MIXED SPECTRUM REPARAMETERIZATIONS OF LINEAR FLOWS ON $\mathbb{T}^2$

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To the memory of great mathematicians of the 20th century Andrei Nikovaevich Kolmogogov and Ivan Georgievich Petrovskii

ABSTRACT. We prove the existence of mixed spectrum C^∞ reparameterizations of any linear flow on $\mathbb{T}^2$ with Liouville rotation number. For a restricted class of Liouville rotation numbers we prove the existence of mixed spectrum real-analytic reparameterizations.

1. INTRODUCTION; FORMULATION OF RESULTS

Consider the translation T_α on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ given by $T_\alpha(x) = x+\alpha$. If $\alpha_1, \dots, \alpha_d, 1$ are rationally independent then T_α is minimal and uniquely ergodic. We define the linear flow on the torus $\mathbb{T}^{d+1}$ by

$$\frac{dx}{dt} = \alpha, \qquad \frac{dy}{dt} = 1,$$

where $x \in \mathbb{T}^d$ and $y \in \mathbb{T}^1$. We denote this flow by $R^t_{(\alpha,1)}$. Notice that this flow can be realized as the time 1 suspension over the translation T_α on $\mathbb{T}^d$. Given a positive continuous function $\phi : \mathbb{T}^{d+1} \to \mathbb{R}^+$ we define the reparameterization of the linear flow by

$$\frac{dx}{dt} = \frac{\alpha}{\phi(x,y)}, \qquad \frac{dy}{dt} = \frac{1}{\phi(x,y)}.$$

The reparameterized flow is still minimal and uniquely ergodic while more subtle properties may change under reparametrization. In particular, the linear flow has discrete (pure point) spectrum with the group of eigenvalues isomorphic to $\mathbb{Z}^{d+1}$. Reparametrizations with continuous time change ϕ may have a wide variety of ergodic properties. This follows from the theory of monotone (Kakutani) equivalence [6] and the fact that every monotome measurable time change is cohomologous to a continuous one [12]; see also [7, Corollary 2.11]. However for sufficiently smooth reparametrizations the possibilities are more limited and they depend on the arithmetic properties of the vector α.

Definition 1. *Given a vector $\alpha \in \mathbb{R}^d$ we say it is Diophantine if there exists and constant $C > 0$ and a number $\sigma \geq d$ such that for all $k \in \mathbb{Z}^d \backslash \{0\}$*

$$\inf_{p\in\mathbb{Z}} | < k, \alpha > -p| \geq \frac{C}{||k||^\sigma}.$$

If the vector is not Diophantine then we call it Liouville.

If α is Diophantine and the function ϕ is C^∞ than the reparametrized flow is smoothly isomorphic to a linear flow . This was first noticed by A. N. Kolmogorov

[9]. Many of the basic questions concerning reparameterizations appeared in Kolmogorov's seminal I.C.M. address in 1954 [10]. M. R. Herman [4] found sharp results of that kind for finite regularity case.

For a Liouvillean α the reparametrized flow often is weakly mixing, i.e. has no eigenfunctions at all. Specifically, M. D. Šklover [13] proved existence of analytic weakly mixing reparametrizations some Liouvillean linear flows on $\mathbb{T}^2$; his result for special flows on which this is based is optimal in that he showed that for any analytic roof function φ other than a trigonometric polynomial there is α such that the special flow under the rotation R_α with the roof function φ is weakly mixing. About the same time A. Katok found a general criterion for weak mixing (see [7, Theorem 5.7]). B. Fayad [3] showed that for a Liouville translation T_α on the torus $\mathbb{T}^d$ the special flow under a generic C^∞ function φ is weak-mixing.

Katok [5] showed that for special flows over irrational rotations and under C^5 functions φ the spectrum is simple, the maximal spectral type is singular, and the flow can not be mixing. The latter conclusion was extended by A. V. Kočergin to functions of bounded variation [8]. The argument is based on a Denjoy–Koksma type estimates which fail in higher dimension. Fayad [3] showed that there exist $\alpha \in \mathbb{R}^2$ and analytic functions φ for which the special flow over the translation T_α and under the function φ is mixing. Recently Kočergin showed that for Hölder reparametrizations of some Diophantine linear flows on $\mathbb{T}^2$ mixing is also possible (oral communication).

In this paper we will show that yet another possibility is realized for smooth reparameterization of linear flows. We will restrict ourselves to the case of flows on $\mathbb{T}^2$ although our methods allow a fairly straightforward generalization to higher dimension.

Theorem 1. *If $\alpha \in \mathbb{T}^1$ is Liouville then for a dense set of $\phi \in C^\infty(\mathbb{T}^2, \mathbb{R}^+)$ the reparameterization of $R^t_{(\alpha,1)}$ by ϕ has mixed spectrum with a group of eigenvalues with a single generator.*

For a given irrational number α we denote by $p_n/q_n,\ n = 1, 2, \dots$ the sequence of best rational approximations coming for the continued fraction expansion.

Theorem 2. *For $\alpha \in \mathbb{T}^1$ with a subsequence $\{q_{s(n)}\}$ of the sequence of denominators of the best approximations $\{q_n\}$ satisfying*

$$q_{s(n)+1} > e^{q^5_{s(n)}}$$

there exists a $\phi \in C^\omega(\mathbb{T}^2, \mathbb{R}^+)$ such that the reparameterization of $R^t_{(\alpha,1)}$ by ϕ has a mixed spectrum with a group of eigenvalues with a single generator.

