WEAK MIXING DISC AND ANNULUS DIFFEOMORPHISMS WITH ARBITRARY LIOUVILLE ROTATION NUMBER ON THE BOUNDARY.

B. FAYAD, M. SAPRYKINA

Abstract. Let M be an m-dimensional differentiable manifold with a nontrivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume μ. For any Liouville number α we construct a sequence of area-preserving diffeomorphisms H_n such that the sequence $H_n \circ S_\alpha \circ H_n^{-1}$ converges to a smooth weak mixing diffeomorphism of M. The method is a quantitative version of the approximation by conjugations construction introduced in [1].

For $m = 2$ and M equal to the unit disc $\mathbb{D}^2 = \{x^2 + y^2 \leq 1\}$ or the closed annulus $\mathbb{A} = \mathbb{T} \times [0, 1]$ this result proves the following dichotomy: $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is Diophantine if and only if there is no ergodic diffeomorphism of M whose rotation number on the boundary equals α (on at least one of the boundaries in the case of $\mathbb{A}$). One part of the dichotomy follows from our constructions, the other is an unpublished result of Michael Herman asserting that if α is Diophantine, then any area preserving diffeomorphism with rotation number α on the boundary (on at least one of the boundaries in the case of $\mathbb{A}$) displays smooth invariant curves arbitrarily close to the boundary which clearly precludes ergodicity or even topological transitivity.

1. Introduction

We present a construction method providing analytic weak mixing diffeomorphisms on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $d \geq 2$, and smooth weak mixing diffeomorphisms on any smooth manifold with a nontrivial circle action preserving a smooth volume μ. The diffeomorphisms obtained are homotopic to the Identity and can be made arbitrarily close to it.

We will effectively work either on the two torus for the analytic constructions or on the closed annulus $\mathbb{A} = \mathbb{T} \times [0, 1]$ for the smooth constructions. In the case of the torus the construction is exactly the same in higher dimensions and we explain in §2.4 how the smooth construction can be transfered from the annulus to general manifolds with a nontrivial circle action.

By smooth diffeomorphisms on a manifold with boundary we mean infinitely smooth in the interior and such that all the derivatives can be continuously extended to the boundary.

We recall that a dynamical system (M, T, μ) is said to be ergodic if and only if there is no nonconstant *invariant* measurable complex function h on (M, μ), i.e. such that $h(Tx) = h(x)$. It is said to be weak mixing if it enjoys the stronger property of not having *eigenfunctions* at all, i.e. if there is no nonconstant measurable complex function h on (M, μ) such that $h(Tx) = \lambda h(x)$ for some constant $\lambda \in \mathbb{C}$.

The construction, on any smooth manifold with a nontrivial circle action (in particular $\mathbb{D}^2$), of volume preserving diffeomorphisms enjoying different ergodic properties (among others, weak mixing) was first undertaken in [1]. For $t \in \mathbb{R}$ denote by S_t the elements of the circle action on M with the normalization $S_{t+1} = S_t$.

Let $\mathcal{A}(M)$ be the closure in the C^∞ topology of the set of diffeomorphisms of the form $h \circ S_t \circ h^{-1}$, with $t \in \mathbb{R}$ and h area preserving C^∞-diffeomorphism of M.

For a given $\alpha \in \mathbb{R}$ we denote by $\mathcal{A}_\alpha(M)$ the restricted space of *conjugacies of the fixed rotation* S_α, namely the closure of the set of C^∞-diffeomorphisms of the form $h \circ S_\alpha \circ h^{-1}$.

It is easy to see that the sets $\mathcal{A}_\alpha(M)$ are disjoint for different α and in [4, section 2.3.1], it was proved for a particular manifold M that $\cup_{\alpha \in \mathbb{R}} \mathcal{A}_\alpha(M) \subsetneq \mathcal{A}(M)$. We do not know if the inclusion remains strict on any manifold.

Anosov and Katok proved in [1] that in $\mathcal{A}(M)$ the set of weak mixing diffeomorphisms is generic (contains a G_δ dense set) in the C^∞ topology. Actually, it also follows from the same paper that the same is true in $\mathcal{A}_\alpha(M)$ for a G_δ dense set of $\alpha \in \mathbb{R}$ although the construction, properly speaking, is achieved in the space $\mathcal{A}(M)$. However, [1] does not give a full description of the set of α for which the result holds in $\mathcal{A}_\alpha(M)$. Indeed, the flexibility of the constructions in [1] comes from the fact that α is constructed inductively at the same time as the conjugations are built, that is: at step n, $\alpha_n = p_n/q_n$ is given, and h_n is constructed that commutes with S_{α_n}; then α_{n+1} is chosen so close to α_n that $f_n = H_n S_{\alpha_{n+1}} H_n^{-1}$ (where $H_n = h_1 \circ \cdots \circ h_n$ and each h_n commutes with S_{α_n}) is sufficiently close to f_{n-1} to guarantee the convergence of the sequence $\{f_n\}_{n \in \mathbb{N}}$. Then step $n+1$ gets started by the choice of h_{n+1} etc... The final α is the limit of α_n. By this procedure, there is no need to put any restrictions on the growth of the C^r norms of H_n since α_{n+1} can always be chosen close enough to α_n to force convergence. The counterpart is that the limit diffeomorphism obtained in this way will lie in $\mathcal{A}_\alpha(M)$ with α having rational approximations at a speed that is not controlled.

Since we want to do the construction inside $\mathcal{A}_\alpha(M)$ for an arbitrary Liouville number α, we are only allowed to make use of the fact that the decay of $|\alpha_{n+1} - \alpha_n|$ is faster than any polynomial in q_n. So we have to construct h_n with a polynomial (in q_n) control on the growth of its derivatives to make sure that the above procedure converges.

Recall that an irrational number α is said to be Diophantine if it is not too well approximated by rationals, namely if there exist strictly positive constants γ and τ such that for any couple of integers (p, q) we have:

$$|q\alpha - p| \geq \frac{\gamma}{q^\tau}.$$

In this paper we work in the restricted spaces $\mathcal{A}_\alpha(M)$ and prove the following for *any* Liouville, i.e. not Diophantine and not rational, frequency α:

Theorem 1.1. *Let M be an m-dimensional ($m \geq 2$) differentiable manifold with a nontrivial circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{R}}$, $S_{t+1} = S_t$ preserving a smooth volume μ. If $\alpha \in \mathbb{R}$ is Liouville, then the set of weak mixing diffeomorphisms is generic in the C^∞ topology in $\mathcal{A}_\alpha(M)$.*

On $M = \mathbb{D}^2$ or $\mathbb{A}$, the weak mixing diffeomorphisms we will construct in $\mathcal{A}_\alpha(M)$ will have S_α as their restriction to the boundary. This clarifies the relation between the ergodic properties of the area preserving diffeomorphisms of $\mathbb{D}^2$ and their rotation number on the boundary, complementing the striking result of M. Herman stating that if f is a smooth diffeomorphism of the disc with a Diophantine rotation number on the boundary, then there exists a set of positive measure of smooth invariant curves in the neighborhood of the boundary, thus f is not ergodic. By KAM theory, this phenomenon was known to happen for Diophantine α as soon as the map f displays some twist features near the boundary. Herman's tour de force was to get rid of the twist condition in the area preserving context. To be more precise, we introduce the following

Definition 1.2. Let M denote either $\mathbb{D}^2$ or $\mathbb{A}$. Given $\alpha \in \mathbb{R}$, we denote by $\mathcal{B}_\alpha(M)$ the set of area preserving C^∞-diffeomorphisms of M whose restriction to the boundary (to at least one of the boundary circles in the case of the annulus) has a rotation number α.

THEOREM 1.3 (Herman). *Let M denote either $\mathbb{D}^2$ or $\mathbb{A}$. For a Diophantine α, let $F \in \mathcal{B}_\alpha(M)$. Then the boundary of M (on which the rotation number is α) is accumulated by a set of positive measure of invariant curves of F.*

In the case of the disc and the annulus, as a corollary of Theorems 1.1 and 1.3, we have the following characterization of Diophantine numbers:

COROLLARY 1.4. *Let M denote either $\mathbb{D}^2$ or $\mathbb{A}$. A number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is Diophantine if and only if there is no ergodic diffeomorphism $f \in \mathcal{B}_\alpha(M)$.*

On $M = \mathbb{T}^2$ and under a more restrictive condition on α, the method of approximation by conjugations can be undertaken in the real analytic topology and with very explicit conjugations. For an arbitrary fixed $\sigma > 0$, for any $n \in \mathbb{N}$, we set:

$$\begin{aligned} \phi_n(\theta, r) &= (\theta, r + q_n^2 \cos(2\pi q_n \theta)), \\ g_n(\theta, r) &= (\theta + [n q_n^\sigma] r, r), \\ h_n &= g_n \circ \phi_n, \quad H_n = h_1 \circ \cdots \circ h_n, \\ f_n &= H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}. \end{aligned} \tag{1.1}$$

Here $[\cdot]$ denotes the integer part of the number and R_t denote the action $(\theta, r) \to (\theta + t, r)$. The convergence of the diffeomorphisms f_n is in the sense of a usual metric $d_\rho(\cdot,\cdot)$, based on the supremum norm of analytic functions over the complex strip of width ρ; see Section 2.2 for the definition. We will prove the following

THEOREM 1.5. *Let $\alpha \in \mathbb{R}$ be such that, for some $\delta > 0$, equation*

$$|\alpha - p_n/q_n| < \exp(-q_n^{1+\delta})$$

has an infinite number of integer solutions p_n, q_n (where p_n and q_n are relatively prime for each n). Take $0 < \sigma < \min\{\delta/3, 1\}$. Then, for all $\rho > 0$, there exists a sequence $\alpha_n = p_n/q_n$ (which is a subsequence of the solutions of the equation above) such that the corresponding diffeomorphisms f_n, constructed in (1.1), converge in the sense of the $d_\rho(\cdot,\cdot)$-metric, and $f = \lim_{n\to\infty} f_n$ is weak mixing.

