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ATTRACTED BY AN ELLIPTIC FIXED POINT

by

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À notre mentor et ami Jean-Christophe Yoccoz

Abstract. — We give examples of symplectic diffeomorphisms of $\mathbb{R}^6$ for which the origin is a non-resonant elliptic fixed point which attracts an orbit.

Résumé **(Attiré par un point fixe elliptique).** — Nous donnons des exemples de difféomorphismes symplectiques de $\mathbb{R}^6$ pour lesquels l'origine est un point fixe elliptique non résonant qui attire une orbite.

1. Introduction

Consider a symplectic diffeomorphism of $\mathbb{R}^{2n}$ (for the canonical symplectic form) with a fixed point at the origin. We say that the fixed point is elliptic of frequency vector $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ if the linear part of the diffeomorphism at the fixed point is conjugate to the rotation map

$$S_\omega : (\mathbb{R}^2)^n \circlearrowleft, \qquad S_\omega(s_1, \dots, s_n) := (R_{\omega_1}(s_1), \dots, R_{\omega_n}(s_n)).$$

Here, for $\beta \in \mathbb{R}$, R_β stands for the rigid rotation around the origin in $\mathbb{R}^2$ with rotation number β. We say that the frequency vector ω is non-resonant if for any $k \in \mathbb{Z}^n - \{0\}$ we have $(k, \omega) \notin \mathbb{Z}$, where $(\cdot\,,\cdot)$ stands for the Euclidean scalar product.

It is easy to construct symplectic diffeomorphisms with orbits attracted by a resonant elliptic fixed point. For instance, the time-1 map of the flow generated by the Hamiltonian function $H(x, y) = y(x^2 + y^2)$ in $\mathbb{R}^2$ has a saddle-node type fixed point, at which the linear part is zero, which attracts all the points on the positive part of the x-axis. The situation is much subtler in the non-resonant case.

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Our goal in this paper is to construct an example of symplectic diffeomorphism with an orbit converging to an elliptic non-resonant fixed point. Note that, in such an example, the inverse symplectomorphism has a Lyapunov unstable fixed point.

The Anosov-Katok construction [**1**] of ergodic diffeomorphisms by successive conjugations of periodic rotations of the disk gives examples of smooth area preserving diffeomorphisms with non-resonant elliptic fixed points at the origin that are Lyapunov unstable. The method also yields examples of ergodic symplectomorphisms with non-resonant elliptic fixed points in higher dimensions. These constructions obtained by the successive conjugation technique have totally degenerate fixed points since they are C^∞-tangent to a rotation S_ω at the origin.

In the non-degenerate case, R. Douady gave examples in [**4**] of Lyapunov unstable elliptic points for smooth symplectic diffeomorphisms for any $n \geq 2$, for which the Birkhoff normal form has non-degenerate Hessian at the fixed point but is otherwise arbitrary. Prior examples for $n = 2$ were obtained in [**5**] (note that by KAM theory, a non-resonant elliptic fixed point of a smooth area preserving surface diffeomorphism that has a non zero Birkhoff normal form is accumulated by invariant quasi-periodic smooth curves—see [**14**]—, hence, for $n = 1$, non-degeneracy implies that the point is Lyapunov stable).

In both of the above examples, there is no claim about the existence of an orbit converging to the fixed point for the forward or backward dynamics. In fact, in the Anosov-Katok examples, a sequence of iterates of the diffeomorphism converges uniformly to Identity, hence every orbit is recurrent and no forward orbit can converge to the origin, besides the origin itself. As for the non-degenerate examples of Douady and Le Calvez, their Lyapunov instability is deduced from the existence of a sequence of points that converge to the fixed point and whose orbits travel along a simple resonance away from the fixed point, not from the existence of one particular orbit.

In this paper, we will construct an example of a Gevrey diffeomorphism possessing an orbit which converges to a fixed point.Recall that, given a real $\alpha \geq 1$, Gevrey-α regularity is defined by the requirement that the partial derivatives exist at all (multi)orders ℓ and are bounded by $CM^{|\ell|}\,|\ell|!^\alpha$ for some C and M (when $\alpha = 1$, this simply means analyticity); upon fixing a real $L > 0$ which essentially stands for the inverse of the previous M, one can define a Banach algebra $\big(G^{\alpha,L}(\mathbb{R}^{2n}), \|\,.\,\|_{\alpha,L}\big)$.

We set $X := (\mathbb{R}^2)^3$ and denote by $\mathscr{U}^{\alpha,L}$ the set of all Gevrey-(α, L) symplectic diffeomorphisms of X which fix the origin and are C^∞-tangent to Id at the origin. We refer to Appendix for the precise definition of $\mathscr{U}^{\alpha,L}$ and of a distance $\operatorname{dist}(\Phi, \Psi) = \|\Phi - \Psi\|_{\alpha,L}$ which makes it a complete metric space. We will prove the following.

Theorem A. — *Fix $\alpha > 1$ and $L > 0$. For each $\gamma > 0$, there exist a non-resonant vector $\omega \in \mathbb{R}^3$, a point $z \in X$, and a diffeomorphism $\Psi \in \mathscr{U}^{\alpha,L}$ such that $\|\Psi - \mathrm{Id}\|_{\alpha,L} \leq \gamma$ and $T = \Psi \circ S_\omega$ satisfies $T^n(z) \underset{n\to+\infty}{\longrightarrow} 0$.*

We do not know how to produce real analytic examples. After the first version of the present work was completed, the first real analytic symplectomorphisms with Lyapunov unstable non-resonant elliptic fixed points were constructed in [**6**] (but with no orbits asymptotic to the fixed point). For other instances of the use of Gevrey regularity with symplectic or Hamiltonian dynamical systems, see e.g., [**15**], [**11**], [**12**], [**13**], [**10**], [**3**].

Our construction easily extends to the case where $X = (\mathbb{R}^2)^n$ with $n \geq 3$, however we do not know how to adapt the method to the case $n = 2$. As for the case $n = 1$, there may well be no regular examples at all. Indeed if the rotation frequency at the fixed point is Diophantine, then a theorem by Herman (see [**7**]) implies that the fixed point is surrounded by invariant quasi-periodic circles, and thus is Lyapunov stable. The same conclusion holds by Moser's KAM theorem if the Birkhoff normal form at the origin is not degenerate [**14**]. In the remaining case of a degenerate Birkhoff normal form with a Liouville frequency, there is evidence from [**2**] that the diffeomorphism should then be rigid in the neighborhood of the origin, that is, there exists a sequence of integers along which its iterates converge to Identity near the origin, which clearly precludes the convergence to the origin of an orbit.

Similar problems can be addressed where one searches for Hamiltonian diffeomorphisms (or vector fields) with orbits whose α-limit or ω-limit have large Hausdorff dimension (or positive Lebesgue measure) and in particular contain families of non-resonant invariant Lagrangian tori instead of a single non-resonant fixed point. A specific example for Hamiltonian flows on $(\mathbb{T} \times \mathbb{R})^3$ is displayed in [**9**], while a more generic one has been announced in [**8**]. In these examples, the setting is perturbative and the Hamiltonian flow is non-degenerate in the neighborhood of the tori. The methods involved there are strongly related to Arnold diffusion and are completely different from ours.

2. Preliminaries and outline of the strategy

From now on we fix $\alpha > 1$ and $L > 0$. We also pick an auxiliary $L_1 > L$. For $z \in \mathbb{R}^2$ and $\nu > 0$, we denote by $B(z, \nu)$ the closed ball relative to $\|\,.\,\|_\infty$ centered at z with radius ν. Since $\alpha > 1$, we have

Lemma 2.1. — *There is a real $c = c(\alpha, L_1) > 0$ such that, for any $z \in \mathbb{R}^2$ and $\nu > 0$, there exists a function $f_{z,\nu} \in G^{\alpha,L_1}(\mathbb{R}^2)$ which satisfies*

(a) $0 \leq f_{z,\nu} \leq 1$,

(b) $f_{z,\nu} \equiv 1$ *on* $B(z, \nu/2)$,

(c) $f_{z,\nu} \equiv 0$ *on* $B(z, \nu)^c$,

(d) $\|f_{z,\nu}\|_{\alpha,L_1} \leq \exp(c\,\nu^{-\frac{1}{\alpha-1}})$.

