AROUND THE STABILITY OF KAM-TORI

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Contents

1. Introduction

Let

$$H(\varphi, r) = \langle \omega_0, r\rangle + \mathcal{O}(r^2) \tag{1.1}$$

be a C^2 function defined for $\varphi \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $r \sim 0 \in \mathbb{R}^d$.

The *Hamiltonian system* associated to H is given by

$$(*)_H \qquad \begin{cases} \dot\varphi = \partial_r H(\varphi, r) \\ \dot r = -\partial_\varphi H(\varphi, r). \end{cases}$$

Clearly the torus $\mathbb{T}^d \times \{0\}$ is invariant under the Hamiltonian flow and the induced dynamics is the translation

$$(t, \varphi) \mapsto \varphi + t\omega_0.$$

Moreover this torus is Lagrangian with respect to the canonical symplectic form $d\varphi \wedge dr$ on $\mathbb{T}^d \times \mathbb{R}^d$.

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The objective of this paper is to investigate the "KAM stability" of the torus $\mathcal{T}_0 := \mathbb{T}^d \times \{0\}$ under different hypothesis on H and ω_0. We first explain what we understand by "KAM stability". In its stronger form, we use this terminology to refer to the classical KAM (after Kolmogorov Arnol'd Moser) phenomenon of accumulation of $\mathcal{T}_0$ by invariant KAM tori whose Lebesgue density in the phase space tend to one in the neighborhood of $\mathcal{T}_0$ and whose frequencies cover a set of positive measure. More precisely, a vector ω is said to be *Diophantine* if there exist $\kappa > 0,\ \tau > d-1$ such that

$$|\langle k, \omega\rangle| \geq \frac{\kappa}{|k|^\tau} \quad \forall k \in \mathbb{Z}^d \setminus \{0\} \tag{1.2}$$

We then use the notation $\omega \in DC(\kappa, \tau)$. We say that a C^r (or smooth, or analytic) invariant Lagrangian torus with an induced flow that is C^r (or smoothly, or analytically) conjugated to a Diophantine translation

$$(t, \varphi) \mapsto \varphi + t\omega$$

is a C^r (smooth, analytic) *KAM-torus* of $(*)_H$ with *translation vector* ω.

We say that $\mathcal{T}_0$ is "KAM stable" in a weak sense if the set of KAM tori that accumulates it has positive Lebesgue measure but not necessarily density one. We also drop the requirement that the frequencies cover a set of positive measure.

When we prove the accumulation of $\mathcal{T}_0$ by invariant KAM tori but we do not know if their measure is positive we simply say that $\mathcal{T}_0$ is accumulated by KAM tori and do not speak of stability.

In this paper we deal essentially with the following situations and results. Unless otherwise mentioned the Hamiltonian H is assumed to be analytic as well as $\mathcal{T}_0$ and the KAM tori that are obtained. The exact statements and notations will be deferred to the next section.

(i) If ω_0 is Diophantine then $\mathcal{T}_0$ is accumulated by KAM tori.
(ii) If ω_0 is Diophantine and if the Birkhoff normal form (BNF) of H satisfies a Rüssmann transversality condition at $\mathcal{T}_0$, then $\mathcal{T}_0$ is KAM stable.
(iii) In two degrees of freedom ($d = 2$), if ω_0 is rationally independent and if H satisfies a Kolmogorov non degeneracy condition of its Hessian matrix at $\mathcal{T}_0$, $\mathcal{T}_0$ is KAM stable. For $d \geq 3$, we get KAM stability for a class of ω_0 that includes all vectors except a meagre set of zero Hausdorff dimension.
(iv) For $d \geq 4$, for any $\omega_0 \in \mathbb{R}^d$, there exists a C^∞ (Gevrey) H as in (1.1) such that $\mathcal{T}_0$ is not KAM stable (no positive measure of accumulating tori).

(v) For $d = 2$, if ω_0 is Diophantine and H is smooth $\mathcal{T}_0$ is KAM stable.

It was conjectured by M. Herman in his ICM98-lecture [H] that in the neighborhood of an analytic KAM-torus, the set of KAM-tori is of positive measure, i.e. KAM stability in a weak sense holds. (i) falls short from proving Herman's conjecture. In the case where we cannot prove that $\mathcal{T}_0$ is KAM stable we actually show that there exists a subvariety of dimension at least $d+1$ that is foliated by analytic KAM tori with frequency ω_0. The proof of (i) is based on a counter term KAM theorem inspired by Herman. For every value $c \sim 0$ of the action variable there exists a unique frequency $\Omega(c)$ that cancels the counter term, and if this frequency is Diophantine this yields an invariant KAM torus with frequency $\Omega(c)$. One can show that the jets of the function $\Omega(c)$ are given by those of the gradient of the Birkhoff normal form when the latter is well defined (which is the case if ω_0 is Diophantine). The following alternative then holds : either the BNF is non degenerate and the function Ω takes Diophantine values on a positive measure set which yields KAM stability (this is (iii)), or the BNF is degenerate and we can use the analytic dependance of the counter term on the action variable to show the existence of a direction (after a coordinate change in the action variable) that spans a subvariety of invariant KAM tori of frequency ω_0.

Point (ii) is a more classical KAM result. Its proof is obtained from the counter term KAM theorem as explained above and it can be adapted to smooth Hamiltonians. The hypothesis ω_0 Diophantine is necessary to guarantee the existence of a BNF.

In (iii) KAM stability is studied in the neighborhood of a Liouville torus. The difficulty is that the BNF may not be defined. This difficulty can be overcome if the Kolmogorov non degeneracy condition is satisfied by H at $\mathcal{T}_0$, and if the rationally independent frequency ω_0 satisfies an arithmetic condition that contains all rationally independent vectors if $d = 2$ and all but a meagre set of Hausdorff dimension 0 if $d \geq 3$. The condition is that the uniform Diophantine exponent of ω_0 denoted by $\widehat{\omega}(\omega_0)$ be finite. We recall that in the case of flows, we define $\widehat{\omega}(\omega_0)$ as the supremum of all real numbers γ such that for any sufficiently large N, there exists $k \in \mathbb{Z}^d$ such that $\|k\| \leq N$ and $|(k, \omega)| \leq N^{-\gamma}$. We do not know whether invariant tori with frequencies ω_0 such that $\widehat{\omega}(\omega_0) = +\infty$ are KAM stable if the Kolmogorov non degeneracy condition is satisfied.

The construction of (iv) is based on the successive conjugation method (Anosov Katok construction) starting from an "infinitely degenerate twist map" of the form $(\varphi, r) \mapsto (\varphi + f(r), r)$ with the frequency map f such that $f(0) = \omega_0$ and $f(r)$ having a fixed Liouville coordinate in small neighborhoods of any r such that $r_d \neq 0$. The construction only applies to the case $d \geq 4$. In case $d = 2$ a smooth version of our KAM counter term theorem yields KAM stability of Diophantine tori just as in Herman's last geometric theorem any Diophantine KAM circle of a smooth diffeomorphism of the annulus is shown to be KAM stable [FK].

2. Statements

2.1. **Analytic KAM tori are never isolated.** Let H be a real analytic function of the form (1.1).

Theorem A. *If ω_0 is Diophantine, the torus $\mathbb{T}^d \times \{0\}$ is accumulated by analytic KAM tori of $(*)_H$ with Diophantine translation vector.*

In fact, we shall prove a more precise result. Let N_H be the *Birkhoff Normal Form* of H, that is a uniquely defined power series in the r variable as soon as ω_0 is Diophantine (see Section 3.1). We say that N_H is j*-degenerate* if there exist j orthonormal vectors $\gamma_1, \ldots, \gamma_j$ such that for every $r \sim 0 \in \mathbb{R}^d$

$$\langle \partial_r N_H(r), \gamma_i \rangle = 0 \quad \forall\ 1 \leq i \leq j,$$

but no $j + 1$ orthonormal vectors with this property. Since $\omega_0 \neq 0$ clearly $j \leq d - 1$. A 0-degenerate N_H is also said to be *non-degenerate*.

Theorem B. *If ω_0 is Diophantine and N_H is j-degenerate, then there exists an analytic (co-isotropic) subvariety of dimension $d + j$ containing $\mathbb{T}^d \times \{0\}$ and foliated by analytic KAM-tori of $(*)_H$ with translation vector ω_0.*

A stronger result is known when N_H is $(d - 1)$-degenerate. Indeed Rüssmann [R] (in a different setting) proved

Theorem (Rüssmann). *If ω_0 is Diophantine and N_H is $(d - 1)$-degenerate, then a full neighborhood of $\mathbb{T}^d \times \{0\}$ is foliated by analytic KAM-tori of $(*)_H$ with translation vector $\in \mathbb{R}\omega_0$.*

Our proof of Theorem B in Section 6 will also yield Rüssmann's result. Theorem A follows from Theorem B in the degenerate case and from a more classical KAM theorem in the non-degenerate theorem that we discuss in the next section.

2.2. **KAM stability under non degeneracy conditions of the BNF.**

Let H be a real analytic function of the form (1.1). We say that H has a normal form N_H if there exists a formal power series N_H and a formal symplectic mapping Z of the form

$$Z(\varphi, r) = (\varphi + \mathcal{O}(r), r + \mathcal{O}^2(r))$$

such that

$$H \circ Z(\varphi, r) = N_H^q(r) + \mathcal{O}^{q+1}(r) \in \mathcal{C}^\omega(\mathbb{T}^d \times \{0\}).$$

Remark. *This is in particular the case when ω_0 is Diophantine – N_H is the classical Birkhoff normal form. Moreover if a normal form is exists and ω is rationally independent, then it is unique.*

Only assuming existence and non-degeneracy of the normal form N_H, we shall prove the following.

Theorem C. *If N_H exists, is unique and is non-degenerate, then in any neighborhood of $\mathbb{T}^d \times \{0\}$ the set of analytic KAM-tori of $(*)_H$ is of positive Lebesgue measure with density one at the torus $\mathbb{T}^d \times \{0\}$. In particular, if ω_0 is Diophantine and if N_H is non degenerate at $\mathcal{T}_0$, then $\mathcal{T}_0$ is KAM stable.*

The condition that N_H is non-degenerate is essentially equivalent to Rüssmann's non-degeneracy condition (see [R2, XYQ]). It is here shown to be sufficient in this singular perturbation situation.

Point (ii) of our introduction corresponds to the second statement of Theorem C.

Hence, the conjecture of M. Herman has an affirmative answer when N_H is non-degenerate (theorem C) or $(d-1)$-degenerate (Rüssmann's theorem). Our theorems do not provide an answer to the conjecture in the intermediates cases.

2.3. **KAM stability in the absence of BNF : Liouville torus with non-degeneracy of Kolmogorov type.** Let H be a real analytic function of the form (1.1). and let

$$M_0 = \int_{\mathbb{T}^d} \frac{\partial^2 H}{\partial r^2}(\varphi, 0) d\varphi.$$

We recall the notation $\widehat{\omega}(\omega_0)$ as the supremum of all real numbers γ such that for any sufficiently large N

$$\min_{0<|k|\leq N} |\langle k, \omega_0 \rangle| \leq N^{-\gamma}.$$

Theorem D. *If $\widehat{\omega}(\omega_0) < +\infty$ and if M_0 is non-singular then in any neighborhood of $\mathbb{T}^d \times \{0\}$ the set of analytic KAM-tori of $(*)_H$ is of positive Lebesgue measure with density one at the torus $\mathbb{T}^d \times \{0\}$. Moreover, the set of frequencies of the KAM tori has positive Lebesgue measure in $\mathbb{R}^d$.*

Since for any rationally independent vector $\omega_0 \in \mathbb{R}^2$ we have that $\widehat{\omega}(\omega_0) = 1$ we see that KAM stability holds at $\mathcal{T}_0$ without any other arithmetic condition when $d = 2$. This is a precise formulation of (iii).

Remark. *We could relax the condition $\widehat{\omega}(\omega_0) < +\infty$ to the existence of sequences $Q_n \to \infty$ and $\epsilon_n \to 0$ such that $|(k, \omega_0)| \geq e^{-\epsilon_n Q_n}$ for any $k \in \mathbb{Z}^d, 0 < |k| \leq Q_n$.*

2.4. **Smooth counterexamples to KAM stability.** In the $\mathcal{C}^\infty$-category the situation is different from that of Theorem A. For $d = 2$, we show in section 7.1 that the same 1 dimensional phenomenon of the frequency map pointed out by Herman (see [FK] for the discrete case) gives a set of positive measure of $\mathcal{C}^\infty$ KAM-tori in any neighborhood of $\mathbb{T}^2 \times \{0\}$. For $d = 3$, we have no results, but for $d \geq 4$ we shall prove

Theorem E. *Let $d \geq 4$. For any $\epsilon > 0, s \in \mathbb{N}$ there exists a function h in $C^\infty(\mathbb{T}^4 \times \mathbb{R}^4)$, satisfying $h(\varphi, r) = \mathcal{O}^\infty(r_4)$ and*

$$\|h\|_{\mathcal{C}^s(\mathbb{T}^4 \times \mathbb{R}^4)} < \epsilon,$$

such that the flow Φ_H^t of $H(\varphi, r) = (\omega_0, r) + h(\varphi, r)$ satisfies

$$\limsup_{t \to \pm\infty} \|\Phi_H^t(\varphi, r)\| = \infty$$

for any (φ, r) satisfying $r_4 \neq 0$.

Remark. *We will see in Section 7 that the construction of Theorem E can actually be carried out in any Gevrey class G^σ with $\sigma > 1$.*

Notice that in the examples of Theorem E the hyperplane $r_4 = 0$ is foliated by KAM tori with translation vector ω_0, so the torus $\mathbb{T}^d \times \{0\}$ is not isolated. Theorem E gives however counter-examples for $d \geq 4$ to the positive measure accumulation by KAM-tori. Indeed, each point that lies outside this hyperplane diffuses to infinity along a sequence of time. As we shall see in Proposition 4.3, its positive and negative semi-orbits actually oscillate between $-\infty$ and $+\infty$ in projection to at least two action coordinates.

It would be interesting to construct smooth examples with an isolated KAM-torus, thus showing that the phenomenon of Theorem A is purely analytic. On the other hand if Herman's conjecture is correct, then the phenomenon of Theorem E cannot be carried to the analytic setting.

It is worth noting that Herman did also announce the existence of counter-examples in the C^∞ category to the positive measure conjecture, provided $d \geq 4$. However, he did not provide any clue to these examples and we are not aware whether the examples he had in mind had any invariant tori accumulating the KAM-torus.

2.5. **Plan of the paper.** The paper is organized in the following way. In section 3 we discuss the Birkhoff normal form and we give a different (from the usual) characterization of it. In section 4

we formulate a KAM counter term theorem which we use to give still another characterization of the Birkhoff normal form. Using this result we derive Theorem B and C and Rüssmann's theorem in sections 5 and 6. In section 7 we prove Theorem E, and in section 8 we give a proof of the KAM counter term theorem used in section 4.

2.6. **Notations.** We denote by $\mathbb{D}^d_\delta$ the polydisk in $\mathbb{C}^d$ with radius δ. More generally if $d = (d_1, \dots, d_n)$ and $\delta = (\delta_1, \dots, \delta_n)$, then

$$\mathbb{D}^d_\delta = \mathbb{D}^{d_1}_{\delta_1} \times \cdots \times \mathbb{D}^{d_n}_{\delta_n}.$$

Let $\mathbb{T}^d_\rho$ be the complex neighbourhood of width ρ of of $\mathbb{T}^d$:

$$(\{z \in \mathbb{C} : |\Im z| < \rho\}/\mathbb{Z})^d.$$

A holomorphich function $f : \mathbb{T}^d_\rho \times \mathbb{D}^e_\delta \to \mathbb{C}$ is *real* if it gives real values to real arguments. We denote by

$$\mathcal{C}^\omega(\mathbb{T}^d_\rho \times \mathbb{D}^e_\delta)$$

the space of such real holomorphic functions which we provide with the norm

$$|f|_{\rho,\delta} = \sup_{(\varphi,z)\in\mathbb{T}^d_\rho\times\mathbb{D}^e_\delta} |f(\varphi, z)|.$$

We let

$$\mathcal{C}^\omega(\mathbb{T}^d \times \{0\}) = \bigcup_{\rho,\delta} \mathcal{C}^\omega(\mathbb{T}^d_\rho \times \mathbb{D}^e_\delta).$$

We denote by $\partial^\alpha_\varphi f$ and $\partial^\alpha_z f$ the partial derivates of f with respect to φ and z respectively, with the usual multi-index notations. If $z = (z', z'')$ we say that

$$f \in \mathcal{O}^j(z')$$

if and only if $\partial^{\alpha'}_{z'} f(\varphi, 0, z'') = 0$ for all $|\alpha'| < j$. We denote by $\partial_\varphi f$ and $\partial_z f$ the *gradient* of f with respect to φ and z, respectively, and by $\partial^2_\varphi f$ and $\partial^2_z f$ the corresponding *Hessian*.

For a function $f \in \mathcal{C}^\omega(\mathbb{T}^d_\rho \times \{0\})$, $\mathcal{M}(f)$ is the *mean value*

$$\int_{\mathbb{T}^d} f(\varphi, z) d\varphi.$$

We shall also use the same notations for $\mathbb{C}^n$-valued functions $f = (f_1, \ldots, f_n)$ with the absolute value replaced by $|f| = \max_i |f_i|$ (or some other norm on $\mathbb{C}^n$).

