ON THE v-PICARD GROUP OF STEIN SPACES

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Abstract. We study the image of the Hodge-Tate logarithm map (in any cohomological degree), defined by Heuer, in the case of smooth Stein varieties. Heuer, motivated by the computations for the affine space of any dimension, raised the question whether this image is always equal to the group of closed differential forms. We show that it indeed always contains such forms but the quotient can be non-trivial: it contains a slightly mysterious $\mathbf{Z}_p$-module that maps, via the Bloch-Kato exponential map, to integral classes in the pro-étale cohomology. This quotient is already non-trivial for open unit discs of dimension strictly greater than 1.

Contents

1. Introduction

Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of positive characteristic p. Let C be the p-adic completion of an algebraic closure of K.

In [12], Heuer constructed a *Hodge-Tate logarithm* map HTlog such that, for any smooth rigid analytic space X over C, there is an exact sequence

$$0 \to \mathrm{Pic}_{\mathrm{an}}(X) \to \mathrm{Pic}_v(X^\diamond) \xrightarrow{\mathrm{HTlog}} \Omega^1(X)(-1), \tag{1.1}$$

where $X^\diamond$ denotes the associated diamond, and he proved the following result:

Theorem 1.2. (Heuer, [12, Th. 1.3, Th. 6.1]) *Let X be a smooth rigid analytic space over C.*

(1) If X is proper or a curve, then the map HTlog *from* (1.1) *is surjective.*

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(2) If X is the affine space $\mathbb{A}^d_C$ of dimension d, then the image of HTlog *is equal to the kernel of the differential, i.e., we have an exact sequence:*

$$0 \to \mathrm{Pic}_{\mathrm{an}}(X) \to \mathrm{Pic}_v(X^\diamond) \xrightarrow{\mathrm{HTlog}} \Omega^1(X)^{d=0}(-1) \to 0. \tag{1.3}$$

Heuer has also raised the question (see [12, Rem. 6.9]) whether we have an analog of the exact sequence (1.3) for any smooth Stein space over C, i.e., whether the image of HTlog is equal to the closed differential forms. Using a simple functoriality argument he has shown that, for all smooth rigid analytic spaces, this image contains all exact forms.

The goal of this paper is to extend Theorem 1.2 and to compute the image of the Hodge-Tate logarithm for more general Stein rigid analytic spaces. More precisely, we show the following:

Theorem 1.4. *Let X be a smooth Stein rigid analytic space over C and let $i \geq 1$. Then, the image of the restriction of the Hodge-Tate logarithm to the cohomology group of principal units*

$$\mathrm{HTlog}_U : H^i_v(X^\diamond, U) \to \Omega^i_X(X)(-i)$$

fits into a short exact sequence of abelian groups

$$0 \to \Omega^i(X)(-i)^{d=0} \to \mathrm{Im}(\mathrm{HTlog}_U) \xrightarrow{\mathrm{Exp}} \mathscr{I}^i(X) \to 0 \tag{1.5}$$

where the $\mathbf{Z}_p$-module $\mathscr{I}^i(X) \subset H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(1))$ is the intersection $\mathrm{Im}(\mathrm{Exp}) \cap \mathrm{Im}(\iota)$*, where $\iota : H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Z}_p(1)) \to H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(1))$ is the canonical map.*

The map Exp in (1.5) is the *Bloch-Kato exponential* map

$$\mathrm{Exp} : (\Omega^i(X)/\mathrm{Ker}\, d)(-i) \hookrightarrow H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(1)) \tag{1.6}$$

from the fundamental diagram of Colmez-Dospinescu-Nizioł computing the p-adic pro-étale cohomology of Stein spaces [5], [7].

In particular, the above result shows that the closed differential forms are all in the image of the morphism HTlog. In the cases where the group $\mathscr{I}^i(X)$ is non-trivial, it also shows that this image is larger than what the result for the affine space was suggesting, since it also includes the differential forms coming from the integral pro-étale cohomology. We show that group $\mathscr{I}^i(X)$ is non-trivial already in the case of the unit open disc of dimension at least 2.

The strategy we follow here is similar to the one of Heuer in the case of the affine space (see [12, Sec. 6.2]): we compare the image of the Hodge-Tate logarithm to the kernel of the map from $\Omega^i(X)(-i)$ to the pro-étale cohomology. In the work of Heuer, this map is defined as the boundary morphism coming from the fundamental exact sequence of p-adic Hodge Theory:

$$0 \to \mathbf{Q}_p(1) \to \mathbb{B}^{\varphi=p} \to \mathbb{B}^+_{\mathrm{dR}}/t \simeq \mathscr{O} \to 0$$

In the case of the affine space, Heuer was able to compute this kernel using the computation of the cohomologies of $\mathbb{B}^{\varphi=p}$ and $\mathbb{B}_{\mathrm{dR}}$ by Le Bras in [18] and the Poincaré Lemma of Scholze [20]. However, for a general Stein space, this is not sufficient: to determine completely the kernel we need to use the Bloch-Kato exponential (1.6) from Colmez-Dospinescu-Nizioł (in the form defined by Bosco in [4]); the fact that this map is injective follows from slope properties of Hyodo-Kato cohomology of X. The equality between the map Exp and the one used by Heuer is checked in Section 3.

We end the paper with a discussion of a number of examples: a torus, a Drinfeld space, analytifications of algebraic varieties, an open disc, where we try to determine the v-Picard group. Having Theorem 1.4, the main difficulty is in proving that the Picard group can be computed using the sheaf of principal units.

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Notation and conventions. Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of characteristic p. Let $\overline{K}$ be an algebraic closure of K and let $\mathscr{O}_{\overline{K}}$ denote the integral closure of $\mathscr{O}_K$ in $\overline{K}$. Let $C = \widehat{\overline{K}}$ be the p-adic completion of $\overline{K}$. Let $W(k)$ be the ring of Witt vectors of k with fraction field F (i.e., $W(k) = \mathscr{O}_F$). Set $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$ and let φ be the absolute Frobenius on $W(\bar{k})$. We will denote by $\widehat{\mathbf{B}}_{\mathrm{st}}, \mathbf{B}_{\mathrm{dR}}$ the semistable and de Rham period rings of Fontaine.

All rigid analytic spaces considered will be over K or C. We assume that they are separated, taut[1], and countable at infinity.

2. Preliminaries

We gather here the basic facts needed later on in the paper.

2.1. **Vector bundles in the v-topology.** We gather here a few facts about v-vector bundles.

Recall that the v-topology on a perfectoid space X is defined as the topology whose covers are generated by all open covers (in the analytic topology) and by all the surjective maps of affinoids (see [22, Lecture 17]). We have that all diamonds are v-sheaves. If Y is a diamond over $\mathrm{Spd}(C)$, a v-sheaf V is a v-vector bundle of rank n, $n \in \mathbf{N}$, on Y if it is a $\mathrm{GL}_n^\diamond$-torsor for the v-topology (where $\mathrm{GL}_n^\diamond$ denotes the diamond associated to the usual rigid space GL_n). If $q : X \to Y$ is a v-cover of diamonds and V a vector bundle on X then every descent datum on V is effective, i.e. the descent datum comes from a v-vector bundle on Y (see [12, Def. 2.5 and Lem. 2.6]). In particular, the v-vector bundles of rank n on a diamond Y (up to isomorphism) are classified by $H^1_v(Y, \mathrm{GL}_n^\diamond)$. In this paper, we are interested in the group of line bundles:

$$\mathrm{Pic}_v(Y) := H^1_v(Y, \mathrm{GL}_1^\diamond).$$

The diamond Y can also be equipped with the étale topology and the quasi-pro-étale topology (see [22, Sec. 9.2]). If Y comes from a rigid space X (i.e. $Y = X^\diamond$), then we have an equivalence $Y_{\text{ét}} \simeq X_{\text{ét}}$ ([22, Th. 10.4.2]) and we have the following inclusion of sites:

$$X_{\mathrm{an}} \subset X_{\text{ét}} \simeq X^\diamond_{\text{ét}} \subset X_{\text{proét}} \subset X^\diamond_{\text{qproét}} \subset X^\diamond_v.$$

If X is an affinoid perfectoid, we know from a result of Kedlaya-Liu [15, Th. 3.5.8] that the notions of vector bundles in all these topologies coincide. The pro-étale, quasi-pro-étale and v-topologies being locally affinoid perfectoid, it follows that for a general smooth rigid space X, we also have that the groups of vector bundles in these three topologies are equal. It is also known ([10, Prop. 8.2.3]) that $\mathrm{Pic}_{\mathrm{an}}(X) \simeq \mathrm{Pic}_{\text{ét}}(X)$. We are left to study the map

$$\mathrm{Pic}_{\text{ét}}(X) \hookrightarrow \mathrm{Pic}_v(X^\diamond). \tag{2.1}$$

2.2. **Topologies on X.** Let X be a smooth rigid space over C and $X^\diamond$ be the associated diamond. In the following we write X_v for the site $X_v^\diamond$. We denote by $\mathscr{O}_{\text{ét}}$, $\mathscr{O}_{\text{proét}}$ and $\mathscr{O}_v$ the structure sheaves for the étale, pro-étale and v-topology. For τ one of these topologies, we also denote by $\mathscr{O}_\tau^\times$ the sheaf of invertible functions, by $\mathscr{O}_\tau^+$ the sheaf of integral elements and by $U_\tau := 1 + \mathfrak{m}_C \mathscr{O}_\tau^+ \subset \mathscr{O}_\tau^\times$ the sheaf of principal units. Let $\overline{\mathscr{O}}^\times$ be the quotient of $\mathscr{O}_\tau^\times$ by U_τ. For $\mathscr{G} \in \{\mathscr{O}_\tau^\times, \overline{\mathscr{O}}_\tau^\times\}$, we write

[1] That is, for all quasi-compact opens V of X, the closure $\overline{V}$ of V in X is quasi-compact.

$\mathscr{G}[\frac{1}{p}]$ for the sheaf of abelian groups $\varinjlim_{x\mapsto x^p}\mathscr{G}$. If X is quasi-compact then $\mathscr{O}_\tau^\times[\frac{1}{p}](X)=\mathscr{O}_\tau^\times(X)[\frac{1}{p}]$. By [12, Lem. 2.16], we have $\overline{\mathscr{O}}_\tau^\times[\frac{1}{p}]\simeq\overline{\mathscr{O}}_\tau^\times$.

We summarize in the following proposition the various equalities that we have between the H^1-groups of these sheaves:

Proposition 2.2. *We denote by $\nu: X_v\to X_{\text{ét}}$ the canonical morphism. It decomposes as $X_v\xrightarrow{\lambda} X_{\text{proét}}\xrightarrow{\mu} X_{\text{ét}}$.*

(1) The sheaf $\mathscr{O}^\times$: we can interchange pro-étale and v-topologies:

$$H^1_{\text{proét}}(X,\mathscr{O}^\times)\xrightarrow{\sim} H^1_v(X,\mathscr{O}^\times).$$

(2) The sheaf $\overline{\mathscr{O}}_v^\times$: we can interchange all three topologies:

$$\lambda_*\overline{\mathscr{O}}_v^\times\simeq\overline{\mathscr{O}}_{\text{proét}}^\times,\quad \mathrm{R}^1\lambda_*\overline{\mathscr{O}}_v^\times=0,\quad \textit{and in particular,}\quad H^1_{\text{proét}}(X,\overline{\mathscr{O}}^\times)\xrightarrow{\sim} H^1_v(X,\overline{\mathscr{O}}^\times)$$
$$\mu_*\overline{\mathscr{O}}_{\text{proét}}^\times\simeq\overline{\mathscr{O}}_{\text{ét}}^\times,\quad \mathrm{R}^1\mu_*\overline{\mathscr{O}}_{\text{proét}}^\times=0,\quad \textit{and in particular,}\quad H^1_{\text{ét}}(X,\overline{\mathscr{O}}^\times)\xrightarrow{\sim} H^1_{\text{proét}}(X,\overline{\mathscr{O}}^\times)$$
$$\nu_*\overline{\mathscr{O}}_v^\times\simeq\overline{\mathscr{O}}_{\text{ét}}^\times,\quad \mathrm{R}^1\nu_*\overline{\mathscr{O}}_v^\times=0,\quad \textit{and in particular,}\quad H^1_{\text{ét}}(X,\overline{\mathscr{O}}^\times)\xrightarrow{\sim} H^1_v(X,\overline{\mathscr{O}}^\times).$$

(3) More generally, we have natural isomorphisms

$$\mathrm{R}\nu_*\overline{\mathscr{O}}^\times=\overline{\mathscr{O}}^\times,\quad \mathrm{R}\mu_*\overline{\mathscr{O}}^\times=\overline{\mathscr{O}}^\times. \tag{2.3}$$

Proof. The first point follows from the result of Kedlaya-Liu [15, Th. 3.5.8], as explained above. For the second point, see the proof of Lemma 2.22 in [12]. The third claim is proved in [13, Th. 1.7, Cor. 2.11]. □

We recall now the exponential and logarithm maps from [12]. They will be used to define the Hodge-Tate logarithm HTlog. The logarithm exact sequence stated below will play an important role in the computation of the cokernel of the map (2.1) as it will allow us to compare it to the p-adic pro-étale cohomology.