Weak mixing diffeomorphisms, given by this theorem, are uniquely ergodic. This can be shown by the same method as in [9].

REMARK 1.6. The result in Theorem 1.5 is actually weaker than what can be obtained by time change, e.g. the existence on $\mathbb{T}^2$ of real analytic weak mixing reparametrizations of $R_{t(1,\alpha)}$ for any irrational α such that $\limsup_{p\in\mathbb{Z}, q\in\mathbb{N}^*} -\frac{\ln|\alpha - p/q|}{q} \neq 0$ [2, 6, 7, 10]. Indeed, such reparametrizations belong *a priori* to $\mathcal{A}_\alpha(\mathbb{T}^2)$ (Cf. [4]). However, we included the constructions on $\mathbb{T}^2$ with explicit successive conjugations as in (1.1) because the proof of weak mixing follows almost immediately from the general criteria we established to treat the general smooth case, and also because these constructions might be generalized to other manifolds where the techniques of reparametrizations are not available.

Acknowledgments. It is a pleasure to thank Håkan Eliasson for useful comments all along the work. We are grateful to Raphaël Krikorian for his help in the main construction of Section 5 and to Anatole Katok, Jean-Paul Thouvenot and Alistair Windsor for many useful conversations. We also thank the referee for many useful recommendations.

2. Preliminaries

2.1. **General scheme of the constructions.** Here we give a general scheme of the construction of the diffeomorphisms as a limit of conjugacies of a given Liouvillean action while §3 outlines the particular choices that will yield the weak mixing property for the limit diffeomorphism. Henceforth, M denotes either the torus $\mathbb{T}^2$ or the annulus $\mathbb{A}$ and we consider polar coordinates (θ, r) on M that denotes either the torus $\mathbb{T}^2$ or the annulus $\mathbb{A}$. By λ and μ we denote the usual Lebesgue measures on $\mathbb{R}$ and on $\mathbb{R}^2$, respectively. The term "measure-preserving" will refer to the measure μ.

For $\alpha \in \mathbb{R}$, we consider the map $S_\alpha : M \to M$, $(\theta, r) \mapsto (\theta + \alpha, r)$. The diffeomorphisms that we shall construct, are obtained as limits of measure preserving transformations

$$f = \lim_{n\to\infty} f_n, \quad \text{where} \quad f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}. \tag{2.1}$$

Here $\alpha_n = p_n/q_n$ is a convergent sequence of rational numbers, such that $|\alpha - \alpha_n| \to 0$ monotonically; H_n is a sequence of measure preserving diffeomorphisms of M. In different constructions, the convergence of f_n will be meant in the C^∞ or real analytic category; the topology in each case is standard, and will be recalled in Sections 2.2 and 2.3.

Each H_n is obtained as a composition

$$H_n = h_1 \circ \cdots \circ h_n, \tag{2.2}$$

where every h_n is a measure preserving diffeomorphism of M satisfying

$$h_n \circ S_{\alpha_n} = S_{\alpha_n} \circ h_n. \tag{2.3}$$

At step n, h_n must display enough stretching to insure an increasing distribution of the orbits of $H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$. However, this stretching must be appropriately controlled with respect to $|\alpha - \alpha_n|$ to guarantee convergence of the construction.

2.1.1. *Decomposition of h_n.* In the subsequent constructions, each h_n will be obtained as a composition

$$h_n = g_n \circ \phi_n, \tag{2.4}$$

where ϕ_n is constructed in such a way that $S_{1/q_n} \circ \phi_n = \phi_n \circ S_{1/q_n}$; the diffeomorphism g_n is a twist map of the form

$$g_n(\theta, r) = (\theta + [nq_n^\sigma]r, r), \tag{2.5}$$

for some $0 < \sigma < 1$ that will be fixed later. The role of g_n is to introduce shear in the "horizontal" direction (the direction of the circle action), while ϕ_n is responsible for the "vertical" motion, i.e. transversal to the circle action. The choice of the shear factor nq_n^σ will be explained in §3.1.

In the real analytic case, ϕ_n will be given by an explicit formula and convergence will follow from an assumption on the rational approximations of α. In the smooth case, ϕ_n will be constructed in Section 5.2 in such a way that its derivatives satisfy estimates of the type:

$$\|D_a\phi_n\|_0 \le c(n,a)q_n^{|a|}, \quad \|D_a\phi_n^{-1}\|_0 \le c(n,a)q_n^{|a|},$$

where $c(n,a)$ is independent of q_n (Cf. §2.3 and §5.2.3 about the notations we adopt). This polynomial growth of the norms of ϕ_n is crucial to insure the convergence of the construction above and is the reason why it can be carried out for an arbitrary Liouville number.

2.2. **Analytic topology.** Let us discuss the topology on the space of real-analytic diffeomorphisms of $\mathbb{T}^2$, homotopic to the identity. All of them have a lift of type $F(\theta, r) = (\theta + f_1(\theta, r), r + f_2(\theta, r))$, where $f_i : \mathbb{R}^2 \to \mathbb{R}$ are real-analytic and $\mathbb{Z}^2$-periodic.

For any $\rho > 0$, consider the set of real analytic $\mathbb{Z}^2$-periodic functions on $\mathbb{R}^2$, that can be extended to holomorphic functions on $A^\rho = \{|\mathrm{Im}\,\theta|, |\mathrm{Im}\,r| < \rho\}$. For a function f in this set, let $\|f\|_\rho = \sup_{A^\rho} |f(\theta, r)|$. We define $C^\omega_\rho(\mathbb{T}^2)$ as a subset of the above set, defined by the condition: $\|f\|_\rho < \infty$.

Consider the space $\mathrm{Diff}^\omega_\rho$ of those diffeomorphisms, for whose lift it holds: $f_i \in C^\omega_\rho(\mathbb{T}^2)$, $i = 1, 2$. For any two diffeomorphisms F and G in this space we can define the distance

$$d_\rho(F, G) = \max_{i=1,2}\{\inf_{p\in\mathbb{Z}} \|f_i - g_i + p\|_\rho\}.$$

For a diffeomorphism T with a lift $T(\theta, r) = (T_1(\theta, r),\, T_2(\theta, r))$ denote

$$\|DT\|_\rho = \max\left\{\left\|\frac{\partial T_1}{\partial\theta}\right\|_\rho, \left\|\frac{\partial T_1}{\partial r}\right\|_\rho, \left\|\frac{\partial T_2}{\partial\theta}\right\|_\rho, \left\|\frac{\partial T_2}{\partial r}\right\|_\rho\right\}.$$

2.3. C^∞**-topology.** Here we discuss the (standard) topology on the space of smooth diffeomorphisms of $M = \mathbb{T}^2$, which we shall use later. The annulus is endowed with the topology in the similar way.

We are interested in convergence in the space of smooth diffeomorphisms of M, homotopic to the identity, and hence having lift of type $\tilde{F}(\theta, r) = (\theta + f_1(\theta, r),\, r + f_2(\theta, r))$, where $f_i : \mathbb{R}^2 \to \mathbb{R}$ are $\mathbb{Z}^2$-periodic. For a continuous function $f : (0, 1) \times (0, 1) \to \mathbb{R}$, denote

$$\|f\|_0 := \sup_{z\in(0,1)\times(0,1)} |f(z)|.$$

For conciseness we introduce the following notation for partial derivatives of a function: for $a = (a_1, a_2) \in \mathbb{N}^2$ we denote $|a| := a_1 + a_2$ and

$$D_a := \frac{\partial^a}{\partial r^{a_1}\partial\theta^{a_2}}.$$

For F, G in the space $\mathrm{Diff}^k(\mathbb{T}^2)$ of k-smooth diffeomorphisms of the torus, let $\tilde{F}$ and $\tilde{G}$ be their lifts. For mappings $F : \mathbb{R}^2 \to \mathbb{R}^2$ denote by F_i the i-th coordinate function. Define the distances between two diffeomorphisms F and G as

$$\tilde{d}_0(F, G) = \max_{i=1,2}\{\inf_{p\in\mathbb{Z}} \|(\tilde{F} - \tilde{G})_i + p\|_0\},$$
$$\tilde{d}_k(F, G) = \max\{\tilde{d}_0(F, G),\ \|D_a(\tilde{F}_i - \tilde{G}_i)\|_0 \mid i = 1, 2,\quad 1 \le |a| \le k\}.$$

We shall use the metric, measuring the distance both between diffeomorphisms and their inverses:

$$d_k(F, G) = \max\{\tilde{d}_k(F, G), \tilde{d}_k(F^{-1}, G^{-1})\}.$$

For $M = \mathbb{D}^2$, the $\mathrm{Diff}^k(M)$ topologies are defined in the natural way with the help of the supremum norm of continuous functions over the disc.

For the smooth topology on M, a sequence of $\mathrm{Diff}^\infty(M)$ diffeomorphisms is said to be convergent in $\mathrm{Diff}^\infty(M)$, if it converges in $\mathrm{Diff}^k(M)$ for all k. The space $\mathrm{Diff}^\infty(M)$, endowed with the metric

$$d_\infty(F, G) = \sum_{k=1}^{\infty} \frac{d_k(F, G)}{2^k(1 + d_k(F, G))},$$

is a compact metric space, hence for any of its closed subspaces, Baire theorem holds.

2.4. **Reduction to the case of the annulus.** Let $(M, \mathcal{S}, \mu)$ denote a system of an m-dimensional smooth manifold with a nontrivial circle action preserving a smooth volume μ.

We denote by F the set of fixed points of the action $\mathcal{S}$. For $q \geq 1$ we denote by F_q the set of fixed points of the map $S_{1/q}$. And by ∂M we denote the boundary of M. Finally we let $B := \partial M \cup F \cup_{q\geq 1} F_q$.