Formal power series. Let $z = (z_1, \ldots, z_n)$. An element

$$f \in \mathcal{C}^\omega(\mathbb{T}^d_\rho)[[z]]$$

is a formal power series

$$f = f(\varphi, z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(\varphi) z^\alpha$$

whose coefficients $a_\alpha \in \mathcal{C}^\omega(\mathbb{T}^d_\rho)$ (possibly vector valued). We denote by

$$[f]_j(\varphi, z) = \sum_{|\alpha| = j} a_\alpha(\varphi) z^\alpha,$$

the homogenous component of degre j, and

$$[f]^j = \sum_{i \le j} [f]_i.$$

The partial derivate $\partial^\alpha_\varphi f$ and $\partial^\alpha_z f$ are well-defined and if $z = (z', z'')$ we define that $f \in \mathcal{O}^j(z')$ in the same way as for functions. The mean value $\mathcal{M}(f)$ is the power series obtained by taking the mean values of the coefficients.

Parameters. Let B be an open subset of some euclidean space. Define

$$\mathcal{C}^{\omega,\infty}(\mathbb{T}^d_\rho \times \mathbb{D}^e_\delta, B)$$

to be the set of $\mathcal{C}^\infty$ functions (possibly vector valued)

$$f : \mathbb{T}^d_\rho \times \mathbb{D}^e_\delta \times B \ni (\varphi, z, \omega) \mapsto f(\varphi, z, \omega)$$

such that for all $\omega \in B$ [1]

$$f_\omega : \mathbb{T}^d_\rho \times \mathbb{D}^e_\delta \ni (\varphi, \omega) \mapsto f(\varphi, z, \omega)$$

is real holomorphic. We define

$$||f||_{\rho,\delta,s} = \sup_{|\alpha| \le s} |\partial^\alpha_\omega f_\omega|_{\rho,\delta}.$$

[1] we appologize for the double use of ω

3. The Birkhoff Normal Form (BNF)

Let

$$H(\varphi, r) = \langle \omega_0, r\rangle + \mathcal{O}^2(r) \ \in \ \mathcal{C}^\omega(\mathbb{T}^d_\rho \times \mathbb{D}^d_\delta)$$

and

$$\omega_0 \in DC(\kappa_0, \tau_0).$$

3.1. **The Birkhoff normal form (BNF).** Let us recall a well-known result.

Proposition. *There exist*

$$\begin{cases} f(\varphi, r) \in \mathcal{C}^\omega(\mathbb{T}^d_\rho)[[r]] \cap \mathcal{O}^2(r) \\ N(r) \in \mathbb{R}[[r]] \end{cases}$$

such that

$$H(\psi, r + \partial_\psi f(\psi, r)) = N(r).$$

Moreover, $N(r)$ is unique and f is uniquely determined by fixing arbitrarily the mean value $\mathcal{M}(f)$.

Remark. *The unique series N is the Birkhoff normal form of H, denoted N_H. It is clear that*

$$N_H(r) = \langle \omega_0, r\rangle + \mathcal{O}^2(r).$$

We say that the unique f for which $\mathcal{M}(f) = 0$ is the generating function of the BNF, denoted f_H.

We know that the generating function f_H is convergent if, and only if, H is integrable [I] (see also [V, N]). It was known to Poincaré that for "typical" (in a sense we would call today generic) H, f_H will be divergent. (Siegel [S55] proved the same thing in a neigbourhood of an elliptic equilibrium with another, and stronger, notion of "typical".)

However, essentially nothing is known about the BNF itself when ω_0 is Diophantien. For example, it is not known:

(i) can N_H be divergent?
(ii) if H is non integrable, can N_H be convergent?

We only have a result of Perez-Marco [P-M] saying that if the BNF N_H is divergent for some H, then N_H is divergent for "typical" (i.e. except for a pluri-polar set) H. More generally, nothing is known about the set of all BNF's

$$\mathcal{B}(\omega_0) = \{N_H : H(\varphi, r) = \langle \omega_0, r\rangle + \mathcal{O}^2(r)\}.^2$$

[2] apart from the fact that N_H has some Gevrey-growth [S05] and that $\mathcal{B}(\omega_0)$ contains all convergent series

Is it a "large" set or a "small" set in the space of all power series? It has been shown [B] that if N_H fulfills a certain condition $\mathcal{G}$, which is prevalent in the space of power series, then the invariant torus $\mathbb{T}^d \times \{0\}$ is *doubly exponentially stable.* However, it is not known if N_H can belong to $\mathcal{G}$ when H is non-integrable.

3.2. **Exact symplectic mappings and generating functions.** Consider the equations

$$\left\{ \begin{array}{l} \varphi = \psi + p(\psi, r) \\ s = r + q(\psi, r) \end{array} \right. \tag{3.3}$$

with

$$p, q \in \mathcal{C}^\omega(\mathbb{T}^d \times \{0\})$$

and

$$\det(I + \partial_\psi p(\psi, r)) \neq 0$$

for all $(\psi, r) \in \mathbb{T}^d \times \{r \sim 0\}$.

These equations can be solved uniquely for (ψ, s) as

$$\left\{ \begin{array}{l} \psi = \varphi + \Phi(\varphi, r) \\ s = r + R(\varphi, r) \end{array} \right. \tag{3.4}$$

with

$$\Phi, R \in \mathcal{C}^\omega(\mathbb{T}^d \times \{0\})$$

and

$$\det(I + \partial_\varphi \Phi(\varphi, r)) \neq 0$$

for all $(\varphi, r) \in \mathbb{T}^d \times \{r \sim 0\}$. Conversely, the equations (3.4), under the two supplementary conditions on Φ, R, can be solved uniquely for (φ, s) as (3.3), with the two supplementary conditions on p, q.

Remark. *It is easy to verify that*

$$p \in \mathcal{O}(r) \quad \textit{and} \quad q \in \mathcal{O}^2(r)$$

if and only if

$$\Phi \in \mathcal{O}(r) \quad \textit{and} \quad R \in \mathcal{O}^2(r).$$

The mapping

$$Z : (\varphi, r) \mapsto (\psi, s)$$

is a real analytic local diffeomeorphism on $\mathbb{T}^d \times \{r \sim 0\}$. It is *symplectic* if and only if the one-form

$$Z^*(rd\varphi) - (rd\varphi)$$

is closed, and it is *exact* if and only if this one-form is exact.

Proposition 3.1. *Z is symplectic if and only if the one-form $pdr+qd\psi$ is closed. Z is exact if and only if the one-form $pdr+qd\psi$ is exact*

If

$$\Phi \in \mathcal{O}(r) \quad \textit{and} \quad R \in \mathcal{O}^2(r),$$

then Z is exact if and only if it is symplectic.

Hence, if Z is exact there is a unique (modulo an additive constant) function f such that $df = pdr + qd\psi$. The function f is said to be a *generating function for Z.*

Proof. We have

$$sd\psi - rd\varphi = (r + q(\psi, r))d\psi - rd\psi - \partial_\psi(rp)d\psi - \sum_{i,j} r_j \partial_{r_i} p_j(\psi, r)dr_i$$

and

$$d(rp) = \partial_\psi(rp)d\psi + pdr + \sum_{i,j} r_j \partial_{r_i} p_j dr_i.$$

Hence

$$sd\psi - rd\varphi = qd\psi + pdr - d(rp)$$

which proves the first two statements.

Finally, if

$$\Phi \in \mathcal{O}(r) \quad \text{and} \quad R \in \mathcal{O}^2(r),$$

then

$$p \in \mathcal{O}(r) \quad \text{and} \quad q \in \mathcal{O}^2(r).$$

Now, $pdr + qd\psi$ is closed if and only if for all i, j

$$\begin{cases} \partial_{r_i} p_j = \partial_{r_j} p_i \\ \partial_{\psi_i} q_j = \partial_{\psi_j} q_i \\ \partial_{\psi_i} p_j = \partial_{r_j} q_i \end{cases}$$

By the symmetry condition on $\partial_r p$ this implies that there exists a unique function $f(\psi, r)$ such that for all j

$$\partial_{r_j} f = p_j, \quad f(\psi, 0) = 0.$$

Then, for all i, j,

$$\partial_{r_j} \partial_{\psi_i} f = \partial_{\psi_i} p_j = \partial_{r_j} q_i$$

and, hence,

$$\partial_{\psi_i} f(\psi, r) = q_i(\psi, r) + h_i(\psi).$$

Since $f, q \in \mathcal{O}(r)$, this implies that $h_i = 0$. $\square$

Corollary 3.2. *If*

$$\begin{array}{rl} Z: & \mathbb{T}^d \times \{r \sim 0\} \to \mathbb{T}^d \times \{r \sim 0\} \\ & (\varphi, r) \mapsto (\varphi + \Phi(\varphi, r), r + R(\varphi, r)) \end{array}$$

is a symplectic real analytic local diffeomorphism such that

$$\Phi \in \mathcal{O}(r) \quad \textit{and} \quad R \in \mathcal{O}^2(r),$$

then

$$N_{H\circ Z} = N_H.$$

Proof. Applying the BNF proposition of section 3.1 to H and $\tilde{H} = H \circ Z$ we find two generating functions

$$f, \tilde{f} \in \mathcal{C}^\omega(\mathbb{T}^d_\rho)[[r]] \cap \mathcal{O}^2(r).$$

By truncating these function at degre n and applying Proposition 3.1 we find two exact symplectic mappings W_n and $\tilde{W}_n$ such that

$$H \circ W_n(\varphi, r) = N_H^n + \mathcal{O}^{n+1}(r)$$

and

$$H \circ Z \circ \tilde{W}_n(\varphi, r) = N_{H\circ Z}^n + \mathcal{O}^{n+1}(r)$$

By Proposition 3.1 again $Z \circ \tilde{W}_n$ has a generating function

$$g_n \in \mathcal{C}^\omega(\mathbb{T}^d_{\rho''} \times \mathbb{D}^d_{\delta''}) \cap \mathcal{O}^2(r).$$

Letting $n \to \infty$, the result now follows from the uniqueness of the BNF proposition of section 3.1. □

3.3. **Another characterization of the BNF.** Let $P(r, c)$ be a power series in r, c. We say that

$$P(r, c) = 0 \mod \mathcal{O}^2(r - c) \quad \text{or} \quad P(r, c) \in \mathcal{O}^2(r - c)$$

if

$$P(r, c) = \langle r - c, Q(r, c)(r - c)\rangle$$

for some matrix valued power series $Q(r, c)$. Using this notation any $P(r, c)$ can be written

$$P(c, c) + \langle \partial_r P(c, c), r - c)\rangle + \mathcal{O}^2(r - c).$$

Proposition 3.3. *Let $c = (c_1, \ldots, c_d)$. There exist*

$$\begin{cases} f(\varphi, r, c) \in \mathcal{C}^\omega(\mathbb{T}^d_\rho)[[r, c]] \cap \mathcal{O}^2(r, c) \\ \Omega(c) \in \mathbb{R}^d[[c]] \\ \Gamma(c) \in \mathbb{R}[[c]] \end{cases}$$

such that

$$H(\psi, r + \partial_\psi f(\psi, r, c)) = \Gamma(c) + \tag{3.5}$$

$$\langle \Omega(c), r-c\rangle + \langle (r-c), F(\psi + \partial_r f(\psi, r, c), r, c)(r-c)\rangle + \mathcal{O}^{q+1}(c)$$

for all q.

Moreover, if (3.5) holds for a specific q, then $\Gamma(c)$ *and* $\Omega(c)$ *are unique mod* $\mathcal{O}^q(c)$ *and*

$$\Gamma(c) = N_H(c) + \mathcal{O}^{q+1}(c)$$

and

$$\Omega(c) = \partial_c N_H(c) + \mathcal{O}^q(c).$$

Proof. We must show not only that there exists at least one solution f, Γ, Ω of this problem, but we must also show that Γ, Ω are the same for all such solutions. Let

$$H_j(\varphi, r) = [H(\varphi, r)]_j$$

be the homogenous component of degre j of $H(\varphi, r)$ and define $f_j(\psi, r, c)$, $\Gamma_j(c)$, $\Omega_j(c)$ and $F_j(\varphi, r, c)$ similarly.

For $j = 1$, the equation becomes

$$\Gamma_1(c) + \langle \Omega_0, r-c\rangle = \langle \omega_0, r\rangle$$

which gives $\Omega_0 = \omega_0$ and $\Gamma_1 = \langle \omega_0, c\rangle$.

For $j = 2$, the equation becomes

$$\begin{aligned} \langle \omega_0, \partial_\psi f_2(\psi, r, c)\rangle + H_2(\psi, r) = \Gamma_2(c) + \\ + \langle \Omega_1(c), r-c\rangle + \langle r-c, F_0(\psi)(r-c)\rangle \end{aligned} .$$

Write $H_2(\psi, r)$

$$= H_2(\psi, c) + \langle \partial_r H_2(\psi, c), r-c\rangle + \langle r-c, Q(\psi)(r-c)\rangle.$$

Then we must have

$$\Gamma_2(c) + \langle \Omega_1(c), r-c\rangle = \mathcal{M}(H_2(\cdot, c) + \langle \partial_r H_2(\cdot, c), r-c\rangle)$$

which determines Γ_2 and Ω_1 uniquely.

If we take $F_0 = Q$, then we get the equation for f_2:

$$\langle \omega_0, \partial_\psi f_2(\psi, r, c)\rangle = -\mathcal{V}\Big(H_2(\psi, c) + \langle \partial_r H_2(\psi, c), r-c\rangle\Big)$$

where $\mathcal{V} = id - \mathcal{M}$. Clearly this equation defines f_2 uniquely modulo a mean value g_2. But we can also add any term of degree two in $\mathcal{O}^2(r-c)$ to f_2 and still get a solution simply by changing the definition of F_0. Hence f_2 is unique modulo a mean value g_2 and modulo $\mathcal{O}^2(r-c)$. (In the sequel we must show, in particular, that the higher order terms of γ and Ω remain the same for these different choices of f_2.)

We now proceed by induction on $j \geq 3$: assume that we have constructed for $2 \leq m \leq j-1$, the homogenous components $f_m(\psi, r, c)$, $\Gamma_m(c)$, $\Omega_{m-1}(c)$ and $F_{m-2}(\varphi, r, c)$ and assume that $f_m(\psi, r, c)$ is unique modulo a meanvalue $g_m(r,c)$ and modulo $\mathcal{O}^2(r-c)$ – we have seen that this induction assumption is true for $j=2$.

For $j\geq 3$, the equation becomes

$$\begin{aligned}\langle\omega_0,\partial_\psi f_j(\psi,r,c)\rangle+G_j(\psi,r,c)&=\Gamma_j(c)+\langle\Omega_{j-1}(c),r-c\rangle+\\&+\langle r-c,(K_{j-2}+F_{j-2})(\psi,r,c)(r-c)\rangle\end{aligned}$$

where $G_j(\psi,r,c)$

$$=[(H_2+\cdots+H_j)(\psi,r+\partial_\psi f_2(\psi,r,c)+\cdots+\partial_\psi f_{j-1}(\psi,r,c),c)]_j$$

and $K_{j-2}(\psi,r,c)$

$$=[(F_0+\cdots+F_{j-3})(\psi+\partial_r f_2(\psi,r,c)+\cdots+\partial_r f_{j-1}(\psi,r,c),c)]_{j-2}.$$

We write $G_j(\psi,r,c)$

$$=G_j(\psi,c,c)+\langle\partial_r G_j(\psi,c,c),r-c\rangle+\langle(r-c),Q(\psi,r,c)(r-c)\rangle$$

and notice that $G_j(\psi,c,c)+\langle\partial_r G_j(\psi,c,c),r-c\rangle$ only depend on $f_2,\ldots,f_{j-1}$ modulo their meanvalues and modulo $\mathcal{O}^2(r-c)$ – hence this term is uniquely determined by $H_2+\cdots+H_j$. Then

$$\Gamma_j(c)+\langle\Omega_{j-1}(c),r-c\rangle=\mathcal{M}(G_j(\cdot,c,c)+\langle\partial_r G_j(\cdot,c,c),r-c\rangle)$$

which determines Γ_j and Ω_{j-1} uniquely.

If we take $F_{j-2}=Q-K_{j-2}$, then we get for f_j the equation

$$\langle\omega_0,\partial_\psi f_j(\psi,r,c)\rangle=-\mathcal{V}\Big(G_j(\psi,c,c)+\langle\partial_r G_j(\psi,c,c),r-c\rangle\Big)\tag{3.6}$$

which has a unique solution modulo a mean value $g_j(r,c)$. But we can also add any term of degre j in $\mathcal{O}^2(r-c)$ to f_j and still get a solution simply by changing the definition of F_{j-2}.

This shows the existence of f,Γ and Ω verifying (3.5) up to any order q, as well as the uniqueness.

By Propositions 3.1 there exists

$$f\in\mathcal{C}^\omega(\mathbb{T}^d)[[r]]\cap\mathcal{O}^2(r)$$

such that

$$H(\psi,r+\partial_\psi f(\varphi,r))=N_H(r).$$

Now

$$N_H(r)=N_H(c)+\langle\partial_r N_H(c),r-c\rangle+\mathcal{O}^2(r-c)$$

and the uniqueness of $\Omega(c)$, mod $\mathcal{O}^q(c)$, gives the final statement. $\square$

4. A KAM counter term theorem and the BNF

Let B be the unit ball centered at ω_0 or, more generally, the intersection of this unit ball with an affine subspace of $\mathbb{R}^d$ through ω_0.

Let $\kappa > 0$ and $\tau > d-1$ be given numbers.

Let $l : \mathbb{R} \to \mathbb{R}$ denote a fixed non-negative C^∞ function such that $|l| \le 1$, and $l(x) = 0$ if $|x| \ge 1/2$ and $l(x) = 1$ if $|x| \le 1/4$.