The usual p-adic exponential and logarithm maps $\exp(x)=\sum_n x^n/n!$ and $\log(x)=(-1)^n(x-1)/n$ define morphisms of sheaves

$$\exp: p'\mathscr{O}^+\to 1+p'\mathscr{O}^+ \text{ and } \log: 1+\mathfrak{m}\mathscr{O}^+\to\mathscr{O}$$

where $p'=p$ if $p>2$ and $p'=4$ if $p=2$, such that $\log(1+p'\mathscr{O}^+)\subset p'\mathscr{O}^+$, $\exp\circ\log=\mathrm{Id}$ on $1+p'\mathscr{O}^+$ and $\log\circ\exp=\mathrm{Id}$ on $\mathscr{O}^+$. We have the following result:

Lemma 2.4. (Heuer, [12, Lem. 2.18, Lem. 2.21]) *Let X be a smooth rigid space over C and $\nu: X_v\to X_{\text{ét}}$ and $\mu: X_{\text{proét}}\to X_{\text{ét}}$ as before.*

(1) For $\tau\in\{v,\text{proét}\}$, there are exact sequences of sheaves on X_τ:

$$\begin{aligned} &1\to(\mathbf{Q}_p/\mathbf{Z}_p)(1)\to U_\tau\xrightarrow{\log}\mathscr{O}_\tau\to 1,\\ &1\to\mathscr{O}_\tau\xrightarrow{\exp}\mathscr{O}_\tau^\times[\tfrac{1}{p}]\to\bar{\mathscr{O}}_\tau^\times\to 1.\end{aligned} \tag{2.5}$$

(2) Let $i\geq 1$. The maps (2.5) *and the isomorphisms* (2.3) *induce natural isomorphisms*

$$\log:\mathrm{R}^i\nu_*U\xrightarrow{\sim}\mathrm{R}^i\nu_*\mathscr{O}\quad \textit{and}\quad \mathrm{R}^i\mu_*U\xrightarrow{\sim}\mathrm{R}^i\mu_*\mathscr{O};$$
$$\exp:\mathrm{R}^i\nu_*\mathscr{O}\xrightarrow{\sim}\mathrm{R}^i\nu_*\mathscr{O}^\times\quad \textit{and}\quad \mathrm{R}^i\mu_*\mathscr{O}\xrightarrow{\sim}\mathrm{R}^i\mu_*\mathscr{O}^\times.$$

2.3. **The Hodge-Tate logarithm.** We recall here the definition of the Hodge-Tate logarithm from [12].

Proposition 2.6 (Hodge-Tate morphisms)**.** *Let X be a smooth rigid space over C, $\nu: X_v\to X_{\text{ét}}$ and $\mu: X_{\text{proét}}\to X_{\text{ét}}$ as before. Then for all $i\geq 0$, there are $\mathscr{O}_X$-linear isomorphisms:*

$$\mathrm{HT}:\mathrm{R}^i\nu_*\mathscr{O}\xrightarrow{\sim}\Omega^i_X(-i),\quad \mathrm{HT}:\mathrm{R}^i\mu_*\mathscr{O}\xrightarrow{\sim}\Omega^i_X(-i)\quad \textit{on } X_{\text{ét}}.$$

For μ, the result is due to Scholze in [20, Cor. 6.19], [21, Prop. 3.23]: the Hodge-Tate morphism HT is defined as the inverse of the connecting morphism in the Faltings extension (see Section 3.1.3). We obtain the result for ν using that $\mathrm{R}\lambda_*\mathscr{O}_v = \mathscr{O}_{\text{proét}}$.

The Leray spectral sequence for the morphism $\nu : X_v \to X_{\text{ét}}$ and the sheaf $\mathscr{O}^\times$

$$E_2^{i,j} = H^i_{\text{ét}}(X, \mathrm{R}^j\nu_*\mathscr{O}^\times) \Rightarrow H^{i+j}_v(X, \mathscr{O}^\times) \tag{2.7}$$

induces an exact sequence:

$$0 \to H^1_{\text{ét}}(X, \nu_*\mathscr{O}^\times) \to H^1_v(X, \mathscr{O}^\times) \to H^0_{\text{ét}}(X, \mathrm{R}^1\nu_*\mathscr{O}^\times) \to H^2_{\text{ét}}(X, \nu_*\mathscr{O}^\times) \to H^2_v(X, \mathscr{O}^\times). \tag{2.8}$$

For $i \geq 1$, we define the Hodge-Tate logarithm

$$\mathrm{HTlog}_i : H^i_v(X, \mathscr{O}^\times) \to H^0_{\text{ét}}(X, \Omega^i_X(-i))$$

as the composition:

$$\mathrm{HTlog}_i : H^i_v(X, \mathscr{O}^\times) \to H^0_{\text{ét}}(X, \mathrm{R}^i\nu_*\mathscr{O}^\times) \xleftarrow[\sim]{\exp} H^0_{\text{ét}}(X, \mathrm{R}^i\nu_*\mathscr{O}) \xrightarrow{\sim} H^0_{\text{ét}}(X, \Omega^i_X(-i)), \tag{2.9}$$

where the first arrow is the edge map of the Leray spectral sequence (2.7), the second one is the exponential map from Lemma 2.4, and the third one is the isomorphism from Proposition 2.6. We obtain an analogous morphism replacing the v-topology by the pro-étale one.

Since $\nu_*\mathscr{O}^\times = \mathscr{O}^\times$ and using the map (2.9), we can rewrite the exact sequence (2.8) as

$$0 \to \mathrm{Pic}_{\text{an}}(X) \to \mathrm{Pic}_v(X) \xrightarrow{\mathrm{HTlog}} \Omega^1(X)(-1) \to H^2_{\text{ét}}(X, \mathscr{O}^\times) \to H^2_v(X, \mathscr{O}^\times)$$

Remark 2.10. By Lemma 2.4, we can also consider the restriction of HTlog to U and use the pro-étale and v-topology interchangeably. For $i \geq 1$, we define the Hodge-Tate logarithm

$$\mathrm{HTlog}_U : H^i_v(X, U) \to H^0_{\text{ét}}(X, \Omega^i_X(-i))$$

similarly as the composition:

$$\mathrm{HTlog_U} : H^i_v(X, U) \to H^0_{\text{ét}}(X, \mathrm{R}^i\nu_*U) \xrightarrow[\sim]{\log} H^0_{\text{ét}}(X, \mathrm{R}^i\nu_*\mathscr{O}) \xrightarrow{\sim} H^0_{\text{ét}}(X, \Omega^i_X(-i)). \tag{2.11}$$

It is compatible with the map HTlog from (2.9).

Remark 2.12. For a Stein space, we know that for a coherent sheaf $\mathscr{F}$, the group $H^i_{\text{ét}}(X, \mathscr{F})$ is zero for $i > 0$. In particular, we obtain that $H^i_{\text{ét}}(X, \mathrm{R}^j\mu_*\mathscr{O}) = 0$, $i \geq 1$, and hence the Leray spectral sequence for $\mu : X_{\text{proét}} \to X_{\text{ét}}$ and the sheaf $\mathscr{O}$, induces an isomorphism

$$H^i_{\text{proét}}(X, \mathscr{O}) \xrightarrow{\sim} H^0_{\text{ét}}(X, \mathrm{R}^i\mu_*\mathscr{O}).$$

In the following, when X is Stein, we still write HT for the composition:

$$\mathrm{HT} : H^i_{\text{proét}}(X, \mathscr{O}) \xrightarrow{\sim} H^0_{\text{ét}}(X, \mathrm{R}^i\mu_*\mathscr{O}) \xrightarrow{\sim} \Omega^i(X)(-i).$$

And similarly for the v-topology:

$$\mathrm{HT} : H^i_v(X, \mathscr{O}) \xrightarrow{\sim} H^0_{\text{ét}}(X, \mathrm{R}^i\nu_*\mathscr{O}) \xrightarrow{\sim} \Omega^i(X)(-i).$$

2.4. **The Leray spectral sequence for Stein spaces.** We show here that, in the case of smooth Stein spaces, the Leray spectral sequence for the projection $\nu : X_v \to X_{\text{ét}}$ and the sheaf $\mathscr{O}^\times$ simplifies enormously.

Proposition 2.13. *Let X be a smooth Stein rigid analytic variety of dimension d over C. Then:*

(1) $H^0_{\text{ét}}(X, \mathscr{O}^\times) \simeq H^0_v(X, \mathscr{O}^\times)$ and $H^i_{\text{ét}}(X, \mathscr{O}^\times) \simeq H^i_v(X, \mathscr{O}^\times)$ for $i \geq d+2$.

(2) We have exact sequences:

$$0 \to H^1_{\text{ét}}(X, \mathscr{O}^\times) \to H^1_v(X, \mathscr{O}^\times) \xrightarrow{\mathrm{HTlog}_1} \mathrm{Ker}(d_2^\times : \Omega^1(X)(-1) \to H^2_{\text{ét}}(X, \mathscr{O}^\times)) \to 0$$

$$0 \to H^i_{\text{ét}}(X, \mathscr{O}^\times)/\mathrm{Im}\, d_i^\times \to H^i_v(X, \mathscr{O}^\times) \xrightarrow{\mathrm{HTlog}_i} \mathrm{Ker}(d_{i+1}^\times : \Omega^i(X)(-i) \to H^{i+1}_{\text{ét}}(X, \mathscr{O}^\times)) \to 0 \textit{ for all } d \geq i \geq 2.$$

(3) $H^{d+1}_{\text{ét}}(X, \mathscr{O}^\times)/\mathrm{Im}\, d_{d+1}^\times \simeq H^d_v(X, \mathscr{O}^\times)$.

Here the maps $d^{\times}_{i+1}:\Omega^i(X)(-i)\to H^{i+1}_{\text{ét}}(X,\mathscr{O}^{\times})$ (for $i\geq 1$) are the maps given by the composition

$$d^{\times}_{i+1}:\quad \Omega^i(X)(-i)\xleftarrow[\sim]{\mathrm{HT}} H^0(X,\mathrm{R}^i\nu_*\mathscr{O})\xrightarrow[\sim]{\exp} H^0(X,\mathrm{R}^i\nu_*\mathscr{O}^{\times})\xrightarrow{d_{i+1}} H^{i+1}_{\text{ét}}(X,\mathscr{O}^{\times}),\tag{2.14}$$

where d_{i+1} is the only differential on the E_{i+1}-page of the Leray spectral sequence (see the proof):

$$E_2^{i,j}=H^i_{\text{ét}}(X,\mathrm{R}^j\nu_*\mathscr{O}^{\times})\Rightarrow H^{i+j}_v(X,\mathscr{O}^{\times})$$

Proof. We analyze the terms of the above spectral sequence.

(i) *The E_2-page.* For all $i,j\geq 1$, we have isomorphisms

$$H^i_{\text{ét}}(X,\mathrm{R}^j\nu_*\mathscr{O}^{\times})\xrightarrow{\sim} H^i_{\text{ét}}(X,\Omega^j_X(-j))$$

and the term on the right is zero since X is Stein. So, on the page E_2, the only non-zero terms will be in the row $j=0$ and column $i=0$ (for $0\leq j\leq d$) and we have:

$$E_2^{i,0}=H^i_{\text{ét}}(X,\mathscr{O}^{\times})\text{ and }E_2^{0,j}=H^0_{\text{ét}}(X,\mathrm{R}^j\nu_*\mathscr{O}^{\times})\text{ for all }i\geq 0,d\geq j\geq 1.$$

There is only one non-zero differential $d_2:H^0_{\text{ét}}(X,\mathrm{R}^1\nu_*\mathscr{O}^{\times})\to H^2_{\text{ét}}(X,\mathscr{O}^{\times})$.

(ii) *The E_3-page.* The terms $E_3^{i,j}$ are equal to $E_2^{i,j}$ except for $(i,j)\in\{(2,0),(0,1)\}$ where they are

$$E_3^{2,0}\simeq H^2_{\text{ét}}(X,\mathscr{O}^{\times})/\operatorname{Im}d_2\text{ and }E_3^{0,1}\simeq\operatorname{Ker}(d_2).$$

There is only one non-zero differential $d_3:H^0_{\text{ét}}(X,\mathrm{R}^2\nu_*\mathscr{O}^{\times})\to H^3_{\text{ét}}(X,\mathscr{O}^{\times})$.

(iii) *The E_∞-page.* Iterating the above computation, we get that the spectral sequence degenerates at $d+2$ and we have:

$$\begin{aligned}
&E_\infty^{0,0}=H^0_{\text{ét}}(X,\mathscr{O}^{\times}),\quad E_\infty^{1,0}=H^1_{\text{ét}}(X,\mathscr{O}^{\times})\\
&E_\infty^{i,0}\simeq H^i_{\text{ét}}(X,\mathscr{O}^{\times})/\operatorname{Im}d_i,\ \text{ for }d+1\geq i\geq 2,\\
&E_\infty^{i,0}\simeq H^i_{\text{ét}}(X,\mathscr{O}^{\times}),\ \text{ for }i\geq d+2,\\
&E_\infty^{0,j}\simeq\operatorname{Ker}(d_{j+1}:H^0_{\text{ét}}(X,\mathrm{R}^j\nu_*\mathscr{O}^{\times})\to H^{j+1}_{\text{ét}}(X,\mathscr{O}^{\times})),\ \text{ for }d\geq j\geq 1,
\end{aligned}$$

as wanted. □

Corollary 2.15. *Let X be a smooth Stein space of dimension d over C. For $d\geq i\geq 2$, there is an exact sequence:*

$$0\to\operatorname{Coker}(\mathrm{HTlog}_{i-1})\to H^i_{\text{ét}}(X,\mathscr{O}^{\times})\xrightarrow{\nu_i^*}\operatorname{Ker}(\mathrm{HTlog}_i)\to 0,\tag{2.16}$$

where ν_i^* is the pullback map $H^i_{\text{ét}}(X,\mathscr{O}^{\times})\to H^i_v(X,\mathscr{O}^{\times})$.