Just as a linear flow on the torus $\mathbb{T}^{d+1}$ can be represented as the constant time suspension flow over a translation on $\mathbb{T}^d$ one can represent the reparameterization of a linear flow by ϕ as a special flow over the same translation on $\mathbb{T}^d$ and under a function φ. The function φ is given by

$$\varphi(x) = \int_0^1 \phi(x + t\alpha, t)\,dt. \tag{1}$$

This is the return time function for the section $x_{d+1} = 0$. Conversely any special flow is differentiably conjugate to a reparametrization (see Lemmas 2 and 3). This will allow as to deal exclusively with special flows.

We call λ an eigenvalue of the flow T^t if there exists a measurable function h for which $h(T^t x) = e^{2\pi i \lambda t} h(x)$. It is a simple matter to calculate the eigenvalues of a constant time suspension. Restricting an eigenfunction to a section on which it is measurable we see an eigenvalue λ of the suspension flow with constant return time C satisfies the equation

$$e^{2\pi i\lambda C} = \sigma_T$$

where σ_T is some eigenvalue of the transformation T in the base. For every eigenvalue $\sigma_T = e^{2\pi i\lambda_T}$ we get an associated eigenvalue of the flow λ_T/C plus we get an additional generator $\lambda = 1/C$ coming from the trivial eigenvalue in the base.

For special flows under functions not cohomologous to a constant the situation is more complex. Eigenvalues of general special flows are determined by a multiplicative cohomological equation. The equation for a suspension flow is a special case of this.

Lemma 1 (Eigenvalue Criterion for Special Flows). [2] *A special flow over an ergodic transformation T on a Lebesgue space L and under a function φ has an eigenvalue λ if and only if the function $e^{2\pi i\lambda\varphi(x)}$ is multiplicatively cohomologous to 1, that is if and only if*

$$h(Tx) = e^{2\pi i\lambda\varphi(x)}h(x) \tag{2}$$

has a non-trivial measurable solution.

Theorem 3. *Let $\beta > 0$. If $\alpha \in \mathbb{T}^1$ is Liouville, then there exists a positive $\varphi \in C^\infty(\mathbb{T}^1, \mathbb{R})$, such that, for the special flow over R_α and under φ, (2) admits solutions only for $\lambda = n\beta^{-1}$.*

Theorem 4. *Let $\beta > 0$. If $\alpha \in \mathbb{T}^1$ admits a subsequence $\{q_{s(n)}\}$ of the sequence of best approximations satisfying*

$$q_{s(n)+1} > e^{q^5_{s(n)}} \tag{3}$$

then there exists a positive $\varphi \in C^\omega(\mathbb{T}^1, \mathbb{R})$, such that, for the special flow over R_α and under φ, (2) admits solutions only for $\lambda = n\beta^{-1}$.

These results for special flows establish that the reparameterized flow has a spectrum whose discrete part has a single generator. Notice that a reparameterized linear flow on $\mathbb{T}^2$ cannot have a discrete spectrum with a single generator. Such a flow would be measurably conjugate to a linear flow on $\mathbb{T}^1$ which is impossible since a linear flow on $\mathbb{T}^1$ has orbits of full measure while the orbit of any reparameterized linear flow on $\mathbb{T}^2$ has zero measure [11].

Notice that for any measure preserving transformation $T : X \to X$ if positive functions φ_1, φ_2 on X are such that

$$\psi(Tx) - \psi(x) = \varphi_1(x) - \varphi_2(x) \tag{4}$$

for some measurable ψ, then the special flows over T with roof functions φ_1 and φ_2 are conjugate. The conjugacy in question is provided by the shift of the "base" along the orbits on the first special flow by time $\psi(x)$.

An expression of the form $\psi(Tx) - \psi(x)$ is called *an additive coboundary*, two functions whose difference is an additive coboundary are called *(additively) cohomologous.*

Notice that for any irrational α any trigonometric polynomial P with zero average is an additive coboundary, i.e.

$$P(x) = Q(x + \alpha) - Q(x)$$

where Q is another trigonometric polynomial. Thus the roof function of a special flow over an irrational rotation can be changed by adding any trigonometric polynomial with zero average without changing properties of the flow.

In particular adding an additive coboundary to φ will not change the group of eigenvalues. Thus, given any φ, we can produce a dense set of functions which give the same eigenvalues by adding trigonometric polynomials with zero average.

2. Some Open Problems

2.1. **Regularity.**

2.1.1. It is interesting to find roof functions (and hence reparametrizations) of finite smoothness which produce mixed spectrum over a Diophantine rotation.

Constructing such functions for certain special Diophantine numbers can probably be achieved by a modification of the method of this paper. The next problem seems more challenging.

2.1.2. What the optimal regularity conditions allowing such behavior would be and how far are they from Herman's conditions?

2.2. **Other Types of Spectra.** We have shown the existence of mixed spectra special flows over Liouville rotations on $\mathbb{T}^1$. Our techniques easily extend to produce the existence of mixed spectra with only a one parameter family of eigenfunctions over some Liouville translations on higher dimensional tori.

2.2.1. It would be interesting to obtain mixed spectra with families of eigenvalues with more parameters.

2.2.2. A more exotic possibility would be a situation where all eigenvalues are retained from the constant time suspension but the spectrum has some continuous part.

These two problems do not seem to be beyond the grasp of the currently available techniques.

2.2.3. Finally there is a question of possibility of exotic discrete spectra where the reparameterization exhibits eigenfrequencies not arising from the original linear flow. Such a situation must appear if the reparametrized flow has eigenfunctions with more frequencies than the dimension of the torus (e.g. three frequencies on $\mathbb{T}^2$).

Currently this problem does not look easily approachable. Let us point out though that in the nonlinear context of the "approximation by conjugation" constructions [1] a similar possibility can be realized.

