i K'_q(X_{i,*}) \simeq K'_q((\operatorname*{proj\,lim}_i X_i)_*).$$

We have the following versions of the localization exact sequence. Their proofs are analogous to the proof of their classical version and the interested reader will find the details of the Kummer log-étale case in Hagihara [14, Theorem 4.5].

Proposition 4.6. *Let X be a Noetherian, equicharacteristic log-scheme, Y a strictly closed subscheme and U its complement. Assume that $(M_X/\mathcal{O}_X^*)_{\overline{x}} \simeq \mathbf{N}^{r(x)}$ for every point $x \in X$. Then we have the canonical long exact sequence*

$$\to K'_i(Y_{\mathrm{k\acute{e}t}}) \to K'_i(X_{\mathrm{k\acute{e}t}}) \to K'_i(U_{\mathrm{k\acute{e}t}}) \to K'_{i-1}(Y_{\mathrm{k\acute{e}t}}) \to$$

Proposition 4.7. *Let X be a Noetherian log-scheme, Y a strictly closed subscheme and U its complement. Assume that $(M_X/\mathcal{O}_X^*)_{\overline{x}} \simeq \mathbf{N}^{r(x)}$ for every point $x \in X$. Then we have the canonical long exact sequence*

$$\to K'_i(Y_{\mathrm{kfl}}) \to K'_i(X_{\mathrm{kfl}}) \to K'_i(U_{\mathrm{kfl}}) \to K'_{i-1}(Y_{\mathrm{kfl}}) \to$$

Recall Hagihara's notion of an M-framed log-scheme. Let $M \simeq \mathbf{N}^r$ be a monoid. An M-framed log-scheme is a pair (X, θ), where $\theta : M \to \Gamma(X, M_X/\mathcal{O}_X^*)$ is a frame such that for all points $x \in X$ the composite $M \to \Gamma(X, M_X/\mathcal{O}_X^*) \to (M_X/\mathcal{O}_X^*)_x$ is isomorphic to a projection $\mathbf{N}^r \to \mathbf{N}^m$, $r \geq m$. Note that the log-structure on X is Zariski. A standard example is given by a regular scheme with the log-structure coming from a strict normal crossing divisor (generate M from the irreducible components of the divisor at infinity).

Proposition 4.8. *(Poincaré isomorphism) Let X be a log-regular, regular quasi-compact log-scheme with a frame M. Then the natural morphism $K_i(X_*) \to K_i'(X_*)$ is an isomorphism for all i and any Kummer topology.*

Proof. We will argue the case of the Kummer log-flat topology. Assume that X has dimension n. Let $\mathcal{F}$ be a log-flat coherent sheaf. By the lemma below we can find a resolution

$$0 \to \mathcal{P} \to \mathcal{E}_{n-1} \to \dots \mathcal{E}_0 \to \mathcal{F} \to 0,$$

where each $\mathcal{E}_i$ is a locally free sheaf and $\mathcal{P}$ is coherent. Zariski localize now on X and take $Y = \operatorname{Spec}(\mathcal{O}_{X,x})$ for a point $x \in X$ with a chart $P \to M_Y$, $P \simeq (M_X/\mathcal{O}_X^*)_x \simeq \mathbf{N}^r$. We may assume that the above long exact sequence pullbacked to Y_{kfl} comes from a Zariski long exact sequence on $B = \mathcal{O}_{X,x} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P^{1/m}]$, for some m. Note that B is log-regular and regular. Hence $\mathcal{P}|\operatorname{Spec}(B)_{\mathrm{kfl}}$ is locally free. This suffices to exhibit a covering $U \to X$ for the Kummer log-flat topology such that $\mathcal{P}|U_{\mathrm{kfl}}$ is locally free, as wanted. $\square$

Lemma 4.9. *For any Kummer log-flat or log-étale coherent sheaf $\mathcal{F}$ there exists a locally free sheaf $\mathcal{E}$ that surjects onto $\mathcal{F}$.*

Proof. Take a point $x \in X$. By interpreting kfl-modules as equivariant modules, we can construct a surjection: $f_x : \mathcal{E}_x \to \mathcal{F}_x$ on $\operatorname{Spec}(\mathcal{O}_{X,x})_{\mathrm{kfl}}$. Note that, by Corollary 3.25, $\mathcal{E}_x$ is a sum of invertible sheaves.

Consider now the following commutative diagram

$$\begin{array}{ccccc}
M^{\mathrm{div}} & \xrightarrow{\theta} & \Gamma(X_{\mathrm{kfl}}, M_X/\mathcal{O}_X^*) & \xrightarrow{\partial} & \operatorname{Pic}(X_{\mathrm{kfl}}) \\
\| & & \downarrow & & \downarrow \\
M^{\mathrm{div}} & \xrightarrow{\theta_x} & \Gamma(\mathcal{O}_{X,x,\mathrm{kfl}}, M_X/\mathcal{O}_X^*) & \xrightarrow{\partial} & \operatorname{Pic}(\mathcal{O}_{X,x,\mathrm{kfl}}),
\end{array}$$

where $M^{\mathrm{div}} = \operatorname{inj\,lim}_n M^{1/n}$ or $M^{\mathrm{div}} = \operatorname{inj\,lim}_{(n,p)=1} M^{1/n}$, $p = char(x)$. By Corollary 3.18, the map $\partial\theta_x$ is surjective. Hence there exists a locally free sheaf $\mathcal{E}$ on X_{kfl} that restricts to $\mathcal{E}_x$. By [14, Lemma 4.11] , there exists an invertible sheaf $\mathcal{L}$ on X_{kfl} such that the map $\mathcal{E}_x \to \mathcal{F}_x$ extends to a map $f : \mathcal{E} \otimes \mathcal{L} \to \mathcal{F}$. The map f is surjective in a neighbourhood U_x of x. We finish by covering X with a finite number of such U_x's and taking a direct sum of the corresponding maps f. $\square$

Remark 4.10. It is easy to see that all of the above holds for the K-theory groups with coefficients: $K_*(X_*, \mathbf{Z}/n)$ and $K_*'(X_*, \mathbf{Z}/n)$.

4.2. Calculations.

picard1 **Proposition 4.11.** *Let $X = \operatorname{Spec}(A)$ be a log-scheme such that A is Noetherian and strictly local. We have the following isomorphisms*

$$\operatorname{Pic}(X_{\mathrm{kfl}}) \simeq (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z}), \quad \operatorname{Pic}(X_{\mathrm{k\acute{e}t}}) \simeq (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})',$$
$$K_0(X_{\mathrm{kfl}}) \simeq \mathbf{Z}[(M_X^{gp}/\mathcal{O}_X^*)_x \otimes \mathbf{Q}/\mathbf{Z}], \quad K_0(X_{\mathrm{k\acute{e}t}}) \simeq \mathbf{Z}[(M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})']$$

where x denotes the closed point of X.