Proof. Let $d\geq i\geq 2$. The second exact sequence from Proposition 2.13 shows that ν_i^* factorizes through $H^i_{\text{ét}}(X,\mathscr{O}^{\times})/\operatorname{Im}d_i^{\times}$ and that its image is equal to $\operatorname{Ker}(\mathrm{HTlog}_i)$. Hence we get the surjectivity on the right in (2.16).

Let us now compute the kernel of ν_i^*. By Proposition 2.13, it is equal to the image of the map

$$d_i^{\times}:\Omega^{i-1}(X)(-i+1)\to H^i_{\text{ét}}(X,\mathscr{O}^{\times}).$$

Using the second exact sequences from Proposition 2.13 but in degree $i-1$, we obtain that the kernel of $d_i^{\times}$ is equal to the image of HTlog_{i-1}. This gives an isomorphism:

$$d_i^{\times}:\operatorname{Coker}(\mathrm{HTlog}_{i-1})\xrightarrow{\sim}\operatorname{Im}(d_i^{\times}),$$

as wanted. □

3. Comparison of two boundary maps

To compute the image of the Hodge-Tate logarithm, we will relate it to the kernel of the boundary morphism appearing in the fundamental diagram from [5, Th. 1.8], [7, Th. 5.14]. In order to check the compatibility between the morphism HTlog and the map from [7], we use the alternative definition of the latter given by Bosco in [3]. We start this section by recalling briefly the construction of the two maps.

3.1. **Hodge-Tate map revisited.** We express here the Hodge-Tate map as a Poincaré Lemma projection map.

3.1.1. *Poincaré Lemma.* We first state the Poincaré Lemma for the de Rham period sheaves $\mathbb{B}_{\mathrm{dR}}^+$ and $\mathbb{B}_{\mathrm{dR}}$ from [20]. We start by recalling the definitions of the various period sheaves and some of their properties. We work here on the pro-étale site of a locally noetherian adic space X over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$.

The Fontaine period sheaf $\mathbb{A}_{\mathrm{inf}}$ is defined as the sheaf $W(\mathscr{O}_{\mathrm{proét}}^{\flat,+})$. It comes with a morphism $\theta : \mathbb{A}_{\mathrm{inf}} \to \mathscr{O}^+$. We write $\mathbb{B}_{\mathrm{inf}} := \mathbb{A}_{\mathrm{inf}}[\frac{1}{p}]$. The morphism θ extends to $\theta : \mathbb{B}_{\mathrm{inf}} \to \mathscr{O}^+$. The de Rham period sheaf

$$\mathbb{B}_{\mathrm{dR}}^+ := \lim_n \mathbb{B}_{\mathrm{inf}}/(\mathrm{Ker}(\theta)^n$$

admits a filtration $\mathrm{Fil}^i\mathbb{B}_{\mathrm{dR}}^+ := \mathrm{Ker}(\theta)^i\mathbb{B}_{\mathrm{dR}}^+$. Let t be a generator of $\mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}}^+$. We set $\mathbb{B}_{\mathrm{dR}} := \mathbb{B}_{\mathrm{dR}}^+[t^{-1}]$ and equip it with the induced filtration. The morphism θ induces an isomorphism $\mathbb{B}_{\mathrm{dR}}^+/t \xrightarrow{\sim} \mathscr{O}$.

For $0 < u \leq v$, we define $\mathbb{A}^{[u,v]}$ as the p-adic completion of the sheaf $\mathbb{A}_{\mathrm{inf}}[\frac{p}{[\alpha]}, \frac{[\beta]}{p}]$ for elements α and β in $\mathscr{O}_{C^\flat}$ such that $v(\alpha) = \frac{1}{v}$ and $v(\beta) = \frac{1}{u}$ and

$$\mathbb{B} := \lim_{0<u\leq v} \mathbb{A}^{[u,v]}.$$

We have the relative fundamental exact sequence of p-adic Hodge theory:

$$\begin{aligned} 0 \to \mathbf{Q}_p \to \mathbb{B}[\tfrac{1}{t}]^{\varphi=1} \to \mathbb{B}_{\mathrm{dR}}/\mathbb{B}_{\mathrm{dR}}^+ \to 0 \\ 0 \to \mathbf{Q}_p(r) \to \mathbb{B}^{\varphi=p^r} \to \mathbb{B}_{\mathrm{dR}}^+/t^r\mathbb{B}_{\mathrm{dR}}^+ \to 0, \end{aligned} \tag{3.1}$$

for all $r \geq 1$. We note that the map $\mathbb{B}^{\varphi=p} \to \mathbb{B}_{\mathrm{dR}}^+/t^r\mathbb{B}_{\mathrm{dR}}^+ \underset{\sim}{\xrightarrow{\theta}} \mathscr{O}$ can be identified, via the identification of $\mathbb{B}^{\varphi=p^r}$ with the C-points of the universal cover of the multiplicative p-divisible group, with the composition (see [18, Prop. 2.20, Rem. 2.21, Example 2.22]):

$$\lim_{x\mapsto x^p}(1+\mathfrak{m}\mathscr{O}) = U^\flat \xrightarrow{x\mapsto x^\sharp} U \xrightarrow{\log} \mathscr{O}, \tag{3.2}$$

where the first map is the sharp map given by the projection on the first factor.

Similarly, for a smooth adic space X over K, we define[2] $\mathscr{O}\mathbb{B}_{\mathrm{inf}} := \mu^*\mathscr{O}_{\mathrm{ét}} \otimes^{\square}_{W(k)} \mathbb{B}_{\mathrm{inf}}$. We still have a map $\theta : \mathscr{O}\mathbb{B}_{\mathrm{inf}} \to \mathscr{O}_{\mathrm{proét}}$. Then we set:

$$\mathscr{O}\mathbb{B}_{\mathrm{dR}}^+ := \lim_n \mathscr{O}\mathbb{B}_{\mathrm{inf}}/\mathrm{Ker}(\theta)^n, \quad \mathscr{O}\mathrm{Fil}^r\mathbb{B}_{\mathrm{dR}}^+ := \mathrm{Ker}(\theta)^r\mathscr{O}\mathbb{B}_{\mathrm{dR}}^+.$$

Finally, for a generator t of $\mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}}^+$, we take $\mathscr{O}\mathbb{B}_{\mathrm{dR}}^+[t^{-1}]$ and equip it with the filtration induced from $\mathscr{O}\mathbb{B}_{\mathrm{dR}}^+$. Let $\mathscr{O}\mathbb{B}_{\mathrm{dR}}$ be the completion of $\mathscr{O}\mathbb{B}_{\mathrm{dR}}^+[t^{-1}]$ with respect to this filtration. [3]

Theorem 3.3 (Poincaré Lemma)**.** [20, Cor. 6.13], [9, Cor. 2.4.2] *Let X be a smooth rigid space of dimension d over K. Then:*

[2]Here the $\square$ refers to the solid tensor product.

[3]This definition of $\mathscr{O}\mathbb{B}_{\mathrm{dR}}$ is due to [9, Def. 2.2.10, Rem. 2.2.11]. The Poincaré Lemma is still valid in this setting and all the arguments of [20], [21] remain essentially unchanged.

(1) *There are exact sequences of pro-étale sheaves on X:*

$$0 \to \mathbb{B}^+_{\mathrm{dR}} \to \mathscr{O}\mathbb{B}^+_{\mathrm{dR}} \xrightarrow{\nabla} \mathscr{O}\mathbb{B}^+_{\mathrm{dR}} \otimes^{\square}_{\mu^*\mathscr{O}} \Omega^1 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathscr{O}\mathbb{B}^+_{\mathrm{dR}} \otimes^{\square}_{\mu^*\mathscr{O}} \Omega^d \to 0, \tag{3.4}$$

$$0 \to \mathrm{Fil}^r\mathbb{B}^+_{\mathrm{dR}} \to \mathrm{Fil}^r\mathscr{O}\mathbb{B}^+_{\mathrm{dR}} \xrightarrow{\nabla} \mathrm{Fil}^{r-1}\mathscr{O}\mathbb{B}^+_{\mathrm{dR}} \otimes^{\square}_{\mu^*\mathscr{O}} \Omega^1 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathrm{Fil}^{r-d}\mathscr{O}\mathbb{B}^+_{\mathrm{dR}} \otimes^{\square}_{\mu^*\mathscr{O}} \Omega^d \to 0$$

for all $r \in \mathbf{Z}$, where $\Omega^i := \mu^*\Omega^i_{\text{ét}}$ for $i \geq 1$ (recall that μ is the canonical projection[4] *$\mu : X_{\text{proét}} \to X_{\text{ét}}$). We have analogues of the exact sequences in* (3.4) *for $\mathbb{B}_{\mathrm{dR}}$ and $\mathscr{O}\mathbb{B}_{\mathrm{dR}}$.*

(2) *For $r \in \mathbf{Z}$, the quotient complex*

$$0 \to \mathrm{gr}^r_F\, \mathbb{B}_{\mathrm{dR}} \to \mathrm{gr}^r_F\, \mathscr{O}\mathbb{B}_{\mathrm{dR}} \xrightarrow{\nabla} \mathrm{gr}^{r-1}_F\, \mathscr{O}\mathbb{B}_{\mathrm{dR}} \otimes^{\square}_{\mu^*\mathscr{O}} \Omega^1 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathrm{gr}^{r-d}_F\, \mathscr{O}\mathbb{B}_{\mathrm{dR}} \otimes^{\square}_{\mu^*\mathscr{O}} \Omega^d \to 0$$

is exact and can be identified with the complex

$$0 \to \mathscr{O}(r) \to \mathscr{O}\mathbb{C}(r) \xrightarrow{\nabla} \mathscr{O}\mathbb{C}(r) \otimes^{\square}_{\mu^*\mathscr{O}} \Omega^1(-1) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathscr{O}\mathbb{C}(r) \otimes^{\square}_{\mu^*\mathscr{O}} \Omega^d(-d) \to 0,$$

where we set $\mathscr{O}\mathbb{C} := \mathrm{gr}^0_F\, \mathscr{O}\mathbb{B}_{\mathrm{dR}}$. We note that, by [20, Ch. 6], *$\mathrm{gr}^i_F\, \mathscr{O}\mathbb{B}_{\mathrm{dR}} \simeq \mathscr{O}\mathbb{C}(i)$.*

We denote by ε the map $\mathrm{gr}^0_F\, \mathbb{B}_{\mathrm{dR}} \to \mathrm{gr}^0_F\, \mathscr{O}\mathbb{B}_{\mathrm{dR}}$. Using that $\mu_*(\operatorname{coker}\varepsilon) \to \Omega^1(-1)$ is an isomorphism, we get a long exact sequence:

$$0 \to \mu_*\mathrm{gr}^0_F\mathbb{B}_{\mathrm{dR}} \to \mu_*\mathrm{gr}^0_F\mathscr{O}\mathbb{B}_{\mathrm{dR}} \to \Omega^1(-1) \to \mathrm{R}^1\mu_*\mathrm{gr}^0_F\mathbb{B}_{\mathrm{dR}} \to \mathrm{R}^1\mu_*\mathrm{gr}^0_F\mathscr{O}\mathbb{B}_{\mathrm{dR}}.$$

We denote by PL^{-1} the connecting morphism:

$$\mathrm{PL}^{-1} : \Omega^1(-1) \to \mathrm{R}^1\mu_*(\mathbb{B}^+_{\mathrm{dR}}/t).$$

It will follow from Lemma 3.13 below that PL^{-1} is an isomorphism and we write PL for its inverse.

The above construction can be made more explicit. Theorem 3.3 has the following useful corollary (see [18, Rem. 3.18] or [3, (6.4), (6.5)] for a more general statement):

Proposition 3.5. *Let X be a smooth rigid space of dimension d over K and let $r \in \mathbf{Z}$. Let $\mu : X_{C,\text{proét}} \to X_{C,\text{ét}}$ be the canonical projection. Then we have the following quasi-isomorphisms (the differentials are $\mathbf{B}_{\mathrm{dR}}$-linear):*

$$\mathrm{R}\mu_*(\mathbb{B}_{\mathrm{dR}}) \xleftarrow{\sim} \big(\mathscr{O}\otimes^{\square}_K\mathbf{B}_{\mathrm{dR}} \xrightarrow{d} \Omega^1\otimes^{\square}_K\mathbf{B}_{\mathrm{dR}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^d\otimes^{\square}_K\mathbf{B}_{\mathrm{dR}}\big)$$

$$\mathrm{R}\mu_*(\mathrm{Fil}^r\mathbb{B}_{\mathrm{dR}}) \xleftarrow{\sim} \big(\mathscr{O}\otimes^{\square}_K\mathrm{Fil}^r\mathbf{B}_{\mathrm{dR}} \xrightarrow{d} \Omega^1\otimes^{\square}_K\mathrm{Fil}^{r-1}\mathbf{B}_{\mathrm{dR}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^d\otimes^{\square}_K\mathrm{Fil}^{r-d}\mathbf{B}_{\mathrm{dR}}\big).$$

These are topological quasi-isomorphisms, i.e., more specifically, quasi-isomorphisms in the ∞-derived category of sheaves with values in solid K-modules. We will denote this category by $\mathscr{D}(X_{\text{ét}}, K_{\square})$. In particular, it follows from this proposition, that the pro-étale cohomologies of $\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}}$ and $\mathbb{B}^+_{\mathrm{dR}}/t$ are computed by the following complexes on $X_{C,\text{ét}}$:

(3.6)

$$\mathrm{R}\mu_*(\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}}) \xleftarrow{\sim} \big(\mathscr{O}\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \xrightarrow{d} \Omega^1\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-1}\mathbf{B}^+_{\mathrm{dR}}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^d\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-d}\mathbf{B}^+_{\mathrm{dR}})\big),$$

$$\mathrm{R}\mu_*(\mathbb{B}^+_{\mathrm{dR}}/t) \xleftarrow{\sim} \big(\mathscr{O}\otimes^{\square}_K C \xrightarrow{0} \Omega^1\otimes^{\square}_K C(-1) \xrightarrow{0} \cdots \xrightarrow{0} \Omega^d\otimes^{\square}_K C(-d)\big).$$

The maps

$$\mathrm{PL}: \quad \mathrm{R}\mu_*(\mathbb{B}^+_{\mathrm{dR}}/t) \to \Omega^i\otimes^{\square}_K C(-i)[-i], \quad \mathrm{PL}: \quad H^i_{\text{proét}}(X_C, \mathbb{B}^+_{\mathrm{dR}}/t) \to \Omega^i(X_C)(-i) \tag{3.7}$$

are given by the canonical projections.