2.3. **Dichotomy.** The mixed spectrum examples we have constructed here all exhibit Fourier coefficients for the roof functions which behave extremely irregularly. They all have blocks of non-zero Fourier coefficients separated from one another on an exponential scale. For functions with more regular decay of coefficients it may well be that exotic spectra are impossible.

2.3.1. It would be interesting to find for which roof functions there is a dichotomy; for any irrational α either the special flow is conjugate to the constant time suspension or it is weak mixing.

2.3.2. A natural conjecture here is that for functions satisfying assumptions of the weak mixing criterion [7, Theorem 5.7] such a dychotomy holds. A simplest example of this kind is the function

$$h(z) = 2 + \sum_{n\neq 0} 2^{-|n|} e^{2\pi i n x} = \frac{8 - 2\cos x}{5 - 2\cos x}.$$

3. Reduction of Reparametrization to Special Flow

Lemma 2. *Given a positive function $\varphi \in C^\infty$ there exists a positive function $\phi \in C^\infty$ which satisfies*

$$\varphi(x) = \int_0^1 \phi(x + t\alpha, t)dt.$$

Proof. Let $\epsilon > 0$. Let $b(y)$ be a C^∞ function satisfying

(1) $b(y) \geq 0$.
(2) $b(y) = 0$ for $y \in [0, \epsilon] \cup [1 - \epsilon, 1]$.
(3) $\int b(y)dy = 1$.

Choose $\delta < \min \varphi(x)$. Set $\phi(x, y) = (\varphi(x - y\alpha) - \delta)b(y) + \delta$. This defines a C^∞ function on $\mathbb{T}^2$ with the required property. □

For the analytic case we will use a slightly different argument.

Lemma 3. *Given a positive real analytic function φ on the circle there exists a trigonometric polynomial Q and a positive real analytic function ϕ on R^2_α such that*

$$\varphi(x) = \int_0^1 \phi(x + t\alpha, t)dt + Q(x + \alpha) - Q(x). \tag{5}$$

Proof. Every real analytic function is cohomologous to the function obtained by substructing any finite number of nonconstant terms in its Fourier expansion. and this cohomology is again given by a trigonometric polynomial. The cohomologous function can be made arbitrary close to a constant with any number of derivatives; in particular one can assume that for the Fourier coefficients

$$\hat{\varphi}(0) > \frac{\pi}{2} \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} |\hat{\varphi}(m)| \tag{6}$$

For such a function equation (1) can be solved in positive functions by elementary Fourier analysis. Namely, let

$$\phi(x, y) = \sum_{m,n\in\mathbb{Z}} c_{m,n} e^{2\pi i(mx+ny)}.$$

Now

$$\int_{\mathbb{T}^1} \phi(x + t\alpha, t)dt = c_{0,0} + \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \left(\sum_{n\in\mathbb{Z}} \frac{c_{m,n}(e^{2\pi i(m\alpha+n)} - 1)}{2\pi(m\alpha + n)} \right) e^{2\pi i m x}.$$

From this it is natural to look for a function ϕ such that for each $m \in \mathbb{Z}$ there is only one nonzero Fourier coefficient $c_{m,n}$, say c_{m,n_m}. From equating Fourier coefficients we get

$$\hat{\varphi}(m) = \begin{cases} c_{m,n_m} \frac{e^{2\pi i(m\alpha+n_m)}-1}{2\pi(m\alpha+n_m)} & (m,n_m) \neq (0,0) \\ c_{0,0} & (m,n_m) = (0,0) \end{cases}$$

which immediately gives

$$c_{m,n_m} = \begin{cases} \hat{\varphi}(m) \frac{2\pi i(m\alpha+n_m)}{e^{2\pi i(m\alpha+n_m)}-1} & (m,n_m) \neq (0,0) \\ \hat{\varphi}(m) & (m,n_m) = (0,0) \end{cases}.$$

Now we choose n_m to be the closest integer to $-m\alpha$. Then we have

$$\left| \frac{2\pi i(m\alpha + n_m)}{e^{2\pi i(m\alpha+n_m)} - 1} \right| < \frac{\pi}{2}$$

From the formula for c_{m,n_m} and the fact φ is a real-function we have $c_{-m,n_{-m}} = c_{-m,-n_m} = \overline{c_{m,n_m}}$ which proves that ϕ is real. Finally

$$\phi(x,y) \geq \hat{\varphi}(0) - \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} |c_{m,n_m}| > \frac{\pi}{2} \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} |\hat{\varphi}(m)|$$

which establishes our last claim. □

Theorems 1 and 2 follow from Theorems Theorems 3 and 4 via Lemmas 2 and 3. In the rest of the paper we prove Theorems 3 and 4.

4. Construction of the Ceiling Function

We will construct the required function φ using an inductive process. The basic element is a smooth “step” function.

Definition 2. *Let $\theta : \mathbb{R} \to [0,1]$ be a C^∞ function satisfying*

$$\begin{aligned} \theta(x) &= 0 \quad x \leq 0 \\ \theta(x) &= 1 \quad x \geq 1 \end{aligned}$$

4.1. **Arithmetic Conditions.** Each step of the construction takes place on a different scale. These scales are related to the arithmetic properties of α. For the smooth construction we choose a sequence of scales $\{p_{s(n)}/q_{s(n)}\}$, from the sequence of best approximations $\{q_n\}$, with the following properties

$$q_{s(n)+1} > q_{s(n)}^n \tag{7}$$

$$q_{s(n+1)} > a_n q_{s(n)}. \tag{8}$$

Such a choice can always be made since we assumed that α was Liouville. For the analytic construction we replace (7) with (3).