Proof. The statement about the Picard groups is simply a reformulation of Corollary 3.17. Since, by Theorem 3.22, every locally free sheaf is a sum of invertible sheaves and, by Proposition 3.29, there are no nontrivial relations we get the statement about K_0-groups. $\square$

field **Proposition 4.12.** *Let $X = \operatorname{Spec}(K)$, for a field K, be a log-scheme with a chart $P \to M_X$, $P \simeq M_X/\mathcal{O}_X^* \simeq \mathbf{N}^r$. Then*

$$K_*'(K_{\mathrm{kfl}}) \simeq K_*'(K_{\mathrm{Zar}}) \otimes_{\mathbf{Z}} \mathbf{Z}[P^{gp} \otimes \mathbf{Q}/\mathbf{Z}], \quad K_*'(K_{\mathrm{k\acute{e}t}}) \simeq K_*'(K_{\mathrm{Zar}}) \otimes_{\mathbf{Z}} \mathbf{Z}[P^{gp} \otimes (\mathbf{Q}/\mathbf{Z})'].$$

Proof. For any m, denote by $F^m\mathcal{M}(X_{\mathrm{kfl}})$ the full subcategory of the category of Kummer log-flat coherent sheaves that become classical on the covering X_m of X. We have $F^m\mathcal{M}(X_{\mathrm{kfl}}) \simeq \mathcal{M}(X_{m,\mathrm{Zar}}, H_m)$, where the group scheme $H_m = \mathrm{Spec}(\mathbf{Z}[P^{1/m,gp}/P^{gp}])$. Here the right hand side denotes the category of H_m-equivariant Zariski coherent sheaves on X_m. By devissage, the natural functor $\mathcal{M}(X_{\mathrm{Zar}}, H_m) \to \mathcal{M}(X_{m,\mathrm{Zar}}, H_m)$ induces an isomorphism on K'-theory groups. Here H_m acts trivially on K.

Consider now the functor

$$\bigoplus_{\xi\in P^{1/m,gp}/P^{gp}} \mathcal{M}(X_{\mathrm{Zar}}) \to \mathcal{M}(X_{\mathrm{Zar}}, H_m); \quad \{\mathcal{F}_\xi\} \mapsto \oplus \mathcal{F}_\xi \otimes \mathcal{L}_\xi,$$

where $\mathcal{L}_\xi$ is the invertible sheaf corresponding to the map $K \to K[P^{1/n,gp}/P^{gp}]$, $a \mapsto a\xi$. Since H_m is diagonalizable this is an equivalence of categories (cf. [33, Exp.I,Prop.4.7.3]). This yields the isomorphism $\bigoplus_{\xi\in P^{1/m,gp}/P^{gp}} K'(K_{\mathrm{Zar}}) \xrightarrow{\sim} K(F^m\mathcal{M}(X_{\mathrm{kfl}}))$ and, by passing to the limit with respect to m, our proposition. $\square$

For a framed log-scheme (X, M) and a prime ideal $\mathfrak{p}$ of M, we write $V(\mathfrak{p}) = \{x \in X | \mathfrak{p} \subset \theta_x((M_X/\mathcal{O}_X^*)_x \setminus \{1\})\}$, where $\theta_x : M \xrightarrow{\theta} \Gamma(X, M_X) \to (M_X/\mathcal{O}_X^*)_x$. $V(\mathfrak{p})$ is a closed subset of X and we equip it with the reduced subscheme structure. We write $M(\mathfrak{p})$ for the unique face of M such that $M(\mathfrak{p}) \oplus (M \setminus \mathfrak{p}) = M$, and set

$$\Lambda[\mathfrak{p}] = \mathbf{Z}[(M(\mathfrak{p})^{gp} \otimes \mathbf{Q}/\mathbf{Z}) \setminus \cup_{\mathfrak{q}\subsetneq\mathfrak{p}}(M(\mathfrak{q})^{gp} \otimes \mathbf{Q}/\mathbf{Z})].$$

We will denote by $\Lambda'[\mathfrak{p}]$ the same group as $\Lambda[\mathfrak{p}]$ but defined using $(\mathbf{Q}/\mathbf{Z})'$ instead of $\mathbf{Q}/\mathbf{Z}$.

structure **Theorem 4.13.** *Let X be a Noetherian M-framed log-scheme. Then*

(1) *if X is equicharacteristic then there is a natural isomorphism*

$$\beta : \bigoplus_{\mathfrak{p},\ \textit{prime of } M} K'_*(V(\mathfrak{p})_{\mathrm{Zar}}) \otimes \Lambda'[\mathfrak{p}] \to K'_*(X_{\mathrm{k\acute{e}t}});$$

(2) *there is a natural isomorphism*

$$\beta : \bigoplus_{\mathfrak{p},\ \textit{prime of } M} K'_*(V(\mathfrak{p})_{\mathrm{Zar}}) \otimes \Lambda[\mathfrak{p}] \to K'_*(X_{\mathrm{kfl}}).$$

Proof. Let us define the map β (in the second case). We fix $\xi \in M(\mathfrak{p})^{gp} \otimes \mathbf{Q}/\mathbf{Z}$. The corresponding map $\beta_\xi : K'_*(V(\mathfrak{p})_{\mathrm{Zar}}) \to K'_*(X_{\mathrm{kfl}})$ is induced by the functor

$$\beta_\xi : \mathcal{M}(V(\mathfrak{p})_{\mathrm{Zar}}) \to \mathcal{M}(X_{\mathrm{kfl}}), \quad \mathcal{F} \mapsto i_*(\varepsilon^*\mathcal{F} \otimes \mathcal{O}_{V(\mathfrak{p})}\{\xi\}),$$

where $i : V(\mathfrak{p}) \hookrightarrow X$ is the natural closed immersion and $\mathcal{O}_{V(\mathfrak{p})}\{\xi\}$ is the coherent sheaf on $V(\mathfrak{p})_{\mathrm{kfl}}$ (see Example 3.19) associated to the locally free sheaf $\mathcal{O}_{V(\mathfrak{p})}(\xi)$ on $V(\mathfrak{p})_{\mathrm{kfl}}$ obtained as the image of ξ (or rather of the minimal lifting of ξ) by the following map

$$M(\mathfrak{p})^{\mathrm{div}} \to M^{\mathrm{div}} \to \Gamma(X_{\mathrm{Zar}}, (M_X/\mathcal{O}_X^*)^{\mathrm{div}}) \to \Gamma(V(\mathfrak{p})_{\mathrm{Zar}}, (M/\mathcal{O}^*)^{\mathrm{div}}) \xrightarrow{\partial} \mathrm{Pic}(V(\mathfrak{p})_{\mathrm{kfl}}).$$

Note here that using $\mathcal{O}_{V(\mathfrak{p})}(\xi)$ instead of $\mathcal{O}_{V(\mathfrak{p})}\{\xi\}$ would tend to give a zero map.

The functor β_ξ is exact (follow [14, 6.2] replacing $\mathrm{Spec}(k)$ by $\mathrm{Spec}(\mathbf{Z})$). The rest of the argument goes as follows. One proves that the map β is compatible with localization sequences and by a limit argument reduces the proof to the case of a field. Then it suffices to evoke Proposition 4.12, and we are done.

Compatibility with localization sequences requires the following lemma (Lemma 9.4 in [14]) that we have to reprove in our setting.