4.1. **A cut-off operator and flat functions.** For $f \in \mathcal{C}^{\omega,\infty}(\mathbb{T}^d_\rho \times \mathbb{D}^e_\delta, B)$, let

$$\mathcal{P}(f)(\varphi, z, \omega) = \sum_{n \in \mathbb{Z}^d \smallsetminus \{0\}} \hat{f}(n, z, \omega) e^{2\pi i \langle n, \varphi \rangle} l(\langle n, \omega \rangle \frac{|n|^\tau}{\kappa}).$$

Remark. *Notice that $\mathcal{P}(f)$ depend on the choice of l, τ and κ. We shall not care about the dependence on the first two factors – all constants will depend on l and τ – but we shall keep careful track on the dependence on κ.*

Notice also that $g = \mathcal{P}(f)$ is a flat function on $DC(\kappa, \tau)$, i.e.

$$\partial^\alpha_\varphi \partial^\beta_z \partial^\gamma_\omega g(\varphi, z, \omega) = 0$$

for all multi-indices α, β, γ whenever $\omega \in DC(\kappa, \tau)$ – a function g with this property is said to be (κ, τ)-*flat*. In particular, if $f = \mathcal{P}(f)$ the f is (κ, τ)-flat.

Lemma 4.1. *We have*

$$\|\mathcal{P}(f)\|_{\rho', \delta, s} \le C_s (\frac{1}{\kappa})^s (\frac{1}{\rho - \rho'})^{(s+1)\tau + d} \|f\|_{\rho, \delta, s}$$

for any $\rho' < \rho$ and any $s \in \mathbb{N}$. The constant C_s only depends, besides s, on τ and l.

Proof. The Fourier coefficients (with respect to φ) verify

$$\left\| \hat{f}(n, \cdot, \cdot) \right\|_{0, \delta, s} \le \|f\|_{\rho, \delta, s} e^{-2\pi |n| \rho}.$$

The functions

$$l_n(\omega) = l(\langle n, \omega \rangle \frac{|n|^\tau}{\kappa})$$

verify

$$\|l_n\|_{0,0,s} \le |n|^{(\tau+1)s} \frac{1}{\kappa^s} \|l\|_{0,0,s}.$$

Hence for $|\alpha| \le s$ and $(\varphi, z, \omega) \in \mathbb{T}^d_{\rho'} \times \mathbb{D}^d_\delta \times B$

$$|\partial^\alpha_\omega \mathcal{P}(f)(\varphi, z, \omega)| \le$$

$$C_s \sum_{n\neq 0} e^{2\pi|n|\rho'} \left(\|\hat{f}(n,\cdot,\cdot)\|_{0,\delta,s} + \|\hat{f}(n,\cdot,\cdot)\|_{0,\delta,0} |n|^{(\tau+1)s} \frac{1}{\kappa^s} \right)$$

which gives the estimate. (Here we have used Proposition 10.1.) □

4.2. **A counter term theorem.**

Proposition 4.2. *Given $0 < \kappa < 1$ and $\tau > d-1$. Then, for all $s \in \mathbb{N}$, there exist non-negative constants (only depending on s and τ)*

$$\alpha(s) \geq (s-t) + \alpha(t), \quad s \geq t \geq 0,$$

such that if

$$H(\varphi, r) = N^q(r) + \mathcal{O}^{q+1}(r) \in \mathcal{C}^\omega(\mathbb{T}^d \times \{0\}), \quad q \geq \alpha(1) + 1,$$

with

$$N^q(r) = \langle \omega_0, r\rangle + \mathcal{O}^2(r),$$

then there exist $\rho, \delta > 0$ and

$$\begin{cases} f = f(\varphi, r, c, \omega) \in \mathcal{C}^{\omega,\infty}(\mathbb{T}^d_\rho \times \mathbb{D}^d_\delta \times \mathbb{D}^d_\delta, B) \cap \mathcal{O}^2(r,c) \\ \Lambda = \Lambda(c,\omega) \in \mathcal{C}^{\omega,\infty}(\mathbb{D}^d_\delta, B) \end{cases}$$

such that

$$H(\psi, r + \partial_\psi f(\psi, r, c, \omega)) + \langle \omega + \Lambda(c,\omega), r + \partial_\psi f(\psi, r, c, \omega)\rangle \\ = \langle \omega, r - c\rangle + \mathcal{O}^2(r-c) + g \tag{4.7}$$

(modulo an additive constant that depends on c,ω) with g (κ,τ)-flat and $g \in \mathcal{O}^2(r,c) \cap \mathcal{O}^q(c)$.

Moreover,

(i) *there exist constants C_s, only depending on s, H, l, τ such that*

$$\|\Lambda + \partial_r N^q\|_{0,\eta,s} + \|f\|_{\rho,\eta,s} \leq C_s \eta^q \left(\frac{1}{\kappa\eta}\right)^{\alpha(s)}$$

for any $\eta < \delta$

(ii) *there exists a constant C, only depending on H, l, τ, such that*

$$\delta \geq \frac{1}{C} \kappa^{\frac{\alpha(1)}{q-\alpha(1)}}$$

(iii) *if*

$$\omega_0 \in DC(2\kappa, \tau)$$

then the mapping

$$\mathbb{D}^{d+1}_{\delta'} \ni (c,\lambda) \mapsto \Lambda(c, (1+\lambda)\omega_0) \in \mathbb{C}^d$$

is real holomorphic for some δ'.

Remark. *Notice that this proposition (except part (iii)) does not require that ω_0 is Diophantine.*

We shall prove this proposition in section 5, but here we shall derive its consequences.

Corollary 4.3. *Given $0 < \kappa < 1$ and $\tau > d-1$ and non-negative constants $\alpha(s)$ as in Proposition 3.3.*

If

$$H(\varphi, r) = N^q(r) + \mathcal{O}^{q+1}(r) \in \mathcal{C}^\omega(\mathbb{T}^d \times \{0\}), \quad q \geq \alpha(1) + 1,$$

with

$$N^q(r) = \langle \omega_0, r \rangle + \mathcal{O}^2(r),$$

then there exists a unique real $\mathcal{C}^\infty$ function Ω, defined for

$$|c| < \eta_0 = \frac{1}{C'} \kappa^{\frac{\alpha(1)}{q-\alpha(1)}},$$

where C' only depends on H, τ, l, such that

$$\Omega(c) + \Lambda(c, \Omega(c)) = 0.$$

Moreover,

(i) *for any $s \in \mathbb{N}$ there exists a constant C'_s such that*

$$\|\Omega - \partial_r N^q\|_{\mathcal{C}^s(|c|<\eta)} \leq C'_s \eta^q (\frac{1}{\kappa\eta})^{\alpha(s)}$$

for any $\eta < \eta_0$

(ii) *the Taylor series of Ω up to degree $q-1$ at $c = 0$ is given by $\partial_r N^q(c)$.*

(iii) *If $\omega_0 \in DC(\kappa, \tau)$, the Taylor series of Ω at $c = 0$ is given by $\partial_r N_H(c)$.*

Remark. *This corollary gives a third characterization of the BNF.*

Proof. We have that $\omega_0 + \Lambda(0, \omega_0) = 0$ (because $f, g \in \mathcal{O}^2(r, c)$) and by (i) of Proposition 4.2

$$|\partial_\omega \Lambda(c, \omega)| \leq C_1 \eta_0^q (\frac{1}{\kappa\eta_0})^{\alpha(1)} \lesssim \frac{1}{2}$$

for $|c| < \eta_0$ and $\omega \in B$. The local existence of Ω follows now by the implicit function theorem. By a Cauchy estimate and (i) of Proposition 4.2

$$|\partial_c \Lambda(c, \omega)| \lesssim C_0 \eta_0^{q-1} (\frac{1}{\kappa\eta_0})^{\alpha(0)} + \left\|\partial_c^2 N^q\right\|_{0,\eta_0,0} \lesssim \bar{C}$$

where $\bar{C}$ only depends on H, τ, l, which implies that Ω is defined for $|c| < \eta_0$, provided C' is sufficiently large depending only on H, τ, l.

Now since $g = \mathcal{O}^q(c)$, (4.7) yields

$$H(\psi, r + \partial_\psi f(\psi, r, c, \Omega(c))) = \Gamma(c, \omega) + \langle \Omega(c), r - c \rangle + \mathcal{O}^2(r - c) + \mathcal{O}^q(c)$$

and we get (ii) from the uniqueness up to $\mathcal{O}^{q-1}(c)$ of Ω a seen in Proposition 3.3.

If $\omega_0 \in DC(\kappa,\tau)$, then (κ,τ)-flatness of g in (4.7) implies

$$H(\psi, r+\partial_\psi f(\psi,r,c,\Omega(c))) = \Gamma(c,\omega) + \langle \Omega(c), r-c\rangle + \mathcal{O}^2(r-c)$$

which by Taylor expansion at $c=0$ and the uniqueness of Ω in Proposition 3.3 implies (iii).

It remains to prove the estimates (i). If we define

$$F(c,\tilde\omega) = \Lambda(c, \tilde\omega + \partial_r N^q(c)) + \partial_r N^q(c)$$

and $\tilde\Omega(c) = \Omega(c) - \partial_r N^q(c)$, then

$$\tilde\Omega(c) + F(c,\tilde\Omega(c)) = 0.$$

Now

$$|\partial_{\tilde\omega} F(c,\tilde\omega)| \lesssim \frac{1}{2},$$

and

$$\|F\|_{\mathcal{C}^s} \leq \tilde C_s \eta^q (\frac{1}{\kappa\eta})^{\alpha(s)},$$

where the $\mathcal{C}^s$-norm is taken over all $|c|<\eta,\ |\tilde\omega|<\frac{1}{2}$.

Then, by an induction,

$$\left\|\tilde\Omega\right\|_{\mathcal{C}^s} \lesssim C'_s \|F\|_{\mathcal{C}^s}.$$

□

It follows immediately under the same hypothesis as in Corollarys 4.3

Corollary 4.4. *If $\Omega(c)\in DC(\kappa,\tau)$, then*

$$H\circ Z_c(\varphi,r) = \Gamma(c) + \langle\Omega(c), r-c\rangle + \mathcal{O}^2(r-c)$$

where Z_c is the exact symplectic mapping generated by $f(\varphi,r,c,\Omega(c))$. Moreover

$$(\varphi,c)\mapsto Z_c(\varphi,c)$$

is a local diffeomorphism.

5. Nondegenerate BNF and KAM stability

This section is devoted to the proof of Theorem C.

5.1. **Transversality.**

Lemma 5.1. *If $N_H(r)$ is non-degenerate, then there exist $p, \sigma > 0$ such that for any $k \in \mathbb{Z}^d \setminus 0$ there exists a unit vector $u_k \in \mathbb{R}^d$ such that the series*

$$f_k(r) = \langle \frac{k}{|k|}, \partial_r N_H(r) \rangle$$

is (p, σ)-transverse in direction u_k, i.e.

$$\max_{0 \le j \le p} |\partial_t^j f_k(tu_k)_{|t=0}| \ge \sigma.$$

Proof. Indeed, if this were not true, there would exist a sequence $k_n \in \mathbb{Z}^d \setminus \{0\}$ such that for any $u \in \mathbb{R}^d$

$$\max_{0 \le j \le n} |\partial_t^j f_{k_n}(tu)_{|t=0}| < \frac{1}{n}.$$

Extracting a subsequence for which $k_{n_j}/|k_{n_j}| \to v \in \mathbb{R}^d$ clearly gives that $\langle v, \partial_r N_H(r) \rangle = 0$, i.e. N_H would be degenerate. □

Consider now these p, σ. Let $\Omega \in \mathcal{C}^p(\{|c| < \eta\})$ and assume

$$\|\Omega - [\partial_r N_H]^p\|_{\mathcal{C}^p(\{|c|<\eta\})} \le \frac{\sigma}{2}.$$

Lemma 5.2. *If N_H is (p, σ)-transverse (in some direction), then*

$$\mathrm{Leb}\{|c| < \eta : |\langle \frac{k}{|k|}, \Omega(c) \rangle| < \varepsilon\} \le C_p (\frac{\varepsilon}{\sigma})^{\frac{1}{p}} \eta^{d-1}$$

for any η, k, ϵ.

Proof. We have, for some $0 \le j \le p$,

$$|\partial_t^j \langle \frac{k}{|k|}, \Omega(c + tu) \rangle| \ge \frac{\sigma}{2}$$

for all $|c + tu| < \eta$. The estimate is now an easy calculation. □

5.2. **Proof of theorem C.** By Lemma 5.1 we are given p and σ that correspond to the transversality of the formal series N_H. We can assume without restriction that $\sigma \le 1$. Fix $q = (1 + 2p)\alpha(p) + 1$. Performing a conjugacy, we can assume without restriction that

$$H(\varphi, r) = N^q(r) + \mathcal{O}^{q+1}(r)$$

We shall apply Proposition 4.2 and Corollaries 4.3–4.4 with

$$\tau = dp + 1 \quad \text{and} \quad \kappa \le \sigma^q \le 1.$$

Now let

$$\eta =: \frac{1}{C''} (\frac{\kappa}{\sigma})^{\frac{1}{2p}}.$$

Since $q \geq (1+2p)\alpha(1)+1$ we have $\eta \leq \eta_0$ for all $C'' \geq C'$, with η_0 and C' defined in Corollary 4.3. Then Ω is defined in $\{|c|<\eta\}$ and
(5.8)
$$\|\Omega - [\partial_r N_H]^p\|_{\mathcal{C}^p(\{|c|<\eta\})} \leq C'_p \eta^q (\frac{1}{\kappa\eta})^{\alpha(p)} + \|[\partial_r N_H]^p - \partial_r N_H^q\|_{\mathcal{C}^p(\{|c|<\eta\})}$$
which is
$$\leq \tilde{C}\eta$$
since $q \geq (1+2p)\alpha(p)+1$ – notice that $\tilde{C}$ is independent of $C'' \geq C'$. Finally if C'' is sufficiently large (depending on p, τ, l, H, thus on q) we have that $\tilde{C}\eta \leq \sigma/2$.

By Lemma 5.2
$$\mathrm{Leb}\{|c|<\eta : |\langle \frac{k}{|k|}, \Omega(c)\rangle| < \varepsilon\} \lesssim (\frac{\varepsilon}{\sigma})^{\frac{1}{p}} \eta^{d-1},$$
hence
$$\begin{aligned}\mathrm{Leb}\{|c|<\eta : \Omega(c) \notin DC(\kappa,\tau)\} &\lesssim (\frac{\kappa}{\sigma})^{\frac{1}{p}} \eta^{d-1} \\ &\lesssim \eta \mathrm{Leb}\{|c|<\eta\}\end{aligned}$$
provided κ is sufficiently small. Hence, the set
$$\{|c|<\eta : \Omega(c) \in DC(\kappa,\tau)\}$$
is of positive measure and density 1 as $\kappa \to 0$. Theorem C now follows from Corollary 4.4.

6. Analytic KAM tori are never isolated. Degenerate BNF and Invariant co-isotropic submanifolds.

This section is devoted to the proof of theorem B and of Rüssmann's theorem.

Let $q = \alpha(1)+1$ and assume, after a conjugacy, that
$$H(\varphi, r) = N^q(r) + \mathcal{O}^{q+1}(r)$$
We shall apply Proposition 4.2 and Corollaries 4.3+4.4 with
$$q = \alpha(1)+1, \quad \tau = \tau_0 \quad \text{and} \quad \kappa = \frac{\kappa_0}{2}.$$

Then
$$\Omega(c) + \Lambda(c, \Omega(c)) = 0$$
and
$$\Omega(c) = \partial_r N_H(c) + \mathcal{O}^\infty(c).$$

Since N_H is j-degenerate we have
$$\partial_v^n N_H(0) = 0 \qquad \forall n \geq 0$$

for any $v \in \mathrm{Lin}(\gamma = (\gamma_1, \ldots, \gamma_j))$, where ∂_v is the directional derivative in direction v. From this we derive that

$$\partial_v^n(\omega_0 + \Lambda(\cdot, \omega_0))_{|c=0} = 0 \qquad \forall n \geq 0.$$

Since $s \mapsto \Lambda(\langle s, \gamma \rangle, \omega_0)$ is an analytic function in $s \in \mathbb{R}^j$, $s \sim 0$, it must be identically 0, hence $\Omega(\langle s, \gamma \rangle)$ is identically ω_0, i.e.

$$\Omega(\langle s, \gamma \rangle) \in DC(\kappa, \tau)$$

for all sufficiently small s.

From Corollary 4.4 it follows that for any $c \in \mathrm{Lin}(\gamma)$ sufficiently small H has a KAM-torus with frequency ω_0 and that the set of all these tori,

$$\bigcup_{c \in \mathrm{Lin}(\gamma)} Z_c(\mathbb{T}^d, c),$$

is a $(d + j)$-dimensional subvariety. This completes the proof of Theorem B.

When N_H is $(d-1)$-degenerate, then

$$\partial_r N_H(c) = \mu(\langle c, \omega_0 \rangle)\omega_0$$

where $\mu(t) = 1 + \mathcal{O}(t)$ is a formal power series in one variable.