Proof. — Use Lemma 3.3 of [**12**]. □

We now fix an arbitrary real $R > 0$ and pick an auxiliary function $\eta_R \in G^{\alpha,L_1}(\mathbb{R})$ which is identically 1 on the interval $[-2R, 2R]$, identically 0 outside $[-3R, 3R]$, and everywhere non-negative. We then define $g_R:\mathbb{R}^2 \to \mathbb{R}$ by the formula

$$g_R(x,y) := xy\,\eta_R(x)\,\eta_R(y). \tag{2.1}$$

The following diffeomorphisms will be of constant use in this paper:

Definition 2.1. — *For $i \neq j \in \{1,2,3\}$, $z \in \mathbb{R}^2$ and $\nu > 0$, we denote by $\Phi_{i,j,z,\nu}$ the time-one map of the Hamiltonian flow generated by the function $\exp(-c\,\nu^{-\frac{2}{\alpha-1}})f_{z,\nu} \otimes_{i,j} g_R$, where $f_{z,\nu} \otimes_{i,j} g_R:X \to \mathbb{R}$ stands for the function*

$$s = (s_1, s_2, s_3) \mapsto f_{z,\nu}(s_i) g_R(s_j).$$

In the above definition, our convention for the Hamiltonian vector field generated by a function H is $X_H = \sum(-\frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i})$. Note that the Hamiltonian $\exp(-c\,\nu^{-\frac{2}{\alpha-1}})f_{z,\nu} \otimes_{i,j} g_R$ can be viewed as a compactly supported function of s_i and s_j, hence it generates a complete vector field and Definition 2.1 makes sense. Actually, any $H \in G^{\alpha,L_1}(X)$ has bounded partial derivatives, hence X_H is always complete; the flow of X_H is made of Gevrey maps for which estimates are given in Appendix A.2. In the case of $\Phi_{i,j,z,\nu}$, for ν small enough we have

$$\Phi_{i,j,z,\nu} \in \mathscr{U}^{\alpha,L} \quad \text{and} \quad \|\Phi_{i,j,z,\nu} - \mathrm{Id}\|_{\alpha,L} \leq K\exp(-c\,\nu^{-\frac{1}{\alpha-1}}), \tag{2.2}$$

with $K := C\|g_R\|_{\alpha,L_1}$, where C is independent from i,j,z,ν and stems from (A.6). Here are the properties which make the $\Phi_{i,j,z,\nu}$'s precious. To alleviate the notations, we state them for $\Phi_{2,1,z,\nu}$ but similar properties hold for each diffeomorphism $\Phi_{i,j,z,\nu}$.

Lemma 2.2. — *Let $z \in \mathbb{R}^2$ and $\nu > 0$. Then $\Phi_{2,1,z,\nu}$ satisfies:*

(a) *For every $(s_1,s_2,s_3) \in X$ such that $s_2 \in B(z,\nu)^c$,*

$$\Phi_{2,1,z,\nu}(s_1,s_2,s_3) = (s_1,s_2,s_3).$$

(b) *For every $x_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}^2$ and $s_3 \in \mathbb{R}^2$,*

$$\Phi_{2,1,z,\nu}((x_1,0),s_2,s_3) = ((\widetilde{x}_1,0),s_2,s_3) \ \textit{with} \ |\widetilde{x}_1| \leq |x_1|.$$

(c) *For every $x_1 \in [-2R,2R]$, $s_2 \in B(z,\nu/2)$ and $s_3 \in \mathbb{R}^2$,*

$$\Phi_{2,1,z,\nu}((x_1,0),s_2,s_3) = ((\widetilde{x}_1,0),s_2,s_3) \ \textit{with} \ |\widetilde{x}_1| \leq \kappa\,|x_1|,$$

where $\kappa := 1 - \frac{1}{2}\exp(-c\,\nu^{-\frac{2}{\alpha-1}})$.

Hence, a map like $\Phi_{2,1,z_2,\nu_2}$ will preserve the x_1-axis and "descend" orbits towards the origin on this axis, while keeping the other two variables frozen (item (b)). However, it is only when the second variable is inside the ball of radius ν_2 around z_2 that $\Phi_{2,1,z_2,\nu_2}$ will effectively bring down a point of the x_1-axis towards the origin (item (c)). Let us roughly summarize this by saying that $\Phi_{2,1,z_2,\nu_2}$ acts as an elevator on the first x-axis, that never goes up and that effectively goes down when the second variable is in some given ball, that we call "activating". Moreover, if the second variable is securely outside the activating ball, then $\Phi_{2,1,z_2,\nu_2}$ is completely inactive

(Identity). Finally, our elevator is close to Identity when the parameter ν_2 is close to zero.

Proof of Lemma 2.2. — The dynamics of the flow generated by $f_{z,\nu} \otimes_{2,1} g_R$ can easily be understood from those of the flows generated by $f_{z,\nu}$ alone on the second factor $\mathbb{R}^2$ and by g_R alone on the first factor $\mathbb{R}^2$. Indeed, for any functions f and g on $\mathbb{R}^2$,

$$\Phi^{f\otimes_{2,1} g}(s_1, s_2, s_3) = \big(\Phi^{f(z_2)g}(s_1), \Phi^{g(z_1)f}(s_2), s_3\big),$$

where Φ^h denotes the time one map associated to the Hamiltonian h, hence

$$\Phi_{2,1,z,\nu}(s_1, s_2, s_3) = \big(\Phi^{\delta f_{z,\nu}(s_2) g_R}(s_1), \Phi^{\delta g_R(s_1) f_{z,\nu}}(s_2), s_3\big)$$

with $\delta := \exp(-c\,\nu^{-\frac{2}{\alpha-1}})$. This immediately yields (a).

Suppose $s_1 = (x_1, 0)$. We get $g_R(s_1) = 0$ and $\Phi^{\tau g_R}(x_1, 0) = \big(\widetilde{x}_1(\tau), 0\big)$ with $\widetilde{x}_1(0) = x_1$ and $\frac{\mathrm{d}\widetilde{x}_1}{\mathrm{d}\tau} = -\widetilde{x}_1 \eta_R(\widetilde{x}_1)$, hence $|\widetilde{x}_1(\tau)| \leq |x_1|$ for any $\tau \geq 0$, and (b) follows using $\tau := \delta f_{z,\nu}(s_2)$.

If moreover $x_1 \in [-2R, 2R]$, then $\tilde{x}_1(\tau) = \mathrm{e}^{-\tau} x_1$. We conclude by observing that $s_2 \in B(z, \nu/2)$ implies $\tau = \delta f_{z,\nu}(s_2) \leq 1$, whence $\mathrm{e}^{-\tau} \leq 1 - \frac{1}{2}\tau$. $\square$

From now on, we denote simply by $|\,.\,|$ the $\|\,.\,\|_\infty$ norm in $\mathbb{R}^2$ or in $X = \mathbb{R}^6$, and by $B(s, \rho)$ the corresponding closed ball centered at s with radius ρ (the context will tell whether it is in $\mathbb{R}^2$ or $\mathbb{R}^6$).

Heuristics of the synchronized attraction scheme. We describe now the attraction mechanism towards the origin, that will be carried out in Section 3. It is based on the use of longer and longer compositions of regularly alternating 'elevators', more precisely compositions of a large number of maps of the form

$$\Phi_{1,3,z_1,\nu_1} \circ \Phi_{3,2,z_3,\nu_3} \circ \Phi_{2,1,z_2,\nu_2}$$

(with an inductive choice of the parameters z_i and ν_i) followed by rigid rotations $S^{(\omega_n)}$, with an appropriate sequence of resonant frequencies ω_n.

Suppose that z_2 is inside the activating ball of some elevator $\Phi_{2,1}$, which is hence actively descending z_1 on the x_1-axis. Suppose also that, simultaneously, some $\Phi_{3,2}$ is descending z_2. At some point, z_2 will exit the activating ball of $\Phi_{2,1}$, which then becomes completely inactive. The variable z_1 stops its descent and will just be rotating due to the rotation $S^{(\omega_n)}$. A $\Phi_{1,3}$ that is active at this height of z_1 can then be used to descend z_3. As z_3 goes down, $\Phi_{3,2}$ becomes inactive and z_2 will henceforth only rotate. This allows to introduce a new $\Phi_{2,1}$ which is active at this new height of z_2. An alternating procedure of the three types of elevators can thus be put in place. At each moment in the attraction procedure, one variable is just rotating and, each time it enters an activating ball, it drives down strictly another variable. The third variable in the meantime must just not go up. This is the content of Lemma 3.1 below, where we see that, when a composition T_1 of $\Phi_{2,1}$ and $\Phi_{1,3}$ is active, we have z_1 strictly going down, z_3 just not going up, and z_2 just rotating.

However, for this description to hold, a fine synchronization between the three components ω_i of the frequency of the rotation is required, guaranteeing for example that when one variable enters an activating ball, the corresponding variable that it should bring down indeed happens to be on its own x-axis. For example, when we use T_1, we take the ω_i's rational, with the denominator of ω_2 being a multiple of the denominator of ω_1 that is itself a multiple of the denominator of ω_3. Of course, this constrains us to deal with resonant frequencies, which is why we use a sequence of resonant $\omega^{(n)}$'s. They will be chosen so as to suit the fine-tuned alternating attraction mechanism that we just tried to convey, while converging to a non-resonant frequency.

Observe that if ω_1 and ω_2 were rationals with the same denominator, then it would be possible to get an attraction scheme by just alternating maps of the form $\Phi_{2,1}$ and $\Phi_{1,2}$. With non-resonant frequencies however, we could not put up the attraction scheme with just two variables and our method, as is, does not yield a statement similar to Theorem A on $\mathbb{R}^4$, let alone $\mathbb{R}^2$.

In summary, Theorem A is obtained by an inductive construction of the required Ψ, z and ω:

— The diffeomorphism Ψ in Theorem A will be obtained as an infinite product (for composition) of diffeomorphisms of the form $\Phi_{i,j,z,\nu}$, with smaller and smaller values of ν so as to derive convergence in $\mathscr{U}^{\alpha,L}$ from (2.2).

— On the other hand, the initial condition z will be obtained as the limit of a sequence contained in the ball $B(0,R) \subset X$.

— As for the non-resonant frequency vector ω in Theorem A, it will be obtained as a limit of vectors with rational coordinates with larger and larger denominators, so as to make possible a kind of "orbit synchronization" at each step of the construction.