Let λ be the product of Lebeasgue measures on $\mathbb{S}^1 \times \mathbb{D}^{m-1}$. Denote by $\mathcal{R}$ the standard "horizontal" action of $\mathbb{S}^1$ on $\mathbb{S}^1 \times \mathbb{D}^{m-1}$. We quote the following proposition of [4] that is similar to corresponding statements in [1, 11]

Proposition 2.1. [4, proposition 5.2] *Let M be an m-dimensional differentiable manifold with a nontrivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$ preserving a smooth volume μ. Let $B := \partial M \cup F \cup (\bigcup_q F_q)$. There exists a continuous surjective map $G : \mathbb{S}^1 \times \mathbb{D}^{m-1} \to M$ with the following properties:*

(1) The restriction of G to the interior $\mathbb{S}^1 \times \mathbb{D}^{m-1}$ is a C^∞ diffeomorphic embedding;
(2) $\mu(G(\partial(\mathbb{S}^1 \times \mathbb{D}^{m-1}))) = 0$;
(3) $G(\partial(\mathbb{S}^1 \times \mathbb{D}^{m-1})) \supset B$;
(4) $G_*(\lambda) = \mu$;
(5) $\mathcal{S} \circ G = G \circ \mathcal{R}$.

We show now how this proposition allows to carry a construction as in the preceding section from $(\mathbb{S}^1 \times \mathbb{D}^{m-1}, \mathcal{R}, \lambda)$ to the general case $(M, \mathcal{S}, \mu)$.

Suppose $f : \mathbb{S}^1 \times \mathbb{D}^{m-1} \to \mathbb{S}^1 \times \mathbb{D}^{m-1}$ is a weak mixing diffeomorphism given, as above, by $f = \lim f_n$, $f_n = H_n \circ R_\alpha \circ H_n^{-1}$ where, moreover, the maps H_n are equal to identity in a neighborhood of the boundary, the size of which can be chosen to decay arbitrarily slowly. Then if we define the diffeomorphisms $\tilde{H}_n : M \to M$

$$\begin{aligned} \tilde{H}_n(x) &= G \circ H_n \circ G^{-1}(x) \text{ for } x \in G(\mathbb{S}^1 \times \mathbb{D}^{m-1}), \text{ and} \\ \tilde{H}_n(x) &= x \text{ for } x \in G(\mathbb{S}^1 \times \partial(\mathbb{D}^{m-1})), \end{aligned}$$

we will have that $\tilde{H}_n \circ S_\alpha \circ \tilde{H}_n^{-1}$ is convergent in the C^∞ topology to the weak mixing diffeomorphism $\tilde{f} : M \to M$ defined by

$$g(x) = G(f(G^{-1}(x)) \text{ for } x \in G(\mathbb{S}^1 \times \mathbb{D}^{m-1}), \text{ and}$$

$$g(x) = S_\alpha(x) \text{ for } x \in G(\mathbb{S}^1 \times \partial(\mathbb{D}^{m-1})).$$

In the sequel, to alleviate the notations, we will assume that $m = 2$ and will do the constructions on the annulus $\mathbb{A} = \mathbb{S}^1 \times [0, 1]$ or on the two torus $\mathbb{T}^2$.

3. Criterion for weak mixing

The goal of this section is to give a simple geometrical criterion involving only the diffeomorphisms $\phi_n \circ R_{\alpha_{n+1}} \circ \phi_n^{-1}$ and insuring the weak mixing property for the diffeomorphism f given by (2.1)–(2.5) in case of convergence. The criterion will be stated in Proposition 3.9 of §3.6.

The following characterization of weak mixing will be used (see, for example, [10]): f is weak mixing if there exists a sequence $m_n \in \mathbb{N}$ such that for any Borel sets A and B we have:

$$|\mu(B \cap f^{-m_n}(A)) - \mu(B)\mu(A)| \to 0. \tag{3.1}$$

3.1. We will now give an overview of the criterion assuming that M is the annulus $\mathbb{T} \times [0,1]$ and denoting by horizontal intervals the sets $I = [\theta_1, \theta_2] \times \{r\}$. We say that a sequence ν_n, consisting for each n of a collection of disjoint sets on M (for example horizontal intervals), *converges to the decomposition into points* if any measurable set B can be approximated as $n \to \infty$ by a union of atoms in ν_n (Cf. §3.2). We denote this by $\nu_n \underset{n\to\infty}{\longrightarrow} \epsilon$.

The first reduction is given by a Fubini Lemma 3.3. Here we decompose B at each step n into a union of small codimension one sets for which a precise version of (3.1) is assumed to hold, see (3.2). For each n these sets are images by a smooth map F_n of a collection η_n of horizontal intervals such that $F_n(\eta_n) \to \epsilon$. Lemma 3.3 shows that (3.2) guarantees weak mixing.

The second step is Lemma 3.4 asserting that under an additional condition of proximity (3.3) between $f_n^{m_n}$ and f^{m_n}, it is enough to check (3.2) for f_n.

Now, we take F_n in the Fubini Lemma equal to $H_{n-1} \circ g_n$. Since H_{n-1} in the construction only depends on q_{n-1}, q_n can be chosen so that $\|DH_{n-1}\|_0 < \ln q_n$. With our choice of g_n ($\sigma < 1$ in (2.5)) this implies that $H_{n-1} \circ g_n(\eta_n) \to \epsilon$ if $\eta_n \to \epsilon$ is a partial partition with horizontal inetrvals of length less than $1/q_n$ (Cf. Lemma 3.5). With the above observations, we are reduced to finding a collection η_n and a sequence m_n with the property that $H_{n-1} \circ g_n \circ \phi_n \circ R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1}(I)$ is almost uniformly distributed in M for $I \in \eta_n$.

The geometrical ingredient of the criterion appears in §3.5 and merely states that if a set (in particular $\phi_n \circ R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1}(I)$) is almost a vertical line going from one boundary of the annulus to the other, then the image of this set by g_n defined in (2.5) is almost uniformly distributed in M. "Almost vertical" is made precise and quantified in Definition 3.6. Actually, the choice of g_n ($\sigma > 0$ in 2.5) gives in addition that $H_{n-1} \circ g_n$ of a an almost vertical segment will be almost uniformly distributed in M, since we impose that $\|DH_{n-1}\|_0 < \ln q_n$.

In conclusion, the criterion for weak mixing (Proposition 3.9) roughly states as follows: Let f be given by (2.1)–(2.5). If for some sequence m_n satisfying the proximity condition (3.3) between $f_n^{m_n}$ and f^{m_n}, there exists a sequence $\eta_n \to \epsilon$ consisting of horizontal intervals of length less than $1/q_n$ such that the image of $I \in \eta_n$ by $\phi_n \circ R_{\alpha_{n+1}} \circ \phi_n^{-1}$ is increasingly almost vertical as $n \to \infty$ then the limit diffeomorphism f is weak mixing.

3.2. **A Fubini Lemma.**

Definition 3.1. A collection of disjoint sets on M will be called *partial decomposition* of M. We say that a sequence of partial decompositions ν_n converges to the decomposition into points (notation: $\nu_n \to \epsilon$) if, given a measurable set A, for any n there exists a measurable set A_n, which is a union of elements of ν_n, such that $\lim_{n\to\infty} \mu(A \triangle A_n) = 0$ (here $\triangle$ denotes the symmetric difference).

In this section we work with $M = \mathbb{T}^2$ or $M = \mathbb{A}$. For these manifolds we formulate the following definition.

Definition 3.2. Let $\hat{\eta}$ be a partial decomposition of $\mathbb{T}$ into intervals, and consider on M the decomposition η consisting of intervals in $\hat{\eta}$ times some $r \in [0,1]$. Decompositions of the above type will be called *standard partial decompositions*. We shall say that ν is the image under a diffeomorphism $F : M \to M$ of a standard decomposition η (notation: $\nu = F(\eta)$), if

$$\nu = \{\Gamma = F(I) \mid I \in \eta\}.$$

Here we formulate a standard criterion for weak mixing. The proof is based on the application of Fubini Lemma.

LEMMA 3.3 (Fubini Lemma). *Let f be a measure μ preserving diffeomorphism of M. Suppose that there exists an increasing sequence m_n of natural numbers, and a sequence of partial decompositions $\nu_n \to \epsilon$ of M, where, for each n, ν_n is the image under a measure-preserving diffeomorphism $F_n : M \to M$ of a standard partial decomposition, with the following property: for any fixed square $A \subset M$ and any $\varepsilon > 0$, for any n large enough we have: for any atom $\Gamma_n \in \nu_n$*

$$\left|\lambda_n(\Gamma_n \bigcap f^{-m_n}(A)) - \lambda_n(\Gamma_n)\mu(A)\right| \leq \varepsilon\lambda_n(\Gamma_n)\mu(A), \tag{3.2}$$

where $\lambda_n = F_n^*(\lambda)$.

Then the diffeomorphism f is weak mixing.