Lemma 4.14. *Let N be a face of M and U an M-framed log-scheme. Assume that the frame of U comes from a chart $M \to M_U$ that maps $N \setminus \{1\}$ to zero in $\Gamma(U, \mathcal{O}_U)$. Then for any exact closed immersion $i : V \hookrightarrow U$ with the induced M-frame and $\xi \in N^{\mathrm{div}}$ we have $i^*\mathcal{O}_U\{\xi\} \simeq \mathcal{O}_V\{\xi\}$.*

Proof. Write $M \simeq \mathbf{N}^m, N \simeq \mathbf{N}^k, M = N \oplus Q$ for a face Q. Let $\xi \in N^{1/n}$. Set $M' = N^{1/n} \oplus Q$. We have $U' = U \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M'] = U \times_{\mathrm{Spec}(\mathbf{Z})} S$, where $S = \mathrm{Spec}(\mathbf{Z} \otimes_{\mathbf{Z}[N]} \mathbf{Z}[N^{1/n}])$. Similarly, $V' = V \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M'] = V \times_{\mathrm{Spec}(\mathbf{Z})} S$. One easily computes

$$\begin{aligned} S &= \mathrm{Spec}(\mathbf{Z}[x_1, \ldots, x_k]/(x_1^n, \ldots, x_k^n)); \\ \mathcal{O}_S(x_I)(S) &= x_I^{-1}(\oplus_J x_J \mathbf{Z}), \quad x_I = x_1^{i_1} \ldots x_k^{i_k}, x_J = x_1^{j_1} \ldots x_k^{j_k}, 0 \leq j_l \leq k-1; \\ \mathcal{O}_S\{x_I\}(S) &= x_I^{-1}(\oplus_J x_J \mathbf{Z}), \quad x_I = x_1^{i_1} \ldots x_k^{i_k}, x_J = x_1^{j_1} \ldots x_k^{j_k}, i_l \leq j_l \leq k-1. \end{aligned}$$

Hence, if we write $\xi = x_I$, then $\mathcal{O}_S\{x_I\}$ is a direct factor of $\mathcal{O}_S(x_I)$ and the cokernel is a free $\mathbf{Z}$-module. It follows that

$$\mathcal{O}_{U'}\{x_I\} = \mathcal{O}_S\{x_I\} \otimes_{\mathbf{Z}} \mathcal{O}_U, \quad \mathcal{O}_{V'}\{x_I\} = \mathcal{O}_S\{x_I\} \otimes_{\mathbf{Z}} \mathcal{O}_V.$$

Thus $\mathcal{O}_{V'}\{x_I\} = i^* \mathcal{O}_{U'}\{x_I\}$, as wanted. □

□

Example 4.15. Let A be a complete discrete valuation ring with residue field k and the log-structure coming from the closed point. Then, by Theorem 4.13 (see the argument below), we have

$$K_*(A_{\mathrm{kfl}}) \simeq K_*(A) \oplus K_*(k) \otimes \mathbf{Z}[\mathbf{Q}/\mathbf{Z} \setminus \{0\}], \quad K_*(A_{\mathrm{k\acute{e}t}}) \simeq K_*(A) \oplus K_*(k) \otimes \mathbf{Z}[(\mathbf{Q}/\mathbf{Z})' \setminus \{0\}].$$

When comparing this with Proposition 4.11 we get that $[A(a)] = [k\{a\}] + [A]$ in $K_*(A) \oplus K_*(k) \otimes \mathbf{Z}[\mathbf{Q}/\mathbf{Z} \setminus \{0\}]$.

Example 4.16. More generally, let X be a regular, log-regular scheme with the log-structure associated to a divisor D with strict normal crossing. Let $\{D_i | i \in I\}$ be the set of the irreducible (regular) components of D. For an index set $J \subset I$ denote by D_J the intersection of irreducible components indexed by J and by $\Lambda_{|J|}$ (resp. $\Lambda'_{|J|}$) the free abelian groups generated by the set $\{(a_1, \ldots, a_{|J|}) | a_i \in \mathbf{Q}/\mathbf{Z} \setminus \{0\}\}$ (resp. the set $\{(a_1, \ldots, a_{|J|}) | a_i \in (\mathbf{Q}/\mathbf{Z})' \setminus \{0\}\}$).

Corollary 4.17. *For any $q \geq 0$ we have the canonical isomorphism*

$$K_q(X_{\mathrm{kfl}}) \simeq \bigoplus_{J \subset I} K_q(D_J) \otimes \Lambda_{|J|}$$

Moreover, if D is equicharacteristic then canonically

$$K_q(X_{\mathrm{k\acute{e}t}}) \simeq \bigoplus_{J \subset I} K_q(D_J) \otimes \Lambda'_{|J|}$$

Proof. The Kummer log-flat statement follows from Theorem 4.13. For the Kummer log-étale note that we do have a localization sequence

$$\to K'_q(D_{\mathrm{k\acute{e}t}}) \to K'_q(X_{\mathrm{k\acute{e}t}}) \to K'_q(U_{\mathrm{k\acute{e}t}}) \to K'_{q-1}(D_{\mathrm{k\acute{e}t}}) \to$$

where $U = X_{\mathrm{tr}}$. This follows just like in the classical situation using the fact that Kummer log-étale coherent sheaves on U are simply the Zariski coherent sheaves and those can be extended to the whole of X. Now the proof of Theorem 4.13 goes through. □

Example 4.18. Again, all of the above holds for the K-theory groups with coefficients. For example, let A be a complete discrete valuation ring of mixed characteristic $(0,p)$. Let X be a smooth A-scheme equipped with the log-structure coming from the special fiber X_0. Then

$$K_*(X_{\mathrm{kfl}}, \mathbf{Z}/p^k) \simeq K_*(X_0, \mathbf{Z}/p^k) \otimes \mathbf{Z}[\mathbf{N} \setminus \{0\}] \oplus K_*(X, \mathbf{Z}/p^k)$$

Since, by Geisser-Levine [13], $K_i(X_0, \mathbf{Z}/p^k) = 0$, for $i \geq \dim X_0$, we get

$$K_i(X_{\mathrm{kfl}}, \mathbf{Z}/p^k) \simeq K_i(X, \mathbf{Z}/p^k) \simeq K_i(X[1/p], \mathbf{Z}/p^k), \quad i \geq \dim X_0.$$

5. Topological K-theory of log-schemes

In this section we initiate the study of topological K-theory of log-schemes.

5.1. **Homotopy theory of simplicial presheaves and sheaves.** The formalism of cohomologies of simplicial presheaves we use here is based on the closed model structures for the category of simplicial presheaves and sheaves on an arbitrary Grothendieck site developed by Jardine [16], [17], [18], [19].