3. The attraction mechanism

From now on, for any three integers q_1, q_2, q_3, we use the notation "$q_1 \mid q_2$" to indicate the existence of $k \in \mathbb{Z}$ such that $q_1 k = q_2$, and "$q_1 \mid q_2 \mid q_3$" when $q_1 \mid q_2$ and $q_2 \mid q_3$.

Starting from a point $z = ((x_1,0),(x_2,0),(x_3,0))$, the mechanism of attraction of the point to the origin is an alternation between bringing closer to zero the x_1,x_2 or x_3 coordinates when all the coordinates of the point come back to the horizontal axes. The main ingredient is the following lemma, where we use shortcut notation $\Phi_{i,j,x,\nu}$ for $\Phi_{i,j,(x,0),\nu}$.

Lemma 3.1. — *Let $\omega = (p_1/q_1, p_2/q_2, p_3/q_3) \in \mathbb{Q}^3$ with p_i, q_i coprime positive integers and*

$$z = ((x_1,0),(x_2,0),(x_3,0)) \in B(0,R).$$

Set

$$\begin{aligned}T_1 &= \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi_{1,3,x_1,q_1^{-3}} \circ S_\omega\\ T_2 &= \Phi_{3,2,x_3,q_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ S_\omega\\ T_3 &= \Phi_{1,3,x_1,q_1^{-3}} \circ \Phi_{3,2,x_3,q_3^{-3}} \circ S_\omega.\end{aligned}$$

Then the following properties hold.

I) *If* $q_3 \mid q_1 \mid q_2$, *and*

$$x_1 \geq 1/q_1, \quad x_2 \geq 0, \quad x_3 \geq 1/q_3,$$

then there exists $N \geq 1$ *such that*

$$T_1^N(z) = ((\hat{x}_1, 0), (\hat{x}_2, 0), (\hat{x}_3, 0))$$

with

$$0 \leq \hat{x}_1 \leq x_1/2, \quad 0 \leq \hat{x}_2 = x_2, \quad 0 \leq \hat{x}_3 \leq x_3,$$

and $|T_1^m(z)_i| \leq x_i$ *for all* $m \in \{0, \dots, N\}$.

II) *If* $q_1 \mid q_2 \mid q_3$, *and*

$$x_1 \geq 0, \quad x_2 \geq 1/q_2, \quad x_3 \geq 1/q_3,$$

then there exists $N \geq 1$ *such that*

$$T_2^N(z) = ((\hat{x}_1, 0), (\hat{x}_2, 0), (\hat{x}_3, 0))$$

with

$$0 \leq \hat{x}_1 \leq x_1, \quad 0 \leq \hat{x}_2 \leq x_2/2, \quad 0 \leq \hat{x}_3 = x_3,$$

and $|T_2^m(z)_i| \leq x_i$ *for all* $m \in \{0, \dots, N\}$.

III) *If* $q_2 \mid q_3 \mid q_1$, *and*

$$x_1 \geq 1/q_1, \quad x_2 \geq 0, \quad x_3 \geq 1/q_3,$$

then there exists $N \geq 1$ *such that*

$$T_3^N(z) = ((\hat{x}_1, 0), (\hat{x}_2, 0), (\hat{x}_3, 0))$$

with

$$0 \leq \hat{x}_1 = x_1, \quad 0 \leq \hat{x}_2 \leq x_2, \quad 0 \leq \hat{x}_3 \leq x_3/2,$$

and $|T_3^m(z)_i| \leq x_i$ *for all* $m \in \{0, \dots, N\}$.

Proof. — We will prove the lemma for T_2 since it will be the first map that we will use in the sequel. The proof for the maps T_1 and T_3 follows exactly the same lines.

The hypothesis $x_2 \geq 1/q_2$ implies that the orbit of $z_2 = (x_2, 0)$ under the rotation R_{ω_2} enters the q_2^{-3} neighborhood of z_2 only at times that are multiples of q_2. Moreover $R_{\omega_2}^{\ell q_2}(z_2) = z_2$. A similar remark holds for z_3.

Since $q_3 \geq q_2$, we consider the action of $\mathscr{T} := \Phi_{2,1,x_2,q_2^{-3}} \circ S_\omega$ first. Since $q_1 \mid q_2$, if $s = (s_1, s_2, s_3)$ with $s_1 = (u_1, 0)$ and $s_2 = (u_2, 0)$, by Lemma 2.2 (a)–(b):

$$\mathscr{T}^m(s) = (s_{1,m}, R_{\omega_2}^m(s_2), R_{\omega_3}^m(s_3)) \quad \text{for all } m \in \mathbb{N},$$

with

$$|s_{1,m}| \leq |s_1|.$$

Consider now the orbit of z under the full diffeomorphism T_2. Since $q_2 \mid q_3$, the previous remark shows that one has to take the effect of $\Phi_{3,2,x_3,q_3^{-3}}$ into account only for the iterates of order $m = \ell q_3$. One therefore gets

$$T_2^m(z) = (z_{1,m}, z_{2,m}, R_{\omega_3}^m(z_3)), \quad \text{for all } m \in \mathbb{N},$$

where in particular $z_{2,\ell q_3} = (x_{2,\ell q_3}, 0)$ with

$$0 < x_{2,(\ell+1)q_3} \leq (1 - \tfrac{1}{2}\exp(-cq_3^{\frac{6}{\alpha-1}}))x_{2,\ell q_3},$$

and where

$$z_{2,\ell q_3+\ell'} = R_{\omega_2}^{\ell'}(z_{2,\ell q_3}), \quad 1 \leq \ell' \leq q_3 - 1,$$

$$|z_{1,m}| \leq x_1, \quad \text{for all } m \in \mathbb{N}.$$

We let L be the smallest integer such that $0 < x_{2,Lq_3} \leq x_2/2$ and get the conclusion with $N = Lq_3$. □

4. Proof of Theorem A

The proof is based on an iterative process (Proposition 4.3), which itself is based on the following preliminary result.

Proposition 4.1. — *Let $\omega = (p_1/q_1, p_2/q_2, p_3/q_3) \in \mathbb{Q}_+^3$ with $q_3 \mid q_1 \mid q_2$ and $z = ((x_1,0),(x_2,0),(x_3,0)) \in B(0,R)$ with $x_1, x_2, x_3 > 0$ and $x_2 \geq 1/q_2$. Then, for any $\eta > 0$, there exist*

(a) *$\overline{\omega} = (\overline{p}_1/\overline{q}_1, \overline{p}_2/\overline{q}_2, \overline{p}_3/\overline{q}_3)$ such that $\overline{q}_3 \mid \overline{q}_1 \mid \overline{q}_2$, the orbits of the translation of vector $\overline{\omega}$ on $\mathbb{T}^3$ are η-dense and $|\overline{\omega} - \omega| \leq \eta$;*

(b) *$\bar{z} = ((\overline{x}_1,0),(\overline{x}_2,0),(\overline{x}_3,0))$ such that $0 < \overline{x}_i \leq x_i/2$ for every $i \in \{1,2,3\}$ and $\overline{x}_2 \geq 1/\overline{q}_2$;*

(c) *$z' \in X$, $\widehat{x}_1 \in (\overline{x}_1 + \frac{1}{\overline{q}_1^3}, x_1)$ and $N \geq 1$, such that $|z' - z| \leq \eta$ and the diffeomorphism*

$$\mathscr{T} = \Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \Phi_{1,3,\widehat{x}_1,\overline{q}_1^{-3}} \circ \Phi_{3,2,x_3,\overline{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ S_{\overline{\omega}}$$

satisfies

$$\mathscr{T}^N(z') = \bar{z}$$

and $|\mathscr{T}^m(z')_i| \leq (1+\eta)x_i$ for $m \in \{0,\ldots,N\}$.

Moreover, $\overline{q}_1$, $\overline{q}_2$ and $\overline{q}_3$ can be taken arbitrarily large.

The proof of Proposition 4.1 will require the following

Lemma 4.2. — *Given $\eta, Q > 0$ and $p/q \in \mathbb{Q}$ with p and q coprime integers, $q \geq 1$, there exists $\widehat{p}/\widehat{q} \in \mathbb{Q}$ with $\widehat{p}$ and $\widehat{q}$ coprime integers such that*

$$q \mid \widehat{q}, \qquad \widehat{q} > Q, \qquad \left|\frac{\widehat{p}}{\widehat{q}} - \frac{p}{q}\right| < \eta.$$

Proof of Lemma 4.2. — According to Dirichlet's Theorem on Primes in Arithmetic Progressions, there are infinitely many prime numbers of the form $\ell p + 1$ with $\ell \in \mathbb{N}^*$. We can thus find an integer $\ell > \max\{Q, 1/\eta\}$ such that $\widehat{p} := \ell p + 1$ is prime, and the conclusion then holds with $\widehat{q} := \ell q$ since $\frac{\widehat{p}}{\widehat{q}} - \frac{p}{q} = \frac{1}{\ell q}$. □

Proof of Proposition 4.1. — We divide the proof into three steps.