Proof. To prove that f is weak mixing, it is enough to show that for any square A and a Borel set B

$$|\mu(B \cap f^{-m_n}(A)) - \mu(B)\mu(A)| \to 0$$

when $n \to \infty$. In the case of the annulus, is even enough to show this for any square A that is strictly contained in the interior of $\mathbb{A}$. By assumption, for any n we have: $\lambda_n(\Gamma_n) = \lambda_n(F_n(I_n)) = \lambda(I_n)$. Then

$$\lambda_n(\Gamma_n \cap f^{-m_n}(A)) = \lambda_n(F_n(I_n \cap F_n^{-1} \circ f^{-m_n}(A))) = \lambda(I_n \cap F_n^{-1} \circ f^{-m_n}(A)).$$

By (3.2), this implies:

$$|\lambda(I_n \cap F_n^{-1} \circ f^{-m_n}(A)) - \lambda(I_n)\mu(A)| \leq \varepsilon\lambda(I_n)\mu(A).$$

Take any Borel set $B \subset \mathbb{T}^2$. Since $\nu_n \to \epsilon$, for any ε, for fixed A and B, there exists n and a measurable set $\hat{B} = \cup_{i\in\sigma}\Gamma_n^i$ (Γ_n^i are elements of ν_n, and σ is an appropriate index set) such that

$$|\mu(\hat{B} \triangle B)| < \varepsilon\mu(B)\mu(A).$$

Consider $\tilde{B} = F_n^{-1}(\hat{B})$ (it is also measurable since F_n is continuous). Then

$$\tilde{B} = \bigcup_{i\in\sigma} F_n^{-1}(\Gamma_n^i) = \bigcup_{i\in\sigma} I_n^i := \bigcup_{0\leq y\leq 1} \bigcup_{i\in\sigma(y)} I_n^i(y) \times \{y\}.$$

We estimate:

$$\begin{aligned}
&|\mu(B \cap f^{-m_n}(A)) - \mu(B)\mu(A)| \\
&= |\mu(F_n^{-1}(B) \cap F_n^{-1} \circ f^{-m_n}(A)) - \mu(B)\mu(A)| \\
&\leq |\mu(\tilde{B} \cap F_n^{-1} \circ f^{-m_n}(A)) - \mu(\tilde{B})\mu(A)| + 2\varepsilon\mu(B)\mu(A) \\
&= \int_0^1 \sum_{i\in\sigma(y)} |\lambda(I_n^i(y) \times \{y\} \cap F_n^{-1} \circ f_n^{-m_n}(A)) - \lambda(I_n^i)\mu(A)| \mathrm{d}y \\
&+ 2\varepsilon\mu(B)\mu(A) \leq 3\varepsilon\mu(B)\mu(A).
\end{aligned}$$

□

3.3. **Reduction from f to f_n.**

LEMMA 3.4 (Reduction to f_n). *If f is the limit diffeomorphism from (2.1), and the sequence m_n in the latter lemma satisfies*

$$d_0(f^{m_n}, f_n^{m_n}) < \frac{1}{2^n}, \tag{3.3}$$

then we can replace the diffeomorphism f in the criterion (3.2) by f_n:

$$\left|\lambda_n(\Gamma_n \bigcap f_n^{-m_n}(A)) - \lambda_n(\Gamma_n)\mu(A)\right| \leq \varepsilon\lambda_n(\Gamma_n)\mu(A), \tag{3.4}$$

and the result of Lemma 3.3 still holds.

Proof. Let us show that the assumptions of this lemma imply (3.2). Fix an arbitrary square $A \subset M$ and $\varepsilon > 0$.

Consider two squares, A_1 and A_2, such that

$$A_1 \subset A \subset A_2, \quad \mu(A \triangle A_i) \leq \frac{\varepsilon}{3}\mu(A).$$

Moreover, if n is sufficiently large, we can guarantee that

$$\operatorname{dist}(\partial A, \partial A_i) > \frac{1}{2^n},$$

(where $\operatorname{dist}(A, B) = \inf_{x\in A,\, y\in B} |x - y|$, and ∂A denotes the boundary of A), and

$$\left|\lambda_n(\Gamma_n \bigcap f_n^{-m_n}(A_i)) - \lambda_n(\Gamma_n)\mu(A_i)\right| \leq \frac{\varepsilon}{3}\lambda_n(\Gamma_n)\mu(A_i).$$

By (3.3), for any x the following holds: $f_n^{m_n}(x) \in A_1$ implies $f^{m_n}(x) \in A$, and $f^{m_n}(x) \in A$ implies $f_n^{m_n}(x) \in A_2$. Therefore,

$$\lambda_n(\Gamma_n \cap f_n^{-m_n}(A_1)) \leq \lambda_n(\Gamma_n \cap f^{-m_n}(A)) \leq \lambda_n(\Gamma_n \cap f_n^{-m_n}(A_2)),$$

which gives the estimate:

$$\left(1 - \frac{\varepsilon}{3}\right)\lambda_n(\Gamma_n)\mu(A_1) \leq \lambda_n(\Gamma_n \cap f^{-m_n}(A)) \leq \left(1 + \frac{\varepsilon}{3}\right)\lambda_n(\Gamma_n)\mu(A_2),$$

implying (3.2). □

3.4. **Reduction from f_n to $h_n \circ R_{\alpha_{n+1}} \circ h_n^{-1}$.** The following is a technical lemma that will allow us to focus in the sequel only on the action of $h_n \circ R^{m_n}_{\alpha_{n+1}} \circ h_n^{-1}$ (more specifically on $g_n \circ \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$) in order to get (3.4):

Lemma 3.5. *Let η_n be a sequence of standard partial decompositions of M into horizontal intervals of length less or equal to $1/q_n$, let g_n be defined by (2.5) with some $0 < \sigma < 1$, and let H_n be a sequence of area-preserving diffeomorphisms of M such that for all n*

$$\|DH_{n-1}\|_0 < \ln q_n. \tag{3.5}$$

Consider partitions $\nu_n = \{\Gamma_n = H_{n-1}g_n(I_n) \mid I_n \in \eta_n\}$.

Then $\eta_n \to \epsilon$ implies $\nu_n \to \epsilon$.

Proof. Let $\sigma < \sigma' < 1$, and consider a partition of the annulus into squares $S_{n,i}$ of side length between $q_n^{-\sigma'}$ and $2q_n^{-\sigma'}$. Since $\eta_n \to \epsilon$, we have for $\varepsilon > 0$ arbitrarilly small, if n is large enough, $\mu(\cup_{I\in\eta_n} I) \geq 1 - \varepsilon$, so that for a collection of atoms S with total measure greater than $1 - \sqrt{\varepsilon}$ we have $\mu(\cup_{I\in\eta_n} I \cap S) \geq (1 - \sqrt{\varepsilon})\mu(S)$. Since $\sigma' < 1$ and any $I \in \eta_n$ has length at most $1/q_n$, we have for the same atoms S as above $\mu(\cup_{I\in\eta_n, I\subset S}) \geq (1 - 2\sqrt{\varepsilon})\mu(S)$ if n is sufficiently large.

Consider now the sets $C_{n,i} = H_{n-1}g_n(S_{n,i})$. In the same way as the squares $S_{n,i}$, a large proportion of these sets can be well approximated by unions of elements of ν_n. But by (3.5), we have:

$$\operatorname{diam}(C_{n,i}) \leq \|DH_{n-1}\|_0 \, \|Dg_n\|_0 \operatorname{diam}(S_{n,i}),$$

which goes to 0 as $n \to \infty$. Therefore, any Borel set B can be approximated by a union of such sets $C_{n,i}$ with any ahead given accuracy, if n is sufficiently large, hence B gets well approximated by unions of elements of ν_n. $\square$

3.5. **Horizontal stretch under g_n.** We shall call by *horizontal interval* any line segment of the form $I \times \{r\}$, where I is an interval on the θ-axis. *Vertical intervals* have the form $\{\theta\} \times J$ where J is an interval on the r-axis. Let π_r and π_θ denote the projection operators onto r and θ coordinate axes, respectively.

The following definition formalizes the notion of "almost uniform distribution" of a horizontal interval in the vertical direction.

Definition 3.6 ($(\gamma, \delta, \varepsilon)$-distribution). We say that a diffeomorphism $\Phi : M \to M$ $(\gamma, \delta, \varepsilon)$-distributes a horizontal interval I (or $\Phi(I)$ is $(\gamma, \delta, \varepsilon)$-distributed), if

- $\pi_r(\Phi(I))$ is an interval J with $1 - \delta \le \lambda(J) \le 1$;
- $\Phi(I)$ is contained in a "vertical strip" of type $[c, c + \gamma] \times J$ for some c;
- for any interval $\tilde{J} \subset J$ we have:

$$\left| \frac{\lambda(I \cap \Phi^{-1}(\mathbb{T} \times \tilde{J}))}{\lambda(I)} - \frac{\lambda(\tilde{J})}{\lambda(J)} \right| \le \varepsilon \frac{\lambda(\tilde{J})}{\lambda(J)}. \tag{3.6}$$

We shall more often write the latter relation in the form

$$|\lambda(I \cap \Phi^{-1}(\mathbb{T} \times \tilde{J}))\lambda(J) - \lambda(I)\lambda(\tilde{J})| \le \varepsilon\lambda(I)\lambda(\tilde{J}).$$

Lemma 3.7. *Let g_n be a diffeomorphism of the form (2.5) with some fixed $0 < \sigma < 1$. Suppose that a diffeomorphism $\Phi : M \to M$ $(\gamma, \delta, \varepsilon)$-distributes a horizontal interval I with $\gamma = 1/(nq_n^\sigma)$, $\delta = 1/n$, $\varepsilon = 1/n$. Denote $\pi_r(\Phi(I))$ by J.*

Then for any square S of side length $q_n^{-\sigma}$, lying in $\mathbb{T} \times J$ it holds:

$$|\lambda(I \cap \Phi^{-1} \circ g_n^{-1}(S))\lambda(J) - \lambda(I)\mu(S)| \le 8/n\lambda(I)\mu(S). \tag{3.7}$$

Lemma 3.7 asserts that, if a diffeomorphism Φ "almost uniformly" distributes I in the vertical direction, then the composition of Φ and the affine map g_n "almost uniformly" distributes I on the whole of M.

To prove Lemma 3.7, we shall need the following preliminary statement: it says that g_n "almost uniformly" distributes on M any sufficiently thin vertical strip.