We begin by recalling basic facts about cohomology of simplicial presheaves. Let us start with some definitions. A *closed model category* is a category $\mathcal{M}$ equipped with three classes of maps called cofibrations, fibrations and weak equivalences, such that the following axioms are satisfied:

(1) $\mathcal{M}$ is closed under all finite limits and colimits.
(2) Given $f : X \to Y$ and $g : Y \to Z$, if any of the two of f, g or gf are weak equivalences, then so is the third.
(3) If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.
(4) Given any commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & {\scriptstyle p}\downarrow \\ V & \longrightarrow & Y \end{array}$$

in $\mathcal{M}$, where i is a cofibration and p is a fibration, then an arrow $V \to X$ exists making this diagram commute assuming that either i or p is a weak equivalence.
(5) Any map $f : X \to Y$ may be factored
 - $f = pi$, where p is a fibration and i is a trivial cofibration, and
 - $f = qj$, where q is a trivial fibration and j is a cofibration.

A *trivial fibration* is a map that is a fibration and a weak equivalence and a *trivial cofibration* is a map that is a cofibration and a weak equivalence. A basic example of a closed model category is the category $\mathbf{S}$ of simplicial sets: the cofibrations of $\mathbf{S}$ are the monomorphisms, the weak equivalences are the maps which induce isomorphisms on all possible homotopy groups of associated realizations, and the fibrations are the Kan fibrations.

A *closed simplicial model category* is a closed model category $\mathcal{M}$ which has a natural function complex $\mathrm{Hom}(U, X)$ in the category $\mathbf{S}$ of simplicial sets for each pair of objects U, X in $\mathcal{M}$. This simplicial set is supposed to satisfy some adjointness properties as well as the following axiom:

- If $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration, then the induced map of simplicial sets

$$\mathrm{Hom}(B, X) \xrightarrow{(i^*, p_*)} \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B, X)$$

 is a Kan fibration, which is trivial if either i or p is trivial.

A closed model category $\mathcal{M}$ is called *proper* if it satisifes the following additional axiom:

- Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow \\ B & \xrightarrow{g} & D \end{array}$$

 (1) if the square is a pullback, j is a fibration and g is a weak equivalence, then f is a weak equivalence.
 (2) if the square is a push out, i is a cofibration and f is a weak equivalence, then g is a weak equivalence.

The category $\mathbf{S}$ of simplicial sets is a proper closed simplicial model category.

Let C be a site and let T be the Grothendieck topos of sheaves on C. Denote by pT (resp. sT) the category of presheaves (resp. sheaves) of simplicial sets on C. When X is a presheaf, we denote by $\pi_0(X)$ the sheaf on T associated to the presheaf

$$U \mapsto \pi_0(X(U)).$$

For an object U in C, we let $X|U$ be the image of X in the site $C|U$. When $n > 0$ is an integer and $x \in X_0(U)$, we denote by $\pi_n(X|U, x)$ the sheaf on $C|U$ associated to the preasheaf

$$V \mapsto \pi_n(X(V), x).$$

Here, for a simplicial set S, we take $\pi_n(S) = \pi_n(|S|)$, where $|S|$ is the geometric realization of S.

Definition 5.1. Let $f : X \to Y$ be a map of presheaves. Then

- f is called a weak equivalence if the induced map $f_* : \pi_0(X) \to \pi_0(Y)$ is an isomorphism, and for all $n > 0$, all objects U in C, and all $x \in X_0(U)$, the natural maps
$$f_* : \pi_n(X|U, x) \to \pi_n(Y|U, f(x))$$
are isomorphisms;
- f is called a cofibration if, for any object U from C, the induced map $f(U) : X(U) \to Y(U)$ is injective;
- f is called a fibration if it satisfies the following lifting property: for any commutative diagram
$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y, \end{array}$$
where i is a trivial cofibration, there exists a map $B \to X$ such that the resulting diagram commutes.

For two simplicial presheaves X and Y, the simplicial set $\mathrm{Hom}(X, Y)$ is defined by

$$n \mapsto \mathrm{Hom}_{pT}(X \times \Delta^n, Y),$$

where Δ^n is the standard n-simplex. We also have the simplicial presheaf $\mathcal{H}\mathrm{om}(X, Y)$ defined by

$$U \mapsto (n \mapsto \mathrm{Hom}_{p(T|U)}(X|U \times \Delta^n, Y|U)).$$

Jardine proves (see Prop. 1.4 in [17]) the following

Theorem 5.2. *With the above definitions the categories sT and pT are proper closed simplicial model categories.*

We can associate to sT and pT the homotopy categories $\mathcal{H}\mathrm{o}(sT)$ and $\mathcal{H}\mathrm{o}(pT)$ by formally inverting all weak equvalences. We have (Prop. 2.8 from [16])

Theorem 5.3. *The associated sheaf functor induces an equivalence*

$$\mathcal{H}\mathrm{o}(pT) \simeq \mathcal{H}\mathrm{o}(sT)$$

between the associated homotopy categories.

What we have just described is the (global) homotopy theory of simplicial presheaves. Recall that there exists also a local theory (see [16]). A *local fibration* is a map $p : F \to G$ that satisfies the local right lifting property, that is, $p(U)$ has the lifting property of the Kan fibration up to a refinement of the object U from C by a covering. An example of a local fibration is a *pointwise Kan fibration*, i.e., a map $p : F \to G$ such that each $p(U) : F(U) \to G(U)$ is a Kan fibration. Not every local fibration is a global fibration. For example, the Eilenberg-MacLane presheaves $K(F, n)$ are not in general globally fibrant. Note that if the site C has enough points than a map p is a local fibration if and only if it is a Kan fibration on all the stalks. Local weak equivalence between locally fibrant simplicial presheaves is defined by shififying the definition of weak equivalence of Kan complexes. If the site C has enough points, then a map $q : F \to G$ of locally fibrant simplicial presheaves is a local weak equivalence if and only if it induces weak equivalences on all the stalks. Jardine (see Prop. 2.8 in [16]) shows that the associated homotopy category is equivalent to that of $\mathcal{H}\mathrm{o}(pT)$.

For simplicial presheaves X and Y, we denote by $[X, Y]$ the set of morphisms from X to Y in the homotopy category. A simplicial presheaf X is called (globally) fibrant if the unique map $X \to *_C$ is a (global) fibration. Here $*_C$ is the final object of the category of presheaves on C. For any simplicial presheaf X the canonical map $X \to *_C$ admits a factorization $X \to X^f \to *_C$, where $X \to X^f$ is a trivial cofibration and X^f is fibrant. Such a map $X \to X^f$ is called a fibrant replacement of X. For two simplicial presheaves X and Y, we have

$$[X, Y] = [X, Y^f] = \pi_0 \operatorname{Hom}(X, Y^f),$$

where $Y \to Y^f$ is a fibrant replacement of Y. That is, the set $[X, Y^f]$ is given by morphisms $X \to Y^f$ modulo simplicial homotopy.

5.1.1. *Cohomology of simplicial presheaves.* Let F be a pointed simplicial presheaf . Define cohomology of C with coefficients in F (see [16, 3]) by

$$H^{-m}(C, F) = [*_C, \Omega^m F] \quad \text{for} \quad m \geq 0.$$

In the case the site C has a final object X we will write $H^{-m}(X, F)$ for $H^{-m}(C, F)$. Note that $H^{-m}(C, F) \simeq [S^m, F]_*$, where the subscript $*$ refers to morphisms in the pointed homotopy category. Here S^m is the simplicial m-sphere $\Delta^m/\partial\Delta^m$. $H^{-m}(C, F)$ is a pointed set for $m = 0$, a group for $m > 0$, and an abelian group for $m > 1$.