1. First use Lemma 4.2 to choose coprime integers $\widehat{p}_3$ and $\widehat{q}_3$ with $\widehat{q}_3$ large multiple of q_2, so that

$$q_1 \mid q_2 \mid \widehat{q}_3, \qquad x_2 \geq \frac{1}{q_2}, \qquad x_3 \geq \frac{1}{\widehat{q}_3}, \qquad \frac{1}{\widehat{q}_3} < \eta \tag{4.1}$$

and the new rotation vector

$$\widehat{\omega} = (p_1/q_1, p_2/q_2, \widehat{p}_3/\widehat{q}_3)$$

satisfies $|\widehat{\omega} - \omega| < \eta$. Set

$$\widehat{T}_2 = \Phi_{3,2,x_3,\widehat{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ S_{\widehat{\omega}}.$$

By Lemma 3.1 **II)**, there exist $\widehat{N} \geq 1$ and $\widehat{z} = ((\widehat{x}_1, 0), (\widehat{x}_2, 0), (\widehat{x}_3, 0))$ such that $\widehat{T}_2^{\widehat{N}}(z) = \widehat{z}$, with

$$\widehat{x}_1 \leq x_1, \quad \widehat{x}_2 \leq x_2/2, \quad \widehat{x}_3 = x_3,$$

and $\left|\widehat{T}_2^m(z)_i\right| \leq x_i$ for all $m \in \{0, \ldots, \widehat{N}\}$.

2. Next, consider a vector of the form

$$\widetilde{\omega} = (\widetilde{p}_1/\widetilde{q}_1, p_2/q_2, \widehat{p}_3/\widehat{q}_3)$$

with coprime $\widetilde{p}_1$ and $\widetilde{q}_1$, and

$$\widehat{q}_3 \mid \widetilde{q}_1, \qquad \widehat{x}_1 > \frac{1}{\widetilde{q}_1}, \qquad \frac{\widehat{q}_3}{\widetilde{q}_1} < \eta, \tag{4.2}$$

so that in particular

$$\frac{\widehat{x}_1}{2} > \frac{1}{\widetilde{q}_1^3}. \tag{4.3}$$

Set

$$\widetilde{T_3} = \Phi_{1,3,\widehat{x}_1,\widetilde{q}_1^{-3}} \circ \Phi_{3,2,x_3,\widehat{q}_3^{-3}} \circ S_{\widetilde{\omega}}.$$

By Lemma 3.1 **III)**, there exist $\widetilde{N} \geq 1$ and $\widetilde{z} = ((\widetilde{x}_1, 0), (\widetilde{x}_2, 0), (\widetilde{x}_3, 0))$ such that $\widetilde{T_3}^{\widetilde{N}}(\widehat{z}) = \widetilde{z}$ with

$$\widetilde{x}_1 = \widehat{x}_1, \quad \widetilde{x}_2 \leq \widehat{x}_2 \leq x_2/2, \quad \widetilde{x}_3 \leq \widehat{x}_3/2 = x_3/2, \tag{4.4}$$

and $\left|\widetilde{T}_3^m(\widehat{z})_i\right| \le \widehat{x}_i$ for all $m \in \{0,\dots,\widetilde{N}\}$.

Define now

$$\mathbf{T} = \Phi_{1,3,\widehat{x}_1,\widetilde{q}_1^{-3}} \circ \Phi_{3,2,x_3,\widehat{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ S_{\widetilde{\omega}}.$$

Choosing $\widetilde{q}_1$ in (4.2) large enough and $\widetilde{p}_1$ properly (using Lemma 4.2), one can assume that $\widetilde{\omega}$ is arbitrarily close to $\widehat{\omega}$, so that $S_{\widetilde{\omega}}^{-1}$ is arbitrarily C^0-close to $S_{\widehat{\omega}}^{-1}$ on the ball $B = B(0,|z|+1)$, and moreover that the inverse of $\Phi_{1,3,\widehat{x}_1,\widetilde{q}_1^{-3}}$ is arbitrarily C^0-close to Id on B. As a consequence, one can assume that $\mathbf{T}^{-1}$ is arbitrarily C^0-close to $\widehat{T}_2^{-1}$ on B. Hence one can choose $\widetilde{\omega}$ with $|\widetilde{\omega}-\omega| < \eta$ such that there exists $\mathbf{z}$ with $|\mathbf{z}-z| < \eta$ which satisfies

$$\mathbf{T}^{\widehat{N}}(\mathbf{z}) = \widehat{z}, \qquad |\mathbf{T}^m(\mathbf{z})_i| \le (1+\eta)x_i \quad \text{for all } m \in \{0,\dots,\widehat{N}\}.$$

Moreover, using Lemma 2.2, one proves by induction that:

$$\mathbf{T}^m(\widehat{z})_2 \in B(x_2,\widehat{q}_2^{-3})^c, \qquad \mathbf{T}^m(\widehat{z}) = \widetilde{T_3}^m(\widehat{z}) \quad \text{for all } m \in \{0,\dots,\widetilde{N}\}.$$

As a consequence

$$\mathbf{T}^{\widehat{N}+\widetilde{N}}(\mathbf{z}) = \widetilde{T_3}^{\widetilde{N}}(\widehat{z}) = \widetilde{z}$$

and $|\mathbf{T}^m(\mathbf{z})_i| < (1+\eta)x_i$ for all $m \in \{0,\dots,\widehat{N}+\widetilde{N}\}$.

3. It remains now to perturb $\mathbf{T}$ in the same way as above to bring the first component of $\widetilde{z}$ closer to the origin. Use again Lemma 4.2 and consider coprime integers $\overline{p}_2$ and $\overline{q}_2$ such that

$$\widetilde{q}_1 \mid \overline{q}_2, \qquad x_2 \ge 1/\overline{q}_2, \qquad \widetilde{x}_2 \ge 1/\overline{q}_2, \qquad \frac{\widetilde{q}_1}{\overline{q}_2} < \eta, \tag{4.5}$$

and such that the vector

$$\overline{\omega} = (\widetilde{p}_1/\widetilde{q}_1, \overline{p}_2/\overline{q}_2, \widehat{p}_3/\widehat{q}_3) \tag{4.6}$$

satisfies $|\overline{\omega}-\omega| < \eta$. Set now

$$\mathscr{T} = \Phi_{2,1,\widetilde{x}_2,\overline{q}_2^{-3}} \circ \Phi_{1,3,\widehat{x}_1,\widetilde{q}_1^{-3}} \circ \Phi_{3,2,x_3,\widehat{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ S_{\overline{\omega}}.$$

As above, a proper choice of $\overline{p}_2$ and $\overline{q}_2$ satisfying (4.5) makes $\mathscr{T}^{-1}$ arbitrarily C^0-close to $\mathbf{T}^{-1}$ and yields the existence of a $z' \in X$ such that $|z'-z| < \eta$, satisfying

$$\mathscr{T}^{\widehat{N}+\widetilde{N}}(z') = \widetilde{z}, \qquad |\mathscr{T}^m(z')_i| < (1+\eta)x_i \quad \text{for all } m \in \{0,\dots,\widehat{N}+\widetilde{N}\}.$$

Set

$$\overline{T}_1 = \Phi_{2,1,\widetilde{x}_2,\overline{q}_2^{-3}} \circ \Phi_{1,3,\widetilde{x}_1,\overline{q}_1^{-3}} \circ S_{\overline{\omega}}.$$

Using Lemma 2.2 and Lemma 3.1 **I)**, one proves by induction that now for $m \ge 0$:

$$\mathscr{T}^m(\widetilde{z})_2 \in B(x_2,\overline{q}_2^{-3})^c, \qquad \mathscr{T}^m(\widetilde{z})_3 \in B(x_3,\overline{q}_3^{-3})^c, \qquad \mathscr{T}^m(\widetilde{z}) = \overline{T}_1^m(\widetilde{z}).$$

By Lemma 3.1 **I)** there exists $\overline{N}$ such that

$$\overline{T}_1^{\overline{N}}(\widetilde{z}) = \overline{z} = ((\overline{x}_1,0),(\overline{x}_2,0),(\overline{x}_3,0))$$

with

$$\overline{x}_1 \le \widetilde{x}_1/2 \le x_1/2, \quad \overline{x}_2 = \widetilde{x}_2 \le x_2/2, \quad \overline{x}_3 \le \widetilde{x}_3 \le x_3/2,$$

and $\left|(\overline{T}_1{}^m(\widetilde{z})_i\right| \leq \widetilde{x}_i \leq x_i$ for all $m \in \{0,\dots,\overline{N}\}$. As a consequence, setting $N = \widehat{N} + \widetilde{N} + \overline{N}$:

$$\mathscr{T}^N(z') = \bar{z}, \qquad |\mathscr{T}^m(z')_i| \leq (1+\eta)x_i \quad \text{for all } m \in \{0,\dots,N\}.$$

We finally change the notation of (4.6) and write

$$\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3) = (\overline{p}_1/\overline{q}_1, \overline{p}_2/\overline{q}_2, \overline{p}_3/\overline{q}_3),$$

so that in particular $\widetilde{q}_1 = \overline{q}_1$, $\widehat{q}_3 = \overline{q}_3$ and $\overline{q}_3 \mid \overline{q}_1 \mid \overline{q}_2$. Hence the orbits of $S_{\overline{\omega}}$ are $\overline{q}_2$-periodic. Moreover, from (4.3) and the equality $\widehat{x}_1 = \widetilde{x}_1$, one deduces

$$\widehat{x}_1 - \overline{x}_1 > \frac{1}{\overline{q}_1^3}.$$

Note finally that the last conditions in (4.1), (4.2) and (4.5) now read

$$\frac{1}{\overline{q}_3} < \eta, \qquad \frac{\overline{q}_3}{\overline{q}_1} < \eta, \qquad \frac{\overline{q}_1}{\overline{q}_2} < \eta.$$