Lemma 3.8. *Suppose that $g : M \to M$ has a lift*

$$g(\theta, r) = (\theta + br, r) \quad \textit{for some} \quad b \in \mathbb{Z},\ |b| \ge 2.$$

For an interval K on the r-axes, $\lambda(K) \le 1$, denote by $K_{c,\gamma}$ a strip

$$K_{c,\gamma} := [c, c + \gamma] \times K.$$

Let $L = [l_1, l_2]$ be an interval on the θ-axes. If $b\lambda(K) > 2$, then for

$$Q := \pi_r(K_{c,\gamma} \cap g^{-1}(L \times K)),$$

it holds:

$$|\lambda(Q) - \lambda(K)\lambda(L)| \le \gamma\lambda(K) + \frac{2\lambda(L)}{b} + \frac{2\gamma}{b}.$$

Proof. By definition, $Q = \{r \in K \mid \exists \theta \in [c, c+\gamma] : \theta + br \in [l_1, l_2]\}$. Then

$$Q = \{r \in K \mid br \in [l_1 - \gamma, l_2] - c\}.$$

To estimate $\lambda(Q)$, note that the interval bK (seen as an interval on the real line) intersects not more than $b\lambda(K)+2$ intervals of type $[i, i+1]$, $i \in \mathbb{Z}$, on the line, and not less than $b\lambda(K)-2$ such intervals. Hence,

$$\lambda(Q) \leq (b\lambda(K)+2)\frac{(l_2-l_1)+\gamma}{b} = \lambda(K)\lambda(L) + \gamma\lambda(K) + \frac{2\lambda(L)}{b} + \frac{2\gamma}{b}.$$

The lower bound is obtained in the same way. □

Proof of Lemma 3.7. Let S be a square in $\mathbb{T} \times J$ of size $q_n^{-\sigma} \times q_n^{-\sigma}$. Denote $\pi_\theta(S)$ by S_θ, $\pi_r(S)$ by S_r. In these notations, $\lambda(S_r) = \lambda(S_\theta) = q_n^{-\sigma}$, and $\lambda(S_\theta)\lambda(S_r) = \mu(S) = q_n^{-2\sigma}$.

Let us study what part of $\Phi(I)$ is sent by g_n into S. Since $\Phi(I)$ is contained in a strip $[c, c+\gamma] \times J$ for some c, by assumption, and g_n preserves horizontals, this part lies in $K_{c,\gamma} := [c, c+\gamma] \times S_r$. Denoting S_θ by $[s_1, s_2]$, define a "smaller" rectangle $S_1 \subset S$: $S_1 = [s_1+\gamma, s_2-\gamma] \times S_r$ (in our assumptions, 2γ is much less than $\lambda(S_\theta)$, so this rectangle is non-empty). Consider two sets:

$$Q := \pi_r(K_{c,\gamma} \cap g_n^{-1}(S)), \quad Q_1 := \pi_r(K_{c,\gamma} \cap g_n^{-1}(S_1)).$$

Then we have:

$$\Phi(I) \cap (\mathbb{T} \times Q_1) \subset \Phi(I) \cap g_n^{-1}(S) \subset \Phi(I) \cap (\mathbb{T} \times Q). \tag{3.8}$$

The second inclusion is evident, the first one comes from the fact that g_n preserves lengths of horizontal intervals.

Lemma 3.8 permits us to estimate $\lambda(Q)$ and $\lambda(Q_1)$. Indeed, to estimate the former one, apply Lemma 3.8 with $b = [nq_n^\sigma]$, $\gamma = (nq_n^\sigma)^{-1}$, $K = S_r$, and $L = S_\theta$. We get:

$$|\lambda(Q) - \mu(S)| \leq \frac{\lambda(S_r)}{nq_n^\sigma} + \frac{2\lambda(S_\theta)}{[nq_n^\sigma]} + \frac{2}{nq_n^\sigma[nq_n^\sigma]} \leq \frac{4}{n}\mu(S).$$

In the same way, applying Lemma 3.8 with the same b, γ, K as above and $L = \pi_\theta S_1 = [s_1+\gamma, s_2-\gamma]$, we get the same estimate (for large n):

$$|\lambda(Q_1) - \mu(S_1)| \leq \frac{4}{n}\mu(S).$$

In particular, this implies $\lambda(Q) \leq 2\mu(S)$, and $\lambda(Q_1) \leq 2\mu(S)$.

Both Q and Q_1 are finite unions of disjoint intervals. Then, using (3.6) with $\varepsilon = \frac{1}{n}$ (which was the assumption of the present lemma), we have:

$$|\lambda(I \cap \Phi^{-1}(\mathbb{T} \times Q))\lambda(J) - \lambda(I)\lambda(Q)| \leq \frac{1}{n}\lambda(I)\lambda(Q) \leq \frac{2}{n}\lambda(I)\mu(S),$$

and the same estimate holds for Q_1 instead of Q. The last preliminary estimates are:

$$\begin{aligned}
&|\lambda(I \cap \Phi^{-1}(\mathbb{T} \times Q))\lambda(J) - \lambda(I)\mu(S)| \leq \\
&|\lambda(I \cap \Phi^{-1}(\mathbb{T} \times Q))\lambda(J) - \lambda(I)\lambda(Q)| + \lambda(I)|\lambda(Q) - \mu(S)| \\
&\leq \frac{2}{n}\lambda(I)\mu(S) + \frac{4}{n}\lambda(I)\mu(S) = \frac{6}{n}\lambda(I)\mu(S);
\end{aligned}$$

and, in the same way, (noting that $\mu(S) - \mu(S_1) = \frac{2}{n}\mu(S)$), one estimates

$$|\lambda(I \cap \Phi^{-1}(\mathbb{T} \times Q_1))\lambda(J) - \lambda(I)\mu(S)| \leq \frac{8}{n}\lambda(I)\mu(S).$$

Now relation (3.8), together with the preliminary estimates above, gives the desired conclusion:

$$
\begin{aligned}
&|\lambda(I \cap \Phi^{-1} \circ g_n^{-1}(S))\lambda(J) - \lambda(I)\mu(S)| \\
&\leq \max\{|\lambda(I \cap \Phi^{-1}(\mathbb{T} \times Q))\lambda(J) - \lambda(I)\mu(S)|, \\
&|\lambda(I \cap \Phi^{-1}(\mathbb{T} \times Q_1))\lambda(J) - \lambda(I)\mu(S)|\} \leq \frac{8}{n}\lambda(I)\mu(S).
\end{aligned}
$$

□

3.6. **Criterion for weak mixing.** We can now state the following

PROPOSITION 3.9 (Criterion for weak mixing). *Assume that $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ is a sequence of diffeomorphisms constructed following (2.2), (2.3), (2.4) and (2.5) with some $0 < \sigma < 1/2$, and that for all n (3.5) holds.*

Suppose that the limit $\lim_{n\to\infty} f_n = f$ exists. If there exist a sequence m_n satisfying (3.3) and a sequence of standard partial decompositions η_n of M into horizontal intervals of length less than $1/q_n$ such that

(1) $\eta_n \to \epsilon$,
(2) for any interval $I_n \in \eta_n$, the diffeomorphism

$$\Phi_n := \phi_n \circ R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1}$$

$(\frac{1}{nq_n^\sigma}, \frac{1}{n}, \frac{1}{n})$-distributes the interval I_n,

then the limit diffeomorphism f is weak mixing.

Proof. We use Lemma 3.4 to prove weak mixing. Consider partitions $\nu_n = \{\Gamma_n = H_{n-1} \circ g_n(I_n) \mid I_n \in \eta_n\}$, and let $\lambda_n = (H_{n-1} \circ g_n)^* \lambda$. By Lemma 3.5, ν_n converges to the decomposition into points.

Let an arbitrary square A and $\varepsilon > 0$ be fixed. In order to be able to apply Lemma 3.4, it is left to check condition (3.4) for any $\Gamma_n \in \nu_n$, with $f_n^{m_n} = H_n \circ S_{\alpha_{n+1}}^{m_n} \circ H_n^{-1} = H_{n-1} \circ g_n \circ \Phi_n \circ g_n^{-1} \circ H_{n-1}^{-1}$. By assumption (2) of the present lemma, for all $I_n \in \eta_n$, $\pi_r(\Phi_n(I_n)) \supset [-1/n, 1 - 1/n]$. Let S_n be a square of side length $q_n^{-\sigma}$, $S_n \subset \mathbb{T} \times [-1/n, 1 - 1/n]$. Consider

$$C_n := H_{n-1}(S_n).$$

Assumption (2) permits to apply Lemma 3.7. Then we have (estimating $\frac{1}{\lambda(J)} \leq 2$):

$$
\begin{aligned}
&|\lambda_n(\Gamma_n \cap f_n^{-m_n}(C_n)) - \lambda_n(\Gamma_n)\mu(C_n)| \\
&= |\lambda(I_n \cap \Phi_n^{-1} \circ g_n^{-1}(S_n)) - \lambda(I_n)\mu(S_n)| \\
&\leq \frac{1}{\lambda(J)}|\lambda(I_n \cap \Phi_n^{-1} \circ g_n^{-1}(S_n))\lambda(J) - \lambda(I_n)\mu(S_n)| + \frac{(1 - \lambda(J))}{\lambda(J)}\lambda(I_n)\mu(S_n) \\
&\leq 2\frac{8}{n}\lambda(I_n)\mu(S_n) + \frac{2}{n}\lambda(I_n)\mu(S_n) = \frac{18}{n}\lambda_n(\Gamma_n)\mu(C_n).
\end{aligned}
$$

By (3.5), we have for n sufficiently large $\operatorname{diam}(C_n) \leq \|D(H_{n-1})\|_0 \operatorname{diam}(S_n) \leq \frac{1}{2^n}$. Hence, for n large enough, one can approximate A by such sets C_n lying in $\mathbb{T} \times [1/n, 1 + 1/n]$. More precisely, for n large enough, there exist two sets, which are unions of sets C_n: $A_1 = \cup_{\sigma_1} C_n$, $A_2 = \cup_{\sigma_2} C_n$ such that

$$A_i \subset \mathbb{T} \times [1/n, 1 - 1/n], \quad A_1 \subset A \cap \mathbb{T} \times [1/n, 1 - 1/n] \subset A_2,$$

$$|\mu(A) - \mu(A_i)| \leq \frac{\varepsilon}{3}\mu(A).$$

Take n so that $\frac{18}{n} < \frac{\varepsilon}{3}$. Then we can estimate:

$$\lambda_n(\Gamma_n \cap f_n^{-m_n}(A)) - \lambda_n(\Gamma_n)\mu(A) \leq \lambda_n(\Gamma_n \cap f_n^{-m_n}(A_2)) - \lambda_n(\Gamma_n)\mu(A_2) + \frac{\varepsilon}{3}\lambda_n(\Gamma_n)\mu(A) \leq \frac{\varepsilon}{3}\lambda_n(\Gamma_n)\mu(A_2) + \frac{\varepsilon}{3}\lambda_n(\Gamma_n)\mu(A) \leq \varepsilon\lambda_n(\Gamma_n)\mu(A).$$