5.1.2. *Change of sites.* This section is based on [19]. Let $f : C \to D$ be a morphism of sites given by a functor $f : D \to C$ that preserves finite limits and sends covers to covers. We have the associated presheaf functors

$$f_* : C^\wedge \to D^\wedge, \qquad f^p : D^\wedge \to C^\wedge,$$

where $C^\wedge$ denotes the category of presheaves on C. The functor f^p is left adjoint to f_*. Both functors are exact and f_* maps sheaves to sheaves. Both f^p and f_* preserve cofibrations and f_* preserves fibrations. In particular, the functor $F \mapsto F(U)$ preserves fibrations. Thus a global fibration is a pointwise fibration hence a local fibration. The functor f^p also preserves weak equivalences.

Jardine proves the following

jar **Theorem 5.4.** *Let $f : C \to D$ be a morphism of sites. Let F be a pointed simplicial presheaf on the site C. Take a global fibrant replacement $F \to F^f$ of F. Then we have an isomorphism*

$$H^m(C, F) \simeq H^m(D, f_* F^f),$$

for all $m \leq 0$.

Proof. We start with the following lemma.

Lemma 5.5. *Suppose that F is a globally fibrant simplicial presheaf on C. Then there is an adjointness isomorphism*

$$[*_D, f_* F] \simeq [*_C, F].$$

Proof. We know that $f_* F$ is also globally fibrant. Hence we have the following sequence of isomorphisms

$$[*_D, f_* F] \simeq \pi_0 \operatorname{Hom}(*_D, f_* F) \simeq \pi_0 \operatorname{Hom}(f^p *_D, F) \simeq \pi_0 \operatorname{Hom}(*_C, F) \simeq [*_C, F],$$

as wanted. □

Since globally fibrant objects are preserved by the loop functor (Corollary 3.2 from [16]), the above lemma gives us the following isomorphisms

$$H^m(C, F) \simeq [*_C, \Omega^m F] \simeq [*_C, \Omega^m F^f] \simeq [*_D, f_* \Omega^m F^f].$$

Since the loop functor commutes with the direct image functor, we also get

$$[*_D, f_* \Omega^m F^f] \simeq [*_D, \Omega^m f_* F^f].$$

This proves our theorem. □

It will be useful for us to identify the homotopy group presheaves of the presheaf $f_* F^f$ from the above theorem.

seq **Proposition 5.6.** *We have*

$$\pi_k f_* F^f(V) \simeq H^{-k}(f(V), F|f(V)).$$

Proof. This follows from the following sequence of isomorphisms

$$\pi_k f_* F^f(V) = \pi_k F^f(f(V)) \simeq [*_{f(V)}, \Omega^k F^f|f(V)] \simeq [*_{f(V)}, \Omega^k F|f(V)].$$

□

5.2. **Topological K-theory.** We base this section on Gillet and Soulé [9, 3.1]. Let $(C, \mathcal{O}_C)$ be a ringed site with enough points. We assume that $\mathcal{O}_C$ is unitary and commutative. For any $n \geq 1$, we consider the following presheaves

$$GL_n : U \mapsto GL_n(\Gamma(U, \mathcal{O}_U)), \qquad BGL_n : U \mapsto BGL_n(\Gamma(U, \mathcal{O}_U)).$$

Here $BGL_n(\Gamma(U, \mathcal{O}_U))$ is the classifying space of $GL_n(\Gamma(U, \mathcal{O}_U))$.

Let F be a simplicial presheaf such that $\pi_0(F) = *$. We define its Bousfield-Kan integral completion $\mathbf{Z}_\infty F$ to be the simplicial presheaf $U \mapsto \mathbf{Z}_\infty F(U)$. The functor $\mathbf{Z}_\infty$ for simplicial sets is defined in [4]. Its basic property gives us that if a map of simplicial presheaves $f : F \to G$ induces an isomorphism of presheaves of integral homology groups $f : H_n(F, \mathbf{Z}) \to H_n(G, \mathbf{Z})$, then the map $\mathbf{Z}_\infty f : \mathbf{Z}_\infty F \to \mathbf{Z}_\infty G$ is a weak equivalence. We set $BGL = \operatorname{inj\,lim}_n BGL_n$ and

$$K = \mathbf{Z} \times \mathbf{Z}_\infty BGL,$$

where the constant presheaf $\mathbf{Z}$ is concentrated in degree zero and pointed by zero.

To compare the above definition with Quillen's K-theory, take, for any ringed site $(C, \mathcal{O}_C)$, the functor $U \mapsto P_C(U)$, where $P_C(U)$ is the category of locally free $\mathcal{O}_C|U$-modules of finite rank. Consider the simplicial presheaf $\Omega BQP_C : U \mapsto \Omega BQP_C(U)$. Here Q is the Quillen Q-construction. Consider also a related simplicial presheaf $\Omega BQP : U \mapsto \Omega BQP(\mathcal{O}_C(U))$, where $P(\mathcal{O}_C(U))$ is the category of finitely generated projective $\mathcal{O}_C(U)$-modules. There is a natural map $\Omega BQP \to \Omega BQP_C$ and, by [8, 2.15], a natural map (in the homotopy category) $\mathbf{Z} \times \mathbf{Z}_\infty BGL \to \Omega BQP_C$. Gillet and Soulé [9, 3.2.1] prove the following

GS **Lemma 5.7.** *If C is locally ringed, then the natural maps of pointed simplicial presheaves*

$$\mathbf{Z} \times \mathbf{Z}_\infty BGL \to \Omega BQP \to \Omega BQP_C$$

are weak equivalences.

Let C be now the Zariski site of some scheme X. Choose a fibrant replacement K^f of $\Omega BQP_{\mathrm{Zar}}$. It defines a map $K_m(X) = \pi_m(\Omega BQP_{\mathrm{Zar}}(X)) \to H^{-m}(X_{\mathrm{Zar}}, K)$. Gillet and Soulé show [9, 3.2.2] that the Mayer-Vietoris property implies the following

Proposition 5.8. *Suppose that X is a Noetherian regular scheme of finite Krull dimension. Then the above map gives an isomorphism*

$$K_m(X) \xrightarrow{\sim} H^{-m}(X_{\mathrm{Zar}}, K), \qquad m \geq 0$$

5.2.1. *Topological K/n-theory.* For a scheme X, write

$$\mathbf{K}(X) = \mathbf{K}(X_{\mathrm{Zar}}) = \{K^0(X), K^1(X), \dots, \}$$

for the Waldhausen spectrum associated to the category of Zariski locally free sheaves (cf. [35, 1.5.2]). Write

$$\mathbf{K}/n(X) = \mathbf{K}/n(X_{\mathrm{Zar}}) = \{K^0/n(X), K^1/n(X), \dots\}$$

for the corresponding mod-n spectrum. Both spectra are connective and contravariant in X. For a site C built from schemes, denote by K and K/n the pointed simplicial presheaves $K : X \mapsto K^0(X)$ and $K/n : X \mapsto K^0/n(X)$. Since, by the $+ = Q$ theorem, the map (of simplicial presheaves) $\mathbf{Z} \times \mathbf{Z}_\infty BGL \to \Omega BQP$ is a weak equivalence and there exists a (local) weak equivalence $\Omega BQP \to (U \mapsto K^0(U))$ [35, 1.11.2] this notation is compatible with the one used above.