Fix $(\theta_1,\theta_2,\theta_3) \in \mathbb{T}^3$ and recall that $\overline{q}_3 \mid \overline{q}_1 \mid \overline{q}_2$. By the first inequality one can first find $\ell_3 \in \mathbb{N}$ such that $R_{\overline{\omega}_3}^{\ell_3}(0)$ is η-close to θ_3. Then, by the second inequality there is an $\ell_1 \in \mathbb{N}$ such that $R_{\overline{\omega}_1}^{\ell_1\overline{q}_3+\ell_3}(0)$ is η-close to θ_1. Finally, by the last inequality there is an $\ell_2 \in \mathbb{N}$ such that $R_{\overline{\omega}_2}^{\ell_2\overline{q}_1+\ell_1\overline{q}_3+\ell_3}(0)$ is η-close to θ_2. This proves that $S_{\overline{\omega}}^{\ell_2\overline{q}_1+\ell_1\overline{q}_3+\ell_3}(0,0,0)$ is η-close to $(\theta_1,\theta_2,\theta_3)$, so that the orbits of $S_{\overline{\omega}}$ are η-dense on $\mathbb{T}^3$. This concludes the proof. □

Definition 4.1. — *Given $z = (z_1,z_2,z_3) \in X$, we say that a diffeomorphism Φ of X is z-admissible if $\Phi \equiv \mathrm{Id}$ on*

$$\{s \in X : |s_i| \leq \tfrac{11}{10}|z_i|, i = 1,2,3\}.$$

Proposition 4.3. — *Let $\omega = (p_1/q_1, p_2/q_2, p_3/q_3) \in \mathbb{Q}_+^3$ with $q_3 \mid q_1 \mid q_2$ and $z = ((x_1,0),(x_2,0),(x_3,0)) \in B(0,R)$ with $x_1,x_2,x_3 > 0$ and $x_2 \geq 1/q_2$. Suppose $\Phi \in \mathscr{U}^{\alpha,L}$ is z-admissible and $\|\Phi_{2,1,x_2,q_2^{-3}} \circ \Phi - \mathrm{Id}\|_{\alpha,L} < \epsilon$, where ϵ is a positive constant depending only on α, L and L_1, and let*

$$T := \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi \circ S_\omega.$$

Assume that $z_0 \in X$ and $M \geq 1$ are such that $T^M(z_0) = z$. Then, for any $\eta > 0$, there exist

(a) *$\overline{\omega} = (\overline{p}_1/\overline{q}_1, \overline{p}_2/\overline{q}_2, \overline{p}_3/\overline{q}_3)$ such that $\overline{q}_3 \mid \overline{q}_1 \mid \overline{q}_2$, the orbits of the translation of vector $\overline{\omega}$ on $\mathbb{T}^3$ are η-dense and $|\overline{\omega} - \omega| \leq \eta$;*

(b) *$\bar{z} = ((\overline{x}_1,0),(\overline{x}_2,0),(\overline{x}_3,0))$ such that $0 < \overline{x}_i \leq x_i/2$ for every $i \in \{1,2,3\}$ and $\overline{x}_2 \geq 1/\overline{q}_2$;*

(c) *$\bar{z}_0 \in X$ such that $|\bar{z}_0 - z_0| \leq \eta$, and $\overline{M} > M$, and $\bar{\Phi} \in \mathscr{U}^{\alpha,L}$ $\bar{z}$-admissible, so that the diffeomorphism*

$$\overline{T} := \Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \bar{\Phi} \circ S_{\overline{\omega}}$$

satisfies $\overline{T}^{\overline{M}}(\bar{z}_0) = \bar{z}$ *and* $\left|\overline{T}^m(\bar{z}_0)_i\right| \le (1+\eta)x_i$ *for all* $m \in \{M, \dots, \overline{M}\}$.

(d) *Moreover,* $\|\Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \bar{\bar{\Phi}} - \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi\|_{\alpha,L} \le \eta$.

Proof of Proposition 4.3. — Take $\overline{\omega}, \bar{z}, N, z', \widehat{x}_1$ as in Proposition 4.1 and let

$$\mathscr{T} = \Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \Phi_{1,3,\widehat{x}_1,\overline{q}_1^{-3}} \circ \Phi_{3,2,x_3,\overline{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ S_{\overline{\omega}}$$

so that $\mathscr{T}^N(z') = \bar{z}$ and $|\mathscr{T}^m(z')_i| \le (1+\eta)x_i$ for all $m \in \{0, \dots, N\}$. If we define

$$\overline{T} = \Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \Phi_{1,3,\widehat{x}_1,\overline{q}_1^{-3}} \circ \Phi_{3,2,x_3,\overline{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi \circ S_{\overline{\omega}}$$

then, since Φ is z-admissible and $|z - z'| < \eta$, we get $\mathscr{T}^m(z') = \overline{T}^m(z')$ for all $m \in \{0, \dots, N\}$, hence $\overline{T}^N(z') = \bar{z}$ and $\left|\overline{T}^m(z')_i\right| \le (1+\eta)x_i$ for all $m \in \{0, \dots, N\}$.

Let

$$\bar{\bar{\Phi}} \coloneqq \Phi_{1,3,\widehat{x}_1,\overline{q}_1^{-3}} \circ \Phi_{3,2,x_3,\overline{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi, \tag{4.7}$$

so that, indeed, $\overline{T} = \Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \bar{\bar{\Phi}} \circ S_{\overline{\omega}}$. Notice that we can write $\Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \bar{\bar{\Phi}} = \Phi^{u_3} \circ \Phi^{u_2} \circ \Phi^{u_1} \circ \Psi$ (notation of Lemma A.2), where $\Psi = \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi$ and the Gevrey-(α, L_1) norms of u_1, u_2, u_3 are controlled by Lemma 2.1; we thus get (d) by taking ϵ as in Lemma A.2 with $n = 3$ and applying (A.8), choosing $\overline{q}_1, \overline{q}_2, \overline{q}_3$ sufficiently large.

Comparing $\overline{T}$ and T in C^0-norm in the ball $B(0, |z_0| + 1)$, since we can take $\overline{\omega}$ arbitrarily close to ω and the $\overline{q}_i$'s arbitrarily large, we can find $\bar{z}_0 \in X$ such that $|\bar{z}_0 - z_0| \le \eta$ and $\overline{T}^M(\bar{z}_0) = z'$. We thus take $\overline{M} = M + N$, so that $\overline{T}^{\overline{M}}(\bar{z}_0) = \bar{z}$ and $\left|\overline{T}^m(\bar{z}_0)_i\right| \le (1+\eta)x_i$ for all $m \in \{M, \dots, \overline{M}\}$.

To finish the proof of (c), just observe that $\bar{\bar{\Phi}} \in \mathscr{U}^{\alpha,L}$ and $\bar{\bar{\Phi}}$ is $\bar{z}$-admissible since $\overline{x}_i \le x_i/2$ and $\overline{q}_1^{-3} \le \widehat{x}_1/10$, $\overline{q}_3^{-3} \le x_3/10$ (possibly increasing $\overline{q}_1$ and $\overline{q}_3$ if necessary). $\square$

Clearly, Proposition 4.3 is tailored so that it can be applied inductively. The gain obtained when going from T to $\overline{T}$ is twofold : on the one hand the orbit of the new initial point $\bar{z}_0$ is pushed further close to the origin, and on the other hand the rotation vector at the origin is changed to behave increasingly like an non-resonant vector.

Proof of Theorem A. — Let $\gamma > 0$. We pick

$$\omega^{(0)} = (p_1^{(0)}/q_1^{(0)}, p_2^{(0)}/q_2^{(0)}, p_3^{(0)}/q_3^{(0)}) \in \mathbb{Q}_+^3$$

with $q_3^{(0)} \mid q_1^{(0)} \mid q_2^{(0)}$, and $x_1^{(0)}, x_2^{(0)}, x_3^{(0)} > R/4$ so that $x_2^{(0)} \ge 1/q_2^{(0)}$ and

$$z_0^{(0)} \coloneqq ((x_1^{(0)}, 0), (x_2^{(0)}, 0), (x_3^{(0)}, 0)) \in B(0, R/2).$$

Let $\Phi^{(0)} \coloneqq \mathrm{Id}$ and $M^{(0)} \coloneqq 0$. Define

$$T^{(0)} \coloneqq \Psi^{(0)} \circ S_{\omega^{(0)}} \quad \text{with} \quad \Psi^{(0)} \coloneqq \Phi_{2,1,x_2^{(0)},1/(q_2^{(0)})^3} \circ \Phi^{(0)}.$$

Choosing $q_2^{(0)}$ sufficiently large, we have $\|\Psi^{(0)} - \mathrm{Id}\|_{\alpha,L} \le \min\{\epsilon/2, \gamma/2\}$ by (2.2). The hypotheses of Proposition 4.3 hold for $z^{(0)} = z_0^{(0)}$.