[4]The notation here is slightly different from the one of [20] since the pro-étale sheaf that we denote by $\mathscr{O}_{\text{proét}}$ is the completion of $\mu^*\mathscr{O}_{\text{ét}}$.

3.1.2. *Stein spaces.* We now apply the above computations to the case when X is a Stein space defined over K. The quasi-isomorphism (3.6) yields the quasi-isomorphism

$$\mathrm{R}\Gamma_{\text{proét}}(X_C,\mathbb{B}^+_{\mathrm{dR}}/t)\simeq\big(\mathscr{O}(X)\otimes^{\square}_K C\xrightarrow{0}\Omega^1(X)\otimes^{\square}_K C(-1)\xrightarrow{0}\cdots\xrightarrow{0}\Omega^d(X)\otimes^{\square}_K C(-d)\big). \tag{3.8}$$

The case of $\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}}$ is a bit subtler because of the usual problems with topological tensor products and limits, respectively colimits. Let $\{X_n\}$, $n\in\mathbf{N}$, be a strictly increasing dagger affinoid covering of X (i.e., we have $X_n\Subset X_{n+1}$: the adic closure of X_n is contained in X_{n+1}). Then a dagger analogue of (3.6) yields a quasi-isomorphism

$$\mathrm{R}\Gamma_{\text{proét}}(X_{n,C},\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})\simeq\big(\mathscr{O}(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}})\to\Omega^1(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-1}\mathbf{B}^+_{\mathrm{dR}})\to\cdots\to\Omega^d(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-d}\mathbf{B}^+_{\mathrm{dR}})\big).$$

Passing to the limit over n we obtain a quasi-ismorphism

$$\mathrm{R}\Gamma_{\text{proét}}(X_C,\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})\simeq\mathrm{R}\lim_n\big(\mathscr{O}(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}})\to\Omega^1(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-1}\mathbf{B}^+_{\mathrm{dR}})\to\cdots\to\Omega^d(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-d}\mathbf{B}^+_{\mathrm{dR}})\big).$$

We used here the fact that $\mathrm{R}\Gamma_{\text{proét}}(X_C,\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})\xrightarrow{\sim}\mathrm{R}\lim_n\mathrm{R}\Gamma_{\text{proét}}(X_{n,C},\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})$.

For $i,n\geq 0$, we have

$$H^i_{\text{proét}}(X_{n,C},\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})=\operatorname{Ker}d_i/\operatorname{Im}d_{i-1},$$
$$\Omega^{i-1}(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-i+1}\mathbf{B}^+_{\mathrm{dR}})\xrightarrow{d_{i-1}}\Omega^i(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-i}\mathbf{B}^+_{\mathrm{dR}})\xrightarrow{d_i}\Omega^{i+1}(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-i-1}\mathbf{B}^+_{\mathrm{dR}}).$$

This yields an exact sequence and an isomorphism

$$0\to\Omega^i(X_n)^{d=0}\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-i}\mathbf{B}^+_{\mathrm{dR}})\to\operatorname{Ker}d_i\to(\Omega^i(X_{n,C})/\operatorname{Ker}d)(-i-1)\to 0,$$
$$\operatorname{Im}d_{i-1}\simeq\operatorname{Im}d\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-i}\mathbf{B}^+_{\mathrm{dR}}).$$

Putting them together we obtain the exact sequence

$$0\to H^i_{\mathrm{dR}}(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-i}\mathbf{B}^+_{\mathrm{dR}})\to H^i_{\text{proét}}(X_{n,C},\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})\to(\Omega^i(X_{n,C})/\operatorname{Ker}d)(-i-1)\to 0 \tag{3.9}$$

We note here that $H^i_{\mathrm{dR}}(X_n)$ is of finite rank over K. Passing to the limit over n we get the exact sequence

$$0\to\lim_n(H^i_{\mathrm{dR}}(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-i}\mathbf{B}^+_{\mathrm{dR}}))\to\lim_n H^i_{\text{proét}}(X_{n,C},\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})\to(\Omega^i(X_C)/\operatorname{Ker}d)(-i-1)\to 0.$$

The exactness on the right follows from the fact that $\mathrm{R}^1\lim_n(H^i_{\mathrm{dR}}(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-i}\mathbf{B}^+_{\mathrm{dR}}))=0$ because the pro-system $\{H^i_{\mathrm{dR}}(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-i}\mathbf{B}^+_{\mathrm{dR}})\}_{n\in\mathbf{N}}$ is Mittag-Leffler. We note that, since $\mathrm{R}^1\lim_n(\Omega^i(X_{n,C})/\operatorname{Ker}d)=0$, we also have

$$H^i_{\text{proét}}(X_C,\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})\xrightarrow{\sim}H^i(\mathrm{R}\lim_n\mathrm{R}\Gamma_{\text{proét}}(X_{n,C},\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}}))\simeq\lim_n H^i_{\text{proét}}(X_{n,C},\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}}).$$

Hence we have obtained an exact sequence

$$0\to\lim_n(H^i_{\mathrm{dR}}(X_n)\otimes^{\square}_K(\mathbf{B}_{\mathrm{dR}}/t^{-i}\mathbf{B}^+_{\mathrm{dR}}))\to H^i_{\text{proét}}(X_C,\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})\xrightarrow{\pi}(\Omega^i(X_C)/\operatorname{Ker}d)(-i-1)\to 0 \tag{3.10}$$

Remark 3.11. If we do not assume that X is the base change of a variety defined over K, we still get the maps PL and π from (3.7) and (3.10):

$$\mathrm{PL}:\quad H^i_{\text{proét}}(X,\mathbb{B}^+_{\mathrm{dR}}/t)\to\Omega^i(X)(-i),$$
$$\pi:\quad H^i_{\text{proét}}(X,\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})\to(\Omega^i(X)/\operatorname{Ker}d)(-i-1).$$

Indeed, for a covering $\{X_n\}_n$ as above, since the X_n's are dagger affinoids, they are defined over finite extensions K_n of K and we still have the exact sequence (3.9) (replacing K by the K_n for each $n\in\mathbf{N}$). By taking limits over n, we obtain the map π:

$$\pi: H^i_{\text{proét}}(X,\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})\to\lim_n H^i_{\text{proét}}(X_n,\mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})\to\lim_n(\Omega^i(X_n)/\operatorname{Ker}d)(-i-1)$$
$$\xleftarrow{\sim}(\Omega^i(X)/\operatorname{Ker}d)(-i-1).$$

Similarly for the map PL. We have a dagger analog of (3.8):

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_n,\mathbb{B}^+_{\mathrm{dR}}/t)\simeq\big(\mathscr{O}(X_{n,K_n})\otimes^{\square}_{K_n}C\xrightarrow{0}\Omega^1(X_{n,K_n})\otimes^{\square}_{K_n}C(-1)\xrightarrow{0}\cdots\xrightarrow{0}\Omega^d(X_{n,K_n})\otimes^{\square}_{K_n}C(-d)\big).$$

This yields the maps $\mathrm{PL}_n: H^i_{\mathrm{pro\acute{e}t}}(X_n,\mathbb{B}^+_{\mathrm{dR}}/t)\to\Omega^i(X_n)(-i)$. Passing to the limit over n, we get the map

$$\mathrm{PL}:\quad H^i_{\mathrm{pro\acute{e}t}}(X,\mathbb{B}^+_{\mathrm{dR}}/t)\to\lim_n H^i_{\mathrm{pro\acute{e}t}}(X_n,\mathbb{B}^+_{\mathrm{dR}}/t)\xrightarrow{\mathrm{PL}_n}\lim_n\Omega^i(X_n)(-i)\xleftarrow{\sim}\Omega^i(X)(-i).$$

3.1.3. *Hodge-Tate morphism revisited.* Let X be a smooth rigid analytic space over K. A consequence of Theorem 3.3 is the following short exact sequence of pro-étale sheaves (called Faltings extension) on $X_{\mathrm{pro\acute{e}t}}$:

$$0\to\mathscr{O}(1)\to\mathrm{gr}^1_F\mathscr{O}\mathbb{B}^+_{\mathrm{dR}}\to\mathscr{O}\otimes^{\square}_{\mu^*\mathscr{O}}\Omega^1\to 0,$$

which yields the boundary map

$$\mathscr{O}\otimes^{\square}_{\mu^*\mathscr{O}}\Omega^1\to\mathscr{O}(1)[1]. \tag{3.12}$$

Then the Hodge-Tate morphism HT from Proposition 2.6 in degree 1 is given by the inverse of the projection of the map (3.12) from the pro-étale to the étale site of X

$$\partial_{\mathbf{B}^+}:\quad\Omega^1\to\mathrm{R}^1\mu_*\mathscr{O}(1);$$

in higher degrees it is the inverse of its wedge product (see the proof of [21, Lem. 3.24]).

We have the following result:

Lemma 3.13. *(1) Let X be a smooth rigid analytic space over K. Let $i\geq 1$. There is a commutative diagram*

$$\begin{tikzcd} H^i_{\mathrm{pro\acute{e}t}}(X,\mathscr{O}) \arrow[r, "\mathrm{HT}", "\sim"'] & \Omega^i(X)(-i) \arrow[dl, "\mathrm{PL}^{-1}"] \\ H^i_{\mathrm{pro\acute{e}t}}(X,\mathbb{B}^+_{\mathrm{dR}}/t) \arrow[u, "\theta", "\wr"'] & \end{tikzcd}$$

In particular, PL^{-1} *is an isomorphism.*

(2) Let X be a smooth Stein space over C. Let $i\geq 1$. There is a commutative diagram

$$\begin{tikzcd} H^i_{\mathrm{pro\acute{e}t}}(X,\mathscr{O}) \arrow[r, "\mathrm{HT}", "\sim"'] & \Omega^i(X)(-i) \\ H^i_{\mathrm{pro\acute{e}t}}(X,\mathbb{B}^+_{\mathrm{dR}}/t). \arrow[u, "\theta", "\wr"'] \arrow[ur, "\mathrm{PL}"'] & \end{tikzcd}$$

In particular, PL *is an isomorphism.*

Proof. We start with the first claim. Let $i=1$. Consider the canonical map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}(1) \arrow[r] \arrow[d, equal] & \mathrm{gr}^1_F\mathscr{O}\mathbb{B}^+_{\mathrm{dR}} \arrow[r] \arrow[d] & \mathscr{O}\otimes^{\square}_{\mu^*\mathscr{O}}\Omega^1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{O}(1) \arrow[r] & \mathrm{gr}^1_F\mathscr{O}\mathbb{B}_{\mathrm{dR}} \arrow[r] & \mathscr{O}\mathbb{C}\otimes^{\square}_{\mu^*\mathscr{O}}\Omega^1 \arrow[r] & 0, \end{tikzcd}$$

where the first sequence is obtained from the $\mathbb{B}^+_{\mathrm{dR}}$-Poincaré Lemma and the second one from its $\mathbb{B}_{\mathrm{dR}}$-version. The latter sequence is exact on the right because the map $\mathrm{gr}^0_F\mathscr{O}\mathbb{B}_{\mathrm{dR}}\to\mathrm{gr}^{-1}_F\mathscr{O}\mathbb{B}_{\mathrm{dR}}$ is zero since $\nabla: F^0\mathscr{O}\mathbb{B}_{\mathrm{dR}}\to F^0\mathscr{O}\mathbb{B}_{\mathrm{dR}}$. (We think of the second sequence as a $\mathbb{B}_{\mathrm{dR}}$-Faltings extension).

By projecting the above diagram to the étale site we obtain a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mu_*\mathscr{O}(1) \arrow[r] \arrow[d, equal] & \mu_*\mathrm{gr}^1_F\mathscr{O}\mathbb{B}^+_{\mathrm{dR}} \arrow[r] \arrow[d, "\wr"] & \Omega^1 \arrow[r, "\partial_{\mathbf{B}^+}"] \arrow[d, equal] & \mathrm{R}^1\mu_*\mathscr{O}(1) \arrow[d, equal] \\
0 \arrow[r] & \mu_*\mathscr{O}(1) \arrow[r] & \mu_*\mathrm{gr}^1_F\mathscr{O}\mathbb{B}_{\mathrm{dR}} \arrow[r] & \Omega^1 \arrow[r, "\partial_{\mathbf{B}}"] & \mathrm{R}^1\mu_*\mathscr{O}(1).
\end{tikzcd}$$

Since, by [20, Prop. 6.16], $\mathrm{R}\mu_*\mathrm{gr}^i_F\mathscr{O}\mathbb{B}_{\mathrm{dR}} \simeq \mathscr{O}(i)$, all the vertical maps are isomorphisms.