Set

$$K_m^C(X) := H^{-m}(X_C, K), \quad K_m^C(X, \mathbf{Z}/n) := H^{-m}(X_C, K/n), \quad m \geq 0.$$

Corollary 5.9. *Suppose that X is a Noetherian regular scheme of finite Krull dimension. Then we have a natural isomorphism*

$$K_m(X, \mathbf{Z}/n) \xrightarrow{\sim} K_m^{\mathrm{Zar}}(X, \mathbf{Z}/n) = H^{-m}(X_{\mathrm{Zar}}, K/n), \qquad m \geq 0$$

Proof. The fibration sequence

$$K^0/n \to K^1 \xrightarrow{n} K^1$$

gives compatible long exact sequences

$$\begin{tikzcd}
H^{-m}(X_{\mathrm{Zar}}, K) \arrow[r, "n"] & H^{-m}(X_{\mathrm{Zar}}, K) \arrow[r] & H^{-m}(X_{\mathrm{Zar}}, K/n) \arrow[r] & H^{-m+1}(X_{\mathrm{Zar}}, K) \\
K_m(X) \arrow[r, "n"] \arrow[u, "\wr"'] & K_m(X) \arrow[r] \arrow[u, "\wr"'] & K_m(X, \mathbf{Z}/n) \arrow[r] \arrow[u] & K_{m-1}(X) \arrow[u, "\wr"']
\end{tikzcd}$$

Our corollary easily follows. □

5.3. **Topological log-étale K/n-theory.** We show in this section that l-adic topological log-étale K-theory of a log-regular scheme computes étale K-theory of the largest open set on which the log-structure is trivial. As the reader will see the log-étale story presented here is very similar to the story of étale K-theory. We will mainly work with schemes S such that

(*) S is separated, Noetherian and regular. The natural number n is invertible on S and $\sqrt{(-1)} \in \mathcal{O}_X$ if n is even. S has finite Krull dimension and a uniform bound on n-torsion étale cohomological dimension of all residue fields. Each residue field of S has a Tate-Tsen filtration.

We quote from Jardine (Theorem 3.9 in [16])

Theorem 5.10. *Suppose that X satisfies the above condition. Then, for $n \geq 0$, we have an isomorphism*

$$[*_{X_{\text{ét}}}, \Omega^m K^1/n] \simeq K_{m-1}^{DF}(X, \mathbf{Z}/n), \quad m \geq 0,$$

where $K_*^{DF}(X, \mathbf{Z}/n)$ is the étale K-theory.

This yields the following

Corollary 5.11. *Suppose that X satisfies the above condition. Then there is an isomorphism*

$$K_m^{\text{ét}}(X, \mathbf{Z}/n) \simeq K_m^{DF}(X, \mathbf{Z}/n), \quad m \geq 0.$$

Proof. The above theorem and the weak equivalence $K^0/n \simeq \Omega K^1/n$ give the following isomorphisms

$$H^{-m}(X_{\text{ét}}, K/n) = [*_{X_{\text{ét}}}, \Omega^m K^0/n] \simeq [*_{X_{\text{ét}}}, \Omega^{m+1} K^1/n] \simeq K_m^{DF}(X, \mathbf{Z}/n),$$

as wanted. □

We will now compute the homotopy groups of K-presheaves. Recall [36, 2.7, 2.7.2] that, for a scheme Y satisfying condition (*) such that $\Gamma(Y, \mathcal{O}_Y)$ contains a primitive n'th root of unity, there are compatible functorial Bott element homomorphisms

$$\beta_n : \mu_n(Y) \to K_2(Y, \mathbf{Z}/n),$$

where $\mu_n(Y)$ denotes the group of n'th roots of unity in $\Gamma(Y, \mathcal{O}_Y)$.

sheaves **Proposition 5.12.** *Suppose that X satisfies condition (*) and that for all $x \in X$, $M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$ is isomorphic to a direct sum of $\mathbf{N}$. Let n be invertible on X. Then the sheaves of homotopy groups of K/n in the Kummer log-étale topology are given by*

$$\tilde{\pi}_q(K/n) \simeq \begin{cases} \mathbf{Z}/n(i) & \text{for } q = 2i \geq 0 \\ 0 & \text{for } q \geq 0, \text{ odd.} \end{cases}$$

Proof. We have a map of sheaves

$$\mathbf{Z}/n(i) \to \tilde{\pi}_{2i}(K/n)$$

induced locally by taking the product of the map $\beta_n \to \pi_2(K/n(Y))$.

It suffices to show that this map is an isomorphism and that, for q odd, the sheaf $\tilde{\pi}_q(K/n)$ is trivial. For that we need to compute the stalks of the presheaves K/n. For any point $x \in X$, consider the natural chart $P \to \mathcal{O}_{X,\overline{x}}$, where $P = M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$. By assumption $P \simeq \mathbf{N}^r$, for some r. We have

$$K/n_{x(log)} = \operatorname{inj}\lim_{U} K/n(U) = \operatorname{inj}\lim_{k} K/n(\mathcal{O}_{X,\overline{x},k}),$$

where the first limit is over the Kummer log-étale neighbourhoods U of the log geometric point $x(log)$ in X, and the second limit is over the base changes $\mathcal{O}_{X,\overline{x},k} = \mathcal{O}_{X,\overline{x}} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P]$ of $\mathcal{O}_{X,\overline{x}}$ by the k-power map $k : P \to P$, k being invertible in $\mathcal{O}_{X,\overline{x}}$. Since $P \simeq \mathbf{N}^r$, the ring $\mathcal{O}_{X,\overline{x},k}$ is local.

By Gabber's rigidity [7] we have the following commutative diagram

$$\begin{tikzcd}
\pi_q(K/n(\mathcal{O}_{X,\overline{x}})) \arrow[r, "\sim"] \arrow[d] & \pi_q(K/n(\overline{k})) \arrow[d, "\wr"] \\
\pi_q(K/n(\mathcal{O}_{X,\overline{x},k})) \arrow[r, "\sim"] & \pi_q(K/n(\overline{k})).
\end{tikzcd}$$

Hence, $\operatorname{inj}\lim_k \pi_q(K/n(\mathcal{O}_{X,\overline{x},k})) \xrightarrow{\sim} \pi_q(K/n(\overline{k}))$. The proposition now follows from the computations of K-theory of separably closed fields. $\square$

The above computation yields the following

sp **Proposition 5.13.** *Suppose that X satisfies condition (*) and that, for all $x \in X$, $M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$ is isomorphic to a direct sum of $\mathbf{N}$. Let n be invertible on X. Then there exists a cohomological spectral sequence $E_r^{p,q}$, $r \geq 2$, such that*

$$E_2^{p,q} = \begin{cases} H^p(X_{\text{két}}, \mathbf{Z}/n(q/2)) & \text{for } q-p \geq 0 \text{ and } q \text{ even} \\ 0 & \text{for } q-p \geq 0 \text{ and } q \text{ odd.} \end{cases}$$