The lower estimate for this difference is obtained in the same way (using A_1). We have shown that, if n is sufficiently large, for an arbitrary $\Gamma_n \in \nu_n$ (3.4) holds. Then, by Lemma 3.4, f is weak mixing. □

4. Analytic case on the torus $\mathbb{T}^2$.

This section is devoted to the analytic construction on the torus $\mathbb{T}^2$. We recall the notations of the Theorem 1.5 that we want to prove. For an arbitrary fixed $\sigma > 0$, for any $n \in \mathbb{N}$:

$$\begin{aligned} \phi_n(\theta, r) &= (\theta, r + q_n^2 \cos(2\pi q_n \theta)), \\ g_n(\theta, r) &= (\theta + [n q_n^\sigma] r, r), \\ h_n &= g_n \circ \phi_n, \quad H_n = h_1 \circ \cdots \circ h_n, \\ f_n &= H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}. \end{aligned} \tag{4.1}$$

4.1. **Proof of convergence.** Let α, δ and σ be as in the statement of Theorem 1.5, and let $\rho > 0$ be fixed. Let $\alpha_n = p_n/q_n$ be a sequence such that $|\alpha - \alpha_n|$ is decreasing and

(P1) For all $n \in \mathbb{N}$,
$$|\alpha - \alpha_n| < \exp(-q_n^{1+3\sigma}).$$

By eventually extracting from α_n we can assume that this sequence also has the following properties:

(P2) Denote the lift of the inverse of the diffeomorphism H_n from (4.1) by $((H_n^{-1})_1, (H_n^{-1})_2)$, and set
$$\rho_n := \max_{i=1,2} \inf_{p \in \mathbb{Z}} \|(H_n^{-1})_i + p\|_\rho, \quad \rho_0 := \rho.$$
Then for all $n \in \mathbb{N}$,
$$q_n^\sigma \geq 4\pi n \rho_{n-1} + \ln(8\pi n q_n^{\sigma+4}).$$

(P3) With the definition of $\|DH\|_\rho$ of Section 2.2, we have for all $n \in \mathbb{N}$, and for all t such that $|t - \alpha| \leq |\alpha_n - \alpha|$,
$$q_n \geq \|D(H_{n-1}) R_t \circ H_{n-1}^{-1}\|_\rho.$$

(P4) For all $n \in \mathbb{N}$
$$\|D(H_{n-1})\|_0 \leq \ln q_n.$$

Properties (P2)–(P4) are possible to guarantee by choosing q_n sufficiently large because H_{n-1} does not depend on q_n.

The first three properties are used to prove the convergence, and the latter one is estimate (3.5), needed for the proof of weak mixing of the limit diffeomorphism, which will be done with the help of Proposition 3.9.

The following statement implies the convergence of the sequence f_n.

Lemma 4.1. *Suppose $\alpha_n = \frac{p_n}{q_n}$ satisfies (P1)–(P3) for some fixed $\sigma > 0$ and $\rho > 0$. Then, for any n large enough, we have:*
(a) the diffeomorphisms defined by (4.1) satisfy:
$$d_\rho(f_n, f_{n-1}) \leq \exp(-q_n);$$

(b) for any $m \leq q_{n+1}$ it holds:

$$d_0(f_n^m, f^m) \leq \frac{1}{2^n}.$$

Proof. With the notations above, using the Mean value theorem and (P3), we have (for some t between α_n and α_{n+1}):

$$\begin{aligned} d_\rho(f_n, f_{n-1}) &\leq \|(DH_{n-1})R_t \circ H_{n-1}^{-1}\|_\rho \|(h_n \circ R_{\alpha_{n+1}} \circ h_n^{-1} - R_{\alpha_n}) \circ H_{n-1}^{-1}\|_\rho \\ &\leq q_n \|h_n \circ R_{\alpha_{n+1}} \circ h_n^{-1} - R_{\alpha_n}\|_{\rho_{n-1}}. \end{aligned} \tag{4.2}$$

Denote $(\cos 2\pi q_n(z + \alpha_{n+1}) - \cos 2\pi q_n z)$ by $R(z)$. For an arbitrary $s \geq 0$, we can write:

$$\begin{aligned} \|R\|_s \leq & \|e^{2\pi i q_n z}\|_s |1 - e^{2\pi i q_n \alpha_{n+1}}| \leq 2\pi q_n \|e^{2\pi i q_n z}\|_s |\alpha_{n+1} - \alpha_n| \\ & \leq 4\pi q_n \|e^{2\pi i q_n z}\|_s |\alpha - \alpha_n|, \end{aligned} \tag{4.3}$$

(we used the estimate $|\alpha_{n+1} - \alpha_n| \leq 2|\alpha - \alpha_n|$). By the definition of h_n,

$$h_n \circ R_{\alpha_{n+1}} \circ h_n^{-1} - R_{\alpha_n} = ([nq_n^\sigma] q_n^2 R(\theta - [nq_n^\sigma]r) + (\alpha_{n+1} - \alpha_n)\, ,\, q_n^2 R(\theta - [nq_n^\sigma]r)).$$

Then

$$\|h_n \circ R_{\alpha_{n+1}} \circ h_n^{-1} - R_{\alpha_n}\|_s \leq 2n q_n^{2+\sigma} \|R(\theta - [nq_n^\sigma]r)\|_s.$$

By (4.3), it is less than

$$8\pi n q_n^{3+\sigma} \|\exp(2\pi i q_n(\theta - [nq_n^\sigma]r))\|_s |\alpha - \alpha_n|. \tag{4.4}$$

Applying (4.2), (4.4), (P2) and (P1) in sequence, we get:

$$\begin{aligned} d_\rho(f_n, f_{n-1}) \leq & q_n \|h_n \circ R_{\alpha_{n+1}} \circ h_n^{-1} - R_{\alpha_n}\|_{\rho_{n-1}} \\ \leq & 8\pi n q_n^{4+\sigma} \exp(4\pi n q_n^{1+\sigma} \rho_{n-1}) |\alpha - \alpha_n| \leq \exp(q_n^{1+2\sigma}) |\alpha - \alpha_n| \\ \leq & \exp(q_n^{1+2\sigma}(1 - q_n^\sigma)) \leq \exp(-q_n^{1+2\sigma}) < \exp(-q_n). \end{aligned}$$

The second part of the claim is proved in the same way. One has to note that $f_n^m = h_n \circ S_{\alpha_{n+1}}^m \circ h_n^{-1} = h_n \circ R_{m\alpha_{n+1}} \circ h_n^{-1}$, and

$$d_0(f^m, f_n^m) = \sum_{j=n}^{\infty} d_0(f_j^m, f_{j+1}^m).$$

□

4.2. **Proof of weak mixing.** For the proof of weak mixing, we shall use Proposition 3.9 that was proved in the previous section. In order to apply the lemma, we choose a sequence (m_n), $m_n \leq q_{n+1}$ (in this case, by Lemma 4.1 (b), (3.3) holds), and a sequence of standard partial decompositions (η_n) consisting of horizontal intervals with length less than $1/q_n$, $\eta_n \to \epsilon$, such that the diffeomorphism

$$\Phi_n := \phi_n \circ R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1} \tag{4.5}$$

$(\frac{1}{nq_n}, 0, \frac{1}{n})$-distributes any interval $I_n \in \eta_n$.

4.2.1. *Choice of the mixing sequence m_n.* We shall assume that

$$q_{n+1} \geq q_n^7.$$

Define

$$m_n = \min \left\{ m \leq q_{n+1} \mid \inf_{k \in \mathbb{Z}} \left| m \frac{q_n p_{n+1}}{q_{n+1}} - 1/2 + k \right| < \frac{q_n}{q_{n+1}} \right\}.$$

Note that the set of numbers m above is non-empty. Indeed, since p_{n+1} and q_{n+1} are relatively prime, the set $\{j \frac{q_n p_{n+1}}{q_{n+1}} \mid j = 0, \ldots q_{n+1}\}$ on the circle contains $\frac{q_{n+1}}{GCD(q_n, q_{n+1})}$, which is at least $\frac{q_{n+1}}{q_n}$, different equally distributed points.

We shall use the following estimate, which follows from the above assumption on the growth of q_n:

$$|m_n q_n \alpha_{n+1} - 1/2| (\mathrm{mod}\, 1) \leq \frac{q_n}{q_{n+1}} \leq q_n^{-6}. \tag{4.6}$$

4.2.2. *Stretching of the diffeomorphisms Φ_n.* Consider the set

$$B_n = \bigcup_{k=0}^{2q_n} \left[\frac{k}{2q_n} - \frac{1}{2q_n^{3/2}} \, , \, \frac{k}{2q_n} + \frac{1}{2q_n^{3/2}} \right]. \tag{4.7}$$

We shall see that Φ_n displays strong stretching in the vertical direction on small horizontal intervals, lying outside B_n. To do this, we shall use the notion of uniform stretch from [3], which we recall here.

Definition 4.2 (Uniform stretch). Given $\varepsilon > 0$ and $k > 0$, we say that a real function f on an interval I is (ε, k)-uniformly stretching on I if for $J = [\inf_I f, \sup_I f]$

$$\lambda(J) \geq k,$$

and for any interval $\tilde{J} \subset J$ we have:

$$\left| \frac{\lambda(I \cap f^{-1}\tilde{J})}{\lambda(I)} - \frac{\lambda(\tilde{J})}{\lambda(J)} \right| \leq \varepsilon \frac{\lambda(\tilde{J})}{\lambda(J)}.$$

The following criterion, that is easy to verify, gives a necessary and sufficient condition for a real function (of class at least C^2) to be uniformly stretching. The proof can be found in [3].