Since

$$\mu(\langle c, \omega_0 \rangle)\omega_0 + \Lambda(c, \mu(\langle c, \omega_0 \rangle)\omega_0) = \mathcal{O}^\infty(c),$$

taking $c = t\omega_0$, we have (assuming ω_0 is a unit vector)

$$\mu(t)\omega_0 + \Lambda(t\omega_0, \mu(t)\omega_0) = 0 \tag{6.9}$$

modulo a term in $\mathcal{O}^\infty(t)$. Since, by Proposition 4.2 (iii), the lefthand side is analytic in $t\omega_0$ and μ we obtain from anyone of the equations (6.9) that $\mu(t)$ is a convergent power series. Then

$$t \mapsto \mu(t)\omega_0 + \Lambda(t\omega_0, \mu(t)\omega_0)$$

is analytic for $t \sim 0$, hence identically zero. We derive from this that

$$\Omega(c) = \mu(\langle c, \omega_0 \rangle)\omega_0,$$

i.e.

$$\Omega(c) \in DC(\kappa, \tau)$$

for all sufficiently small c. Rüssmann's theorem now follows from Corollary 4.4 as in the proof of Theorem B.

7. Smooth non KAM stable Diophantine tori

7.1. **The smooth case in $d = 2$ degrees of freedom.** We let H be as in the introduction but we only assume that H is of class C^∞. The results of sections 3 and 5 remain valid but we will only have C^∞ instead of analytic functions. For example we will not be able to use the analyticity dependance of Λ on the first varaible, that is crucial in the degenerate situation as shown in section 6.

But let us examine the frequency function $\Omega(c)$ given by Corollary 4.3. It is a smooth function from a neighborhood of 0 in $\mathbb{R}^2$ to $\mathbb{R}^2$, such that $\Omega(0) = \omega_0 \in DC(\kappa_0, \tau_0)$. We restrict to a neighborhood where $\omega_{0,i}/2 \leq \Omega_i(c) \leq 2\omega_{0,i}$. A vector $\Omega(c) = (\omega_1, \omega_2) \in \mathbb{R}^2$ then satisfies a Diophantine condition for flows as in (1.2) as soon as $\alpha(c) = \omega_1/\omega_2$ satisfies a Diophantine condition for diffeomorphisms, of the form $|k\alpha + l| \geq C\kappa_0/|k|^{\tau_0}$, with C some constant that only depends on ω_0.

If we restrict $\alpha(\cdot)$ to any segment I that goes through 0 we get a smooth real function such that $\alpha(0)$ satisfies the latter Diophantine condition for diffeomorphisms. As explained in Proposition 3 of [FK], the one dimensional phenomenon here is that, provided κ_0 and τ_0 are relaxed to $\kappa < \kappa_0/2$ and $\tau = \tau_0 + 1$, then for a positive measure set of points in I, α satisfies a Diophantine condition for diffeomorphisms. Indeed $\alpha(0)$ is a density point in $DC(C\kappa, \tau)$ and the two alternative for α are *(i) : α is locally constant $\alpha \sim \alpha_0 \in DC(C\kappa, \tau)$ on a neighborhood of 0 in I, or (ii) : α is not locally constant and it takes a positive measure set of values in $DC(C\kappa, \tau)$ on a positive measure set of points in I.*

We conclude that for a positive measure set of c in any neighborhood of 0, $\Omega(c) \in DC(\kappa, \tau)$, so that Corollary 4.4 yields the following result, that can be coined *Herman's last geometric theorem* since it is just the flow version of the disc diffeomorphisms theorem treated in [FK].

Theorem. *Let $H \in C^\infty(\mathbb{T}^2 \times \mathbb{R}^2)$ and assume that $\mathbb{T}^2 \times \{0\}$ is a KAM torus. Then $\mathbb{T}^2 \times \{0\}$ is accumulated by a positive measure set of smooth KAM tori with Diophantine translation vectors.*

7.2. **A smooth counter-example in $d \geq 4$ degrees of freedom.** A vector $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ is said to be *Liouville* if $(k, \alpha) = 0 \implies k = (0, 0)$ and if for any $N > 0$ there exists $k \in \mathbb{Z}^2 - \{0, 0\}$ such that $|(k, \alpha)| < \|k\|^{-N}$.

We call a sequence of intervals (open or closed or halfopen) $I_n = (a_n, b_n) \subset]0, \infty[$ an increasing cover of the half line if :

(1) $\lim_{n\to-\infty} a_n = 0$
(2) $\lim_{n\to+\infty} a_n = +\infty$

(3) $a_n < b_{n-1} < a_{n+1} < b_n$

Proposition 7.1. *Let* $(\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ *be fixed. For every* $\epsilon > 0$ *and every* $s \in \mathbb{N}$*, there exists an increasing cover* (I_n) *of* $]0, \infty[$ *and functions* $f_i \in C^\infty(\mathbb{R}, (0,1))$*,* $i = 1, 2, 3$*, such that* $\|f_i\|_s < \epsilon$ *and*

- *For each* $n \in \mathbb{Z}$*, the functions* f_1 *and* f_2 *are constant on* I_{3n} *:*
$$f_{1|I_{3n}} \equiv \bar{f}_{1,n}, \quad f_{2|I_{3n}} \equiv \bar{f}_{2,n}$$
- *For each* $n \in \mathbb{Z}$*, the functions* f_1 *and* f_3 *are constant on* I_{3n+1} *:*
$$f_{1|I_{3n+1}} \equiv \bar{f}_{1,n}, \quad f_{3|I_{3n+1}} \equiv \bar{f}_{3,n}$$
- *For each* $n \in \mathbb{Z}$*, the functions* f_2 *and* f_3 *are constant on* I_{3n+1} *:*
$$f_{2|I_{3n-1}} \equiv \bar{f}_{2,n}, \quad f_{3|I_{3n-1}} \equiv \bar{f}_{3,n-1}$$
- *The vectors* $(\bar{f}_{1,n}+\omega_1, \bar{f}_{2,n}+\omega_2)$*,* $(\bar{f}_{1,n}+\omega_1, \bar{f}_{3,n}+\omega_3)$ *and* $(\bar{f}_{2,n}+\omega_2, \bar{f}_{3,n}+\omega_3)$ *are Liouville.*

Remark. *It follows that* f_1, f_2, f_3 *are* $\mathcal{C}^\infty$*-flat at zero.*

Proof. We want to construct $f_1(\cdot)$ such that f_1 is constant equal to $\bar{f}_{1,n}$ on $[a_{3n}, b_{3n+1}]$ for every $n \in \mathbb{Z}$. The crucial observation in the construction of f_1 is that the segments $[a_{3n}, b_{3n+1}]$ are mutually disjoint.

We will then construct similarly f_2 and f_3 and explain why the Liouville conditions can also be required in addition.

Fix $\zeta \in C^\infty(\mathbb{R}, [0,1])$ be such that $\zeta(x) = 0$ if $x \le -1$ and $\zeta(x) = 1$ if $x \ge 0$. Define a sequence $u_n > 0$ such that $a_{3n} - u_n > b_{3n-2}$ and $b_{3n+1} + u_n < a_{3n+3}$. Observe that
$$g_n(x) := \zeta(u_n^{-1}(x - a_{3n})) - \zeta(u_n^{-1}(x - b_{3n+1} - u_n))$$
satisfies $g_n(x) = 1$ if $x \in [a_{3n}, b_{3n+1}]$ and $g_n(x) = 0$ for $x > a_{3n+3} > b_{3n+1} + u_n$ and for $x < b_{3n-2} < a_{3n} - u_n$. Hence the function
$$f_1 = \sum_{n \in \mathbb{Z}} \bar{f}_{1,n} g_n$$
solves our problem and by just requiring the bound $(B_\eta)(n) : |\bar{f}_{1,n}| \le \eta u_n^n$ for every n and supposing that $\sum |u_n| < \infty$ we get that for any s and any ϵ one can choose η to guarantee that the resulting function $\|f_1\|_s < \epsilon/3$. We define the other functions similarly and then add the Liouville constraints without any problem since the condition $(B_\eta)(n)$ is open. $\square$

Given a cover (I_n) as in Proposition 7.1, we can define a define another cover (I'_n) such that I'_n is strictly contained in $(I)_n$ for every n.

7.3. **Proof of theorem E.** Define

$$H_0(\varphi, r) = \langle \omega_0, r\rangle + f_1(r_4)r_1 + f_2(r_4)r_2 + f_3(r_4)r_3$$

where f_1, f_2, f_3 are as in Proposition 7.1 and $\omega_0 = (\omega_1, \omega_2, \omega_3, \omega_4)$.

Notice that as a consequence of Proposition 7.1 we have that on each I_n two of the coordinates of $(f_1+\omega_1, f_2+\omega_2, f_3+\omega_3)$ are constant and form a Liouville vector. We denote $\hat{I}_n = \mathbb{T}^4 \times \mathbb{R}^3 \times I_n$. Let $\mathcal{H}$ be the set of $H \in C^\infty(\mathbb{T}^4 \times \mathbb{R}^4)$ such that H does not depend on φ_4. For $H \in \mathcal{H}$ the flow Φ_H^t leaves r_4 invariant. We will show how to make arbitrarily small perturbations inside $\mathcal{H}$ of H_0 on any $\hat{I}_n$ that create huge oscillations of the corresponding flow in two of the three directions r_1, r_2, r_3. These perturbations will actually be compositions inside H_0 by exact symplectic maps obtained from suitably chosen generating functions. Iterating the argument gives a construction by successive conjugations scheme similar to [AK]. The difference here is that the conjugations will be applied in a "diagonal" procedure to include more and more intervals I_n into the scheme. Rather than following this diagonal scheme which would allow to define the conjugations explicitly at each step, we will actually adopt a $\mathcal{G}^\delta$-type construction *à la Herman* (see [FH]) that makes the proof much shorter and gives slightly more general results.

Let $\mathcal{U}$ be the set of exact symplectic diffeomorphisms U of $\mathbb{T}^4 \times \mathbb{R}^4$ such that $U(\varphi, r) = (\psi, s)$ satisfies $s_4 = r_4$. In particular if $U \in \mathcal{U}$ implies that $U(\hat{I}_n) = \hat{I}_n$ for any $n \in \mathbb{Z}$.

Proposition 7.2. *Let $I = I_n$ for some n. For any $\epsilon > 0, s \in \mathbb{N}, \Delta > 0, A > 0$ and any $V \in \mathcal{U}$, there exist $U \in \mathcal{U}$ and $T > 0$ such that there exist $(i_1, i_2) \in \{1, 2, 3\}$, distinct, such that for $i = i_1$ and $i = i_2$ we have*

(1) $U = \mathrm{Id}$ on $\hat{I}^c$
(2) $\|H_0 \circ U \circ V - H_0 \circ V\|_s < \epsilon$
(3) $\sup\limits_{0<t<T} |(\Phi^t_{H_0\circ U\circ V}(p))_{4+i_1}| > A$, *for any $p \in \widehat{I'}$ such that $\|p\| \le \Delta$*
(4) $\sup\limits_{0<t<T} |(\Phi^{-t}_{H_0\circ U\circ V}(p))_{4+i_2}| > A$, *for any $p \in \widehat{I'}$ such that $\|p\| \le \Delta$*

Proof. Since V preserves $\hat{I}$ and since $\phi^t_{H_0\circ U\circ V}$ is conjugate to $\Phi^t_{H_0\circ U}$ it is sufficient to prove the proposition for $V = \mathrm{Id}$. Indeed, given V such that $V\hat{I} = \hat{I}$, and applying the Proposition with $V = \mathrm{Id}$ and with constants $\epsilon' \ll \epsilon$ and $A' \gg A$ yields 2 and 3 including V.

Assume hereafter that $I = I_{3n}$, the other cases being exactly similar. Let $a \in C^\infty(\mathbb{R})$ be such that $a(\xi) = 0$ if $\xi \notin I$ and $a(\xi) = 1$ if $\xi \in I'$ (remember that I' is strictly included in I).

Let $\bar{f}_1 := f_{1|I}, \bar{f}_2 := f_{2|I}$ and $\bar{F}_1 = \bar{f}_1 + \omega_1$, $\bar{F}_2 = \bar{f}_2 + \omega_2$. Let $(q_1, q_2) \in \mathbb{Z}^2 - \{0, 0\}$ such that $|q_1| > A + \Delta$ and $|q_2| > A + \Delta$ and $|q_1\bar{F}_1 + q_2\bar{F}_2| < \eta \min(q_1^{-2s}, q_2^{-2s})$ where $\eta = \epsilon/((2\pi)^{s+1}\|a\|_s)$.

Define the following generating function $k \in \mathcal{H}$

$$k(\psi, r) = a(r_4)\sin(2\pi(q_1\psi_1 + q_2\psi_2))$$

and let $U = (\Phi, R) \in \mathcal{U}$ be the symplectic diffeomorphism associated to k. Then $R(\varphi, r)$ equals

$$(r_1+2\pi q_1 a(r_4)\cos(2\pi(q_1\varphi_1+q_2\varphi_2)), r_2+2\pi q_2 a(r_4)\cos(2\pi(q_1\varphi_1+q_2\varphi_2)), r_3, r_4)$$

so that $U = \mathrm{Id}$ on $\hat{I}^c$ and $H_0 \circ U(\varphi, r)$ equals

$$H_0(r)+2\pi a(r_4)\left(q_1(f_1(r_4) + \omega_1) + q_2(f_2(r_4) + \omega_2)\right) s\cos(2\pi(q_1\varphi_1+q_2\varphi_2)).$$

Hence $H_0 \circ U(\varphi, r) - H_0(r) = h(r, \varphi)$ with $h \equiv 0$ if $r_4 \notin I$ and if $r_4 \in I$ we have that $h(r, \varphi) = 2\pi a(r_4)(q_1\bar{F}_1 + q_2\bar{F}_2)\cos(2\pi(q_1\varphi_1 + q_2\varphi_2))$ thus the required $\|h\|_s < \epsilon$.

On the other hand we have that on $\hat{I}$ the flow $\Phi^t_{H_0}$ is completely integrable with tori $\mathcal{T}_r = \{r\}\times\mathbb{T}^4$ carrying the frequencies $(\bar{F}_1, \bar{F}_2, F_3(r_4), \omega_4)$. Recall that $\bar{F}_1$ and $\bar{F}_2$ are independent over $\mathbb{Z}$, that is, the dynamics of the translation flow $T^t_{\bar{F}_1,\bar{F}_2}$ is minimal. But under the change of variable U the torus $\mathcal{T}_r$ for $r_4 \in I'$ becomes $\mathcal{T}'_r = \{(r_1 - 2\pi q_1 \cos(2\pi(q_1\varphi_1 + q_2\varphi_2)), r_2 - 2\pi q_2 \cos(2\pi(q_1\varphi_1 + q_2\varphi_2)), r_3, r_4) : (\varphi_1, \ldots, \varphi_4) \in \mathbb{T}^4\}$. Also, the change of variable is such that $(\Phi(\varphi, r))_j = \varphi_j$ for $j = 1, 2, 3$. All this implies the third claim of Proposition 7.2 since we took $|q_1| > A+\Delta$ and $|q_2| > A + \Delta$. $\square$

It is easy now to deduce Theorem E and in fact a stronger version of it. Define for this purpose $\mathcal{U}_0$ the subset of $U \in \mathcal{U}$ such that $U - \mathrm{Id} = \mathcal{O}^\infty(r_4)$ and $\mathcal{H}_0$ to be the set of hamiltonians of the form $H_0\circ U, U \in \mathcal{U}_0$. Finally we denote $\bar{\mathcal{H}_0}$ the closure in the C^∞ topology of $\mathcal{H}_0$.

Proposition 7.3. *Let $\mathcal{D}$ be the set of hamiltonians $H \in \bar{\mathcal{H}_0}$ such that*

$$\limsup \|\Phi^t_H(p)\| = \infty \tag{7.10}$$

for any $p = (\varphi, r)$ satisfying $r_4 \neq 0$. More precisely, for each p such that $p_8 \neq 0$ we have that there exist $(i_1, i_2) \in \{1, 2, 3\}$,distinct, such that for $i = i_1$ and $i = i_2$ it holds that

$$\limsup_{t\to\pm\infty}(\phi^t_H(p))_{4+i_1} = +\infty, \quad \liminf_{t\to\pm\infty}(\phi^t_H(p))_{4+i_2} = -\infty \tag{7.11}$$

Then $\mathcal{D}$ is a dense (in the C^∞ topology) $\mathcal{G}^\delta$ subset of $\bar{\mathcal{H}_0}$

Proof. For $n, \Delta, A, T \in \mathbb{N}^*$ and $1 \leq i_1 < i_2) \in \leq 3$ let $\mathcal{D}(n, \Delta, A, T, i_1, i_2)$ be the set

$$\left\{ H \in \bar{\mathcal{H}}_0 : \sup_{0<t<T} \min_{i=i_1,i_2; j=1,-1} \min_{p \in \widehat{I'_n} \cap \{\|p\| \leq \Delta\}} |(\phi_H^{jt}(p))_{4+i}| > A \right\}.$$

It is clear that $\mathcal{D}(n, \Delta, A, T, i_1, i_2)$ are open subsets of $\bar{\mathcal{H}}_0$ in any C^s topology. On the other hand we have that

$$\mathcal{D} = \bigcap_{A \in \mathbb{N}^*} \bigcap_{n \in \mathbb{N}^*} \bigcap_{\Delta \in \mathbb{N}^*} \bigcup_{T \in \mathbb{N}^*} \bigcup_{(i_1,i_2) \in \{(1,2),(1,3),(2,3)\}} \mathcal{D}(n, \Delta, A, T, i_1, i_2)$$

but Proposition 7.2 precisely states that

$$\bigcup_{T \in \mathbb{N}^*} \bigcup_{(i_1,i_2) \in \{(1,2),(1,3),(2,3)\}} \mathcal{D}(n, \Delta, A, T, i_1, i_2)$$

is dense in $\bar{\mathcal{H}}_0$ in any C^s topology, which ends the proof of the theorem. □

The same result of Proposition 7.3 holds in any Gevrey class G^σ, for any $\sigma > 1$. The proof of the latter fact follows exactly the same line as the C^∞ case with the following simple modifications.