We apply Proposition 4.3 repeatedly by choosing inductively a sequence $(\eta^{(n)})_{n\ge 1}$ such that

$$\eta^{(n)} \le \min\Big\{\frac{\epsilon}{2^{n+1}}, \frac{\gamma}{2^{n+1}}, \frac{R}{2^{n+3}}, 1/10\Big\}, \qquad \sum_{k=n+1}^{\infty} \eta^{(k)} \le \frac{\eta^{(n)}}{q_2^{(n)}}$$

(where $q_2^{(n)}$ is determined at the nth step of the induction). We get sequences $(\omega^{(n)})_{n\ge 0}$, $(z_0^{(n)})_{n\ge 0}$, $(z^{(n)})_{n\ge 0}$, $(T^{(n)})_{n\ge 0}$, $(M^{(n)})_{n\ge 0}$, with

$$z^{(n)} = ((x_1^{(n)}, 0), (x_2^{(n)}, 0), (x_3^{(n)}, 0)), \quad 0 < x_i^{(n+1)} \le x_i^{(n)}/2$$

and $T^{(n)} = \Psi^{(n)} \circ S_{\omega^{(n)}}$ with $\Psi^{(n)} = \Phi_{2,1,x_2^{(n)},1/(q_2^{(n)})^3} \circ \Phi^{(n)} \in \mathscr{U}^{\alpha,L}$, so that

$$\begin{aligned} |\omega^{(n+1)} - \omega^{(n)}| &\le \eta^{(n+1)}, \\ |z_0^{(n+1)} - z_0^{(n)}| &\le \eta^{(n+1)}, \\ \|\Psi^{(n+1)} - \Psi^{(n)}\|_{\alpha,L} &\le \eta^{(n+1)}. \end{aligned} \tag{4.8}$$

We also have

$$|(T^{(n+1)^m}(z_0^{(n+1)}))_i| \le 1.1 x_i^{(j)} \text{ for all } m \in \{M^{(j)}, \dots, M^{(j+1)}\} \text{ with } j \le n. \tag{4.9}$$

In view of (4.8), the sequences $(z_0^{(n)})$, $(\omega^{(n)})$ and $(\Psi^{(n)})$ are Cauchy. We denote their limits by z_0^∞, ω^∞ and Ψ^∞. Notice that $\|\Psi^\infty - \mathrm{Id}\|_{\alpha,L} \le \gamma$ and $z_0^\infty \ne 0$ (because $|z_0^\infty - z_0^{(0)}| \le R/8$).

We now check that $\mathbf{T} \coloneqq \Psi^\infty \circ S_{\omega^\infty}$ satisfies $|\mathbf{T}^m(z_0^\infty)| \underset{m\to+\infty}{\longrightarrow} 0$. When restricted to $B(0,R)$, $S_{\omega^{(n)}}^{(n)}$ converges uniformly to S_{ω^∞} (by compactness) thus $T^{(n)}$ converges uniformly to $\mathbf{T}$, moreover $B(0,R)$ is invariant by $T^{(n)}$ and contains all the points $z_0^{(n)}$; hence we can use the following elementary lemma (the verification of which is left to the reader):

Lemma 4.4. — *Let E be a metric space and (T_n) a sequence of self-maps which converges uniformly to a limit T. Then, for any sequence $(z^{(n)})$ which converges to a point z in E, we have $T_n^m(z^{(n)}) \underset{n\to+\infty}{\longrightarrow} \mathbf{T}^m(z)$ for each $m \in \mathbb{N}$.*

We thus get $T^{(n)^m}(z_0^{(n)}) \underset{n\to+\infty}{\longrightarrow} \mathbf{T}^m(z_0^\infty)$ for each m. Letting n tend to ∞ in (4.9), we get $|(\mathbf{T}^m(z_0^\infty))_i| \le 1.1 x_i^{(j)}$ for all j and m such that $M^{(j)} \le m \le M^{(j+1)}$. Since $x_i^{(j)} \downarrow 0$ and $M^{(j)} \uparrow \infty$ as j tends to ∞, this yields $|\mathbf{T}^m(z_0^\infty)| \underset{m\to+\infty}{\longrightarrow} 0$.

The orbits of the translation of vector $\omega^{(n)}$ on $\mathbb{T}^3$ being $\eta^{(n)}$-dense and $q_2^{(n)}$-periodic, we see that ω^∞ defines a minimal translation on $\mathbb{T}^3$. Indeed, given $\theta \in \mathbb{T}^3$ and $\epsilon > 0$, we can choose $n, m \in \mathbb{N}$ so that $\eta^{(n)} \leq \epsilon/2$, $\operatorname{dist}(m\omega^{(n)} - \theta, \mathbb{Z}^3) \leq \eta^{(n)}$ and $m < q_2^{(n)}$. Then,

$$\operatorname{dist}(m\omega^\infty - \theta, \mathbb{Z}^3) \leq \eta^{(n)} + m|\omega^\infty - \omega^{(n)}| \leq \eta^{(n)} + q_2^{(n)} \sum_{k=n+1}^{\infty} \eta^{(k)} \leq 2\eta^{(n)},$$

which is $\leq \epsilon$. Hence the orbit of 0 under the translation of vector ω^∞ is ϵ-dense for every ϵ, which entails that ω^∞ is non-resonant.

The proof of Theorem A is thus complete. □

Appendix

Gevrey functions, maps and flows

A.1. Gevrey functions and Gevrey maps. — We follow Section 1.1.2 and Appendix B of [**10**], with some simplifications stemming from the fact that here we only need to consider functions satisfying uniform estimates on the whole of a Euclidean space.

The Banach algebra of uniformly Gevrey-(α, L) functions. — Let $N \geq 1$ be integer and $\alpha \geq 1$ and $L > 0$ be real. We define

$$G^{\alpha,L}(\mathbb{R}^N) := \{f \in C^\infty(\mathbb{R}^N) \mid \|f\|_{\alpha,L} < \infty\}, \|f\|_{\alpha,L} := \sum_{\ell \in \mathbb{N}^N} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^N)}.$$

We have used the standard notations $|\ell| = \ell_1 + \cdots + \ell_N$, $\ell! = \ell_1! \ldots \ell_N!$, $\partial^\ell = \partial_{x_1}^{\ell_1} \ldots \partial_{x_N}^{\ell_N}$, and

$$\mathbb{N} := \{0, 1, 2, \ldots\}.$$

The space $G^{\alpha,L}(\mathbb{R}^N)$ turns out to be a Banach algebra, with

$$\|fg\|_{\alpha,L} \leq \|f\|_{\alpha,L} \|g\|_{\alpha,L} \tag{A.1}$$

for all $f, g \in G^{\alpha,L}(\mathbb{R}^N)$, and there are "Cauchy-Gevrey inequalities": if $0 < L' < L$, then all the partial derivatives of f belong to $G^{\alpha,L'}(\mathbb{R}^N)$ and, for each $p \in \mathbb{N}$,

$$\sum_{m \in \mathbb{N}^N;\ |m| = p} \|\partial^m f\|_{\alpha,L'} \leq \frac{p!^\alpha}{(L - L')^{p\alpha}} \|f\|_{\alpha,L} \tag{A.2}$$

(see [**11**]).

The Banach space of uniformly Gevrey-(α, L) maps. — Let $N, M \geq 1$ be integer and $\alpha \geq 1$ and $L > 0$ be real. We define

$$G^{\alpha,L}(\mathbb{R}^N, \mathbb{R}^M) := \{F \in C^\infty(\mathbb{R}^N, \mathbb{R}^M) \mid \|F\|_{\alpha,L} < \infty\},$$
$$\|F\|_{\alpha,L} := \|F_{[1]}\|_{\alpha,L} + \cdots + \|F_{[M]}\|_{\alpha,L}.$$

This is a Banach space.

When $N = M = 2n$, we denote by $\mathrm{Id} + G^{\alpha,L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ the set of all maps of the form $\Psi = \mathrm{Id} + F$ with $F \in G^{\alpha,L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. This is a complete metric space for the distance $\mathrm{dist}(\mathrm{Id} + F_1, \mathrm{Id} + F_2) = \|F_2 - F_1\|_{\alpha,L}$. We use the notation

$$\mathrm{dist}(\Psi_1, \Psi_2) = \|\Psi_2 - \Psi_1\|_{\alpha,L}$$

as well. We then define

$$\mathscr{U}^{\alpha,L} \subset \mathrm{Id} + G^{\alpha,L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$$

as the subset consisting of all Gevrey-(α, L) symplectic diffeomorphisms of $\mathbb{R}^{2n}$ which fix the origin and are C^∞-tangent to Id at the origin. This is a closed subset of the complete metric space $\mathrm{Id} + G^{\alpha,L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$.

Composition with close-to-identity Gevrey-(α, L) maps. — Let $N \geq 1$ be integer and $\alpha \geq 1$ and $L > 0$ be real. We use the notation $(\mathbb{N}^N)^* := \mathbb{N}^N \smallsetminus \{0\}$ and define

$$\mathscr{N}^*_{\alpha,L}(f) := \sum_{\ell \in (\mathbb{N}^N)^*} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^N)},$$

so that $\|f\|_{\alpha,L} = \|f\|_{C^0(\mathbb{R}^N)} + \mathscr{N}^*_{\alpha,L}(f)$.