The number a_n is a parameter which will be chosen at the n-th step of our construction. The full strength of (7) and (8) will be used at only two points in the proof, elsewhere we use the weaker assumption

$$q_{s(n)} > e^n q_{s(1)}. \tag{9}$$

The choice of $q_{s(1)}$ will be made later in order to ensure that φ satisfies (6).

We use function θ to construct a function φ_n with two bumps on the interval $[0, q_{s(n)}^{-1}]$ each with width approximately $k_n q_{s(n)+1}^{-1}$. We choose the sequence of widths

$$k_n = \frac{q_{s(n)+1}}{q_{s(n)} e^n} \tag{10}$$

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FIGURE 1. The function φ_n on $[0, q_{s(n)}^{-1}]$. φ_n is 0 on the rest of $\mathbb{T}^1$.

and use the function θ to construct a function φ_n with two bumps on the interval $[0, q_{s(n)}^{-1}]$ each with width approximately $k_n q_{s(n)+1}^{-1}$

Definition 3. *Let $B_n : \mathbb{R} \to \mathbb{R}$ be the C^∞ bump function of height 1 given by*

$$B_n(x) = \theta\Big(8q_{s(n)}e^{2n}(x + \frac{e^{-2n}}{8q_{s(n)}})\Big) - \theta\Big(8q_{s(n)}e^{2n}(x - \frac{k_n}{q_{s(n)+1}})\Big)$$

Using this bump function we write

$$R_n(x) = B_n\Big(x - \frac{1}{8q_{s(n)}}\Big) - B_n\Big(x - \frac{7}{8q_{s(n)}}\Big).$$

Since $R_n(0) = R_n(1)$ and $R_n(x)$ is C^∞ flat at $x = 0$ and $x = 1$ we can view R_n as a C^∞ function on $\mathbb{T}^1$. Finally

$$\varphi_n(x) = \frac{\beta}{k_n} R_n(x)$$

As the rotation number α is Liouville we don't know that φ_n is a additive coboundary. In order to ensure that we have a coboundary we will truncate φ_n to get a trigonometric polynomial. Trigonometric polynomials are always coboundaries.

Definition 4. *Define a new function*

$$\tilde{R}_n(x) = \sum_{|m| \le q_{s(n)}^4} \hat{R}_n(m) e^{2\pi i m x}.$$

Finally we will define

$$\tilde{\varphi}_n(x) = \frac{\beta}{k_n} \tilde{R}_n(x)$$

The function φ is constructed from the truncated functions $\tilde{\varphi}_n$. We will need information on the derivatives of the $\tilde{\varphi}_n$ in order to prove that φ is C^∞. Later in the proof of the main technical Proposition we will need estimates on the difference between φ_n and $\tilde{\varphi}_n$.

Lemma 4. *The functions φ_n and $\tilde{\varphi}_n$ satisfy the following estimates*

$$\|\varphi_n - \tilde{\varphi}_n\|_0 < C \frac{e^{5n}}{q_{s(n)} q_{s(n)+1}} \qquad C = \frac{32\beta \|\theta''\|_0}{\pi^2}$$

$$\|\tilde{\varphi}_n^{(r)}\|_0 < C(r) \frac{e^{(2r+1)n} q_{s(n)}^{r+5}}{q_{s(n)+1}} \qquad C(r) = 2\beta 8^r \|\theta^{(r)}\|_0$$

where $\|\cdot\|_0$ is the supremum norm and $f^{(}r)$ denotes the r-th derivative of the function f. In particular,

$$\|\tilde{\varphi}_n\|_0 < \frac{\beta}{k_n} + C \frac{e^{5n}}{q_{s(n)} q_{s(n)+1}}.$$

Proof. We begin by observing that

$$R_n(x) - \tilde{R}_n(x) = \sum_{|m|>q_{s(n)}^4} c_m e^{2\pi i m x}$$

and that $|c_m| < \frac{\|R_n''\|_0}{4\pi^2 m^2}$. Summing and putting $\|R_n''\|_0 = (8q_{s(n)}e^{2n})^2\|\theta''\|_0$ we get

$$\begin{aligned}\|R_n - \tilde{R}_n\|_0 &< \frac{2}{4\pi^2 q_{s(n)}^4} 64 q_{s(n)}^2 e^{4n} \|\theta''\|_0 \\ &< \frac{32\|\theta''\|_0}{\pi^2} \frac{e^{4n}}{q_{s(n)}^2}.\end{aligned}$$

We produce a crude estimate on the derivatives of $\tilde{R}_n$ by noticing that

$$(\tilde{R}_n^{(r)})\hat{}(m) = (R_n^{(r)})\hat{}(m) \qquad |m| < q_{s(n)}^4$$

where $(\cdot)\hat{}(m)$ denotes the m-th Fourier coefficient. We estimate

$$(R_n^{(r)})\hat{}(m) \leq \|R_n^{(r)}\|_0 = (8q_{s(n)}e^{2n})^r \|\theta^{(r)}\|_0.$$

Multiplying by βk_n^{-1}, and using the definition of k_n given in (10) we get the required estimates. ☐

Lemma 4 is sufficient to prove that φ is C^∞ and satisfies (6).