Consider now the following map of exact sequences (obtained from the $\mathbb{B}_{\mathrm{dR}}$-Poincaré Lemma)

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}(1) \arrow[r] & \mathrm{gr}^1_F\mathscr{O}\mathbb{B}_{\mathrm{dR}} \arrow[r] & \mathscr{O}\mathbb{C}\otimes^{\square}_{\mu^*\mathscr{O}}\Omega^1 \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{O} \arrow[r, "\varepsilon"] \arrow[u, "t"] & \mathrm{gr}^0_F\mathscr{O}\mathbb{B}_{\mathrm{dR}} \arrow[r] \arrow[u, "t"] & \mathscr{O}\mathbb{C}\otimes^{\square}_{\mu^*\mathscr{O}}\Omega^1(-1). \arrow[u, "t"] &
\end{tikzcd}$$

By projecting it to the étale site we obtain a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mu_*\mathscr{O}(1) \arrow[r] & \mu_*\mathrm{gr}^1_F\mathscr{O}\mathbb{B}_{\mathrm{dR}} \arrow[r] & \Omega^1 \arrow[r, "\partial_{\mathbf{B}}"] & \mathrm{R}^1\mu_*\mathscr{O}(1) \\
0 \arrow[r] & \mu_*\mathscr{O} \arrow[r, "\sim"] \arrow[u, "t"] & \mu_*\mathrm{gr}^0_F\mathscr{O}\mathbb{B}_{\mathrm{dR}} \arrow[r, "0"] \arrow[u, "t"] & \Omega^1(-1) \arrow[r, "\mathrm{PL}^{-1}"] \arrow[u, "t"] & \mathrm{R}^1\mu_*\mathscr{O}. \arrow[u, "t"]
\end{tikzcd}$$

We used here that the canonical map $\mu_*\operatorname{coker}\varepsilon \to \Omega^1(-1)$ is an isomorphism. This proves the first claim of our lemma for $i=1$. The case of $i\geq 1$ is obtained by taking wedge products.

For the second claim of the lemma, choose a Stein covering of X by Stein spaces $\{X_n\}$ such that each X_n is defined over a finite extension K_n of K (to do that you may start with a Stein affinoid covering and then take the naive interiors of these affinoids containing the previous affinoids). The wanted diagram is obtained by taking the limit over n of the diagram in claim (1) (note that $\mathrm{R}^1\lim_n$ is trivial for all the terms of the diagram). □

3.2. **The Bloch-Kato exponential.** We restrict our attention now to smooth Stein spaces. We will introduce here the Bloch-Kato exponential and show how it can be obtained, via the filtered $\mathbb{B}_{\mathrm{dR}}$-Poincaré Lemma, from a boundary map induced by a fundamental exact sequence.

3.2.1. *The definition of the map* Exp. We first recall how the geometric p-adic pro-étale cohomology of Stein spaces can be computed. In [5, Th. 1.8], [8, Th. 5.14], Colmez-Dospinescu-Nizioł proved the following theorem:

Theorem 3.14. *Let X be a Stein smooth rigid analytic space over C. For $i\geq 0$, there is a map of exact sequences in $\mathscr{D}(\mathbf{Q}_{p,\square})$:*

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{i-1}(X)/\operatorname{Ker} d \arrow[r, "\mathrm{Exp}"] \arrow[d, equal] & H^i_{\text{proét}}(X,\mathbf{Q}_p(i)) \arrow[r] \arrow[d, "d\mathrm{Log}"] & (H^i_{\mathrm{HK}}(X)\otimes^{\square}_{\breve{F}}\widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^i} \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}\otimes\theta"] & 0 \\
0 \arrow[r] & \Omega^{i-1}(X)/\operatorname{Ker} d \arrow[r, "d"] & \Omega^i(X)^{d=0} \arrow[r] & H^i_{\mathrm{dR}}(X) \arrow[r] & 0.
\end{tikzcd} \tag{3.15}$$

If X is defined over K, this map is Galois equivariant.

Remark 3.16. The maps in diagram (3.15) were constructed using a comparison of p-adic pro-étale cohomology with syntomic cohomology of Bloch-Kato type. We call the map Exp the "Bloch-Kato exponential"; this is supposed to suggest the Bloch-Kato exponential map from [2].

The cohomology $H^r_{\mathrm{HK}}(X)$ appearing on the right of the first exact sequence is the Hyodo-Kato cohomology as defined by Colmez-Nizioł in [7, Sec. 4] (it is built from the logarithmic crystalline cohomology $\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L,0}/W(k_L)^0)$ for L/K a finite extension with residue field k_L, where

$\mathscr{X}_{\mathscr{O}_L} \to \mathrm{Spf}(\mathscr{O}_L)$ is a semistable formal scheme and $W(k_L)^0$ denotes the formal scheme $\mathrm{Spf}(W(k_L))$ equipped with the log-structure induced by $\mathbf{N} \to W(k_L), 1 \mapsto 0$). It is a $(\varphi, N, \mathscr{G}_K)$-module over $\breve{F}$ equipped with a Hyodo-Kato isomorphism $\iota_{\mathrm{HK}} : H^i_{\mathrm{HK}}(X) \otimes^{\square}_{\breve{F}} C \xrightarrow{\sim} H^i_{\mathrm{dR}}(X)$.

An alternative construction of diagram (3.15) was given by Bosco[5] in [4, Th.7.7]. His construction is closely related to the subject of this paper and we will now briefly describe how to get the top row in (3.15)[6]. The starting point is the exact sequence

$$0 \to \mathbf{Q}_p \to \mathbb{B}_e \to \mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}} \to 0$$

of pro-étale sheaves on X, where we set $\mathbb{B}_e := \mathbb{B}[1/t]$. It yields an exact sequence

$$H^{i-1}_{\mathrm{pro\acute{e}t}}(X, \mathbb{B}_e) \xrightarrow{\alpha_{i-1}} H^{i-1}_{\mathrm{pro\acute{e}t}}(X, \mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}}) \to H^i_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p) \to H^i_{\mathrm{pro\acute{e}t}}(X, \mathbb{B}_e) \xrightarrow{\alpha_i} H^i_{\mathrm{pro\acute{e}t}}(X, \mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}})$$

Because of limit considerations it is better to work with the dagger analog of the above exact sequence. Let $\{X_n\}$, $n \in \mathbf{N}$, be a strictly increasing dagger affinoid covering of X. Each affinoid X_n is the base change to C of an affinoid X_{n,K_n} defined over a finite extension K_n of K.

For $n \in \mathbf{N}$, we have an exact sequence

$$H^{i-1}_{\mathrm{pro\acute{e}t}}(X_n, \mathbb{B}_e) \xrightarrow{\alpha_{i-1}} H^{i-1}_{\mathrm{pro\acute{e}t}}(X_n, \mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}}) \to H^i_{\mathrm{pro\acute{e}t}}(X_n, \mathbf{Q}_p) \to H^i_{\mathrm{pro\acute{e}t}}(X_n, \mathbb{B}_e) \xrightarrow{\alpha_i} H^i_{\mathrm{pro\acute{e}t}}(X_n, \mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}}).$$

Since, by [4, Th. 4.1] and (3.9), we have an isomorphism and an exact sequence

$$H^i_{\mathrm{pro\acute{e}t}}(X_{n,C}, \mathbb{B}_e) \simeq (H^i_{\mathrm{HK}}(X_{n,C}) \widehat{\otimes}^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}}[\tfrac{1}{t}])^{N=0,\varphi=1},$$

$$0 \to H^i_{\mathrm{dR}}(X_{n,K_n}) \otimes^{\square}_{K_n} (\mathbf{B}_{\mathrm{dR}}/t^{-i}\mathbf{B}^+_{\mathrm{dR}}) \to H^i_{\mathrm{pro\acute{e}t}}(X_{n,C}, \mathbb{B}_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}}) \to (\Omega^i(X_{n,C})/\mathrm{Ker}\, d)(-i-1) \to 0,$$

it suffices to show that the map α_{i-1} surjects onto $H^i_{\mathrm{dR}}(X_{n,K_n}) \otimes^{\square}_{K_n} (\mathbf{B}_{\mathrm{dR}}/t^{-i}\mathbf{B}^+_{\mathrm{dR}})$ and

$$\mathrm{Ker}\, \alpha_i \simeq (H^i_{\mathrm{HK}}(X_{n,C}) \widehat{\otimes}^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^i}.$$

But this follows from the analysis of the slopes of Frobenius on the Hyodo-Kato cohomology.

For all $n \geq 0$, we have constructed compatible exact sequences

(3.17)

$$0 \longrightarrow \Omega^{i-1}(X_{n,C})/\mathrm{Ker}\, d \xrightarrow{\mathrm{Exp}} H^i_{\mathrm{pro\acute{e}t}}(X_{n,C}, \mathbf{Q}_p(i)) \longrightarrow (H^i_{\mathrm{HK}}(X_{n,C}) \widehat{\otimes}^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^i} \longrightarrow 0.$$

We obtain the top row in (3.15) by passing to the limit over n and using the isomorphism $H^i_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(i)) \xrightarrow{\sim} \lim_n H^i_{\mathrm{pro\acute{e}t}}(X_{n,C}, \mathbf{Q}_p(i))$.

3.2.2. *Comparison of two boundary maps.* The purpose of this section is to prove the following comparison result:

Proposition 3.18. *Let X be a rigid analytic space, which is Stein and smooth over C. Let $i \geq 1$.*

(1) There is a commutative diagram:

$$\begin{tikzcd} \Omega^i(X)(-i) \arrow[r, "\mathrm{Exp}(-i)"] & H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(1)) \\ H^i_{\mathrm{pro\acute{e}t}}(X, \mathbb{B}^+_{\mathrm{dR}}/t), \arrow[u, "\mathrm{PL}", "\wr"'] \arrow[ur, "\partial_{\mathrm{BdR}}"'] & \end{tikzcd}$$

where ∂_{BdR} is the edge map coming from the exact sequence of pro-étale sheaves (3.1) *and* $\mathrm{Exp}(-i)$ *is the $(-i)$-Tate twist of the map from* (3.15)*.*

(2) We have the exact sequence

$$0 \to \Omega^i(X)^{d=0}(-i) \to H^i_{\mathrm{pro\acute{e}t}}(X, \mathbb{B}^+_{\mathrm{dR}}/t) \xrightarrow{\partial_{\mathrm{BdR}}} H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(1)).$$

[5] Bosco's construction is given for Stein spaces that are base change to C of varieties defined over K but as we will see here, the result is still valid when it is not the case.

[6] We did not check that the map Exp constructed in [4] is the same as the one constructed in [8] but we will not need it.

Proof. The second claim follows immediately from the first one and diagram (3.15). For the first claim, note that the exact sequences (3.1) fit into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Q}_p(1) \arrow[r] & \mathbb{B}_e(1) \arrow[r] & (\mathbb{B}_{\mathrm{dR}}/\mathbb{B}_{\mathrm{dR}}^+)(1) \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{Q}_p(1) \arrow[r] \arrow[u, "\wr"', "\mathrm{Id}"] & \mathbb{B}^{\varphi=p} \arrow[r] \arrow[u, "t^{-1}"] & \mathbb{B}_{\mathrm{dR}}^+/t \arrow[r] \arrow[u, "t^{-1}"] & 0.
\end{tikzcd}$$

This yields that the outer square in the following diagram commutes.

$$\begin{tikzcd}
H^i_{\text{proét}}(X, (\mathbb{B}_{\mathrm{dR}}/\mathbb{B}_{\mathrm{dR}}^+)(1)) \arrow[rr, "\partial"] \arrow[dr, "\pi"] & & H^{i+1}_{\text{proét}}(X, \mathbf{Q}_p(1)) \\
& (\Omega^i(X)/\mathrm{Ker}\, d)(-i) \arrow[ur, hook, "\mathrm{Exp}(-i)"] & \\
& \Omega^i(X)(-i) \arrow[u, two heads, "\mathrm{can}"] \arrow[uur, "\mathrm{Exp}(-i)"'] & \\
H^i_{\text{proét}}(X, \mathbb{B}_{\mathrm{dR}}^+/t) \arrow[uuu, "t^{-1}"] \arrow[ur, "\mathrm{PL}", "\sim"'] \arrow[rr, "\partial_{\mathrm{BdR}}"] & & H^{i+1}_{\text{proét}}(X, \mathbf{Q}_p(1)). \arrow[uuu, "\mathrm{Id}"', "\wr"]
\end{tikzcd}$$

The map π is the one from (3.10) and Remark 3.11. The top triangle commutes by the construction of the map Exp described in Section 3.2.1. Using the computations in Section 3.1.2, it is easy to check that the left trapezoid commutes. This gives us claim (1) of the proposition. $\square$

4. The image of HTlog

The goal of this section is to prove the following result:

Theorem 4.1. *Let X be a smooth Stein rigid space over C. For $i \geq 1$, the image of the restriction of the Hodge-Tate logarithm to the group of principal units*

$$\mathrm{HTlog}_U : H^i_v(X, U) \to \Omega^i(X)(-i)$$

fits into a short exact sequence of solid $\mathbf{Z}_p$-modules

$$0 \to \Omega^i(X)^{d=0}(-i) \to \mathrm{Im}(\mathrm{HTlog}_U) \xrightarrow{\mathrm{Exp}} \mathscr{I}^i(X) \to 0$$

where $\mathscr{I}^i(X) \subset H^{i+1}_{\text{proét}}(X, \mathbf{Q}_p(1))$ is the intersection

$$\mathrm{Im}(\mathrm{Exp}) \cap \mathrm{Im}(\iota^{i+1}) = \mathrm{Im}(\mathrm{Exp}) \cap \mathrm{Ker}(\pi^{i+1}),$$

with

$$\iota^j : H^j_{\text{proét}}(X, \mathbf{Z}_p(1)) \to H^j_{\text{proét}}(X, \mathbf{Q}_p(1)), \quad \pi^j : H^j_{\text{proét}}(X, \mathbf{Q}_p(1)) \to H^j_{\text{proét}}(X, \mathbf{Q}_p/\mathbf{Z}_p(1)).$$

Remark 4.2. Alternatively, using diagram (3.15), the group $\mathscr{I}^i(X)$ can be seen as exact forms in $\Omega^{i+1}(X)$ coming from $H^{i+1}_{\text{proét}}(X, \mathbf{Z}_p(1))$.