This spectral sequence converges strongly to $K^{\text{két}}_{q-p}(X, \mathbf{Z}/n)$ for $q-p \geq 1$. The differential d_r in the above spectral sequence maps $E_r^{p,q}$ to $E_r^{p+r,q+r-1}$.

comp **Theorem 5.14.** *Let X be a log-regular, regular scheme satisfying condition (*). Let n be a natural number invertible on X. Then the open immersion $j : U \hookrightarrow X$, where $U = X_{\text{tr}}$ is the maximal open set of X on which the log-structure is trivial, induces an isomorphism*

$$j^* : K^{\text{két}}_m(X, \mathbf{Z}/n) \xrightarrow{\sim} K^{\text{ét}}_m(U, \mathbf{Z}/n), \quad m \geq 0.$$

Proof. Let $K/n \to K^f/n$ be a globally fibrant replacement. By Theorem 5.4,

$$H^{-m}(U_{\text{ét}}, \mathbf{Z}/n) \simeq H^{-m}(U_{\text{két}}, \mathbf{Z}/n) \simeq H^{-m}(X_{\text{két}}, j_*K^f/n).$$

It suffices to show that the natural map of presheaves on $X_{\text{két}}$,

$$K/n \to j_*(K^f/n)$$

is a weak equivalence. Or that the induced map on all the log-geometric stalks is a weak equivalence. By Proposition 5.13, $\pi_q(K/n_{x(log)})$ is trivial for q odd and isomorphic to $\mathbf{Z}/n(i)$ for $q = 2i$. From Proposition 5.6,

$$\pi_q((j_*(K/n))_{x(log)}) \simeq \operatorname{inj}\lim_{Y} K^{\text{ét}}_q(Y_U, \mathbf{Z}/n),$$

where the limit is over the Kummer log-étale neighbourhoods Y of $x(log)$ in X. Consider now the composition

$$\begin{aligned}
\pi_q((K^f/n)_{x(log)}) = \operatorname{inj}\lim_{Y} K_q(Y, \mathbf{Z}/n) &\overset{j^*}{\simeq} \operatorname{inj}\lim_{Y} K_q(Y_U, \mathbf{Z}/n) \\
&\xrightarrow{\rho} \operatorname{inj}\lim_{Y} K^{\text{ét}}_q(Y_U, \mathbf{Z}/n) \simeq \pi_q((j_*(K^f/n))_{x(log)}).
\end{aligned}$$

By Proposition 5.15 below, the map j^* is an isomorphism. By Thomason [34, 11.5], the map ρ is an isomorphism after inverting the Bott element. This yields the isomorphism

$$\pi_*((K/n)_{x(log)})[\beta_n^{-1}] \xrightarrow{\sim} \pi_*((j_*(K^f/n))_{x(log)})[\beta_n^{-1}].$$

Since the Bott element is invertible on both sides, we get the isomorphism

$$\pi_q((K/n)_{x(log)}) \xrightarrow{\sim} \pi_q((j_*(K^f/n))_{x(log)}),$$

as wanted. □

local **Proposition 5.15.** *Let X be a log-regular, regular scheme. Let n be a natural number invertible on X. For any point $x \in X$, the natural map*

$$\operatorname*{inj\,lim}_Y K_q(Y, \mathbf{Z}/n) \to \operatorname*{inj\,lim}_Y K_q(Y_U, \mathbf{Z}/n)$$

is an isomorphism. Here, the limit is taken over the Kummer log-étale neighbourhoods of $x(log)$ in X.

Proof. Looking étale locally, we may assume that $X = \operatorname{Spec}(\mathcal{O}_{X,\overline{x}})$ (abusing notation a bit), and we have a chart $P \to \mathcal{O}_{X,\overline{x}}$, for $P = M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}} \simeq \mathbf{N}^k$. Consider the closed subscheme of X

$$Z = X \otimes_{\mathbf{Z}[X_1,\dots,X_k]} \mathbf{Z}[X_1,\dots,X_k]/(X_1 \dots X_k).$$

Up to reindexing, Z can be covered by closed subschemes

$$Z_i = X \otimes_{\mathbf{Z}[X_1,\dots,X_k]} \mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i).$$

We will need the following lemma

Lemma 5.16. *Consider the cartesian diagram*

$$\begin{array}{ccc} Z_i' & \longrightarrow & \operatorname{Spec}(\mathbf{Z}[X_1,\dots,X_k]/(X_1^r,\dots,X_i^r)) \\ \big\downarrow{\scriptstyle m_r} & & \big\downarrow{\scriptstyle m_r} \\ Z_i & \longrightarrow & \operatorname{Spec}(\mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i)), \end{array}$$

where the map m_r is defined by sending X_l to X_l^r. The pullback map $m_r^* : K'_*(Z_i, \mathbf{Z}/n) \to K'_*(Z_i', \mathbf{Z}/n)$ is trivial for r large enough and invertible on X.

Proof. We can filter the ring $\mathbf{Z}[X_1,\dots,X_k]/(X_1^r,\dots,X_i^r)$ (as an $\mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i)$ module) with r^i graded pieces isomorphic to $\mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i) \simeq \mathbf{Z}[X_{i+1},\dots,X_k]$. Now, we can do the same for the ring $\mathcal{O}_{Z_i'}$ assuming that there is enough flatness, i.e., that

$$\operatorname{Tor}_j^{\mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i)}(\mathcal{O}_{Z_i}, \mathbf{Z}[X_1,\dots,X_k]/(X_1^{a_1},\dots,X_i^{a_i})) = 0, \quad j > 0, \quad a_1,\dots,a_i \geq 1.$$

But that follows from the results of Kato [22, 6.1] in the following way

$$\begin{aligned} &\operatorname{Tor}_j^{\mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i)}(\mathcal{O}_{Z_i}, \mathbf{Z}[X_1,\dots,X_k]/(X_1^{a_1},\dots,X_i^{a_i})) = \\ &\operatorname{Tor}_j^{\mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i)}(\mathcal{O}_X \otimes_{\mathbf{Z}[X_1,\dots,X_k]} \mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i), \mathbf{Z}[X_1,\dots,X_k]/(X_1^{a_1},\dots,X_i^{a_i})) \xrightarrow{\sim} \\ &\operatorname{Tor}_j^{\mathbf{Z}[X_1,\dots,X_k]}(\mathcal{O}_X, \mathbf{Z}[X_1,\dots,X_k]/(X_1^{a_1},\dots,X_i^{a_i})) = 0. \end{aligned}$$

Hence we have a filtration of $\mathcal{O}_{Z_i'}$ by $\mathcal{O}_{Z_i}$ modules. This filtration has length r^i and the graded pieces are isomorphic to $\mathcal{O}_{Z_i} \otimes_{\mathbf{Z}[X_{i+1},\dots,X_k]} \mathbf{Z}[X_{i+1},\dots,X_k]$, where the tensor product is over the map m_r (sending X_l to X_l^r). Since the map m_r is flat, this yields (by devissage) that the map $K'_*(Z_i, \mathbf{Z}/n) \to K'_*(Z_i', \mathbf{Z}/n)$ is zero for $r^i \geq n$. Clearly $r = n$ will do. □