Proposition 1. *The function*

$$\varphi = \beta + \sum_{n=1}^{\infty} \tilde{\varphi}_n$$

is a positive C^∞-function on $\mathbb{T}^1$ satisfying (6). *Furthermore if α satisfies* (3) *then φ is analytic.*

Proof for the C^∞ Case. It suffices to show that for every r the sum $\sum_{n=0}^{\infty} \|\tilde{\varphi}_n^{(r)}\|_0$ converges. Fix r. Using the derivative estimate in Lemma 4 and our Liouville assumption (7)

$$\|\varphi_n^{(r)}\|_0 < C(r) e^{(2r+1)n} q_{s(n)}^{r+5-n} < e^{(3r+6)n-n^2}$$

which is summable. Finally using the estimate for $\|\tilde{\varphi}_n\|_0$ from Lemma 4 and (9) we can choose $q_{s(1)}$ such that

$$\frac{\pi}{2}\sum_{n=1}^{\infty} \|\tilde{\varphi}_n\|_0 < \beta$$

which proves that φ satisfies (6). ☐

For the analytic case we need information on the Fourier coefficients of $\tilde{\varphi}_n$.

Lemma 5. *If α admits a sequence of best approximations satisfying* (3) *then*

$$\left|(\tilde{\varphi}_n)\hat{}(m)\right| < C\frac{e^{-|m|}}{2^n}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$.

Proof. First we observe that by construction we have $(\tilde{\varphi}_n)^\wedge(m) = 0$ for all $|m| > q^4_{s(n)}$ so we only need to consider $|m| \leq q^4_{s(n)}$. Combining the estimates of Lemma 7 and 4 we get

$$\left|(\tilde{\varphi}_n)^\wedge(m)\right| < \|\tilde{\varphi}_n\|_0 < \beta\frac{q_{s(n)}e^n}{q_{s(n)+1}} + \frac{Ce^{5n}}{q_{s(n)+1}q_{s(n)}}.$$

Using (9) we can reduce this to

$$\left|(\tilde{\varphi}_n)^\wedge(m)\right| < (\beta + C)\frac{q^4_{s(n)}}{q_{s(n)+1}}$$

which gives us the required result since using (3) we immediately get that

$$\left|(\tilde{\varphi}_n)^\wedge(m)\right| < (\beta + C)\frac{e^{-q^4_{s(n)}}}{2n} < (\beta + C)\frac{e^{-m}}{2n}.$$

□

This lemma gives us the analyticity of φ.

Proof for the Analytic Case. The Fourier coefficients of $\varphi$are given by

$$\hat{\varphi}(m) = \sum_{n=1}^{\infty}(\tilde{\varphi}_n)^\wedge(m).$$

From Lemma 5 we have immediately that

$$|\hat{\varphi})(m)| < e^{-|m|}$$

which suffices to show that φ is analytic. □

The remaining part of the proof, showing (2) admits solutions only for $\lambda = n\beta^{-1}$, is the same in both cases. The stronger assumption (3) is necessary only to show that φ is analytic.

5. Proof of Theorems 1 and 2

The function $\tilde{\varphi}_n$ is a trigonometric polynomial with zero mean. Hence there is a solution to

$$\psi_n\big(R_\alpha(x)\big) - \psi_n(x) = \tilde{\varphi}_n(x), \tag{11}$$

which is in fact a trigonometric polynomial given by

$$\psi_n(x) = \hat{\psi_n}(0) + \sum_{0<|m|\leq q^4_{s(n)}} \frac{\widehat{\tilde{\varphi}_n}(m)}{e^{2\pi i m\alpha} - 1}e^{2\pi i m x}. \tag{12}$$

Notice that ψ_n is determined only up to an additive constant. Our construction of $\tilde{\varphi}_n$ is such that the transfer function ψ_n is essentially like a step function (with two discontinuities) taking the values 0 and β, see Figure 2. The crucial consequence is that while $\sum\psi_n$ diverges, the product of the functions $e^{2\pi i\lambda\psi_n(x)}$ will be well-defined if and only if λ is a multiple of β^{-1}.

5.1. **The shape of the function ψ_n.** Define

$$\begin{aligned}
B_1 &= \bigcup_{k \le q_{s(n)}-1} \Big[\frac{k}{q_{s(n)}}, \frac{k}{q_{s(n)}} + \frac{1-e^{-2n}}{8q_{s(n)}}\Big],\\
B_2 &= \bigcup_{k \le q_{s(n)}-1} \Big[\frac{k}{q_{s(n)}} + \frac{1+e^{-2n}}{8q_{s(n)}}, \frac{k}{q_{s(n)}} + \frac{7-e^{-2n}}{8q_{s(n)}}\Big],\\
B_3 &= \bigcup_{k \le q_{s(n)}-1} \Big[\frac{k}{q_{s(n)}} + \frac{7+e^{-2n}}{8q_{s(n)}}, \frac{k+1}{q_{s(n)}}\Big].
\end{aligned}$$

We have

$$\mu(\mathbb{T}^1 - \cup_{i=1}^3 B_i) = \frac{e^{-2n}}{2}. \tag{13}$$

We will state now the central proposition for the proof of Theorems 1 and 2.

Proposition 2 (Structure of the Transfer Function)**.** *Let ψ_n be given by* (12) *with $\psi_n(0) = 0$. There exists a summable sequence $\{\epsilon_n\}$ such that*

(1) *For any $x \in B_1 \cup B_3$,* $\quad |\psi_n(x)| \le \epsilon_n$,

(2) *For any $x \in B_2$,* $\quad |\psi_n(x) - \beta| \le \epsilon_n$.