- The compactly supported function ζ of Proposition 7.1 is taken to be in G^σ, as well as the function a in the proof of Proposition 7.2, and C^s norms are replaced with *Gevrey* norms.
- The conditions $(B_\eta)(n) : |\bar{f}_{j,n}| \leq \eta u_n^n$ are replaced by $|\bar{f}_{j,n}| \leq \eta u_n^{u_n^{-n}}$.
- The Liouville condition on the vectors $(\bar{F}_1, \bar{F}_2) = (\bar{f}_{1,n} + \omega_1, \bar{f}_{2,n} + \omega_2)$ (as well as on $(\bar{f}_{1,n} + \omega_1, \bar{f}_{3,n} + \omega_3)$ and $(\bar{f}_{2,n} + \omega_2, \bar{f}_{3,n} + \omega_3)$) is replaced by a "super-Liouville" condition of the type $|q_1 \bar{F}_1 + q_2 \bar{F}_2| \leq e^{-q_1 - q_2}$ for infinitely many (q_1, q_2).

8. Proof of the KAM counter term theorem

The proof of the counter term theorem (Proposition 4.2) is based on an inductive procedure and will occupy this whole section.

Let $\mathcal{P} = \mathcal{P}_{\kappa,\tau}$ be the cut-off operator defined in section 4.1. We take $\tau > d-1$ and $0 < \kappa < 1$. The operator $\mathcal{P}$ depends on a cut-off function l and constants in this section will, in general without saying, depend on l. Recall that a function g is (κ, τ)-*flat* if

$$\partial_\varphi^\alpha \partial_z^\beta \partial_\omega^\gamma g(\varphi, z, \omega) = 0$$

for all multi-indices α, β, γ whenever $\omega \in DC(\kappa, \tau)$.

Let B be a ball centered at ω_0 or, more generally, the intersection of this unit ball with an affine subspace of $\mathbb{R}^d$ through ω_0.

8.1. **A linear operator.** Define now

$$\mathcal{L}(f) = u$$

through

$$\left\{ \begin{array}{l} \langle \omega, \partial_\varphi u\rangle = f - \mathcal{P}(f) - \mathcal{M}(f) \\ \mathcal{M}(u) = \mathcal{P}(u) = 0. \end{array} \right. \tag{8.12}$$

Lemma 8.1.

$$\|\mathcal{L}(f)\|_{\rho',\delta,s} \leq C_s(\frac{1}{\kappa})^{s+1}(\frac{1}{\rho-\rho'})^{(\tau+1)(s+1)}\|f\|_{\rho,\delta,s}$$

for any $\rho' < \rho$. The constant C_s only depends, besides s, on τ and l.

Proof. We give a proof with the exponent $(\tau+1)s+\tau+d$ – the improved exponent $(\tau+1)s+\tau+1$ requires some more subtle considerations originally due to Rüssmann – see for example [E]. Equation (8.12) is equivalent to $\hat{u}(0,z,\omega) = 0$ and, for $n \in \mathbb{Z}^d - \{0\}$,

$$\hat{u}(n,z,\omega) = \hat{f}(n,z,\omega)l_n(\omega)$$

where

$$l_n(\omega) = \frac{1}{i2\pi\langle n,\omega\rangle}(1 - l(\langle n,\omega\rangle\frac{|n|^\tau}{\kappa})).$$

Since

$$\left\|\hat{f}(n,\cdot,\cdot)\right\|_{0,\delta,s} \leq \|f\|_{\rho,\delta,s} e^{-2\pi|n|\rho}.$$

and

$$\|l_n\|_{0,0,s} \leq C_s|n|^{(\tau+1)s+\tau}\frac{1}{\kappa^{s+1}}\|l\|_{0,0,0} + |n|^\tau\frac{1}{\kappa}\|l\|_{0,0,s},$$

we get (by Proposition 10.1), for $|\alpha| \leq s$ and $(\varphi,z,\omega) \in \mathbb{T}^d_{\rho'} \times \mathbb{D}^d_\delta \times B$,

$$|\partial^\alpha_\omega u(\varphi,z,\omega)| \leq C_s\sum_{n\neq 0} e^{2\pi|n|\rho'} \times$$

$$\times\left(\|\hat{f}(n,\cdot,\cdot)\|_{0,\delta,s}|n|^\tau\frac{1}{\kappa} + \|\hat{f}(n,\cdot,\cdot)\|_{0,\delta,0}|n|^{(\tau+1)s+\tau}\frac{1}{\kappa^{s+1}}\right)$$

which gives the estimates by standard arguments. □

8.2. **The counter term theorem.** Let $0 < \rho, \delta < 1$. Denote by $\mathcal{C}^{\omega,\infty}_{\rho,\delta}$ the set of functions $f \in C^{\omega,\infty}(\mathbb{T}^d_\rho \times \mathbb{D}^d_\delta \times \mathbb{D}^d_\delta, B)$ such that

$$f(\varphi,r,c,\omega) \in \mathcal{O}^2(r,c).$$

Any function $f \in \mathcal{C}^{\omega,\infty}_{\rho,\delta}$ can be written uniquely as

$$a(\varphi,c,\omega) + \langle B(\varphi,c,\omega), r-c\rangle + \frac{1}{2}\langle r-c, F(\varphi,r,c,\omega)(r-c)\rangle^3$$

[3] we applogize for the double use of B

modulo $\mathcal{O}^3(r-c)$ with $a=\mathcal{O}^2(c)$ and $B=\mathcal{O}(c)$.

We say that f is *of order* q if $a\in\mathcal{O}^q(c)$ and $B\in\mathcal{O}^q(c)$.

We define the pseudo-norm

$$[f]_{\rho,\delta,s}=\max(\|a\|_{\rho,\delta,s}\,,\|B\|_{\rho,\delta,s}\,,\|\partial_\varphi\mathcal{L}a\|_{\rho,\delta,s}\,,\|\partial_\varphi\mathcal{L}B\|_{\rho,\delta,s})$$

and the vector

$$M_f=\mathcal{M}\left(B-F\partial_\varphi\mathcal{L}a\right),$$

where, we recall, that $\mathcal{M}(g)$ is the *mean value* $\int_{\mathbb{T}^d}g(\varphi,z)d\varphi$. We denote by $\mathcal{E}^{\omega,\infty}_{\rho,\delta}$ the set of exact symplectic local diffeomorphisms defined on a neighborhood of $\mathbb{T}^d\times\{0\}$ of the form

$$Z_{c,\omega}(\varphi,r)=\begin{pmatrix}\varphi+\Phi(\varphi,c,\omega)\\ r+R_1(\varphi,c,\omega)+R_2(\varphi,c,\omega)(r-c)\end{pmatrix}$$

with $\Phi,R_1,R_2\in\mathcal{C}^{\omega,\infty}(\mathbb{T}^d_\rho\times\mathbb{D}^d_\delta\times\mathbb{D}^d_\delta,B)$ and $R_1=\mathcal{O}^2(c)$, $\Phi,R_2=\mathcal{O}(c)$. If Z' is another mapping in $\mathcal{E}^{\omega,\infty}_{\rho,\delta}$ then we define

$$[Z-Z']_{\rho,\delta,s}=$$

$$\max_i(\|\Phi-\Phi'\|_{\rho,\delta,s}\,,\|R_i-R_i'\|_{\rho,\delta,s}\,,\|\partial_\varphi\mathcal{L}(\Phi-\Phi')\|_{\rho,\delta,s}\,,\|\partial_\varphi\mathcal{L}(R_i-R_i')\|_{\rho,\delta,s})$$

and

$$(Z\circ Z')_{c,\omega}(\varphi,r)=Z_{c,\omega}(Z'_{c,\omega}(\varphi,r)).$$

The goal of this section is to prove the following

Proposition 8.2. *For all $s\in\mathbb{N}$, there exist constants $\epsilon>0$ and $\alpha(s)\geq 0$, only depending on τ, such that if $H\in\mathcal{C}^{\omega,\infty}_{\rho,\delta}$ is independent of ω and satisfies, for some $h<\min(\rho/2,\delta/2)$ and some $\sigma<\epsilon(\tau)$,*

$$[H]_{\rho,\delta,0}\leq\sigma\frac{1}{(1+\|\partial_r^2H\|_{\rho,\delta,0})^7}\kappa^{11}h^{10(\tau+d)+11},\tag{8.13}$$

then there exist $\Lambda\in\mathcal{C}^{\omega,\infty}_{0,\delta-h}$ and $W\in\mathcal{E}^{\omega,\infty}_{\rho-h,\delta-h}$, $H'\in\mathcal{C}^{\omega,\infty}_{\rho-h,\delta-h}$, with $[H']_{\rho-h,\delta-h,0}=0$, and a (κ,τ)-flat function $g\in\mathcal{C}^{\omega,\infty}_{\rho-h,\delta-h}$ such that

$$(H+\langle\omega+\Lambda(c,\omega),\cdot\rangle)\circ W_{c,\omega}(\varphi,r)=\langle\omega,r-c\rangle+H'(r,\varphi,c,\omega)+g(\varphi,r,c,\omega)\tag{8.14}$$

(modulo an additive constant that depends on c,ω) with, for all s,

$$\max\left(\|\Lambda\|_{0,\delta-h,s}\,,[W-\mathrm{id}]_{\rho-h,\delta-h,s},\|g\|_{\rho-h,\delta-h,s}\,,\left\|\partial_r^2(H'-H)\right\|_{\rho-h,\delta-h,s},\right)$$
$$<\sigma\left(\frac{\|\partial_r^2H\|_{\rho,\delta,0}+1}{\kappa h}\right)^{\alpha(s)}(\left\|\partial_r^2H\right\|_{\rho,\delta,0}+[H]_{\rho,\delta,0}+1).\tag{8.15}$$

Moreover, if H is of order q, then $g \in \mathcal{O}^q(c)$.
Furthermore, if

$$\omega_0 \in DC(2\kappa, \tau)$$

and H is analytic on the segment $I_\delta = B_\delta(\omega_0) \cap \mathbb{R}\omega_0$, then Λ, W and H' are analytic on $I_{\delta'}$ for some $0 < \delta' \leq \delta$ and $g = 0$ on $I_{\delta'}$.

We shall first prove the main part of this proposition, then we will explain what modifications are required in order to obtain the final analyticity statement.

The proof of Proposition 8.2 is based on an inductive KAM scheme. In each step of the scheme we conjugate a Hamiltonian of the form

$$\langle \omega, r - c\rangle + a(\varphi, c, \omega) + \langle B(\varphi, c, \omega), r - c\rangle + \frac{1}{2}\langle r - c, F(\varphi, r, c, \omega)(r - c)\rangle.$$

and reduce quadratically the terms a and B. To do so, we look for a conjugacy using a generating function of the form $\langle r, \psi\rangle + u_0(\psi) + \langle u_1(\psi), r\rangle$ and we solve a triangular cohomological system in u_0 and u_1 to reduce a and B. This is only possible up to a (κ, τ)-flat function g and also requires that the constant terms in the cohomological equations vanish and this is why we have to add the counter term $\langle \Lambda, \cdot\rangle$ and a constant.

The inductive step of the scheme is enclosed in Proposition 8.5. To add clearness to the presentation we split the proof of the latter proposition into two parts : in the first part we suppose the constant terms in the cohomological equations do vanish and build the conjugacy (this is the content of Lemma 8.3) and in the second one we show that adding counter terms allows to zero the constant terms in the cohomological equations (this is the content of Lemma 8.4). Proposition 8.5 is a direct consequence of Lemmas 8.3 and 8.4.

We will finally conclude in Sections 8.5 and 8.6 showing that the iteration scheme based on the inductive step of Proposition 8.5 does converge if the initial bound (8.13) is satisfied.

8.3. **Reduction lemmas.** In this section we first fix $\rho, \delta < 1$ and a number h less than $\min(\rho/2, \delta/2) < \frac{1}{2}$ and we set

$$\xi_s = \kappa^{(s+1)} h^{(\tau+1)(s+1)+d}.$$

We fix $H \in \mathcal{C}^{\omega,\infty}_{\rho,\delta}$, which may depend on ω, and let

$$\epsilon_s = [H]_{\rho,\delta,s} \quad \text{and} \quad \zeta_s = \left\|\partial_r^2 H\right\|_{\rho,\delta,s} + 1.$$

Lemma 8.3. *There exist positive constants $\varsigma=\varsigma(\tau)$ and $C_s=C_s(\tau)$, such that if $M_H=0$ and*

$$\epsilon_1<\varsigma\frac{1}{\zeta_1}\kappa^4 h^{4\tau+2d+6}, \tag{8.16}$$

then there exist $Z\in\mathcal{E}^{\omega,\infty}_{\rho-h,\delta-h}$, $H'\in\mathcal{C}^{\omega,\infty}_{\rho-h,\delta-h}$ and a (κ,τ)-flat function g' such that

$$(H+\langle\omega,\cdot\rangle)\circ Z_{c,\omega}(\varphi,r)=\langle\omega,r-c\rangle+H'(\varphi,c,\omega)+g'(\varphi,r,c,\omega),$$

(modulo an additive constant that depends on c,ω) with, for all $s\in\mathbb{N}$,

$$[Z-\mathrm{id}]_{\rho-h,\delta-h,0}<h,$$
$$[H']_{\rho-h,\delta-h,s}\le\nu_s\epsilon_0$$

and

$$\max([Z-\mathrm{id}]_{\rho-h,\delta-h,s},\left\|\partial_r^2(H'-H)\right\|_{\rho-h,\delta-h,s},\|g'\|_{\rho-h,\delta-h,s})\le\nu_s,$$

for

$$\nu_s=C_s(\frac{1}{h\xi_s})^3\zeta_0(\zeta_0\epsilon_s+\zeta_s\epsilon_0).$$

Moreover, if H is of order q, then H' is of order q and $g'\in\mathcal{O}^q(c)$.

Proof. We introduce $G(\varphi,r,c,\omega)\in\mathcal{O}^3(r-c)$ for

$$H(\varphi,r,c,\omega)-a(\varphi,c,\omega)-\langle B(\varphi,c,\omega),r-c\rangle-\frac{1}{2}\langle r-c,F(\varphi,c,\omega)(r-c)\rangle.$$

Notice that

$$\left\|\partial_r^2 G\right\|_{\rho,\delta,s}\lesssim\left\|\partial_r^2 H\right\|_{\rho,\delta,s}$$

and

$$\left\|\partial_r^3 G\right\|_{\rho,\delta-h,s}\lesssim\frac{1}{h}\left\|\partial_r^2 H\right\|_{\rho,\delta,s}.$$

We look for the diffeomorphism $Z(\varphi,r,c,\omega)=(\psi,s)$ *via* a generating function of the form $\langle\psi,r\rangle+U(\psi,c,\omega)$,

$$U(\psi,r,c,\omega)=u_0(\psi,c,\omega)+\langle u_1(\psi,c,\omega),r-c\rangle\in\mathcal{C}^{\omega,\infty}_{\rho-h,\delta-h,s},$$

i.e.

$$\begin{cases} s=r+\partial_\psi u_0+\langle\partial_\psi u_1,r-c\rangle\\ \varphi=\psi+u_1(\psi).\end{cases}$$

All our functions depend, besides ψ, on c,ω and we shall in the sequel suppress this dependence in the notations.

We have, modulo an additive constant,

$$\begin{aligned}(H+\langle\omega,\cdot\rangle)\circ Z(\varphi,r)&-\langle\omega,r-c\rangle\\ &=(I)+(II)+(III)+G(\psi,r+\partial_\psi U(\psi,r)),\end{aligned} \tag{8.17}$$

where

$$\begin{aligned}(I) &= \langle \omega, \partial_\psi u_0\rangle + a + \langle B, \partial_\psi u_0\rangle + \frac{1}{2}\langle F\partial_\psi u_0, \partial_\psi u_0\rangle - \mathcal{M}(a)\\ (II) &= \langle\langle \omega, \partial_\psi u_1\rangle, r-c\rangle + \langle B, r-c+\partial_\psi u_1(r-c)\rangle\\ &\quad + \langle F\partial_\psi u_0, r-c+\partial_\psi u_1(r-c)\rangle\\ (III) &= \frac{1}{2}\langle r-c, F(r-c)\rangle + \langle F(r-c), \partial_\psi u_1(r-c)\rangle\\ &\quad + \frac{1}{2}\langle F\partial_\psi u_1(r-c), \partial_\psi u_1(r-c)\rangle.\end{aligned}$$

The homological equation. To kill as much as possible of a and B in $(H + \langle \omega, \cdot\rangle) \circ Z$ we take

$$\begin{cases} u_0 = -\mathcal{L}(a)\\ u_1 = -\mathcal{L}(B + F\partial_\psi u_0)\end{cases}$$

Observe that if $a, B \in \mathcal{O}^q(c)$, then $u_0, u_1 \in \mathcal{O}^q(c)$. By Lemma 8.1 we have

$$[U]_{\rho-h,\delta,s} + [\partial_r U]_{\rho-h,\delta,s} \le C_s e_s = C_s \frac{1}{h\xi_s^2}(\zeta_s\epsilon_0 + \zeta_0\epsilon_s).^4$$

(Here and elsewhere we use Proposition 10.1 to estimate products.)