Lemma A.0. — *Let $L_1 > L$. There exists $\epsilon_c = \epsilon_c(N, \alpha, L, L_1)$ such that, for any $f \in G^{\alpha,L_1}(\mathbb{R}^N)$ and $F = (F_{[1]}, \ldots, F_{[N]}) \in G^{\alpha,L}(\mathbb{R}^N, \mathbb{R}^N)$, if*

$$\mathscr{N}^*_{\alpha,L}(F_{[1]}), \ldots, \mathscr{N}^*_{\alpha,L}(F_{[N]}) \leq \epsilon_c,$$

then $f \circ (\mathrm{Id} + F) \in G^{\alpha,L}(\mathbb{R}^N)$ and $\|f \circ (\mathrm{Id} + F)\|_{\alpha,L} \leq \|f\|_{\alpha,L_1}$.

Proof. — Since $L < L_1$, we can pick $\mu > 1$ such that $\mu L^\alpha < L_1^\alpha$; we then choose $a > 0$ such that $(1 + a)^{\alpha-1} \leq \mu$ and set $\lambda := \big(N(1 + 1/a)\big)^{\alpha-1}$. We will prove the lemma with $\epsilon_c := (L_1^\alpha - \mu L^\alpha)/\lambda$.

Let f and F be as in the statement, and $g := f \circ (\mathrm{Id} + F)$. We first derive a formula for $\dfrac{1}{k!}\partial^k g$ by computing the Taylor expansion of $g(x + h) = f(x + h + F(x + h))$ at $h = 0$: since the Taylor expansion of $x_i + h_i + F_{[i]}(x + h)$ is

$$x_i + F_{[i]}(x) + h_i + \mathscr{S}_i \quad \text{with } \mathscr{S}_i := \sum_{k \in (\mathbb{N}^N)^*} \frac{1}{k!} \partial^k F_{[i]}(x) h^k,$$

by composition of the Taylor series, we obtain that the formal series

$$\sum_{k \in \mathbb{N}^N} \frac{1}{k!} \partial^k g(x) h^k \in \mathbb{R}[[h_1, \ldots, h_N]]$$

is given by

$$\sum_{r \in \mathbb{N}^N} \partial^r f\big(x + F(x)\big) \frac{(h_1 + \mathscr{S}_1)^{r_1} \cdots (h_N + \mathscr{S}_N)^{r_N}}{r_1! \cdots r_N!} =$$

$$\sum_{n, \ell \in \mathbb{N}^N} \partial^{n+\ell} f\big(x + F(x)\big) \frac{h_1^{n_1} \mathscr{S}_1^{\ell_1} \cdots h_N^{n_N} \mathscr{S}_N^{\ell_N}}{n_1! \ell_1! \cdots n_N! \ell_N!}.$$

Writing $\mathscr{S}_i^L = \sum_{K^1,\dots,K^L\in(\mathbb{N}^N)^*} \frac{1}{K^1!\cdots K^L!} h^{K^1+\cdots+K^L} \prod_{1\le p\le L} \partial^{K^p} F_{[i]}(x)$, we get

$$\sum_{k\in\mathbb{N}^N} \frac{1}{k!}\partial^k g(x) h^k = \sum_{n,\ell\in\mathbb{N}^N} \frac{\partial^{n+\ell} f\big(x+F(x)\big)}{n!\ell!} \sum_{k^1,\dots,k^{|\ell|}\in(\mathbb{N}^N)^*} \frac{h^{n+k^1+\cdots+k^{|\ell|}}}{k^1!\cdots k^{|\ell|}!}\mathscr{P}$$

with $\mathscr{P} := \prod_{i=1}^N \prod_{\ell_1+\cdots+\ell_{i-1}<p\le\ell_1+\cdots+\ell_i} \partial^{k^p} F_{[i]}(x)$. Thus, for each $k\in\mathbb{N}^N$,

$$\frac{1}{k!}\partial^k g = \sum_{\substack{\ell,m,n\in\mathbb{N}^N\\ m+n=k}} \frac{(\partial^{\ell+n} f)\circ(\mathrm{Id}+F)}{\ell!\,n!} \sum_{\substack{k^1,\dots,k^{|\ell|}\in(\mathbb{N}^N)^*\\ k^1+\cdots+k^{|\ell|}=m}} \frac{\prod_{i=1}^N \prod_{\ell_1+\cdots+\ell_{i-1}<p\le\ell_1+\cdots+\ell_i} \partial^{k^p} F_{[i]}}{k^1!\cdots k^{|\ell|}!}$$

with the convention that an empty sum is 0 and an empty product is 1.

Note that if $\ell = 0$, then necessarily $m = 0$ and the corresponding contribution to the sum is $\frac{1}{k!}(\partial^k f)\circ(\mathrm{Id}+F)$, whereas $\ell\neq 0$ implies $m\neq 0$ and $k\neq 0$. Thus, $\|g\|_{C^0(\mathbb{R}^N)} \le \|f\|_{C^0(\mathbb{R}^N)}$ and, for each $k\in(\mathbb{N}^N)^*$,

$$\frac{1}{k!}\ \|\partial^k g\|_{C^0} \le \frac{1}{k!}\ \|\partial^k f\|_{C^0} + \sum_{\substack{\ell,m,n\in\mathbb{N}^N\\ \ell\neq 0,\ m+n=k}} \frac{\|\partial^{\ell+n} f\|_{C^0}}{\ell!\,n!} \sum_{\substack{k^1,\dots,k^{|\ell|}\in(\mathbb{N}^N)^*\\ k^1+\cdots+k^{|\ell|}=m}} \frac{P}{k^1!\cdots k^{|\ell|}!}$$

with $P := \prod_{i=1}^N \prod_{\ell_1+\cdots+\ell_{i-1}<p\le\ell_1+\cdots+\ell_i} \|\partial^{k^p} F_{[i]}\|_{C^0}$. Multiplying by $L^{|k|\alpha}/k!^{\alpha-1}$ and taking the sum over k, we get

$$\|g\|_{\alpha,L} \le \sum_{k\in\mathbb{N}^N} \frac{L^{|k|\alpha}}{k!^\alpha}\|\partial^k f\|_{C^0} + S \tag{A.3}$$

with

$$S := \sum_{\ell\in(\mathbb{N}^N)^*,\ m,n\in\mathbb{N}^N} \frac{L^{|m+n|\alpha}\|\partial^{\ell+n} f\|_{C^0}}{\ell!n!(m+n)!^{\alpha-1}} \sum_{\substack{k^1,\dots,k^{|\ell|}\in(\mathbb{N}^N)^*\\ k^1+\cdots+k^{|\ell|}=m}} \frac{P}{k^1!\cdots k^{|\ell|}!} \tag{A.4}$$

with the same P as above.

Inequality (A.7) from **[11]** says that, if $s\ge 1$ and $k^1,\dots,k^s\in(\mathbb{N}^N)^*$ with $k^1+\cdots+k^s = m$, then $k^1!\cdots k^s! \le N^s m!/s!$. Hence, in each term of the sum S, we can compare $D := \ell!n!(m+n)!^{\alpha-1}k^1!\cdots k^{|\ell|}!$ and $\tilde{D} := \ell!n!(\ell+n)!^{\alpha-1}k^1!^\alpha\cdots k^{|\ell|}!^\alpha$: we have

$$\frac{\tilde{D}}{D} = \Big(\frac{k^1!\cdots k^{|\ell|}!(\ell+n)!}{(m+n)!}\Big)^{\alpha-1} \le \Big(\frac{N^{|\ell|}m!(\ell+n)!}{|\ell|!(m+n)!}\Big)^{\alpha-1}$$
$$\le \Big(\frac{N^{|\ell|}(\ell+n)!}{\ell!\,n!}\Big)^{\alpha-1} \le \lambda^{|\ell|}\mu^{|n|},$$

where the last inequality stems from our choice of λ and μ, using $\frac{(\ell+n)!}{\ell!\,n!} \le (1+1/a)^{|\ell|}(1+a)^{|n|}$. Inserting $\dfrac{1}{D} \le \dfrac{\lambda^{|\ell|}\mu^{|n|}}{\tilde{D}}$ in (A.4), we obtain

$$S \le \sum_{\ell\in(\mathbb{N}^N)^*,\ n\in\mathbb{N}^N} \frac{L^{|n|\alpha}\lambda^{|\ell|}\mu^{|n|}\|\partial^{\ell+n}f\|_{C^0}}{\ell!n!(\ell+n)!^{\alpha-1}} \sum_{k^1,\dots,k^{|\ell|}\in(\mathbb{N}^N)^*} \frac{L^{|k^1+\cdots+k^{|\ell|}|\alpha}P}{k^1!^\alpha\cdots k^{|\ell|}!^\alpha}.$$

The inner sum over $k^1,\dots,k^{|\ell|}\in(\mathbb{N}^N)^*$ coincides with the product

$$\mathscr{N}^*_{\alpha,L}(F_{[1]})^{\ell_1}\cdots\mathscr{N}^*_{\alpha,L}(F_{[N]})^{\ell_N},$$

which is $\le \epsilon_{\mathrm{c}}^{|\ell|}$ by assumption. Hence, coming back to (A.3), we get

$$\|g\|_{\alpha,L} \le \sum_{\ell,n\in\mathbb{N}^N} \frac{(\mu L^\alpha)^{|n|}(\lambda\epsilon_{\mathrm{c}})^{|\ell|}\|\partial^{\ell+n}f\|_{C^0}}{\ell!n!(\ell+n)!^{\alpha-1}} = \sum_{k\in\mathbb{N}^N}\frac{(\mu L^\alpha+\lambda\epsilon_{\mathrm{c}})^{|k|}\|\partial^k f\|_{C^0}}{k!^\alpha}$$

(we have used $\mu\ge 1$ to absorb the first term of the right-hand side of (A.3) in the contribution of $\ell=0$). The conclusion follows from our choice of ϵ_{c}. □

A.2. Estimates for Gevrey flows. — We need some improvements with respect to **[11]** and **[10]** for the estimates of the flow of a small Gevrey vector field.