Lemma 4.3 (Criterion for uniform stretch). *If f satisfies:*

$$\begin{aligned} \inf_{x \in I} |f'(x)| \lambda(I) &\geq k, \\ \sup_{x \in I} |f''(x)| \lambda(I) &\leq \varepsilon \inf_I |f'(x)|, \end{aligned}$$

then f is (ε, k)-uniformly stretching on I.

Lemma 4.4. *Under the conditions of Theorem 1.5, the transformation Φ_n has a lift of the form:*

$$\Phi_n(\theta, r) = (\theta + m_n \alpha_{n+1}, \; r + \psi_n(\theta)),$$

where ψ_n satisfies:

$$\inf_{\mathbb{T} \setminus B_n} |\psi_n'| \geq q_n^{5/2}, \quad \sup_{\mathbb{T} \setminus B_n} |\psi_n''| \leq 9\pi^2 {q_n}^4. \tag{4.8}$$

Proof. By definition, Φ_n has the desired form with

$$\psi_n = q_n^2(\cos(2\pi(q_n\theta + m_n q_n \alpha_{n+1})) - \cos(2\pi q_n\theta)) = -2q_n^2\cos(2\pi q_n\theta) + \sigma_n,$$

where

$$\sigma_n = q_n^2(\cos(2\pi(q_n\theta + m_n q_n\alpha_{n+1})) - \cos(2\pi(q_n\theta + 1/2))).$$

With the help of the Mean value theorem and estimate (4.6), one easily verifies that $|\sigma_n'| < 1$, and $|\sigma_n''| < 1$.

Note that B_n are chosen in such a way that

$$\inf_{\mathbb{T}\setminus B_n} |\sin(2\pi q_n\theta)| \geq q_n^{-1/2}.$$

The statement follows by calculation. □

4.2.3. *Choice of the decompositions η_n.* Let us define a standard partial decompositions η_n of $\mathbb{T}^2$, meeting the conditions of Proposition 3.9.

Let $\hat\eta_n = \{I_n\}$ be the partial decomposition of $\mathbb{T}\setminus B_n$, containing all the intervals I_n such that

$$\psi_n(I_n) = [0,1) \mod 1.$$

We define $\eta_n = \{I \times \{r\} \mid I \in \hat\eta_n,\ r \in \mathbb{T}\}$. Note that, for any $I_n \in \eta_n$, we have: $\pi_r(\Phi(I_n)) = \mathbb{T}$.

Lemma 4.5. *Let η_n be defined as above. Then, for any $I_n \in \eta_n$,*

$$\lambda(I_n) \leq q_n^{-5/2},$$

and $\eta_n \to \epsilon$.

Proof. By Lemma 4.4, $\inf_{\mathbb{T}\setminus B_n} |\psi_n'| \geq q_n^{5/2}$. Therefore, $\lambda(I_n) \leq q_n^{-5/2}$ for any $I_n \in \eta_n$.

Since the diameter of the atoms of η_n goes to zero when n grows, it is enough to show that the total measure of the decompositions goes to 1 when n grows. The total measure of η_n equals:

$$\begin{aligned}\sum_{I_n\in\hat\eta_n} \lambda(I_n) &\leq 1 - \lambda(B_n) - 4q_n \max_{I_n\in\hat\eta_n}\lambda(I_n)\\ &\leq 1 - 2q_n(q_n^{-3/2} + 2q_n^{-5/2}) < 1 - 3q_n^{-1/2} \to 1.\end{aligned}$$

□

4.2.4. *Proof of weak mixing.* To prove weak mixing of f, we shall apply Proposition 3.9. Since (3.3) holds by Lemma 4.1, estimate (3.5) holds by Property (P4), the sequence of decompositions $\eta_n \to \epsilon$ by the lemma above, it is left to verify condition (2) of Proposition 3.9, which we pass to.

Lemma 4.6. *Let $I_n \in \eta_n$, Φ_n be as in (4.5). Then $\Phi_n(I_n)$ is $(\frac{1}{nq_n}, 0, \frac{1}{n})$-distributed.*

Proof. By the choice of η_n, $\pi_r(\Phi_n(I_n)) = \mathbb{T}$, and hence, δ in the definition of $(\gamma,\delta,\varepsilon)$-distribution can be taken equal to 0.

We have seen that Φ_n has a lift $\Phi_n(r,\theta) = (\theta + m_n\alpha, r + \psi_n(\theta))$. Hence, $\Phi_n(I_n)$ is contained in the vertical strip $(I_n + m_n\alpha)\times\mathbb{T}$. By the lemma above, $\lambda(I_n) \leq \frac{1}{q_n^{5/2}} < \frac{1}{nq_n}$ for any $I_n \in \eta_n$. Hence, we can take $\gamma = \frac{1}{nq_n}$.

Our fixed I_n has the form $I \times \{r\}$ for some $r \in \mathbb{T}$ and $I \in \hat\eta_n$. For any $J \subset \mathbb{T}$, the fact that $\Phi_n(\theta, r) \in \mathbb{T}\times J$ is equivalent to $\psi_n(\theta) \in J - r$. Lemma 4.4 implies the estimate:

$$\frac{\sup_{I_n\in\eta_n}|\psi_n''|}{\inf_{I_n\in\eta_n}|\psi_n'|}\lambda(I_n) \leq \frac{9\pi^2}{q_n} < \frac{1}{n}.$$

Then, by Lemma 4.3 (Criterion for uniform stretch), ψ_n is $(\frac{1}{n}, 1)$-uniformly stretching. Hence, for any interval $J \subset \mathbb{T}$, the following holds:

$$\begin{aligned}|\lambda(I_n \cap \Phi_n^{-1}(\mathbb{T} \times J)) - \lambda(I_n)\lambda(J)| =& |\lambda(I \cap \psi_n^{-1}(J - r)) - \lambda(I_n)\lambda(J)| \\ \leq& \frac{1}{n}\lambda(I_n)\lambda(J),\end{aligned}$$

and we take $\varepsilon = \frac{1}{n}$ in the definition of $(\gamma, \delta, \varepsilon)$-distribution. □

We have shown that Φ_n and η_n verify the conditions of Proposition 3.9. It implies that f is weak mixing.

5. C^∞-case on the torus, annulus and disc

Sections 5.1–5.4 are devoted to $M = \mathbb{A}$ and $M = \mathbb{T}^2$. The case of the disc $\mathbb{D}^2$ is studied in Section 5.5.

5.1. Statement of the result.

Take any $0 < \sigma < 1$. On $M = \mathbb{A}$, consider the following transformations:

$$\begin{aligned}&g_n(x, y) = (x + [nq_n^\sigma]y,\ y), \\ &h_n = g_n \circ \phi_n, \quad H_n = h_1 \circ \ldots \circ h_n, \\ &f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1};\end{aligned} \tag{5.1}$$

where the sequence $\alpha_n = p_n/q_n$, converging to α, and the diffeomorphisms ϕ_n, satisfying

$$R_{\frac{1}{q_n}} \circ \phi_n = \phi_n \circ R_{\frac{1}{q_n}}, \tag{5.2}$$

will be constructed in Section 5.2 below so that

Theorem 5.1. *For any Liouville number α, there exists a sequence α_n of rationals and a sequence ϕ_n of measure preserving diffeomorphisms satisfying (5.2) such that the diffeomorphisms f_n, constructed as in (5.1), converge in the sense of the* $\mathrm{Diff}^\infty(M)$ *topology, the limit diffeomorphism $f = \lim_{n\to\infty} f_n$ being weak mixing and $f \in \mathcal{A}_\alpha(M)$. Moreover, for any $\varepsilon > 0$, the parameters can be chosen so that*

$$d_\infty(f, R_\alpha) < \varepsilon.$$

Remark 5.2. This result implies Theorem 1.1. Indeed, it follows directly from Theorem 5.1, that weak mixing diffeomorphisms are dense in $\mathcal{A}_\alpha(M)$. It is a general fact (see [5]) that, in this case, weak mixing diffeomorphisms are generic in $\mathcal{A}_\alpha(M)$ with our topology.

5.2. Construction of ϕ_n.

We begin by constructing a "standard diffeomorphism" on the square $[-1, 1] \times [-1, 1] = [-1, 1]^2$, from which ϕ_n will be obtained by a rescaling of the domain of definition.

5.2.1. *Preliminary construction.* For a fixed $\varepsilon < 1/2$, consider the squares $\Delta = [-1, 1]^2$, $\Delta(\varepsilon) = [-1 + \varepsilon, 1 - \varepsilon]^2$ and $\Delta(2\varepsilon)$.

Lemma 5.3. *For any $\varepsilon < 1/2$ there exists a smooth measure-preserving diffeomorphism $\varphi = \varphi(\varepsilon)$ of $\mathbb{R}^2$, equal to the identity outside $\Delta(\varepsilon)$ and rotating the square $\Delta(2\varepsilon)$ by $\pi/2$.*

Proof. Let $\psi = \psi(\varepsilon)$ be a smooth transformation satisfying

$$\psi(\theta, r) = \begin{cases} (\theta, r) & \text{on } \mathbb{R}^2 - \Delta(\varepsilon), \\ (\theta/5, r/5) & \text{on } \Delta(2\varepsilon), \end{cases}$$

and η be a smooth transformation, such that

$$\eta(\theta, r) = \begin{cases} (r, -\theta) & \text{on } \{\theta^2 + r^2 \leq 1/3\}, \\ (\theta, r) & \text{on } \{\theta^2 + r^2 \geq 2/3\}. \end{cases}$$

Then the composition

$$\tilde{\varphi} := \psi^{-1} \eta \psi$$

provides the desired geometry. Moreover, it preserves the Lebesgue measure on the set

$$U = (\mathbb{R}^2 - \Delta(\varepsilon)) \cup \Delta(2\varepsilon).$$

However, it does not have to preserve the area on the whole of Δ. We describe now a deformation argument following Moser [8] that provides an *area-preserving* diffeomorphism φ on Δ, coinciding with $\tilde{\varphi}$ on U.