Estimation of Z. Since, by (8.16),

$$e_1 \lesssim h^2$$

Proposition 10.3 implies that the mapping $\varphi = \tilde{f}(\psi) = \psi + u_1(\psi)$ is invertible with inverse satisfying

$$\left\|\tilde{f}^{-1} - \mathrm{id}\right\|_{\rho-2h,\delta-h,s} \le C_s e_s,$$

and Proposition 10.2 gives

$$\|Z - \mathrm{id}\|_{\rho-3h,\delta-2h,s} \le C_s \frac{e_s}{h}.$$

It follows that $Z \in \mathcal{E}^{\omega,\infty}_{\rho-4h,\delta-3h}$ and Lemma 8.1 implies that

$$[Z - \mathrm{id}]_{\rho-4h,\delta-3h,s} \le C_s \frac{1}{h\xi_s}\frac{e_s}{h^2}.$$

The function g'. Let

$$h = \mathcal{P}(a) + \langle \mathcal{P}(B - F\partial_\psi u_0), r-c\rangle$$

and $g'(\varphi, r) = h(\psi, r)$. Then, by Lemma 4.1,

$$\|h\|_{\rho-2h,\delta,s} \le C_s e_s$$

[4] the constant C_s will differ from line to line

and, by Proposition 10.2,

$$\|g'\|_{\rho-3h,\delta-2h,s} \leq C_s e_s.$$

We also note that since $a, B, u_0 \in \mathcal{O}^q(c)$, then $g' \in \mathcal{O}^q(c)$. *Checking that H' is of order q.* If H is of order q, we saw that $u_0, u_1 = \mathcal{O}^q(c)$. Hence the terms (I) and (II) in the RHS of (8.17) are $\mathcal{O}^q(c)$. Now $H'(\varphi, r)$ equals

$$I(\varphi+\Phi(\varphi,c),c)+II(\varphi+\Phi(\varphi,c),r-c,c)+III(\varphi+\Phi(\varphi,c),r-c,c)$$

and since $III \in \mathcal{O}^2(r-c)$ we conclude that H' is of order q.

Estimation of H'. We set

$$G_1(\varphi,r) = \langle \partial_r G(\varphi,r), \partial_\psi U(\varphi,r)\rangle$$

and

$$G_2(\varphi,r) = G(\psi, r+\partial_\psi U(\psi,r)) - G(\varphi,r) - \langle \partial_r G(\varphi,r), \partial_\psi U(\varphi,r)\rangle.$$

Then $G_1 \in \mathcal{O}^2(r-c)$ and the RHS of (8.17) satsifies

$$RHS - h = (I) + (II) - h + G_2 + \mathcal{O}^2(r-c)$$

as well as

$$\partial_r^2(RHS - H) = \partial_r^2((III) + G_1 + G_2) - F$$

because $G = \mathcal{O}^3(r-c)$. Now we have that

$$[(I)+(II)-h]_{\rho-2h,\delta,s} \leq C_s \frac{1}{h\xi_s}[(\epsilon_s e_0 + \epsilon_0 e_s)\frac{1}{h} + (\zeta_s e_0 + \zeta_0 e_s)\frac{e_0}{h}] \tag{8.18}$$

(here we use that $M_H = 0$) and by Proposition 10.2(ii),

$$\|G_2\|_{\rho-2h,\delta-h,s} \leq C_s(\zeta_s e_0 + \zeta_0 e_s)\frac{e_0}{h^4} \tag{8.19}$$

since $e_0 \lesssim h^2$. The inequality (8.19) implies in particular that

$$[G_2]_{\rho-3h,\delta-2h,s} \leq C_s \frac{1}{h\xi_s}(\zeta_s e_0 + \zeta_0 e_s)\frac{e_0}{h^5} \tag{8.20}$$

and

$$\left\|\partial_r^2 G_2\right\|_{\rho-2h,\delta-2h,s} \leq C_s(\zeta_s e_0 + \zeta_0 e_s)\frac{e_0}{h^6}. \tag{8.21}$$

It follows from (8.18) and (8.20) that the right hand side of (8.17) verifies

$$[RHS - h]_{\rho-3h,\delta-2h,s} \leq C_s \frac{1}{h\xi_s}[(\epsilon_s e_0 + \epsilon_0 e_s)\frac{1}{h} + C_s(\zeta_s e_0 + \zeta_0 e_s)\frac{e_0}{h^5}].$$

On the other hand, since

$$\left\|\partial_r^2 G_1\right\|_{\rho-2h,\delta-h,s} \leq C_s(\zeta_s e_0 + \zeta_0 e_s)\frac{1}{h^2} \tag{8.22}$$

and

$$\left\|\partial_r^2(III)-F\right\|_{\rho-2h,\delta,s}\le C_s(\zeta_s e_0+\zeta_0 e_s)\frac{1}{h}. \tag{8.23}$$

it follows that

$$\left\|\partial_r^2(RHS-H)\right\|_{\rho-2h,\delta-2h,s}\le C_s(\zeta_s e_0+\zeta_0 e_s)\frac{e_0}{h^6}$$

– by (8.21+8.22+8.23).

Since $H'(\varphi,r)=RHS(\psi,r)-g'(\varphi,r)$ we get by Proposition 10.2 that

$$[H']_{\rho-4h,\delta-3h,s}\le C_s\frac{1}{h\xi_s}[(\epsilon_s e_0+\epsilon_0 e_s)\frac{1}{h}+C_s(\zeta_s e_0+\zeta_0 e_s)\frac{e_0}{h^5}]$$

and

$$\left\|\partial_r^2(H'-H)\right\|_{\rho-4h,\delta-3h,s}\le C_s(\zeta_s e_0+\zeta_0 e_s)\frac{e_0}{h^6}.$$

This completes the proof of the lemma. □

Let $W\in\mathcal{E}^{\omega,\infty}_{\rho,\delta}$ and denote

$$\eta_s=[W-\mathrm{id}]_{\rho,\delta,s}.$$

Lemma 8.4. *There exist positive constants $\varsigma=\varsigma(\tau)$ and $C_s=C_s(\tau)$, such that if*

$$\eta_0<\varsigma\frac{1}{\zeta_0}, \tag{8.24}$$

then there exists $\Lambda\in\mathcal{C}^{\omega,\infty}_{\delta}$ such that

$$\tilde H=H+\langle\Lambda,\cdot\rangle\circ W$$

verifies $M_{\tilde H}=0$ and $\partial_r^2\tilde H=\partial_r^2 H$.

Also, if H is of order q, then $\tilde H$ is of order q.

Moreover, for all $s\in\mathbb{N}$,

$$\|\Lambda\|_{\delta,s}\le C_s\zeta_0(\zeta_0\epsilon_0\eta_s+\zeta_0\epsilon_s+\zeta_s\epsilon_0)$$

and

$$[\tilde H-H]_{\rho,\delta,s}\le C_s\zeta_0(\zeta_0\epsilon_0\eta_s+(\zeta_0\epsilon_s+\zeta_s\epsilon_0)(\eta_0+1)).$$

Proof. Write $H(\varphi,r,c,\omega)$ as

$$a(\varphi,c,\omega)+\langle B(\varphi,c,\omega),r-c\rangle+\frac{1}{2}\langle r-c,F(\varphi,c,\omega)(r-c)\rangle$$

and $\tilde H(\varphi,r,c,\omega)$ as

$$\tilde a(\varphi,c,\omega)+\langle \tilde B(\varphi,c,\omega),r-c\rangle+\frac{1}{2}\langle r-c,\tilde F(\varphi,c,\omega)(r-c)\rangle$$

modulo $\mathcal{O}^3(r-c)$.

Observe that

$$W_{c,\omega}(\varphi, r) = \begin{pmatrix} \varphi + \Phi(\varphi, c, \omega) \\ r + R_1(\varphi, c, \omega) + R_2(\varphi, c, \omega)(r - c) \end{pmatrix},$$

so

$$\tilde{a} = a + \langle \Lambda, R_1 + c \rangle \tag{8.25}$$

$$\tilde{B} = B + (I + {}^tR_2)\Lambda \tag{8.26}$$

and $\tilde{F} = F$. We want to choose Λ so that $M_{\tilde{H}} = 0$, i.e.

$$\mathcal{M}\left(\left[I + {}^tR_2 - F\partial_\varphi \mathcal{L} R_1\right]\Lambda - F\partial_\varphi \mathcal{L} a + B\right) = 0.$$

If $X = -\mathcal{M}({}^tR_2 - F\partial_\varphi \mathcal{L} R_1)$ and $Y = \mathcal{M}(-B + F\partial_\varphi \mathcal{L} a)$, then this amounts to

$$\Lambda = \sum_n X^n Y. \tag{8.27}$$

Observe that if $a, B \in \mathcal{O}^q(c)$, then $Y \in \mathcal{O}^q(c)$, thus $\Lambda \in \mathcal{O}^q(c)$ and $\tilde{H}$ is of order q.

We have

$$\|X\|_{\delta,s} \le C_s(\zeta_s \eta_0 + \zeta_0 \eta_s)$$

and

$$\|Y\|_{\delta,s} \le C_s(\zeta_s \epsilon_0 + \zeta_0 \epsilon_s).$$

By assumption (8.24), $\|X\|_{\delta,0} \le 1/2$, which gives the existence and the estimates on Λ and $\tilde{H}$ by Proposition 10.1 and (8.25)–(8.27). □

8.4. **The inductive step.** Combining Lemma 8.3 and Lemma 8.4 we immediately get the following proposition that constitutes the inductive step of our KAM scheme for the proof of Proposition 8.2. For the needs of the inductive application, we will consider that at each step we have a Hamiltonian $H \in \mathcal{C}^{\omega,\infty}_{\rho,\delta}$ as well as $g \in \mathcal{C}^{\omega,\infty}_{\rho,\delta}$ (κ, τ)-flat, and $W \in \mathcal{E}^{\omega,\infty}_{\rho,\delta}$.

As in the previous section we assume $h < \min(\rho/2, \delta/2)$, and we set

$$\xi_s = \kappa^{(s+1)} h^{(\tau+1)(s+1)+d}$$

and

$$\epsilon_s = [H]_{\rho,\delta,s} \quad \text{and} \quad \zeta_s = \left\|\partial_r^2 H\right\|_{\rho,\delta,s} + \|g\|_{\rho,\delta,s} + 1$$

and

$$\eta_s = [W - id]_{\rho,\delta,s}.$$

Proposition 8.5. *There exist $\varsigma = \varsigma(\tau)$ and $C_s = C_s(\tau)$ such that, if*

$$\eta_0 < \varsigma \frac{1}{\zeta_0} \tag{8.28}$$

and

$$\epsilon_1 < \varsigma \frac{1}{\zeta_1^2(1+\eta_1)} \kappa^4 h^{4\tau+2d+6}, \tag{8.29}$$

then there exist $\Lambda \in \mathcal{C}^{\omega,\infty}_{0,\delta}$, $Z' \in \mathcal{E}^{\omega,\infty}_{\rho-h,\delta-h}$, $H' \in \mathcal{C}^{\omega,\infty}_{\rho-h,\delta-h}$ *and a* (κ,τ)*-flat function* g' *such that*

$$\begin{aligned}(H + g + \langle \omega, \cdot\rangle + \langle \Lambda(c,\omega), \cdot\rangle \circ W) \circ Z'_{c,\omega}(\varphi, r) = \\ \langle \omega, r - c\rangle + H'(\varphi, c, \omega) + g'(\varphi, r, c, \omega)\end{aligned} \tag{8.30}$$

(modulo an additive constant that depends on c, ω*) with*

$$[Z' - \mathrm{id}]_{\rho-h,\delta-h,0} < h, \tag{8.31}$$

$$[H']_{\rho-h,\delta-h,s} \le \nu_s \epsilon_0 \tag{8.32}$$

and

$$\begin{aligned}\max(\|\Lambda\|_{0,\delta-h,s}, \left\|\partial_r^2(H' - H)\right\|_{\rho-h,\delta-h,s}, \|g' - g\|_{\rho-h,\delta-h,s}, \\ [Z' - \mathrm{id}]_{\rho-h,\delta-h,s}, [W \circ Z' - W]_{\rho-h,\delta-h,s}) \le \nu_s,\end{aligned} \tag{8.33}$$

where

$$\nu_s = C_s (h\xi_s)^{-5} \zeta_0^5 (\epsilon_s + \zeta_s \epsilon_0 + \eta_s \epsilon_0). \tag{8.34}$$

Moreover, if H *is of order* q *and* $g \in \mathcal{O}^q(c)$*, then* H' *is of order* q *and* $g' \in \mathcal{O}^q(c)$.

Remark. *Notice that the assumption (8.28) follows if*

$$\eta_1 < \varsigma \frac{1}{\zeta_1},$$

and that

$$\nu_s \epsilon_0 \le \nu_s \epsilon_1$$

and

$$\nu_s \le C_s (h\xi_s)^{-5} \zeta_1^5 (\epsilon_s + \zeta_s \epsilon_1 + \eta_s \epsilon_1).$$

Proof. By Lemma 8.4 there exists $\Lambda \in \mathcal{C}^{\omega,\infty}_{\delta}$,

$$\|\Lambda\|_{0,\delta,s} \le C_s \zeta_0 (\zeta_0 \epsilon_0 \eta_s + \zeta_0 \epsilon_s + \zeta_s \epsilon_0),$$

such that

$$\tilde{H} = H + \langle \Lambda, \cdot\rangle \circ W$$

verifies $M_{\tilde{H}} = 0$,

$$[\tilde{H}]_{\rho,\delta,s} \le \epsilon_s + C_s \zeta_0 (\zeta_0 \epsilon_0 \eta_s + \zeta_0 \epsilon_s + \zeta_s \epsilon_0) = \tilde{\epsilon}_s$$

and

$$\tilde{\zeta}_s = \left\|\partial_r^2 \tilde{H}\right\|_{\rho,\delta,s} + \|g\|_{\rho,\delta,s} + 1 = \zeta_s.$$

Since

$$\tilde{\epsilon}_1 \leq \varsigma \frac{1}{\tilde{\zeta}_1} \kappa^4 h^{4\tau+2d+6},$$

Lemma 8.3 gives $Z' \in \mathcal{E}^{\omega,\infty}_{\rho-h,\delta-h}$, $H' \in \mathcal{C}^{\omega,\infty}_{\rho-h,\delta-h}$ and a (κ,τ)-flat function g'' such that

$$(\tilde{H} + \langle \omega, \cdot \rangle) \circ Z'(\varphi, r, c, \omega) = \langle \omega, r - c\rangle + H'(\varphi, c, \omega) + g''(\varphi, r, c, \omega),$$

(modulo an additive constant that depends on c, ω) with

$$[Z' - \mathrm{id}]_{\rho-h,\delta-h,0} < h,$$

$$[H']_{\rho-h,\delta-h,s} \leq \tilde{\nu}_s \tilde{\epsilon}_0$$

and

$$\max([Z' - \mathrm{id}]_{\rho-h,\delta-h,s}, \left\| \partial_r^2 (H' - \tilde{H}) \right\|_{\rho-h,\delta-h,s}, \|g''\|_{\rho-h,\delta-h,s}) \leq \tilde{\nu}_s,$$

for

$$\tilde{\nu}_s = C_s (h\xi_s)^{-3} \zeta_0 (\zeta_0 \tilde{\epsilon}_s + \zeta_s \tilde{\epsilon}_0).$$

Since $g' = g \circ Z' + g''$ we get that g' is flat and Proposition 10.2 implies that

$$\|g' - g\|_{\rho-2h,\delta-2h,s} \leq C_s h^{-1} \zeta_s \tilde{\nu}_s$$

If we write $W = \mathrm{id} + f$ and $Z' = \mathrm{id} + f'$, then

$$W \circ Z' - W = f' + (f \circ (\mathrm{id} + f') - f).$$

We've already seen that

$$[f']_{\rho-h,\delta-h,s} \leq \tilde{\nu}_s$$

and

$$[f \circ (\mathrm{id} + f') - f]_{\rho-2h,\delta-2h,s} \leq (h\xi_s)^{-1} \tilde{\nu}_s$$

by Proposition 10.2. Let now $\nu_s = C_s (h\xi_s)^{-1} \zeta_0 \tilde{\nu}_s$. □

8.5. **Convergence of the KAM scheme.** We will show in Section 8.6 that the inductive application of Proposition 8.5 yields Proposition 8.2. Before this, we show in the current section two computational lemmas that will allow, under condition (8.13) of Proposition 8.2, to apply inductively Proposition 8.5 by checking conditions (8.28) and (8.29) at each step, and get the required estimates of Proposition 8.2. The first lemma deals with C^1 norms relative to ω, while the second one contains the estimates relative to the higher order norms.

Lemma 8.6. *Fix $0 < h < \frac{1}{2}$ and let $h_n = h2^{-n-1}$. Let a, b, c and $C \geq 0$ and let there be given four non negative sequences $\nu_n, \zeta_n, \eta_n, \epsilon_n$ such that*

$$\nu_n \leq C\kappa^{-2b} h_n^{-2a} \zeta_n^c (\zeta_n + \eta_n)\epsilon_n \tag{8.35}$$

for $n \geq 0$ and

$$\zeta_n \leq \zeta_{n-1} + \nu_{n-1} \qquad \zeta_0 = \zeta \geq 1 \tag{8.36}$$

$$\eta_n \leq \eta_{n-1} + \nu_{n-1} \qquad \eta_0 = 0 \tag{8.37}$$

$$\epsilon_n \leq \nu_{n-1}\epsilon_{n-1} \qquad \epsilon_0 = \epsilon \tag{8.38}$$

for $n \geq 1$.