In particular, we have the following immediate corollary:

Corollary 4.3. *Let X be a smooth Stein space over C. Then,*

1. *The image by* HTlog *of* $\mathrm{Pic}_v(X)$ *contains all the closed differentials. More generally, the image by* HTlog *of* $H^i_v(X, \mathscr{O}^\times)$, $i \geq 1$, *contains all the closed differentials.*
2. *If the map* $H^1_v(X, U) \to \mathrm{Pic}_v(X)$ *is surjective then there is an exact sequence of solid* $\mathbf{Z}_p$*-modules*

$$0 \to \Omega^1(X)^{d=0}(-1) \to \mathrm{Im}(\mathrm{HTlog}) \to \mathscr{I}^1(X) \to 0.$$

Concerning the first claim of the corollary, note that Heuer already proved in [12, Cor. 4.4] that for any smooth rigid space, the image by HTlog of $\mathrm{Pic}_v(X)$ contains all the $df \in \Omega^1(X)$, for f in $\mathscr{O}(X)$.

Remark 4.4. (1) There is no integral cohomology in degree 2 for the affine space. This is why the extra term $\mathscr{I}^1(X)$ does not appear in Heuer's computation. In fact this holds in any degree and we have

$$\Omega^i(\mathbb{A}^n_C)^{d=0}(-i) \xrightarrow{\sim} \mathrm{Im}(\mathrm{HTlog}_U).$$

(2) As for Stein curves, we have that the rational cohomology $H^i_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p)$, $i \geq 2$, is zero, hence $\mathscr{I}^1(X)$ is also trivial in that case. Moreover, $\Omega^1(X)^{d=0} \xrightarrow{\sim} \Omega^i(X)$.

(3) Let $i \geq 1$. Since we have an exact sequence:

$$0 \to \Omega^i(X)/\mathrm{Ker}\, d \xrightarrow{\mathrm{Exp}} H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(i+1)) \to (H^{i+1}_{\mathrm{HK}}(X) \otimes^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0, \varphi=p^{i+1}} \to 0$$

we see that the intersection $\mathscr{I}^i(X)$ is zero when the map from $H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Z}_p(i+1))/T$, where T is the maximal torsion subgroup, to the Hyodo-Kato term above is injective. This will be the case when X is a torus or, more generally, the analytification of an algebraic variety, or the Drinfeld upper half space (see below).

In fact, in general, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathscr{I}^i}(X)(i) \arrow[r] \arrow[d] & H^{i+1}_{\mathrm{\acute{e}t}}(X, \mathbf{Z}_p(i+1))/T \arrow[r] \arrow[d, "\mathrm{can}"] & (H^{i+1}_{\mathrm{HK}}(X) \otimes^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0, \varphi=p^{i+1}} \arrow[d, equal] & \\
0 \arrow[r] & \Omega^i(X)/\mathrm{Ker}\, d \arrow[r, "\mathrm{Exp}"] & H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(i+1)) \arrow[r] & (H^{i+1}_{\mathrm{HK}}(X) \otimes^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0, \varphi=p^{i+1}} \arrow[r] & 0,
\end{tikzcd}$$

where the left square is cartesian. The group $\widetilde{\mathscr{I}^i}(X)$ surjects onto $\mathscr{I}^i(X)$. We like to think of the Hyodo-Kato term as carrying ℓ-adic information, for $\ell \neq p$. Then $\widetilde{\mathscr{I}^i}(X)$ can be seen as a genuinely p-adic phenomena.

(4) The groups $\mathscr{I}^j(X)$ need not be zero in general. This is the case for open unit discs D^d_C of dimension $d > 1$ over C. See Section 5.1.7 below.

Proof of Theorem 4.1. Recall from Section 2, that we can pass from the v-topology to the pro-étale one without changing the groups $H^i_?(X, U)$ and $H^i_?(X, \mathscr{O}^\times)$. We then work on the pro-étale site. As in [12, Sec. 6.2], we start from the logarithmic exact sequence (point (1) of Lemma 2.4) on the pro-étale site:

$$0 \to \mathbf{Q}_p/\mathbf{Z}_p(1) \to U \xrightarrow{\log} \mathscr{O} \to 0.$$

It induces a commutative diagram:

$$\begin{tikzcd}
H^i_{\mathrm{pro\acute{e}t}}(X, U) \arrow[r, "\log"] \arrow[dr, "\mathrm{HTlog}_U"'] & H^i_{\mathrm{pro\acute{e}t}}(X, \mathscr{O}) \arrow[r, "\partial_{\log}"] \arrow[d, "\mathrm{HT}", "\wr"'] & H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p/\mathbf{Z}_p(1)) \\
& \Omega^i(X)(-i), \arrow[ur, "\partial_{\log} \circ \mathrm{HT}^{-1}"'] &
\end{tikzcd} \tag{4.5}$$

where the first row is exact. We have used here that X is Stein (the isomorphism HT is the one from Remark 2.12). We deduce from the diagram that the image of HTlog_U is equal to the kernel of $\partial_{\log} \circ \mathrm{HT}^{-1}$. We prove in Lemma 4.7 below that the following square commutes:

$$\begin{tikzcd}
\Omega^i(X)(-i) \arrow[r, "\partial_{\log} \circ \mathrm{HT}^{-1}"] \arrow[d] & H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p/\mathbf{Z}_q(1)) \\
\Omega^i(X)/\mathrm{Ker}\, d(-i) \arrow[r, hook, "\mathrm{Exp}"] & H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(1)). \arrow[u]
\end{tikzcd} \tag{4.6}$$

It immediately follows that we have an inclusion $\Omega^i(X)^{d=0}(-i) \subset \mathrm{Ker}(\partial_{\log} \circ \mathrm{HT}^{-1})$. Now, since the kernel of the right vertical map is given by the image of $\iota^{i+1} : H^{i+1}_{\text{proét}}(X, \mathbf{Z}_p(1)) \to H^{i+1}_{\text{proét}}(X, \mathbf{Q}_p(1))$, we get an exact sequence:

$$0 \to \Omega^i(X)^{d=0}(-i) \to \mathrm{Ker}(\partial_{\log} \circ \mathrm{HT}^{-1}) \xrightarrow{\mathrm{Exp}} \mathrm{Im}(\mathrm{Exp}) \cap \mathrm{Im}(\iota^{i+1}) \to 0$$

and this concludes the proof of Theorem 4.1. □

Lemma 4.7. *The diagram* (4.6) *is commutative.*

Proof. It suffices to show that we have the commutative diagram

$$\begin{tikzcd}
\Omega^i(X)(-i) \arrow[d, equal] & H^i_{\text{proét}}(X, \mathscr{O}) \arrow[l, "\mathrm{HT}"', "\sim"] \arrow[r, "\partial_{\log}"] & H^{i+1}_{\text{proét}}(X, \mathbf{Q}_p/\mathbf{Z}_q(1)) \\
\Omega^i(X)(-i) \arrow[d] & H^i_{\text{proét}}(X, \mathbb{B}^+_{\mathrm{dR}}/t) \arrow[l, "\mathrm{PL}"', "\sim"] \arrow[u, "\theta", "\wr"'] \arrow[r, "\partial_{\mathrm{BdR}}"] & H^{i+1}_{\text{proét}}(X, \mathbf{Q}_p(1)) \arrow[u, "\pi^{i+1}"] \\
\Omega^i(X)/\mathrm{Ker}\, d(-i) \arrow[urr, "\mathrm{Exp}"'] & &
\end{tikzcd}$$

as the outer trapecoid is exactly the diagram in question. The commutativity of the upper left square follows from the definition of the morphism HT (it is defined using the Poincaré Lemma, see Remark 3.1.3). The right square comes from the map of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Q}_p/\mathbf{Z}_p(1) \arrow[r] & U \arrow[r, "\log"] & \mathscr{O} \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{Q}_p(1) \arrow[r] \arrow[u] & \mathbb{B}^{\varphi=p} \arrow[r] \arrow[u] & \mathbb{B}^+_{\mathrm{dR}}/t\mathbb{B}^+_{\mathrm{dR}} \arrow[r] \arrow[u, "\theta"', "\wr"] & 0,
\end{tikzcd}$$

where the map in the middle is the sharp map appearing in (3.2) (by [18, Example 2.22], it makes the right square commutative). Hence, it is commutative. The triangle commutes by Proposition 3.18. Therefore the outer trapezoid is commutative as well, which is what we wanted. □

5. Examples

Let us look now at some examples of computations of $H^i_\tau(X, \mathbb{G}_m)$, for $i = 1, 2$, for certain smooth Stein space X over C.

5.1. **Picard group.** We start with the Picard group.

5.1.1. *Curves.* Smooth Stein rigid analytic varieties X of dimension 1 were already treated in [12, Sec. 4.1]: as $H^2_{\text{ét}}(X, \mathscr{O}^\times)$ vanishes in this case, the exact sequence (2.8) becomes

$$0 \to \mathrm{Pic}_{\mathrm{an}}(X) \to \mathrm{Pic}_v(X) \xrightarrow{\mathrm{HTlog}} \Omega^1(X) \to 0.$$

Moreover, if such a curve X is defined over K, we have $H^2_{\text{proét}}(X, \mathbf{Q}_p) = 0$ (because $H^2_{\mathrm{dR}}(X) = 0$) hence $\mathscr{I}^1(X) = 0$ and, by Theorem 4.1, $\mathrm{Pic}_v(X)$ surjects onto $\Omega^1(X)$, as desired.

5.1.2. *Affine space.* The case of the affine space was treated in two different ways in [12, Sec. 6]. Our approach here is similar to the one presented in [12, Sec. 6.2]. Let $\mathbb{A}^d_C$ be the rigid analytic affine space of dimension d over C. For $i \geq 1$, since $H^i_{\mathrm{dR}}(\mathbb{A}^d_C) = 0$ and, hence, $H^i_{\mathrm{HK}}(\mathbb{A}^d_C) = 0$, by diagram (3.15), we have an isomorphism

$$\Omega^i(\mathbb{A}^d_C)/\mathrm{Ker}\, d \xrightarrow{\sim} H^{i+1}_{\text{proét}}(\mathbb{A}^d_C, \mathbf{Q}_p(i+1)).$$

Since $H^{i+1}_{\text{proét}}(\mathbb{A}^d_C,\mathbf{Z}_p(i+1))=0$ (by comparison with the algebraic case), we have $\mathscr{I}^i(\mathbb{A}^d_C)=0$. Thus, by Theorem 4.1, we have an isomorphism

$$\mathrm{Im}(\mathrm{HTlog_U}: H^i_v(\mathbb{A}^d_C,U)\to\Omega^i(\mathbb{A}^d_C)(-i))\xleftarrow{\sim}\Omega^i(\mathbb{A}^d_C)^{d=0}(-i).$$

Moreover, by [12, Lem. 6.6], the map from $H^1_v(\mathbb{A}^d_C,U)$ to the v-Picard group is surjective. Thus we obtain an exact sequence

$$0\to\mathrm{Pic_{an}}(\mathbb{A}^d_C)\to\mathrm{Pic}_v(\mathbb{A}^d_C)\to\Omega^1(\mathbb{A}^d_C)^{d=0}(-1)\to 0.$$

Since the analytic Picard group of the affine space is trivial[7], this implies that the Hodge-Tate logarithm is an isomorphism

$$\mathrm{HTlog}:\quad \mathrm{Pic}_v(\mathbb{A}^d_C)\xrightarrow{\sim}\Omega^1(\mathbb{A}^d_C)^{d=0}(-1).$$

5.1.3. *Torus.* Consider the rigid analytic torus $\mathbb{G}^d_{m,C}$ of dimension d over C. This case is similar to the case of affine space because the analytic Picard group is trivial (see [14, Th. A]) but also different because the de Rham cohomology is non-trivial (though of finite rank).