Unfortunately it is very difficult to obtain information about the shape of ψ_n from (12). Instead, we will estimate the values of ψ_n along the first $q_{s(n)+1}$ points of the orbit of 0 using (11) and then interpolate between these points. Indeed, for every $m \ge 0$ one has

$$\psi_n(R_\alpha^m(0)) = \sum_{i=0}^{m-1} \tilde{\varphi}_n\big(R_\alpha^i(0)\big). \tag{14}$$

The function $\tilde{\varphi}_n$ being an approximation of the function φ_n (see Definition 3, we introduce the sequence

$$\bar{\psi}_n\big(R_\alpha^m(0)\big) := \sum_{i=0}^{m-1} \varphi_n\big(R_\alpha^i(0)\big). \tag{15}$$

From Lemma 4 and the fact that we sum over at most $q_{s(n)+1}$ points we get for every $m \le q_{s(n)+1}$

$$\Big|\bar{\psi}_n\big(R_\alpha^m(0)\big) - \psi_n\big(R_\alpha^m(0)\big)\Big| < \frac{Ce^{5n}}{q_{s(n)}}. \tag{16}$$

The analysis of the dynamics of R_α and the shape of the function φ_n will enable us to prove the following estimates on the $\bar{\psi}_n\big(R_\alpha^m(0)\big)$:

Lemma 6 (Consequence of the dynamics of R_α)**.** *We have for $m \le q_{s(n)+1}$:*

(1) *If $R_\alpha^m(0) \in B_1 \cup B_3$, then* $\quad |\bar{\psi}_n\big(R_\alpha^m(0)\big)| \le e^{-n}\beta$,

(2) *If $R_\alpha^m(0) \in B_2$, then* $\quad |\bar{\psi}_n\big(R_\alpha^m(0)\big) - \beta| \le e^{-n}\beta$.

In the next lemma we will describe the shape of φ_n (see Fig. 1).

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FIGURE 2. The function ψ_n on $[0, q_{s(n)}^{-1}]$. A_i and L are defined in Definition 5.

Definition 5. *Define the following covering of* $\mathbb{T}^1$

$$A_1 = \Big[0, \frac{1-e^{-2n}}{8q_{s(n)}}\Big]$$
$$A_2 = \Big[\frac{1+e^{-2n}}{8q_{s(n)}} + \frac{k_n}{q_{s(n)+1}}, \frac{7-e^{-2n}}{8q_{s(n)}}\Big],$$
$$A_3 = \Big[\frac{7+e^{-2n}}{8q_{s(n)}} + \frac{k_n}{q_{s(n)+1}}, \frac{1}{q_{s(n)}}\Big],$$
$$L = \Big(\frac{1-2e^{-2n}}{8q_{s(n)}}, \frac{1+2e^{-2n}}{8q_{s(n)}} + \frac{k_n}{q_{s(n)+1}}\Big) \cup \Big(\frac{7-2e^{-2n}}{8q_{s(n)}}, \frac{7+2e^{-2n}}{8q_{s(n)}} + \frac{k_n}{q_{s(n)+1}}\Big),$$
$$I = [\frac{1}{q_{s(n)}}, 1].$$

We summarize the crucial properties of φ_n:

Lemma 7. *φ_n is a C^∞ function on $\mathbb{T}^1$ with the following properties*

5.1 $\|\varphi_n\|_0 = \dfrac{\beta}{k_n}$.

5.2 $\varphi_n(x) = 0$ *for* $x \in A_1 \cup A_2 \cup A_3 \cup I$.

5.3 $\varphi_n(x) = \frac{\beta}{k_n}$ *for* $x \in \left[\frac{1}{8q_{s(n)}}, \frac{1}{8q_{s(n)}} + \frac{k_n}{q_{s(n)+1}}\right]$.

5.4 $\varphi_n(x) = -\frac{\beta}{k_n}$ *for* $x \in \left[\frac{7}{q_{s(n)}}, \frac{7}{8q_{s(n)}} + \frac{k_n}{q_{s(n)+1}}\right]$.

The proof of this lemma is straightforward from the Definition 3 of φ_n.

Proof of Lemma 6. Since $q_{s(n)}$ is a best approximation of α we have if we assume that $q_{s(n)}$ is even (the other case being similar)

$$\alpha = \frac{p_{s(n)}}{q_{s(n)}} + \frac{1}{q_{s(n)}q_{s(n)+1}} + h.o.t. \tag{17}$$

where *h.o.t.* stands for higher order terms bounded by $(q_{s(n)+1})^{-2}$. From (17) and Lemma 7 we deduce (a)–(d) below:

(a) For $k < (1-e^{-2n})q_{s(n)+1}/8$, $R_\alpha^k(0) \in A_1 \cup I$, hence $\varphi_n(R_\alpha^k(0)) = 0$.

(b) For $\frac{q_{s(n)+1}}{8} < k < \frac{q_{s(n)+1}}{8} + k_n q_{s(n)}$, we have $R_\alpha^k(0) \in [\frac{1}{8q_{s(n)}}, \frac{1}{8q_{s(n)}} + \frac{k_n}{q_{s(n)+1}}]$ if $k = pq_{s(n)}$ and $R_\alpha^k(0) \in I$ if not.

(c) If $k \in [\frac{1-e^{-2n}}{8} q_{s(n)+1}, \frac{q_{s(n)+1}}{8}] \cup [\frac{q_{s(n)+1}}{8} + k_n q_{s(n)}, \frac{1+e^{-2n}}{8} q_{s(n)+1} + k_n q_{s(n)}]$, then $R_\alpha^k(0) \in L$ if $k = pq_{s(n)}$ and $R_\alpha^k(0) \in I$ if not.

(d) If $\frac{1+e^{-2n}}{8} q_{s(n)+1} + k_n q_{s(n)} < k < \frac{7-e^{-2n}}{8} q_{s(n)+1}$, then $R_\alpha^k(0) \in A_2 \cup I$ and $\varphi_n(R_\alpha^k(0)) = 0$.