Then there exists $C' = C'(C, a, b, c) > 0$ such that if for $\varsigma \leq 1$

$$\epsilon < \frac{\varsigma}{C'} \kappa^{2b+1} h^{2a+1} \zeta^{-c-2} \tag{8.39}$$

then

$$\epsilon_n \leq (\kappa h \zeta^{-1})^{2^n - 1}\epsilon \tag{8.40}$$

$$\eta_n < \varsigma \zeta_n^{-1}. \tag{8.41}$$

Proof. Assume $\zeta_n \leq A = 2\zeta$ and $\eta_n \leq 1$ for all n. Then

$$\nu_n \leq BD^n \epsilon_n$$

with $B = C\kappa^{-2b} h^{-2a} 2^{2a+1} A^{c+1}$ and $D = 4^a$. Hence for $n \geq 1$ we have

$$\epsilon_n \leq BD^{n-1}\epsilon_{n-1}^2 \leq B^{2^n-1} D^{2^n-n-1} \epsilon^{2^n} = \frac{1}{BD^{n+1}} (BD\epsilon)^{2^n}$$

which shows (8.40) if C' is sufficiently large.

From (8.40) we get that

$$\sum_n \nu_n \leq \frac{\varsigma}{2\zeta},$$

and the assumptions $\zeta_n \leq A = 2\zeta$ and $\eta_n \leq 1$ — actually (8.41), now follow by induction. $\square$

Lemma 8.7. *Fix $0 < h < \frac{1}{2}$ and let $h_n = h2^{-n-1}$. Let $a, b, c \geq 0$ and suppose ϵ_n is a sequence satisfying (8.40) with $\epsilon = \epsilon_0$ verifying (8.39) of Lemma 8.6. Assume that $C_s \geq 0$, and that four sequences $\nu_{s,n}, \zeta_{s,n}, \eta_{s,n}, \epsilon_{s,n}$ satisfy*

$$\nu_{s,n} \leq C_s \kappa^{-b(s+1)} h_n^{-a(s+1)} \zeta^c (\epsilon_{s,n} + \zeta_{s,n}\epsilon_n + \eta_{s,n}\epsilon_n) \tag{8.42}$$

for all $n \geq 0$, *and*

$$\zeta_{s,n} \leq \zeta_{s,n-1} + \nu_{s,n-1} \qquad \zeta_{s,0} = \zeta \geq 1 \tag{8.43}$$

$$\eta_{s,n} \leq \eta_{s,n-1} + \nu_{s,n-1} \qquad \eta_{s,0} = 0 \tag{8.44}$$

$$\epsilon_{s,n} \leq \nu_{s,n-1}\epsilon_{n-1} \qquad \epsilon_{s,0} = \epsilon \tag{8.45}$$

for all $n \geq 1$. *Then*

$$\sum_{n\geq 0} \nu_{s,n} \leq \sigma(\kappa^{-1}h^{-1}\zeta)^{\alpha(s)}(\zeta+\epsilon) \tag{8.46}$$

where $\sigma := \frac{\varsigma}{C'}$ *of* (8.39) *and* $\alpha(s)$ *is some increasing function in* s *depending on* C_s *and* a, b, c.

Proof. By replacing $\zeta_{s,n}$ by $\zeta_{s,n} + \eta_{s,n}$ we see that it is enough to consider the case $\eta_{s,n} = 0$ for all n.

If we let $U_s = \bar{C}\kappa^{-b(s+1)}h^{-a(s+1)}\zeta^c$, with $\bar{C}(s,a,b,c,C) > 0$ sufficiently large, then it is immediate by induction that

$$\max(\epsilon_{s,n}, \zeta_{s,n} - \zeta, \nu_{s,n}) \leq \sigma U_s^{n+1}(\zeta+\epsilon) \tag{8.47}$$

Thus, if $n \geq N(s) \gg \max(\log(s+1), \log C_s)$, (8.45) and (8.47) and (8.40) imply that

$$C_s\kappa^{-b(s+1)}h_n^{-a(s+1)}\zeta^c\epsilon_{s,n} \leq \sigma U_s^{2n+1}(\kappa h\zeta^{-1})^{2^{n-1}-1}\epsilon(\zeta+\epsilon) \leq \frac{\sigma}{2^n}(\zeta+\epsilon)$$

$$C_s\kappa^{-b(s+1)}h_n^{-a(s+1)}\zeta^c\zeta_{s,n}\epsilon_n \leq \sigma U_s^{2n+1}(\kappa h\zeta^{-1})^{2^{n}-1}\epsilon(\zeta+\epsilon) \leq \frac{\sigma}{2^n}(\zeta+\epsilon)$$

hence, for $n \geq N(s)$ we get that

$$\nu_{s,n} \leq \frac{\sigma}{2^{n-1}}(\zeta+\epsilon) \tag{8.48}$$

and (8.46) follows, with $\alpha(s) = (s+1)(N(s))^2$, if we sum (8.47) for n from 0 to $N(s)$ and (8.48) for $n \geq N(s)$. □

8.6. **Proof of Proposition 8.2.** We now prove Proposition 8.2 from an inductive application of Proposition 8.5.

Let $h < \min(\rho,\delta)/2$, $h_n = h2^{-n-1}$ and define

$$\rho_n = \rho - \sum_{i<n} h_i, \quad \delta_n = \delta - \sum_{i<n} h_i$$

and

$$\xi_{s,n} = \kappa^{(s+1)}h_n^{(\tau+1)(s+1)+d}.$$

We start by setting $H_0 = H$, $\Lambda_0 = 0$, $g_0 = 0$ and $W_0 = \mathrm{id}$, and we shall define inductively H_n, Λ_n, Z_n and $W_n = Z_0 \circ Z_1 \circ \cdots \circ Z_n$. In light of (8.34), let

$$\epsilon_{s,n} = [H_n]_{\rho_n,\delta_n,s}, \quad \zeta_{s,n} = \left\|\partial_r^2 H_n\right\|_{\rho_n,\delta_n,s} + \|g_n\|_{\rho_n,\delta_n,s} + 1$$

$\eta_{s,n} = [W_n - \mathrm{id}]_{\rho_n,\delta_n,s}, \quad \nu_{s,n} = C_s(h_n\xi_{s,n})^{-5}\zeta_{0,n}^5(\epsilon_{s,n} + \zeta_{s,n}\epsilon_{0,n} + \eta_{s,n}\epsilon_{0,n})$

where C_s is given by Proposition 8.5. We fix hereafter $a = 5(\tau + 1 + d), b = 5, c = 5$ and we will apply Lemmas 8.6 and 8.7 with these values and with C_s as in Proposition 8.5, while C of Lemma 8.6 is just C_1. As for ς, we will take it as $\varsigma(\tau)$ of Proposition 8.5.

Note also that for $s = 0$, the fact that $H_0 = H$ does not depend on ω, hence $\zeta_{s,0} = \zeta_{0,0} = \zeta$, $\epsilon_{s,0} = \epsilon_{0,0} = \epsilon$ as required by Lemma 8.7. By assumption also we have $\eta_{s,0} = 0$. To finish with the initial conditions, it follows from (8.13) that ϵ verifies conditions (8.39), provided $\epsilon(\tau)$ of Proposition 8.2 is taken sufficiently small.

Based on (8.32–8.34), we assume by induction that, for $j = 0, \dots, n$, $\nu_{s,j}, \zeta_{s,j}, \eta_{s,j}, \epsilon_{s,j}$ verify (8.35–8.38) for $s = 1$, and (8.42–8.45) for $s \geq 1$.

Then, by (8.40) and (8.41) of Lemma 8.6 we verify that, at each step n, conditions (8.28) and (8.29) of Proposition 8.5 are satisfied so that we can apply the latter proposition and get $\Lambda_{n+1} \in \mathcal{C}^{\omega,\infty}_{0,\delta_{n+1}}$, $Z_{n+1} \in \mathcal{E}^{\omega,\infty}_{\rho_{n+1},\delta_{n+1}}$, $H_{n+1} \in \mathcal{C}^{\omega,\infty}_{\rho_{n+1},\delta_{n+1}}$ and $g_{n+1}\mathcal{C}^{\omega,\infty}_{\rho_{n+1},\delta_{n+1}}$ (κ, τ)-flat such that

$$\begin{aligned}(H_n + \langle\omega, \cdot\rangle + \langle\Lambda_{n+1}, \cdot\rangle \circ W_n + g_n) \circ Z_{n+1}(\varphi, r, c, \omega) = \\ \langle\omega, r - c\rangle + H_{n+1}(\varphi, c, \omega) + g_{n+1}(\varphi, r, c, \omega)\end{aligned}$$

(modulo an additive constant). Moreover, by (8.32–8.34) we have that

$$\begin{aligned}(8.49)\quad &\max(\|\Lambda_n\|_{\rho_n,\delta_n,s}, \left\|\partial_r^2(H_n - H_{n-1})\right\|_{\rho_n,\delta_n,s}, \\ &\|g_n - g_{n-1}\|_{\rho_n,\delta_n,s}, [Z_n - \mathrm{id}]_{\rho_n,\delta_n,s}, [W_{n-1} \circ Z_n - W_{n-1}]_{\rho_n,\delta_n,s}) \leq \nu_{s,n}\end{aligned}$$

and that $\nu_{s,n+1}, \zeta_{s,n+1}, \eta_{s,n+1}, \epsilon_{s,n+1}$ satisfy (8.35–8.38) for $s = 1$, and (8.42–8.45) for $s \geq 1$. Finally, (8.46) gives that

$$\sum_{n\geq 0} \nu_{s,n} \leq \sigma(\kappa^{-1}h^{-1}\zeta)^{\alpha(s)}(\zeta + \epsilon)$$

which together with (8.49) show that $\sum \Lambda_l = \Lambda \in \mathcal{C}^{\omega,\infty}_{\delta-h}$, W_n converges to $W \in \mathcal{E}^{\omega,\infty}_{\rho-h,\delta-h}$, and H_n converges to $H' \in \mathcal{C}^{\omega,\infty}_{\rho-h,\delta-h}$, and g_n converges to a (κ, τ)-flat function $g \in \mathcal{C}^{\omega,\infty}_{\rho-h,\delta-h}$ such that : $[H']_{\rho-h,\delta-h} = 0$ and Λ, W, H', g satisfy (8.14) and (8.15) of Proposition 8.2.

This completes the proof of Proposition 8.2 – except for the last analyticity statement. However, if

$$\omega_0 \in DC(2\kappa, \tau)$$

and H is analytic on the segment $I_\delta = B_\delta(\omega_0) \cap \mathbb{R}\omega_0$, then the same proof, for $s = 0$, applied to functions in $\mathcal{C}^\omega_{\rho,\delta,\delta}$, i.e. functions $f \in$

$C^\omega(\mathbb{T}^d_\rho \times \mathbb{D}^d_\delta \times \mathbb{D}^d_\delta \times \mathbb{D}_\delta)$ such that $f(\varphi, r, c, \omega) \in \mathcal{O}^2(r,c)$ yields the analyticity of Λ, W and H' on $I_{\delta'}$ for some $0 < \delta' \leq \delta$.

8.7. **Proposition 8.2 implies Proposition 4.2.** Denote by $\tilde\alpha(s)$ the sequence of constants in Proposition 8.2 – we can assume without restriction that

$$\tilde\alpha(s) \geq (s-t) + \tilde\alpha(t), \quad s \geq t,$$

– and let

$$\alpha(s) = \tilde\alpha(s) + 1 + \gamma, \quad \gamma = 10(\tau + d) + 11.$$

Let

$$H(\varphi, r) = N^q(r) + \mathcal{O}^{q+1}(r), \quad q \geq 1 + \alpha(1)$$

with $N^q(r) = \langle \omega_0, r\rangle + \mathcal{O}^2(r)$. Then

$$\begin{aligned}\tilde H(\varphi, r, c) =: H(\varphi, r) - N^q(c) - \langle \partial_r N^q(c), r - c\rangle \\ = a(\varphi, c) + \langle B(\varphi, c), r - c\rangle + \mathcal{O}^2(r-c)\end{aligned}$$

with $a \in \mathcal{O}^{q+1}(c)$ and $B \in \mathcal{O}^q(c)$ (which means in particular that H is of order q) Now there exist $3\rho \geq 3\delta > 0$ such that

$$[\tilde H]_{3\rho, 3\eta, 0} < C\eta^q$$

for all $\eta \leq \delta$ — the constants ρ, δ and C only depend on H.

Let

$$C\eta^q = \sigma \frac{1}{(1 + \left\|\partial_r^2 \tilde H\right\|_{3\rho, 3\eta, 0})^7} \kappa^{11} \eta^\gamma$$

— this defines $\sigma = \sigma(\eta)$. Now there is a constant $C' = C'(H, \tau)$ such that if

$$\eta \leq C' \kappa^{\frac{11}{q-\gamma}},$$

then $\sigma \leq \epsilon(\tau)$.

By Proposition 8.2 there exist now $\tilde\Lambda \in \mathcal{C}^{\omega,\infty}_{0,2\eta}$ and $W \in \mathcal{E}^{\omega,\infty}_{2\rho,2\eta}$, and a (κ,τ)-flat function $g \in \mathcal{C}^{\omega,\infty}_{2\rho,2\eta}$ such that $g \in \mathcal{O}^q(c)$

$$(\tilde H + \langle \omega + \tilde\Lambda(c,\omega), \cdot\rangle) \circ W_{c,\omega}(\varphi, r) = \langle \omega, r - c\rangle + \mathcal{O}^2(r-c) + g(\varphi, r, c, \omega)$$

(modulo an additive constant that depends on c, ω). Moreover, for all $s \in \mathbb{N}$, (8.15) implies

$$\max(\left\|\tilde\Lambda\right\|_{0,2\eta,s}, [W - \mathrm{id}]_{2\rho,2\eta,s}) < C_s \frac{\eta^{q-\gamma}}{\kappa^{11}} (\frac{1}{\kappa\eta})^{\tilde\alpha(s)}. \quad (8.50)$$

[5]

Hence if we set $\Lambda(c,\omega) = \tilde\Lambda(c,\omega) - \partial_r N^q(r)$ we get that

$$(H + \langle \omega + \Lambda(c,\omega), \cdot\rangle) \circ W_{c,\omega}(\varphi, r) = \langle \omega, r - c\rangle + \mathcal{O}^2(r-c) + g(\varphi, r, c, \omega)$$

[5] the value of C_s will change from line too line

and, for all $s \in \mathbb{N}$,

$$\|\Lambda + \partial_r N^q\|_{0,2\eta,s} \leq C_s \frac{\eta^{q-\gamma}}{\kappa^{11}} (\frac{1}{\kappa\eta})^{\tilde{\alpha}(s)} \leq C_s \eta^q (\frac{1}{\kappa\eta})^{\alpha(s)}. \tag{8.51}$$

The generating function. By Proposition 3.1, the diffeomorphism

$$W(\varphi, r, c, \omega) = (\varphi + \Phi(\varphi, c, \omega), r + R_1(\varphi, c, \omega) + R_2(\varphi, c, \omega)(r - c))$$

has a generating function $f(\psi, r, c, \omega) = f_0(\psi, c, \omega) + \langle f_1(\psi, c, \omega), r - c\rangle$

$$\begin{cases} s = r + \partial_\psi f \\ \varphi = \psi + \partial_r f = \psi + f_1. \end{cases}$$

If

$$\eta \leq C''(H, \tau)\kappa^{\frac{11+\tilde{\alpha}(1)}{q-(1+\gamma+\tilde{\alpha}(1))}},$$

then (8.50) implies

$$\|\Phi\|_{2\rho,2\eta,1} \leq C_1 \frac{\eta^{q-\gamma}}{\kappa^{11}} (\frac{1}{\kappa\eta})^{\tilde{\alpha}(1)} \lesssim \eta,$$

and, by Proposition 10.3,

$$\|f_1\|_{\rho,\eta,s} \leq C_s \frac{\eta^{q-\gamma}}{\kappa^{11}} (\frac{1}{\kappa\eta})^{\tilde{\alpha}(s)}.$$

Moreover, by Proposition 10.2,

$$\|f_0\|_{\rho,\eta,s} \leq C_s \frac{\eta^{q-\gamma}}{\kappa^{11}} (\frac{1}{\kappa\eta})^{\tilde{\alpha}(s)},$$

so

$$\|f\|_{\rho,\eta,s} \leq C_s \frac{\eta^{q-\gamma}}{\kappa^{11}} (\frac{1}{\kappa\eta})^{\tilde{\alpha}(s)}. \tag{8.52}$$

To conclude we observe that

$$\frac{\eta^{q-\gamma}}{\kappa^{11}} (\frac{1}{\kappa\eta})^{\tilde{\alpha}(s)} \leq \eta^q (\frac{1}{\kappa\eta})^{\alpha(s)},$$

and that

$$\kappa^{\frac{\alpha(1)}{q-\alpha(1)}} \leq \min(\kappa^{\frac{11}{q-\gamma}}, \kappa^{\frac{11+\tilde{\alpha}(1)}{q-(1+\gamma+\tilde{\alpha}(1))}}).$$

Finally, point (iii) of Proposition 4.2 is implied by the last statement of Proposition 8.2. □

9. KAM stability for Liouville tori

In this section we give a sketch of the proof of Theorem D which claims KAM stability of a Liouville torus with a non-degeneracy condition of Kolmogorov type. Notice that since the frequency vector is Liouville we don't have any Birkhoff normal form in general.