For $i\geq 0$, by (3.15), we have the exact sequence:

$$0\to\Omega^i(\mathbb{G}^d_{m,C})/\mathrm{Ker}\,d\to H^{i+1}_{\text{proét}}(\mathbb{G}^d_{m,C},\mathbf{Q}_p(i+1))\to\wedge^{i+1}\mathbf{Q}_p^d\to 0. \tag{5.1}$$

Since we have $H^{i+1}_{\text{ét}}(\mathbb{G}^d_{m,C},\mathbf{Z}_p(i+1))\simeq\wedge^{i+1}\mathbf{Z}_p^d$ (compare with the étale cohomology of the algebraic torus), we see that the map from the integral cohomology to the Hyodo-Kato term is injective. We used here that the projection from pro-étale cohomology to the Hyodo-Kato term is compatible with products and symbol maps: this is because the comparison theorem between the pro-étale cohomology and syntomic cohomology and the projection from syntomic cohomology to the Hyodo-Kato term both satisfy these compatibilities. We obtain that the intersection between the elements coming from $\Omega^1(\mathbb{G}^d_{m,C})/\mathrm{Ker}\,d$ and the ones coming from the integral pro-étale cohomology is trivial. Thus, by Theorem 4.1, we have an isomorphism

$$\mathrm{Im}(\mathrm{HTlog_U}: H^i_v(\mathbb{G}^d_{m,C},U)\to\Omega^i(\mathbb{G}^d_{m,C})(-i))\xleftarrow{\sim}\Omega^i(\mathbb{G}^d_{m,C})^{d=0}(-i). \tag{5.2}$$

Moreover, we have:

Lemma 5.3. *The map from $H^1_v(\mathbb{G}^d_{m,C},U)$ to $H^1_v(\mathbb{G}^d_{m,C},\mathscr{O}^\times)$ is surjective.*

Proof. We will show that $H^1_v(\mathbb{G}^d_{m,C},\overline{\mathscr{O}}^\times)$ is zero. Let $\{X_n\}_{n\in\mathbf{N}}$ be the Stein covering of $\mathbb{G}^d_{m,C}$ from [14, Proof of Th. 7.1]. On each X_n the sheaf $\mathscr{O}^\times_{\text{an}}$ is acyclic. We have the exact sequence

$$0\to \mathrm{R}^1\lim_n H^0_v(X_n,\overline{\mathscr{O}}^\times)\to H^1_v(\mathbb{G}^d_{m,C},\overline{\mathscr{O}}^\times)\to\lim_n H^1_v(X_n,\overline{\mathscr{O}}^\times)\to 0 \tag{5.4}$$

But, by [13, Lem. 2.14], [14, Proof of Th. 7.1] we have

$$H^0_v(X_n,\overline{\mathscr{O}}^\times)=H^0_{\text{an}}(X_n,\overline{\mathscr{O}}^\times)[1/p]=M_n[1/p], \tag{5.5}$$

where M_n is an abelian group of finite type. Moreover, the maps $M_{n+1}\to M_n$ are surjective, and they remain so after inverting p. We get $\mathrm{R}^1\lim_n H^0_v(X_n,\overline{\mathscr{O}}^\times)=0$.

We also claim that $H^1_v(X_n,\overline{\mathscr{O}}^\times)=0$, for all n. Indeed, by the point (2) of Proposition 2.2, we see that it suffices to check this for the étale topology. Using the exponential sequence (point (1) of Lemma (2.4)), it is enough to show that

$$H^1_{\text{ét}}(X_n,\mathscr{O}^\times[\frac{1}{p}])=0 \text{ and } H^2_{\text{ét}}(X_n,\mathscr{O})=0.$$

The second equality follows from the fact that X_n is an affinoid. For the first one, we use that $\mathrm{Pic_{ét}}(X_n)=\mathrm{Pic_{an}}(X_n)=0$ and since X_n is quasi-compact, we have $H^1_{\text{ét}}(X_n,\mathscr{O}^\times[\frac{1}{p}])=H^1_{\text{ét}}(X_n,\mathscr{O}^\times)[\frac{1}{p}]=0$. Our claim follows.

Hence $\lim_n H^1_v(X_n,\overline{\mathscr{O}}^\times)=0$, and, by (5.4), $H^1_v(\mathbb{G}^d_{m,C},\overline{\mathscr{O}}^\times)=0$, as wanted. ☐

[7] In fact, by [11, Ch. V.3, Prop. 2], every analytic vector bundle on $\mathbb{A}^d_C$ is trivial.

Thus we obtain an exact sequence

$$0 \to \mathrm{Pic}_{\mathrm{an}}(\mathbb{G}^d_{m,C}) \to \mathrm{Pic}_v(\mathbb{G}^d_{m,C}) \to \Omega^1(\mathbb{G}^d_{m,C})^{d=0}(-1) \to 0.$$

Since the analytic Picard group of the torus is trivial, this means that the Hodge-Tate logarithm is an isomorphism

$$\mathrm{HTlog}: \quad \mathrm{Pic}_v(\mathbb{G}^d_{m,C}) \xrightarrow{\sim} \Omega^1(\mathbb{G}^d_{m,C})^{d=0}(-1). \tag{5.6}$$

Remark 5.7. Let $d, n \geq 0$. Combining (5.6) and [12, Th. 6.1], we obtain an isomorphism

$$\mathrm{HTlog}: \quad \mathrm{Pic}_v(\mathbb{G}^d_{m,C} \times \mathbb{A}^n_C) \xrightarrow{\sim} \Omega^1(\mathbb{G}^d_{m,C} \times \mathbb{A}^n_C)^{d=0}(-1). \tag{5.8}$$

This isomorphism can be also obtained arguing as above, in the case of the torus. More precisely, since de Rham cohomology satisfies a Künneth formula, replacing $\mathbb{G}^d_{m,C}$ with $\mathbb{G}^d_{m,C} \times \mathbb{A}^n_C$ yields an analogue of the exact sequence (5.1) and then also an analogue of the isomorphism (5.2). The rest of the argument goes through yielding (5.8).

5.1.4. *Analytification of algebraic varieties.* The examples of the affine space and the torus generalize. Let X be the analytification of an affine smooth algebraic variety X^{alg} over C. Then X is Stein. In this case both the algebraic and the analytic de Rham cohomologies are of finite rank (they are functorially isomorphic but not as filtered objects). Let $i \geq 1$. We have a commutative diagram (see Remark 4.4)

$$\begin{tikzcd}
 & 0 \arrow[r] & H^{i+1}_{\text{ét}}(X^{\mathrm{alg}}, \mathbf{Z}_p(i+1))/T \arrow[r] \arrow[d, "\mathrm{can}", "\wr"'] & (H^{i+1}_{\mathrm{HK}}(X^{\mathrm{alg}}) \otimes^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^{i+1}} \arrow[d, "\mathrm{can}", "\wr"'] & \\
0 \arrow[r] & \widetilde{\mathscr{I}^i}(X)(i) \arrow[r] \arrow[d] & H^{i+1}_{\text{ét}}(X, \mathbf{Z}_p(i+1))/T \arrow[r] \arrow[d, "\mathrm{can}"] & (H^{i+1}_{\mathrm{HK}}(X) \otimes^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^{i+1}} \arrow[d, equal] & \\
0 \arrow[r] & \Omega^i(X)/\mathrm{Ker}\, d \arrow[r, "\mathrm{Exp}"] & H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(i+1)) \arrow[r] & (H^{i+1}_{\mathrm{HK}}(X) \otimes^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^{i+1}} \arrow[r] & 0.
\end{tikzcd}$$

The rows are exact. For the top row this follows from the algebraic p-adic comparison theorems [1]. The top square commutes by the compatibility of the algebraic and analytic p-adic comparison morphisms. This fact and the proof of the isomorphism between the algebraic and the analytic Hyodo-Kato cohomologies can be found in [19]. It follows that $\widetilde{\mathscr{I}^i}(X) = 0$ and hence $\mathscr{I}^i(X) = 0$. We have proved:

Corollary 5.9. *Let X be the analytification of an affine smooth algebraic variety over C. Let $i \geq 1$. Then $\mathscr{I}^i(X) = 0$ and we have an isomorphism*

$$\mathrm{Im}(\mathrm{HTlog}_{\mathrm{U}}: H^i_v(X, U) \to \Omega^i(X)(-i)) \xleftarrow{\sim} \Omega^i(X)^{d=0}(-i).$$

5.1.5. *Almost proper varieties.* Even more generally, let X be a smooth rigid analytic variety over C that is of the form $X = Y \setminus Z$, where Y is a proper and smooth rigid analytic variety over C and Z is a closed rigid analytic subvariety of Y. Assume that X is Stein. Let $i \geq 1$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathscr{I}^i}(X)(i) \arrow[r] \arrow[d] & H^{i+1}_{\text{ét}}(X, \mathbf{Z}_p(i+1))/T \arrow[r, "\alpha"] \arrow[d, "\mathrm{can}"] & (H^{i+1}_{\mathrm{HK}}(X) \otimes^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^{i+1}} \arrow[d, equal] & \\
0 \arrow[r] & \Omega^i(X)/\mathrm{Ker}\, d \arrow[r, "\mathrm{Exp}"] & H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(i+1)) \arrow[r] & (H^{i+1}_{\mathrm{HK}}(X) \otimes^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^{i+1}} \arrow[r] & 0.
\end{tikzcd}$$

The rows are exact. The étale cohomology group $H^{i+1}_{\text{ét}}(X, \mathbf{Z}_p(i+1))/T$ is of finite type, by [16, Th. 1.3], and the map α is injective since we have the standard (almost proper) p-adic comparison theorem [19]. It follows that $\widetilde{\mathscr{I}^i}(X)=0$ and hence $\mathscr{I}^i(X)=0$. We have proved an analog of Corollary 5.9 for X.

5.1.6. *Drinfeld space.* In this example, the analytic Picard group is also trivial (see [14, Th. A]) but the de Rham cohomology is not of finite rank anymore. For d an integer, the Drinfeld space over K of dimension d is defined by

$$\mathbb{H}^d_K := \mathbb{P}^d_K \setminus \bigcup_{H\in\mathscr{H}} H$$

where $\mathscr{H} := \mathbb{P}((K^{d+1})^\times)$ denotes the set of the K-rational hyperplanes in the rigid-analytic projective space $\mathbb{P}^d_K$.

For Λ a topological ring, we denote by $\mathrm{Sp}_r(\Lambda)$ the associated generalized Steinberg representation and by $\mathrm{Sp}_r(\Lambda)^*$ its dual. Recall that we have the following computation:

Proposition 5.10. [5, Th. 1.3][6, Th. 1.1] *Let $i\geq 0$. Then:*

(1) There is an exact sequence:

$$0\to\Omega^{i-1}(\mathbb{H}^d_C)/\operatorname{Ker} d\to H^i_{\text{proét}}(\mathbb{H}^d_C,\mathbf{Q}_p(i))\to \mathrm{Sp}_i(\mathbf{Q}_p)^*\to 0.$$

(2) There are isomorphisms:

$$H^i_{\text{ét}}(\mathbb{H}^d_C,\mathbf{Z}_p(i))\simeq \mathrm{Sp}_i(\mathbf{Z}_p)^* \quad \text{and} \quad H^i_{\text{ét}}(\mathbb{H}^d_C,\mathbf{Q}_p(i))\simeq \mathrm{Sp}_i^{\text{cont}}(\mathbf{Q}_p)^*.$$

(3) The above morphisms are compatible, i.e. there is a commutative diagram

$$\begin{tikzcd}
H^i_{\text{ét}}(\mathbb{H}^d_C,\mathbf{Z}_p(i))\otimes_{\mathbf{Z}_p}\mathbf{Q}_p \arrow[r, "\sim"] \arrow[d, "\wr"] & H^i_{\text{ét}}(\mathbb{H}^d_C,\mathbf{Q}_p(i)) \arrow[r] \arrow[d, "\wr"] & H^i_{\text{proét}}(\mathbb{H}^d_C,\mathbf{Q}_p(i)) \arrow[d] \\
\mathrm{Sp}_i(\mathbf{Z}_p)^*\otimes_{\mathbf{Z}_p}\mathbf{Q}_p \arrow[r, "\sim"] & \mathrm{Sp}_i^{\text{cont}}(\mathbf{Q}_p)^* \arrow[r, hook] & \mathrm{Sp}_i(\mathbf{Q}_p)^*.
\end{tikzcd}$$

As in the case of torus, we see that the map from $H^i_{\text{proét}}(\mathbb{H}^d_C,\mathbf{Z}_p(i))$ to the Hyodo-Kato term is injective[8], and we deduce that the intersection $\mathscr{I}^i(\mathbb{H}^d_C)$ is zero. Moreover, we have:

Lemma 5.11. *The map from $H^1_v(\mathbb{H}^d_C,U)$ to $H^1_v(\mathbb{H}^d_C,\mathscr{O}^\times)$ is surjective.*

Proof. Analogous to the proof of Lemma 5.3. $\square$

Using Lemma 5.11, we get an exact sequence:

$$0\to\mathrm{Pic}_{\text{an}}(\mathbb{H}^d_C)\to\mathrm{Pic}_v(\mathbb{H}^d_C)\to\Omega^1(\mathbb{H}^d_C)^{d=0}(-1)\to 0.$$

Since the analytic Picard group of the Drinfeld space is trivial, finally, we obtain an isomorphism

$$\mathrm{HTlog}:\quad \mathrm{Pic}_v(\mathbb{H}^d_C)\xrightarrow{\sim}\Omega^1(\mathbb{H}^d_C)^{d=0}(-1). \tag{5.12}$$

5.1.7. *Open disc.* Let now $d>1$. Consider D^d_C, the the open unit disc D^d of dimension d over C. We will prove that the intersection $\mathscr{I}^1(D^d)$ is nonzero, which shows that the image of the Hodge-Tate logarithm need not be reduced to the closed differentials in general.