From (17) we get that $R_\alpha^m(0) \in B_1$ if $m < (1-e^{-2n})q_{s(n)+1}/8$, but (a) then implies that $\bar{\psi}_n\big(R_\alpha^m(0)\big) = 0$, which proves the first part of (1) in Lemma 6.

When $R_\alpha^m(0) \in B_2$, i.e. when $m \in [(1+e^{-2n})q_{s(n)+1}/8 + k_n q_{s(n)}, (7-e^{-2n})q_{s(n)+1}/8]$ we have from (a)–(d) all together with Lemma 7 (5.1–5.3) that

$$k_n\frac{\beta}{k_n} - 2\frac{4e^{-2n}}{8}\frac{q_{s(n)+1}}{q_{s(n)}}\frac{\beta}{k_n} \le \sum_{k=0}^{m-1}\varphi_n(R_\alpha^k(0)) \le k_n\frac{\beta}{k_n} + 2\frac{4e^{-2n}}{8}\frac{q_{s(n)+1}}{q_{s(n)}}\frac{\beta}{k_n},$$

which finishes the proof of (2) in Lemma 6. We prove the second part of (1) in a similar way using in addition (5.4) of Lemma 7 .

□

To obtain Proposition 2 from Lemma 6 we will need the following information on the derivative of ψ_n:

Lemma 8. *There exists $K > 0$ such that*

$$\|\psi_n'\|_0 < Ke^{2n+1}q_{s(n)}^{10}$$

Proof. Differentiating (12) gives

$$\psi_n'(x) = \sum_{0<|m|<q_{s(n)}^4} \frac{\widehat{\tilde{\varphi}_n'}(m)}{e^{2\pi i m\alpha}-1}e^{2\pi i m x}.$$

Now $|e^{2\pi i m\alpha}-1| > 4\inf_{p\in\mathbb{Z}}|m\alpha - p|$. However since $p_{s(n)+1}/q_{s(n)+1}$ is a best return and $m < q_{s(n)+1}$ we have $\inf_{p\in\mathbb{Z}}|m\alpha-p| > (2q_{s(n)+1})^{-1}$. We can estimate $\widehat{\tilde{\varphi}_n'}(m)$ by $\|\tilde{\varphi}_n'\|_0$. Applying the estimate for $\|\tilde{\varphi}_n'\|_0$ from Lemma 4 yields the result. □

Proof of Proposition 2. Observe first that for any $x \in \mathbb{T}^1$, there exists an $m \le q_{s(n)+1}$ such that $|x - R_\alpha^m(0)| \le 2/q_{s(n)+1}$. Then for $x \in B_1 \cup B_3$, we obtain from Lemma 6 and (16) that

$$\begin{aligned} |\psi_n(x)| &\le |\psi_n(x) - \psi_n(R_\alpha^m(0))| + |\psi_n(R_\alpha^m(0)) - \bar{\psi}_n\big(R_\alpha^m(0)\big)| + |\bar{\psi}_n\big(R_\alpha^m(0)\big)| \\ &\le 2/q_{s(n)+1}Ke^{2n+1}q_{s(n)}^{10} + \frac{Ce^{5n}}{q_{s(n)}} + e^{-n}\beta, \end{aligned}$$

which proves the first part of Proposition 2, the other part being obtained in a similar fashion.

□

5.2. **Proof of the existence of an eigenfunction.** As a corollary of our technical proposition on the shape of the transfer function ψ_n we can establish that β^{-1} is an eigenvalue of the special flow over R_α and under φ.

Proposition 3. *The sequence*

$$h_n(x) = \prod_{i=1}^{n}\exp\big(2\pi i\beta^{-1}\psi_n(x)\big)$$

converges in L_1 to a solution to (2) *for $\lambda = \beta^{-1}$.*

Proof. For the convergence of h_n it suffices to show that $\{h_n\}$ is Cauchy.

$$\|h_n(x) - h_{n+1}(x)\|_1 = \|1 - \exp\big(2\pi i\beta^{-1}\psi_{n+1}(x)\big)\|_1.$$

Hence it suffices to show that $\|1 - \exp\big(2\pi i\beta^{-1}\psi_{n+1}(x)\big)\|_1$ is summable. We have

$$|1 - \exp\big(2\pi i\beta^{-1}\psi_n(x)\big)| \le 2\pi \inf_{p\in\mathbb{Z}} |\beta^{-1}\psi_n(x) - p|.$$

Proposition 2 implies for $x \in B_1 \cup B_2 \cup B_3$ that

$$\inf_{p\in\mathbb{Z}} |\beta^{-1}\psi_n(x) - p| < (1 + \beta^{-1})\epsilon_n. \tag{18}$$

For $x \in \mathbb{T}^1 - \cup_{i=1}^3 B_i$ we use

$$|1 - \exp\big(2\pi i\beta^{-1}\psi_n(x)\big)| \le 2, \tag{19}$$

and recall (13), that is $\mu(\mathbb{T}^1 - \cup_{i=1}^3 B_i) \le \frac{1}{2}e^{-2n}$.