By assumption there exist a $\gamma > 0$ and an increasing sequence Q_n such that

$$|\langle k, \omega_0 \rangle| \geq \frac{1}{|Q_n|^\tau} \quad \forall k \in \mathbb{Z}^d \setminus \{0\}, \ |k| \leq Q_n.$$

Lemma 9.1. *Let $H \in \mathcal{C}^\omega(\mathbb{T}^d_\rho \times \mathbb{D}^e_\delta)$ be of the form (1.1) and let q be fixed.*

For any n sufficiently large (depending on H and q) , there exists an exact symplectic local diffeomorphism

$$Z(\varphi, r) = (\varphi + \mathcal{O}(r), r + \mathcal{O}^2(r))$$

defined in $T^d_{\rho'} \times \mathbb{D}^e_{\delta'}$ where

$$\rho' \geq \rho/4 \quad \text{and} \quad \delta' \geq Q_n^{-2\gamma}$$

such that

$$H \circ Z(\varphi, r) = N^q(r) + F(\varphi, r) + R$$

with $N^q(r) = \langle \omega_0, r \rangle + \mathcal{O}^2(r)$ and $F \in \mathcal{O}^{q+1}(r)$ and

$$|F|_{\rho',\delta'} + |N^q|_{\delta'} \leq Q_n^{2\gamma q} \tag{9.53}$$

$$|R|_{\rho',\delta'} \leq e^{-\sqrt{Q_n}} \tag{9.54}$$

$$\left\| \frac{\partial^2 N}{\partial r^2}(0) - M_0 \right\| \leq e^{-\sqrt{Q_n}} \tag{9.55}$$

Proof. Truncate the Fourier coefficients of H at order $|k| \leq Q'_n = \frac{Q_n}{q}$ to get $\tilde{H}$ and $H = \tilde{H} + R$. Then

$$|\tilde{H}|_{\rho/2,\delta} \leq C$$

and

$$|R|_{\rho/2,\delta} \leq C e^{-Q'_n \rho/2}$$

which is $\leq e^{-\sqrt{Q'_n}}$ if n is large enough.

Apply now Birkhoff reduction up to order q to $\tilde{H}$, for example as in Proposition 3.3 with $c = r$:

- notice that
$$|\tilde{H}_j|_{\rho/2,\delta} \le C'_j$$
and
$$\min_{0<|k|\le Q_n} |(k,\omega_0)| \ge Q_n^{-\gamma};$$
- it follows by a finite induction that
$$|\Gamma_j|_{\rho/2,\delta},\ |\Omega_j|_{\rho/2,\delta},\ |G_j|_{\rho/2,\delta} \le C''_j Q_n^{(j-1)\gamma}$$
and for $j \le q$
$$|f_j|_{\rho/2,\delta} \le C''_j Q_n^{j\gamma};$$
- then Z, implicitly defined by
$$\begin{cases} \varphi = \psi + \frac{\partial f}{\partial r}(\psi, r) \\ s = r + \frac{\partial f}{\partial \psi}(\psi, r) \end{cases} \qquad f = f_2 + \cdots + f_q,$$
is defined in $T^d_{\rho/4} \times \mathbb{D}^e_{\delta'}$ where
$$\delta' \ge C Q_n^{-\gamma};$$
- moreover $N_2^q(r) = \mathcal{M}(\tilde{H}_2(\cdot, r))$ which implies (9.55).

$\square$

9.1. **Proof of theorem D.** Fix $q = 60(2d+\alpha(1)+5) - \alpha(1)$ being the exponent that appears in (8.15) of Proposition 8.2 – and apply Lemma 9.1 to find
$$\bar{H}(\varphi, r) = H \circ Z(\varphi, r) = N^q(r) + F(\varphi, r) + R.$$
Write
$$\begin{aligned} \tilde{H}(\varphi, r, c) =: H \circ (\varphi, r) - N^q(c) - \langle \partial_r N^q(c), r - c\rangle \\ = a(\varphi, c) + \langle B(\varphi, c), r - c\rangle + \mathcal{O}^2(r - c) \end{aligned}$$
with $a \in \mathcal{O}^2(c)$ and $B \in \mathcal{O}(c)$, i.e. $\tilde{H}$ is of order 1.

Observe that, with $\delta_n = Q_n^{-\gamma q^2}$, we have
$$[\tilde{H}]_{\rho',\delta_n,0} \le C(\delta_n^q Q_n^{2\gamma(q+1)} + e^{-\sqrt{Q_n}})$$
which is $\le Q_n^{-\gamma q^3/2} = \delta_n^{q/2}$ if n is large enough.

If $\kappa_n = \delta_n^2$ and τ is $= d$, say, then

$$[\tilde{H}]_{\rho',\delta_n,0} \le \delta_n^{q/3} \kappa_n^{11} \delta_n^{\,10(\tau+d)+11} \frac{1}{(1 + \left\|\partial_r^2 \tilde{H}\right\|_{\rho',\delta_n,0})^7} \tag{9.56}$$

provided n is sufficiently large. That is, (8.13) is satisfied by $\tilde{H}$ with $\sigma \le \delta_n \le \epsilon(\tau)$ when n is large enough.

Hence Proposition 8.2 applies with our choice of κ_n, δ_n and $h = \delta_n/2$, yielding $\Lambda \in \mathcal{C}^{\omega,\infty}_{0,\delta_n/2}$ and $W \in \mathcal{E}^{\omega,\infty}_{\rho/2,\delta_n/2}$, and a (κ_n, τ)-flat function $g \in \mathcal{C}^{\omega,\infty}_{\rho/2,\delta_2/2}$ such that

$$(\bar{H} + \langle \omega + \bar{\Lambda}(c,\omega), \cdot \rangle) \circ W_{c,\omega}(\varphi, r) = \langle \omega, r - c \rangle + \mathcal{O}^2(r-c) + g(\varphi, r, c, \omega) \tag{9.57}$$

(modulo an additive constant that depends on c, ω), where we have set

$$\bar{\Lambda}(c,\omega) = \Lambda(c,\omega) - \partial_r N^q(c).$$

Notice that $\Lambda(0,\omega) = \omega$ and that, from (8.15) and the fact that $\sigma \leq \delta_n^{q/3}$, we get

$$\left\| \bar{\Lambda} + \partial_r N^q \right\|_{0,\delta_n/2,1} \leq \delta_n^2. \tag{9.58}$$

Let $\Psi(\omega, c) = \omega + \bar{\Lambda}(c,\omega)$. Then $\Psi(\omega_0, 0) = 0$ and from (9.58) we have that

$$\left\| \frac{\partial \Psi}{\partial \omega}(\omega_0, 0) - I \right\| \leq \delta_n^2 \tag{9.59}$$

$$\left\| \frac{\partial \Psi}{\partial c}(\omega_0, 0) - M_0 \right\| \leq 2\delta_n \tag{9.60}$$

By the implicit function theorem, there exists a constant $C(M_0)$ (that only depends on M_0) and a function $S : B(\omega_0, C(M_0)\delta_n) \to B(0, \delta_n/2)$, such that

$$\Psi(\omega, S(\omega)) = 0$$

Moreover S is of class C^1 and $dS \sim M_0^{-1}$. A simple computation shows that the set of frequencies in $B(\omega_0, C(M_0)\delta_n)$ that are (κ_n, τ)-Diophantine has measure larger than $(1-\delta_n)\mathrm{Leb}(B(\omega_0, C(M_0)\delta_n))$ (recall that we took $\kappa_n = \delta_n^2$).

This concludes the proof of Theorem D because (9.57) and the (κ_n, τ)-flatness of g imply that for any $\omega \in B(\omega_0, C(M_0)\delta_n) \cap \mathrm{CD}(\kappa_n, \tau)$, $\mathbb{T}^d \times \{S(\omega)\}$ is an invariant KAM torus for $\bar{H} \circ W_{S(\omega),\omega}$.

10. Appendix. Composition and inversion estimates.

In this Appendix we give the useful estimates for our KAM scheme.

10.1. **Convexity estimates.**

Proposition 10.1. *Let $f, g \in \mathcal{C}^{\omega,\infty}(\mathbb{T}^d_\rho \times \mathbb{D}^{d'}_\delta, B)$. Then*

(i)

$$||f||_{\rho,\delta,s} \leq C_{s_1,s_2} ||f||^{a_1}_{\rho,\delta,s_1} ||f||^{a_2}_{\rho,\delta,s_2}$$

for all non-negative numbers a_1, a_2, s_1, s_2 *such that*

$$a_1 + a_2 = 1, \quad s_1 a_1 + s_2 a_2 = s.$$

(ii)

$$||fg||_{\rho,\delta,s} \ \leq \ C_s(||f||_{\rho,\delta,s}||g||_{\rho,\delta,0} + ||f||_{\rho,\delta,0}||g||_{\rho,\delta,s})$$

for all non-negative numbers s.

Proof. A classical result – see the appendix of [Ho] □

Corollary. *Let* $f, g \in \mathcal{C}^{\omega,\infty}(\mathbb{T}^d_\rho \times \mathbb{D}^{d'}_\delta, B)$. *Then*

(i)

$$||f||^{n+1}_{\rho,\delta,1}||f||_{\rho,\delta,s-n} \leq C_s||f||^{n+1}_{\rho,\delta,0}||f||_{\rho,\delta,s+1}$$

for all non-negative numbers s, n

(ii)

$$||f^n||_{\rho,\delta,s} \ \leq \ C_s^{\log(n)}||f||_{\rho,\delta,s}||f||^{n-1}_{\rho,\delta,0}$$

for all non-negative numbers s, n.

Proof. A computation. □

10.2. **Composition.**

Proposition 10.2. *Let* $f, g \in \mathcal{C}^{\omega,\infty}(\mathbb{T}^d_\rho \times \mathbb{D}^{d'}_\delta, B)$ *and assume that*

$$\|g\|_{\rho,\delta,0} \leq \frac{h}{2} \leq \frac{1}{2}\min(\rho, \delta).$$

Then

$$x \mapsto f(x + g(x, \omega), \omega)$$

belongs to $\mathcal{C}^{\omega,\infty}(\mathbb{T}^d_{\rho-h} \times \mathbb{D}^{d'}_{\delta-h}, B)$ *and*

(i) $h(x, \omega) = f(x + g(x, \omega), \omega) - f(x, \omega)$ *verifies*

$$\|h\|_{\rho-h,\delta-h,s} \leq C_s\frac{1}{h}(\|f\|_{\rho,\delta,0}\|g\|_{\rho,\delta,s} + \|f\|_{\rho,\delta,s}\|g\|_{\rho,\delta,0}).$$

(ii) $k(x, \omega) = f(x + g(x, \omega), \omega) - f(x, \omega) - \langle \partial_x f(x, \omega), g(x)\rangle$ *verifies*

$$\|k\|_{\rho-h,\delta-h,s} \leq C_s\frac{1}{h^2}(\|f\|_{\rho,\delta,0}\|g\|_{\rho,\delta,s} + \|f\|_{\rho,\delta,s}\|g\|_{\rho,\delta,0})\|g\|_{\rho,\delta,0}.$$

Proof. We will prove the statements when x and $g(x, \omega)$ are scalars. Notice

$$f(x + g(x, \omega), \omega) = \sum_{n=0}^{\infty} \frac{\partial^n f}{\partial x^n}(x, \omega)\frac{g^n(x, \omega)}{n!}.$$

By Cauchy estimates we have for $n \geq 0$

$$\left\|\frac{\partial^n f}{\partial x^n}\right\|_{\rho-h,\delta-h,s} \leq \frac{1}{h^n}\,\|f\|_{\delta,s}\, n!$$

and, by the Hadamard estimates we have that

$$\|g^n\|_{\rho,\delta,s} \le C_s^{\log(n)} \|g\|_{\rho,\delta,0}^{n-1} \|g\|_{\rho,\delta,s}.$$

Hence, for $j \ge 1$,

$$\| \sum_{n=j}^{\infty} \frac{\partial^n f}{\partial x^n}(x,\omega) \frac{g^n(x,\omega)}{n!} \|_{\rho-h,\delta-h,s} \le$$

$$C_s \|f\|_{\rho,\delta,s} \sum_{n\ge j} \left(\frac{\|g\|_{\rho,\delta,0}}{h}\right)^n + \|f\|_{\rho,\delta,0} \frac{\|g\|_{\rho,\delta,s}}{h} \sum_{n\ge j-1} C_s^{\log(n+2)} \left(\frac{\|g\|_{\rho,\delta,0}}{h}\right)^n.$$

□

10.3. **Inversion.**

Proposition 10.3. *Let $f \in \mathcal{C}^{\omega,\infty}(\mathbb{T}^d_\rho \times \mathbb{D}^{d'}_\delta, B)$ and assume that*

$$\|f\|_{\rho,\delta,1} \lesssim \frac{h}{2} \le \frac{1}{2}\min(\rho,\delta).$$

Then

$$\mathbb{T}^d_\rho \times \mathbb{D}^{d'}_\delta \ni x \mapsto \tilde f(x,\omega) = x + f(x,\omega)$$

is invertible for all $\omega \in B$ with an inverse $\mathbb{T}^d_{\rho-h} \times \mathbb{D}^{d'}_{\delta-h} \ni y \mapsto \tilde g(y,\omega) = y + g(y,\omega)$ satisfying

$$\|g\|_{\rho-h,\delta-h,s} \lesssim C_s \|f\|_{\rho,\delta,s}$$

for all $s \in \mathbb{N}$.

Proof. It is clear by the implicit function theorem that g exists and that

$$\|g\|_{\rho-h,\delta-h,0} \lesssim \|f\|_{\rho,\delta,0} \le h.$$

Since $g(y,\omega) + f(y + g(y,\omega),\omega) = 0$, it follows that

$$\partial_\omega g + (\partial_x f)\circ \tilde g \cdot \partial_\omega g + (\partial_\omega f)\circ \tilde g = 0$$

and, hence,

$$\|g\|_{\rho-h,\delta-h,1} \lesssim \|f\|_{\rho,\delta,1}.$$

Moreover, for $n \ge 1$

$$\partial_\omega^{n+1} g + \partial_\omega^n((\partial_x f)\circ \tilde g \cdot \partial_\omega g)) + \partial_\omega^n((\partial_\omega f)\circ \tilde g) = 0,$$

from which we derive

$$\|g(y,\omega)\|_{\rho-h,\delta-h,n+1} \lesssim \|(\partial_x f)\circ \tilde g \cdot \partial_\omega g\|_{\rho-h,\delta-h,n} + \|(\partial_\omega f)\circ \tilde g\|_{\rho-h,\delta-h,n},$$

and, by Proposition 10.1,

$$\|g(y,\omega)\|_{\rho-h,\delta-h,n+1} \le C_n \|(\partial_x f)\circ \tilde g\|_{\rho-h,\delta-h,n} \|f\|_{\rho,\delta,1} + \|(\partial_\omega f)\circ \tilde g\|_{\rho-h,\delta-h,n}.$$

By Proposition 10.2(i)

$$\|g\|_{\rho-h,\delta-h,n+1} \le C_n\Big(\frac{1}{h}\|f\|_{\rho,\delta,1}\|f\|_{\rho,\delta,n} + \|f\|_{\rho,\delta,n+1} + \|f\|_{\rho,\delta,1}\|g\|_{\rho-h,\delta-h,n}\Big).$$

By assumption $\|f\|_{\rho,\delta,1} \lesssim h$, so

$$\|g\|_{\rho-h,\delta-h,n+1} \le C_n(\|f\|_{\rho,\delta,n+1} + \|g\|_{\rho-h,\delta-h,n})$$

and the result follows by a finite induction. □

References

[AK] D. V. Anosov and A. B. Katok, New examples in smooth ergodic theory. Ergodic diffeomorphisms, Transactions of the Moscow Mathematical Society 23, 1–35, 1970.

[B] A. Bounemoura, Generic super-exponential stability of invariant tori in Hamiltonian systems, Ergodic Theory Dynam. Systems 31, 1287–1303, 2011.

[E] L. H. Eliasson, Perturbations of stable invariant tori for Hamiltonian systems, Ann. Sc. Norm. Sup. Pisa 15, 115–147, 1988.

[FH] A. Fathi and M. Herman, Existence de difféomorphismes minimaux, Asterisque 49, 39–59, 1977.

[FK] B. Fayad and R. Krikorian, Herman's last geometric theorem, Ann. Sci. Éc. Norm. Supér. 42, 193–219, 2009.

[H] M. Herman, Some open problems in dynamical systems, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), Doc. Math. 1998 Extra Vol. II, 797–808, 1998.

[Ho] L. Hormander, Note on Holder estimates. The boundary problem of physical geodesy. Arch. Rational Mech. Anal. 62 (1976), 1–52.

[I] H. Ito, Convergence of Birkhoff normal forms for integrable systems, Comment. Math. Helv. 64, 412–61, 1989.

[N] Z. Nguyen Tien, Convergence versus integrability in Birkhoff normal form, Ann. of Math. 161, 141–156, 2005.

[P-M] R. Perez-Marco, Convergence or generic divergence of the Birkhoff normal form, Ann. of Math. 157, 557–74, 2003.

[R] H. Rüssmann, Über die Normalform analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtslösung, Math. Ann. 169, 55–72, 1967.

[R2] H. Rüssmann, H. Rüssmann. Invariant tori in non-degenerate nearly integrable Hamiltonian systems. Regular and Chaotic Dynamics 6:2(2001), 119-204. MR1843664 (2002g:37083)

[S55] C. L. Siegel, Über die Existenz einer Normalform analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtslösung, Math. Ann 128, 144–70, 1954.

[S05] L. Stolovitch, A KAM phenomenon for singular holomorphic vector fields, Publ. Mat. Inst. Hautes Études Sci. 102, 99–165, 2005.

[V] J. Vey, Sur certains systèmes dynamiques séparables, Amer. J. Math. 100, 591–614, 1978.

[XYQ] Xu, Junxiang, You, Jiangong, Qiu, Qingjiu, Invariant tori for nearly integrable Hamiltonian systems with degeneracy. Math. Z. 226 (1997), 375Ð387.

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