Write $D^d=D_1\times_C D_2$, where D_1, D_2 are open unit discs of dimension 1 and $d-1$, respectively. Choose functions $f_i\in\mathscr{O}^*(D_{i,C})$. We have $d\log f_i\in\Omega^1(D_i)$, $\omega := d\log f_1\wedge d\log f_2\in\Omega^2(D)$, and clearly $d\omega=0$, i.e., $\omega\in\Omega^2(D)^{d=0}$. We note that, since de Rham cohomology of D is trivial in positive degrees, we have the isomorphism

$$d:\Omega^1(D^d)/\operatorname{Ker} d\xrightarrow{\sim}\Omega^2(D^d)^{d=0}.$$

[8] The same argument concerning compatibilities applies.

and, from diagram 3.15, the commutative diagram

$$\begin{tikzcd}
\Omega^1(D^d)/\mathrm{Ker}\, d \arrow[r, "\mathrm{Exp}", "\sim"'] \arrow[d, equal] & H^2_{\mathrm{pro\acute{e}t}}(D^d, \mathbf{Q}_p(2)) \arrow[d, "\mathrm{dLog}", "\wr"'] \\
\Omega^1(D^d)/\mathrm{Ker}\, d \arrow[r, "d", "\sim"'] & \Omega^2(D^d)^{d=0}.
\end{tikzcd}$$

Let now $\delta(f_1), \delta(f_2)$ be the images by the Kummer maps $\delta : \mathscr{O}^*(D_i) \to H^1_{\mathrm{\acute{e}t}}(D_i, \mathbf{Z}_p(1))$ of f_1, f_2. Then $\delta(f_1) \cup \delta(f_2) \in H^2_{\mathrm{\acute{e}t}}(D, \mathbf{Z}_p(2))$. (Here we abuse the notation slightly.) We claim that the image of $\delta(f_1) \cup \delta(f_2)$ in $\Omega^2(D^d)^{d=0}$ is equal to ω. Indeed, we compute

$$dLog(\delta(f_1) \cup \delta(f_2)) = dLog(\delta(f_1)) \cup dLog(\delta(f_2)) = d\log f_1 \cup d\log f_2 = \omega. \tag{5.13}$$

The first equality follows from the fact that the map dLog commutes with cup products: it is defined using comparison with syntomic cohomology and the composition

$$H^2_{\mathrm{pro\acute{e}t}}(D^d, \mathbf{Q}_p(2)) \to H^2(F^2\mathrm{R}\Gamma_{\mathrm{dR}}(D^d/\mathbf{B}^+_{\mathrm{dR}})) \xrightarrow{\theta} \Omega^2(D^d)^{d=0};$$

both of which commute with cup products. The second equality in (5.13) is the compatibility of the étale and de Rham symbol maps: the symbol maps are induced by the first Chern class maps and the passage from étale cohomology to syntomic one as well as the projection from the latter to the filtered de Rham cohomology are both compatible with the Chern class maps.

Now, it suffices to make sure that $\omega \neq 0$. But, for that it is enough to choose nonconstant functions f_1, f_2. An analogous argument will show that $\mathscr{I}^i(D^d)$ is nonzero for all $d-1 \geq i \geq 1$.

Remark 5.14. This example is a curious one: de Rham cohomology is trivial in positive degrees but the analytic Picard group depends on the ground field (it will be non-trivial in our case). More precisely, recall that we have the following result (see [23, Prop. 3.5]):

Proposition 5.15. ([11, Ch. V, Prop. 2]) *Let L be a complete, non-archimedean, non-trivially valued field. Let $r \in (|L| \cup \{\infty\})^d$ and let X_r be an open polydisc of polyradius r:*

$$X_r = \bigcup_{|\eta_i| = s_i < r_i} \mathrm{Sp}(L < \eta_1^{-1}T_1, \cdots, \eta_d^{-1}T_d >).$$

The Picard group $\mathrm{Pic}(X_r)$ is trivial if and only if one of the following holds:

(1) the field L is spherically complete or
(2) the polyradius is $r = (\infty, ..., \infty)$, that is, $X_r = \mathbb{A}^n_L$ is the analytic affine space.

The "only if" part was shown by Lazard [17, Prop. 6]: Assume that L is not spherically complete and let $r \in |L|$; then Lazard constructs a divisor on X_r which is not a principal divisor. This implies that open discs X_r which are bounded in at least one direction have non-trivial line bundles (see [11, p. 87, Rem. 2]).

We have:

Lemma 5.16. *The canonical map $H^1_v(D^d, U) \to \mathrm{Pic}_v(D^d)$ is surjective.*

Proof. We will show that $H^1_v(D^d, \overline{\mathscr{O}}^\times)$ is zero. Take $\{X_n\}_{n\in\mathbf{N}}$ – a Stein covering of D^d by closed balls X_n. We have the exact sequence

$$0 \to \mathrm{R}^1 \lim_n H^0_v(X_n, \overline{\mathscr{O}}^\times) \to H^1_v(D^d, \overline{\mathscr{O}}^\times) \to \lim_n H^1_v(X_n, \overline{\mathscr{O}}^\times) \to 0 \tag{5.17}$$

But, by [12, Lem. 6.5]),

$$H^0_v(X_n, \overline{\mathscr{O}}^\times) = C^\times/(1 + \mathfrak{m}\mathscr{O}_C), \quad H^1_v(X_n, \overline{\mathscr{O}}^\times) = 0. \tag{5.18}$$

Hence

$$\mathrm{R}^1 \lim_n H^0_v(X_n, \overline{\mathscr{O}}^\times) = 0, \quad \lim_n H^1_v(X_n, \overline{\mathscr{O}}^\times) = 0.$$

Thus, by (5.17), $H^1_v(D^d, \overline{\mathscr{O}}^\times) = 0$, as wanted. $\square$

Hence we have exact sequences of non-trivial groups:

$$0 \to \mathrm{Pic}_{\mathrm{an}}(D^d) \to \mathrm{Pic}_v(D^d) \to \mathrm{Im}(\mathrm{HTlog}) \to 0,$$
$$0 \to \Omega^1(D^d)^{d=0}(-1) \to \mathrm{Im}(\mathrm{HTlog}) \xrightarrow{\mathrm{Exp}} \mathscr{I}^1(D^d)(-1) \to 0.$$

We note that $\Omega^1(D^d)^{d=0} \xleftarrow{\sim} \mathscr{O}(D^d)/C$.

5.2. **The group $H^2_{\text{ét}}(X, \mathbb{G}_m)$.** The above computations allow us to deduce a little bit about the structure of the groups $H^2_{\text{ét}}(X, \mathbb{G}_m)$.

Let X be a smooth Stein space of dimension d over C. Let $d \geq i \geq 2$. By Corollary 2.15, we have an exact sequence

$$0 \to \mathrm{Coker}(\mathrm{HTlog}_{i-1}) \to H^i_{\text{ét}}(X, \mathscr{O}^\times) \xrightarrow{\nu_i^*} \mathrm{Ker}(\mathrm{HTlog}_i) \to 0. \tag{5.19}$$

We have $\mathrm{Coker}(\mathrm{HTlog}_{i-1}) = \Omega^{i-1}(X)(-i+1)/\mathrm{Im}(\mathrm{HTlog}_{i-1})$. Since $\mathrm{Im}(\mathrm{HTlog}_{i-1}) \supset \mathrm{Im}(\mathrm{HTlog}_{U,i-1})$, we have $\mathrm{Coker}(\mathrm{HTlog}_{U,i-1}) \twoheadrightarrow \mathrm{Coker}(\mathrm{HTlog}_{i-1})$. In the case that the inclusion map above is an isomorphism, by Theorem 4.1, we have an exact sequence

$$0 \to \Omega^{i-1}(X)^{d=0}(-i+1) \to \mathrm{Im}(\mathrm{HTlog}_{i-1}) \to \mathscr{I}^{i-1}(X) \to 0,$$

which, in combination with the exact sequence (5.19), yields the exact sequence[9]

$$0 \to \Omega^{i-1}(X)(-i+1)/[\mathrm{Ker}\, d - \mathscr{I}^{i-1}(X)] \to H^i_{\text{ét}}(X, \mathscr{O}^\times) \xrightarrow{\nu_i^*} \mathrm{Ker}(\mathrm{HTlog}_i) \to 0. \tag{5.20}$$

Hence, by Sections 5.1.2, 5.1.3, and 5.1.6, if X is an affine space, a torus, or a Drinfeld space (base changed to C), we have the exact sequence

$$0 \to \Omega^1(X)(-1)/\mathrm{Ker}\, d \to H^2_{\text{ét}}(X, \mathbb{G}_m) \xrightarrow{\nu_i^*} \mathrm{Ker}(\mathrm{HTlog}_i) \to 0. \tag{5.21}$$

Moreover, in the case $H^1_{\text{ét}}(X, \overline{\mathscr{O}}^\times) = 0$, we have the injection

$$\mathrm{Ker}(\mathrm{HTlog}_{U,i}) \hookrightarrow \mathrm{Ker}(\mathrm{HTlog}_i).$$

Again, this is the case for an affine space, a torus, or a Drinfeld space.

5.2.1. *Comparison with p-adic cohomology.* To get a handle on $\mathrm{Ker}(\mathrm{HTlog}_{U,i})$, we can use the logarithmic exact sequence

$$0 \to (\mathbf{Q}_p/\mathbf{Z}_p)(1) \to U \xrightarrow{\log} \mathscr{O} \to 0$$

and diagram (4.5) to obtain the bottom sequence in the following commutative diagram with exact rows:

(5.22)

$$\begin{tikzcd}
0 \arrow[r] & (\Omega^{i-1}(X)/\mathrm{Ker}\, d)(1-i) \arrow[r, "\mathrm{Exp}"] \arrow[d, "f_1"] & H^i_{\text{proét}}(X, \mathbf{Q}_p(1)) \arrow[r] \arrow[d, "f_2"] & \mathrm{HK}^i(X)(1) \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & \mathrm{Coker}(\mathrm{HTlog}_{U,i-1}) \arrow[r] & H^i_{\text{proét}}(X, \mathbf{Q}_p/\mathbf{Z}_p(1)) \arrow[r] & \mathrm{Ker}(\mathrm{HTlog}_{U,i}) \arrow[r] & 0,
\end{tikzcd}$$

where we set $\mathrm{HK}^i(X) := (H^i_{\mathrm{HK}}(X) \otimes^{\square}_{\breve{F}} \widehat{\mathbf{B}}^+_{\mathrm{st}})^{N=0,\varphi=p^i}$. The top row comes from diagram 3.15. The left square commutes by diagram (4.6). Hence if we are in a case where:

(1) we have the isomorphisms $\mathrm{Coker}(\mathrm{HTlog}_U^{i-1}) \xrightarrow{\sim} \mathrm{Coker}(\mathrm{HTlog}^{i-1})$ and $\mathrm{Ker}(\mathrm{HTlog}_U^i) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{HTlog}^i)$;
(2) the map f_1 in (5.22) is an isomorphism

the right square in (5.22) is bicartesian and we can compute $\mathrm{Ker}(\mathrm{HTlog}^i)$ using p-adic cohomologies.

[9] The expression in the denominator denotes an extension of the two terms.

5.2.2. *Examples of* $\mathrm{Ker}(\mathrm{HTlog}_{U,i})$. We present here the following computation:

Proposition 5.23. *For $i \geq 1$, we have the natural isomorphisms:*

$$\begin{aligned}\mathrm{Ker}(\mathrm{HTlog}_{U,i}(\mathbb{A}^d_C)) &= 0,\\ \mathrm{Ker}(\mathrm{HTlog}_{U,i}(\mathbb{G}^d_{m,C})) &\cong \wedge^i(\mathbf{Q}_p/\mathbf{Z}_p)^d,\\ \mathrm{Ker}(\mathrm{HTlog}_{U,i}(\mathbb{H}^d_C)) &\cong \mathrm{Sp}_i(\mathbf{Z}_p)^\vee,\end{aligned}$$

denoting by $(-)^\vee$ the Pontryagin dual.

Proof. Let us start with the affine space $X = \mathbb{A}^d_C$. Since $H^i_{\text{proét}}(\mathbb{A}^d_C, \mathbf{Z}_p) = 0$, for $i \geq 1$, the canonical map $H^i_{\text{proét}}(\mathbb{A}^d_C, \mathbf{Q}_p) \to H^i_{\text{proét}}(\mathbb{A}^d_C, \mathbf{Q}_p/\mathbf{Z}_p)$ is an isomorphism. Since $\mathrm{HK}^i(\mathbb{A}^d_C) = 0$, from diagram (5.22), we obtain indeed that $\mathrm{Ker}(\mathrm{HTlog}^i_U) = 0$.

In the case of the torus $X = \mathbb{G}^d_{m,C}$, we know that the map $H^i_{\text{proét}}(\mathbb{G}^d_{m,C}, \mathbf{Z}_p) \to \mathrm{HK}^i(\mathbb{G}^d_{m,C})$ is injective. This implies that the map f_2 in diagram (5.22) is surjective and so is the map f_3. It follows that

$$\mathrm{Ker}(\mathrm{HTlog}_{U,i}(\mathbb{G}^d_{m,C})) \simeq \mathrm{HK}^i(\mathbb{G}^d_{m,C})/H^i_{\text{proét}}(\mathbb{G}^d_{m,C}, \mathbf{Z}_p) \simeq \wedge^i(\mathbf{Q}_p/\mathbf{Z}_p)^d,$$

as wanted.

Finally for the Drinfeld space $X = \mathbb{H}^d_C$, the argument is analogous to the one in the case of the torus and we get

$$\mathrm{Ker}(\mathrm{HTlog}_{U,i}(\mathbb{H}^d_C)) \simeq \mathrm{HK}^i(\mathbb{H}^d_C)/H^i_{\text{proét}}(\mathbb{H}^d_C, \mathbf{Z}_p) \simeq \mathrm{Sp}_i(\mathbf{Q}_p)^*/\mathrm{Sp}_i(\mathbf{Z}_p)^* \simeq \mathrm{Sp}_i(\mathbf{Z}_p)^\vee,$$

as claimed. □

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