Mayer-Vietoris for K'-theory and the above lemma yield that the map $m_r : X \to X$ defined by $X_l \mapsto X_l^r$ kills $K'_*(Z, \mathbf{Z}/n)$ for some $r = n^j$. Since m_r is Kummer log-étale, this gives the isomorphism in our proposition (note that we can assume all the schemes Y in the limits to be regular). □

logetale **Corollary 5.17.** *Let X be a log-regular scheme satisfying condition (*). Let n be a natural number invertible on X. Then the open immersion $j : U \hookrightarrow X$, where $U = X_{\rm tr}$ is the maximal open set of X on which the log-structure is trivial, induces an isomorphism*

$$j^* : K_m^{\text{vét}}(X, \mathbf{Z}/n) \stackrel{\sim}{\to} K_m^{\text{ét}}(U, \mathbf{Z}/n), \quad m \geq 0.$$

Proof. By Theorem 5.4,

$$H^{-m}(U_{\text{ét}}, \mathbf{Z}/n) \simeq H^{-m}(U_{\text{vét}}, \mathbf{Z}/n) \simeq H^{-m}(X_{\text{vét}}, j_* K^f/n).$$

It suffices to show that the natural map of presheaves on $X_{\text{vét}}$,

$$K/n \to j_*(K^f/n)$$

induces a weak equivalence on the stalks at a conservative family of valuative log-geometric points. Recall (section 2.2) that, for $x \in U$, $U \to \operatorname{Spec}(\mathbf{Z}[P])$, a valuative log-geometric point over x can be described as a compatible system of log-geometric points of certain log-blow-ups U_J of U. Since X is log-regular, all the log-blow-ups U_J can be assumed to be regular (see [30, Thm 5.5]). On each U_J, the computations in the proof of Theorem 5.14, show that the map

$$\pi_q(K/n_{x(log)}) \to \pi_q((j_*(K^f/n))_{x(log)})$$

is an isomorphism. This finishes our proof. □

Similarly, Proposition 5.12 implies the following two corollaries.

Corollary 5.18. *Suppose that X is log-regular and satisfies condition (*). Let n be invertible on X. Then the sheaves of homotopy groups of K/n in the log-étale topology are given by*

$$\tilde{\pi}_q(K/n) \simeq \begin{cases} \mathbf{Z}/n(i) & \text{for } q = 2i \geq 0 \\ 0 & \text{for } q \geq 0, \text{ odd.} \end{cases}$$

spectral **Corollary 5.19.** *Suppose that X is log-regular and satisfies condition (*). Let n be invertible on X. Then there exists a cohomological spectral sequence $E_r^{p,q}$, $r \geq 2$, such that*

$$E_2^{p,q} = \begin{cases} H^p(X_{\text{vét}}, \mathbf{Z}/n(q/2)) & \text{for } q - p \geq 0 \text{ and } q \text{ even} \\ 0 & \text{for } q - p \geq 0 \text{ and } q \text{ odd.} \end{cases}$$

This spectral sequence converges strongly to $K_{q-p}^{\text{vét}}(X, \mathbf{Z}/n)$ for $q - p \geq 1$.

Remark 5.20. Let X_* be one of the Kummer sites studied in this paper. Consider the presheaves $K_*^0 : X \mapsto K^0(X_*)$ and $K_*^0/n : X \mapsto K^0/n(X_*)$. They are weakly equivalent to the presheaves K and K/n. Choose their fibrant resolutions $K^f, K^f/n$. For $m \geq 0$ they define functorial maps from the algebraic K-theory to topological K-theory

$$\begin{aligned} \rho_m : \quad & K_m(X_*) = \pi_m(K_*^0(X)) \to \pi_m(K^f(X)) = K_m^*(X), \\ \rho_m : \quad & K_m(X_*, \mathbf{Z}/n) = \pi_m(K_*^0/n(X)) \to \pi_m(K^f/n(X)) = K_m^*(X, \mathbf{Z}/n) \end{aligned}$$

The above yields that for a log-regular regular scheme X satisfying condition (*), a number n invertible on X, and $m \geq 0$, the map

$$\rho_m : \quad K_m(X_{\text{két}}, \mathbf{Z}/n) \to K_m^{\text{két}}(X, \mathbf{Z}/n)$$

factors through the projection

$$\pi : \quad K_m(X_{\text{két}}, \mathbf{Z}/n) \to K_m(X_{\text{két}}, \mathbf{Z}/n)/K'_m(Z_{\text{két}}, \mathbf{Z}/n),$$

where Z is the divisor at infinity. Indeed, we have the following commutative diagram

$$\begin{array}{ccc} K_m(X_{\text{két}}, \mathbf{Z}/n) & \xrightarrow{j^*} & K_m(U_{\text{ét}}, \mathbf{Z}/n) = K_m(U, \mathbf{Z}/n) \\ \downarrow{\scriptstyle \rho_m} & & \downarrow{\scriptstyle \rho_m} \\ K_m^{\text{két}}(X, \mathbf{Z}/n) & \xrightarrow[\sim]{j^*} & K_m^{\text{két}}(U, \mathbf{Z}/n), \end{array}$$

where $j : U = X_{\mathrm{tr}} \hookrightarrow X$ is the natural immersion. And our claim follows now from the localization sequence and Theorem 5.14.

Remark 5.21. Corollary 5.17 is closely related to the following absolute log-purity conjecture (see [15, 3.4.2]).

Conjecture 5.22. *Let X be a log-scheme, locally Noetherian. Assume that X is log-regular and let $j : U \hookrightarrow X$ be the open set of triviality of the log-structure of X. Assume that n is invertible on X. Then the adjunction map*

$$\mathbf{Z}/n(q) \to Rj_*j^*\mathbf{Z}/n(q)$$

is an isomorphism for any q.

Indeed, the log-purity conjecture coupled with the spectral sequences 5.19 for X and U implies Corollary 5.17. On the other hand, the usual computation with Adams operations on the spectral sequences 5.19 for X and U should imply their degeneration up to small torsion. Hence the absolute log-purity conjecture (up to small torsion).

Since log-regular schemes can be desingularized by a log-blow-up, the absolute log-purity conjecture follows easily from the following absolute purity conjecture in étale cohomology.

Conjecture 5.23. *Let $i : Y \hookrightarrow X$ be a closed immersion of Noetherian, regular schemes of pure codimension d. Let n be an integer invertible on X. Then*

$$\mathcal{H}^q_Y(X_{\text{ét}}, \mathbf{Z}/n) \simeq \begin{cases} 0 & \text{for } q \neq 2d \\ \mathbf{Z}/n(-d) & \text{for } q = 2d \end{cases}$$

This conjecture was proved by Gabber [11]. Thus to prove Corollary 5.17, we could have used spectral sequences 5.19 and evoke the purity conjecture in étale cohomology.

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