Lemma A.1. — *Suppose $\alpha\ge 1$ and $0<L<L_1$.*

(i) *For every integer $N\ge 1$, there exists $\epsilon_{\mathrm{f}}=\epsilon_{\mathrm{f}}(N,\alpha,L,L_1)$ such that, for every vector field $X\in G^{\alpha,L_1}(\mathbb{R}^N,\mathbb{R}^N)$, if $\|X\|_{\alpha,L_1}\le\epsilon_{\mathrm{f}}$, then the time-1 map Φ of the flow generated by X belongs to $\mathrm{Id}+G^{\alpha,L}(\mathbb{R}^N,\mathbb{R}^N)$ and*

$$\|\Phi-\mathrm{Id}\|_{\alpha,L}\le\|X\|_{\alpha,L_1}. \tag{A.5}$$

(ii) *For every integer $n\ge 1$, there exists $\epsilon_{\mathrm{H}}=\epsilon_{\mathrm{H}}(n,\alpha,L,L_1)$ such that, for every $u\in G^{\alpha,L_1}(\mathbb{R}^{2n})$, if $\|u\|_{\alpha,L_1}\le\epsilon_{\mathrm{H}}$, then the time-1 map Φ^u of the Hamiltonian flow generated by u belongs to $\mathrm{Id}+G^{\alpha,L}(\mathbb{R}^{2n},\mathbb{R}^{2n})$ and*

$$\|\Phi^u-\mathrm{Id}\|_{\alpha,L}\le 2^\alpha(L_1-L)^{-\alpha}\|u\|_{\alpha,L_1}. \tag{A.6}$$

Building upon the previous result, we get

Lemma A.2. — *Suppose $\alpha\ge1$ and $0<L<L_1$. Then there exist $C=C(n,\alpha,L,L_1)$ and $\epsilon=\epsilon(n,\alpha,L,L_1)$ such that, if $r\ge1$, $u_1,\dots,u_r\in G^{\alpha,L_1}(\mathbb{R}^{2n})$, $\Psi\in\mathrm{Id}+G^{\alpha,L}(\mathbb{R}^{2n},\mathbb{R}^{2n})$ and*

$$\|\Psi-\mathrm{Id}\|_{\alpha,L}+C\big(\|u_1\|_{\alpha,L_1}+\cdots+\|u_r\|_{\alpha,L_1}\big)\le\epsilon, \tag{A.7}$$

then

$$\|\Phi^{u_r}\circ\cdots\circ\Phi^{u_1}\circ\Psi-\Psi\|_{\alpha,L}\le C\big(\|u_1\|_{\alpha,L_1}+\cdots+\|u_r\|_{\alpha,L_1}\big) \tag{A.8}$$

(with the same notation as in Lemma A.1(ii) *for the Φ^{u_i}'s).*

Proof of Lemma A.1. — (i) Let us pick $L' \in (L, L_1)$. We will prove the statement with $\epsilon_{\mathrm{f}} := \epsilon_{\mathrm{c}}(N, \alpha, L, L')$ (notation from Lemma A.0).

Let X be as in the statement. We write the restriction of its flow to the time-interval $[0,1]$ in the form $\Phi(t) = \mathrm{Id} + \xi(t)$, with $t \in [0,1] \mapsto \xi(t) \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ characterized by

$$\xi(t) = \int_0^t X \circ \big(\mathrm{Id} + \xi(\tau)\big)\, \mathrm{d}\tau \quad \text{for all } t \in [0,1].$$

We will show that ξ belongs to

$$\mathscr{B} := \{\, \psi \in C^0\big([0,1], G^{\alpha,L}(\mathbb{R}^N, \mathbb{R}^N)\big) \mid \|\psi\| \le \|X\|_{\alpha,L_1} \,\},$$

which is a closed ball in a Banach space.

Lemma A.0 shows that the formula $\mathscr{T}(\psi)(t) := \int_0^t X \circ \big(\mathrm{Id} + \psi(\tau)\big)\, \mathrm{d}\tau$ defines a map from $\mathscr{B}$ to $\mathscr{B}$. Moreover, if $\psi, \psi^* \in \mathscr{B}$ satisfy

$$\|\psi^*(t) - \psi(t)\|_{\alpha,L} \le A(t) \ \text{ for all } t \in [0,1],$$

where $t \in [0,1] \mapsto A(t)$ is continuous, then for each t and i,

$$\mathscr{T}(\psi^*)(t)_{[i]} - \mathscr{T}(\psi)(t)_{[i]} = \int_0^t \mathrm{d}\tau \sum_{j=1}^N \int_0^1 \mathrm{d}\theta$$
$$\partial_{x_j} X_{[i]} \circ \big(\mathrm{Id} + (1-\theta)\psi(\tau) + \theta\psi^*(\tau)\big)\big(\psi^*(\tau)_{[j]} - \psi(\tau)_{[j]}\big),$$

whence

$$\|\mathscr{T}(\psi^*)(t) - \mathscr{T}(\psi)(t)\|_{\alpha,L} \le K \int_0^t A(\tau)\, \mathrm{d}\tau \ \text{ with } K := \max_{i,j} \|\partial_{x_j} X_{[i]}\|_{\alpha,L'}$$

(we have $K < \infty$ by (A.2) and we have used Lemma A.0 and (A.1)). Iterating this, we get

$$\|\mathscr{T}^p(\psi^*) - \mathscr{T}^p(\psi)\| \le \frac{K^p}{p!} \|\psi^* - \psi\| \ \text{ for all } p \in \mathbb{N},$$

which shows that $\mathscr{T}^p$ is a contraction for p large enough. The map $\mathscr{T}$ thus has a unique fixed point in $\mathscr{B}$, and this fixed point is ξ.

(ii) Let $L' := (L + L_1)/2$. For any $u \in G^{\alpha,L_1}(\mathbb{R}^{2n})$, inequality (A.2) with $p = 1$ reads

$$\sum_{m \in \mathbb{N}^{2n};\ |m|=1} \|\partial^m u\|_{\alpha,L'} \le (L_1 - L')^{-\alpha} \|u\|_{\alpha,L_1}.$$

The left-hand side is precisely the (α, L')-Gevrey norm of the Hamiltonian vector field generated by u. Therefore, point (i) shows that the conclusion holds with $\epsilon_{\mathrm{H}} = (L_1 - L')^{\alpha} \epsilon_{\mathrm{f}}(2n, \alpha, L, L')$.

Proof of Lemma A.2. — Let us pick $L' \in (L, L_1)$. We will show the statement with

$$C \coloneqq 2^\alpha (L_1 - L')^{-\alpha}, \quad \epsilon \coloneqq \min\left\{\epsilon_{\mathrm{c}}(2n, \alpha, L, L'), C\epsilon_{\mathrm{H}}(n, \alpha, L', L_1)\right\}$$

by induction on r.

The induction is tautologically initialized for $r = 0$. Let us take $r \geq 1$ and assume that the statement holds at rank $r-1$. Given $u_1, \ldots, u_r \in G^{\alpha, L_1}(\mathbb{R}^{2n})$ and $\Psi \in \mathrm{Id} + G^{\alpha,L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ satisfying (A.7), we set $\chi \coloneqq \Phi^{u_{r-1}} \circ \cdots \circ \Phi^{u_1} \circ \Psi$, which satisfies

$$\|\chi - \Psi\|_{\alpha,L} \leq C\big(\|u_1\|_{\alpha,L_1} + \cdots + \|u_{r-1}\|_{\alpha,L_1}\big)$$

by the induction hypothesis, and observe that we also have

$$\|\Phi^{u_r} - \mathrm{Id}\|_{\alpha,L'} \leq C\|u_r\|_{\alpha,L_1}$$

since $\|u_r\|_{\alpha,L_1} \leq \epsilon_{\mathrm{H}}(n, \alpha, L', L_1)$. Now

$$\begin{aligned}\|\Phi^{u_r} \circ \cdots \circ \Phi^{u_1} \circ \Psi - \Psi\|_{\alpha,L} &\leq \|(\Phi^{u_r} - \mathrm{Id}) \circ \chi\|_{\alpha,L} + \|\chi - \Psi\|_{\alpha,L} \\ &\leq \|\Phi^{u_r} - \mathrm{Id}\|_{\alpha,L'} + \|\chi - \Psi\|_{\alpha,L}\end{aligned}$$

since

$$\begin{aligned}\|\chi - \mathrm{Id}\|_{\alpha,L} &\leq \|\Psi - \mathrm{Id}\|_{\alpha,L} + \|\chi - \Psi\|_{\alpha,L} \\ &\leq \|\Psi - \mathrm{Id}\|_{\alpha,L} + C\big(\|u_1\|_{\alpha,L_1} + \cdots + \|u_{r-1}\|_{\alpha,L_1}\big) \leq \epsilon_{\mathrm{c}}(2n, \alpha, L, L'),\end{aligned}$$

and we are done.

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