From (18) and (19) we deduce that $\|1 - \exp\big(2\pi i\beta^{-1}\psi_{n+1}(x)\big)\|_1 \le e^{-2n} + (1 + \beta^{-1})\epsilon_n$. In conclusion, we have L_1 convergence of h_n to some function h. Moreover, by definition of ψ_n we have for every n

$$h_n(R_\alpha(x)) = e^{2\pi i\beta^{-1}\sum_{i=1}^n \tilde{\varphi}_i(x)} h_n(x).$$

Since $\beta^{-1}\varphi(x) = 1 + \beta^{-1}\sum_{i=1}^\infty \tilde{\varphi}_i(x)$ (with convergence in the C^∞ topology) we go to the limit as n goes to infinity and get the required condition

$$h(R_\alpha(x)) = e^{2\pi i\beta^{-1}\varphi(x)} h(x).$$

□

5.3. **Proof of the uniqueness of the one parameter group of eigenvalues.** There is a classical criterion used to show that a certain cohomological equation has no measurable solutions:

Lemma 9. *Let T^t be a special flow over an ergodic transformation T on a Lebesgue space L and under a function φ. Suppose $\{m_n\}$ is a sequence of times such that*

$$T^{m_n} \to \mathrm{Id}$$

in probability. If

$$e^{2\pi i\lambda S_{m_n}\varphi(x)} \not\to 1$$

in probability then λ is not an eigenvalue of T^t. Here $S_m\varphi(x) = \sum_{i=0}^{m-1}\varphi(T^i x)$.

Proof. if λ is an eigenvalue for T^t then our earlier condition shows

$$e^{2\pi i\lambda\varphi(x)} = \frac{h(Tx)}{h(x)}$$

for some measurable h. Iterating these expressions gives us

$$e^{2\pi i\lambda S_m\varphi(x)} = \frac{h(T^m x)}{h(x)}.$$

Now along the subsequence $\{m_n\}$ the right-hand side converges to 1 in probability. □

We use the foregoing criterion and Proposition 2 to prove the absence of eigenvalues other than the multiples of β^{-1}.

Proposition 4. *Let* $m_n = \lceil \frac{q_{s(n)+1}}{8q_{s(n)}} \rceil q_{s(n)}$. *For* λ *which is not a multiple of* β^{-1}

$$e^{2\pi i \lambda S_{m_n}\varphi(x)} \not\to 1$$

in probability.

The sequence m_n satisfies the hypothesis of Lemma 9 since for any $x \in \mathbb{T}^1$, one has

$$d\big(R_\alpha^{m_n}(x), x\big) < \frac{1}{4q_{s(n)}}. \tag{20}$$

Thus Proposition 4 proves that there a no eigenvalues of the special flow over R_α and under φ other than multiples of β^{-1}.

Proof. We begin by observing that

$$\big\|e^{2\pi i \lambda S_{m_n}\varphi(x)} - 1\big\|_1 \geq 4\big\|\inf_{p\in\mathbb{Z}} |\lambda S_{m_n}\varphi(x) - p|\big\|_1$$

Next, we break $S_{m_n}\varphi(x)$ up into three pieces (plus a constant) as follows

$$S_{m_n}\varphi(x) = m_n\beta + \sum_{i=1}^{n-1} S_{m_n}\tilde{\varphi}_i(x) + S_{m_n}\tilde{\varphi}_n + \sum_{i=n+1}^{\infty} S_{m_n}\tilde{\varphi}_i(x).$$

We have

$$\begin{aligned}\inf_{p\in\mathbb{Z}} |\lambda S_{m_n}\varphi(x) - p| > \inf_{p\in\mathbb{Z}}&\Big|m_n\lambda\beta + \lambda S_{m_n}\tilde{\varphi}_n(x) - p\Big| \\ &- \lambda\Big\|\sum_{i=1}^{n-1} S_{m_n}\tilde{\varphi}_i\Big\|_0 - \lambda\Big\|\sum_{i=n+1}^{\infty} S_{m_n}\tilde{\varphi}_i\Big\|_0.\end{aligned}$$

By construction $\|\tilde{\varphi}_i\|_0$ is comparable to $k_i^{-1} = o(1/q_{s(i)})$. Hence, if we choose a_n in (8) sufficiently large we can ensure that

$$\Big\|\sum_{i=n+1}^{\infty} S_{m_n}\tilde{\varphi}_i\Big\|_0 \leq m_n \sum_{i=n+1}^{\infty} \frac{1}{q_{s(i)}} \leq e^{-n}.$$

To estimate the norm of the lower frequencies we use the relation

$$S_{m_n}\tilde{\varphi}_i(x) = \psi_i\big(R_\alpha^{m_n}(x)\big) - \psi_i(x)$$

which comes from (11).

Lemma 8 and (20) imply that for every $x \in \mathbb{T}^1$ we have

$$\Big|\sum_{i=1}^{n-1}\Big(\psi_i\big(R_\alpha^{m_n}(x)\big) - \psi_i(x)\Big)\Big| < nKe^{2n-1}q_{s(n-1)}\frac{1}{4q_{s(n)}},$$

again, the choice of a_{n-1} in (8) sufficiently large ensures that this is less than e^{-n}.

Finally, in light of Proposition 2 and the fact that $|||m_n\alpha|||$ is equivalent to $1/8q_{s(n)}$ we deduce that $S_{m_n}\tilde{\varphi}_n(x) = \psi_n\big(R_\alpha^{m_n}(x)\big) - \psi_n(x)$ takes essentially each of the values 0, β and $-\beta$ on more than a proportion 1/8 of $\mathbb{T}^1$. Therefore

$$\big\|\inf_{p\in\mathbb{Z}} |m_n\lambda\beta + \lambda S_{m_n}\tilde{\varphi}_n(x) - p|\big\|_1 \geq \frac{1}{8}\inf_{p\in\mathbb{Z}} |\lambda\beta - p|,$$

which is non-zero when λ is not a multiple of β^{-1}. Thus

$$\big\|e^{2\pi i\lambda S_{m_n}\varphi(x)} - 1\big\|_1 \not\to 0.$$

□

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