Let Ω_0 denote the usual volume form on $\mathbb{R}^2$, and consider $\Omega_1 := \tilde{\varphi}^* \Omega_0$. We shall find a diffeomorphism ν equal to the identity on the set U, and such that $\nu^* \Omega_1 = \Omega_0$.

Let $\Omega' = \Omega_1 - \Omega_0$, and note that $\Omega' = d(\omega_0 - \tilde{\varphi}^* \omega_0)$, where ω_0 is the standard 1-form $\frac{1}{2}(\theta dr - r d\theta)$. Consider the volume form

$$\Omega_t = \Omega_0 + t\Omega'.$$

Since it is non-degenerate, there exists a unique vector field X_t such that

$$\Omega_t(X_t, \cdot) = (\omega_0 - \tilde{\varphi}^* \omega_0)(\cdot). \tag{5.3}$$

One can integrate the obtained vector field to get the one-parameter family of diffeomorphisms $\{\nu_t\}_{t \in [0,1]}$, $\dot{\nu}_t = X_t(\nu_t)$, $\nu_0 = id$. Then $\nu = \nu_1$ is the desired coordinate change. Indeed, one verifies by calculation that

$$\frac{d}{dt} \nu_t^* \Omega_t = 0.$$

Hence, $\nu_1^* \Omega_1 = \nu_0^* \Omega_0 = \Omega_0$.

By an explicit verification, one obtained that $\tilde{\varphi}^*$ preserves the form ω_0 on U (for this note that $\tilde{\varphi}$ on U is an explicit linear transformation). Then on U equation (5.3) writes as $\Omega_t(X_t, \cdot) = 0$. Since Ω_t is non-degenerate, this implies that $X_t = 0$ on U, hence $\nu = \nu_0 = id$ on U, as claimed. The desired area-preserving diffeomorphism is

$$\varphi = \nu \tilde{\varphi}.$$

□

5.2.2. *Construction of ϕ_n.* Let us first define ϕ_n on the fundamental domain $D_n = [0, 1/q_n] \times [0, 1]$. The line $\theta = 1/2q_n$ divides D_n into halves: $D_n^1 = [0, 1/(2q_n)] \times [0, 1]$ and $D_n^2 = (1/(2q_n), 1/q_n) \times [0, 1]$. On D_n^1, consider the affine transformation $C_n(\theta, r) = (4q_n\theta - 1, 2r - 1)$, sending D_n^1 onto the square $\Delta = [-1, 1]^2$. Let φ_n be the diffeomorphism given by Lemma 5.3 with $\varepsilon = 1/(3n)$, and set

$$\phi_n := C_n^{-1} \circ \varphi_n \circ C_n. \tag{5.4}$$

We define $\phi_n = Id$ on D_n^2. Note that ϕ_n is smooth and area-preserving on D_n, and equals identity on the boundary of D. We extend it periodically to the whole $\mathbb{R}^2$ by the formula:

$$\phi_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ \phi_n, \quad \phi_n(\theta, r + 1) = \phi_n(\theta, r) + (0, 1).$$

The transformation ϕ_n, defined in this way, becomes a diffeomorphism both on $\mathbb{T}^2$ and on $\mathbb{A}$ in a natural way.

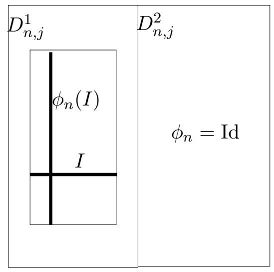

FIGURE 1. Action of ϕ_n

For a fixed n, let us denote by $D_{n,j}$ and $D^i_{n,j}$ (for $i = 1, 2,\ j \in \mathbb{Z}$) the shifts of the fundamental domain D_n of ϕ_n:

$$D_{n,j+q_n} = D_{n,j} = R_{\frac{j}{q_n}}(D_n), \text{ and } D^i_{n,j+q_n} = D^i_{n,j} = R_{\frac{j}{q_n}}(D^i_n).$$

5.2.3. *Notation.* For a diffeomorphism F of M (not necessarily homotopic to the identity), we shall denote by the same letter its lift of the form:

$$F(x, y) = (ax + by + f_1(x, y), cx + dy + f_2(x, y)),$$

where $f_i : \mathbb{R}^2 \to \mathbb{R}$ are, in the case of the torus, $\mathbb{Z}^2$-periodic with the property $\|f_i\|_0 = \inf_{p\in\mathbb{Z}} \|f_i + p\|_0$; and for the case of the annulus, f_i are $\mathbb{Z}$-periodic in the first component, and such that $\|f_1\|_0 = \inf_{p\in\mathbb{Z}} \|f_1 + p\|_0$. Note that the diffeomorphisms in our constructions are defined by their lifts, satisfying this property. For k-smooth diffeomorphisms $F : \mathbb{R}^2 \to \mathbb{R}^2$ we define by F_i the i-th coordinate function, and denote

$$|||F|||_k := \max\{\|D_a F_i\|_0,\ \|D_a(F^{-1})_i\|_{C^0} \mid i = 1, 2,\quad 0 \le |a| \le k\}.$$

5.2.4. *Discussion of the properties of ϕ_n.* We have constructed ϕ_n so that ϕ_n equals identity on $D^2_{n,j}$, $j \in \mathbb{Z}$, and on $D^1_{n,j}$ the image of any interval $I_{n,j} \times \{r\}$, where $r \in [1/(3n), 1 - 1/(3n)]$, and

$$I_{n,j} = \left[\frac{j}{q_n} + \frac{1}{6nq_n}, \frac{j}{q_n} + \frac{1}{2q_n} - \frac{1}{6nq_n}\right], \tag{5.5}$$

with $j = 0, \dots q_n - 1$, both under ϕ_n and ϕ_n^{-1}, is an interval of type $\{\theta\} \times [1/(3n), 1 - 1/(3n)]$ for some $\theta \in I_{n,j}$ (see Figure 1).

Moreover, the following holds:

LEMMA 5.4. *For all $k \in \mathbb{N}$ the diffeomorphisms ϕ_n constructed above satisfy:*

$$|||\phi_n|||_k \le c(n, k) q_n^k,$$

where $c(n, k)$ is independent of q_n.

Proof. The desired estimate follows from (5.4) by the product rule (it is important that φ_n is independent of q_n). □

REMARK 5.5. For any n, the construction implies that $\phi_n(\theta, r) = Id$ in the domains $0 \le r < 1/(6n)$ and $1 - 1/(6n) < r \le 1$. It is easy to verify that in the same domains diffeomorphisms f_n from (5.1) equal $R_{\alpha_{n+1}}$.

5.3. **Proof of convergence.** In the proof we shall use the following lemma:

LEMMA 5.6. *Let $k \in \mathbb{N}$, and h be a diffeomorphism of M. Then for all $\alpha, \beta \in \mathbb{R}$ we obtain*

$$d_k(hR_\alpha h^{-1}, hR_\beta h^{-1}) \leq C_k \|\!|h|\!\|_{k+1}^{k+1} |\alpha - \beta|, \tag{5.6}$$

where C_k only depends on k, and $C_0 = 1$.

Proof. We give the proof for the case $M = \mathbb{T}^2$; for the annulus, the proof is obtained by minor modifications. Note that $D_a h_i$ for $|a| \geq 1$ is $\mathbb{Z}^2$-periodic. Hence, for any $g : \mathbb{R}^2 \to \mathbb{R}^2$, we have: $\sup_{0<x,y<1} |(D_a h_i)(g(x,y))| \leq \|\!|h|\!\|_{|a|}$.

For $k = 0$, the statement of the lemma follows directly from the Mean value theorem.

We claim that for j with $|j| = k$ the partial derivative $D_j(h_i R_\alpha h^{-1} - h_i R_\beta h^{-1})$ will consist of a sum of terms with each term being the product of a single partial derivative

$$\big(D_a h_i\big)(R_\alpha h^{-1}) - \big(D_a h_i\big)(R_\beta h^{-1}) \tag{5.7}$$

with $|a| \leq k$, and at most k partial derivatives of the form

$$D_b h_j^{-1} \tag{5.8}$$

with $|b| \leq k$. This clearly holds for $k = 1$. We proceed by induction.

By the product rule we need only consider the effect of differentiating (5.7) and (5.8). Applying D_c with $|c| = 1$ to (5.7) we get:

$$\sum_{|b|=1} \Big(\big(D_b D_a h_i\big)(R_\alpha h^{-1}) - \big(D_b D_a h_i\big)(R_\beta h^{-1}) \Big) D_c h_b^{-1},$$

which increases the number of terms of the form (5.8) in the product by 1. Differentiating (5.8) we get another term of the form (5.8) but with $|b| \leq k + 1$.

Now we estimate:

$$\|\big(D_a h_i\big)(R_\alpha h^{-1}) - \big(D_a h_i\big)(R_\beta h^{-1})\|_0 \leq \|\!|h|\!\|_{|a|+1} |\alpha - \beta|,$$
$$\|D_c h_j^{-1}\|_0 \leq \|\!|h|\!\|_{|c|}.$$

Taking the inverse maps and applying the result we just proved gives (5.6). ☐

LEMMA 5.7. *For an arbitrary $\varepsilon > 0$, let k_n be a growing sequence of natural numbers, such that $\sum_{n=1}^\infty 1/k_n < \varepsilon$. Suppose that, in construction (5.1), we have: $|\alpha - \alpha_1| < \varepsilon$ and for any n*

$$|\alpha - \alpha_n| < \frac{1}{2k_n C_{k_n} \|\!|H_n|\!\|_{k_n+1}^{k_n+1}}, \tag{5.9}$$

where C_{k_n} are the constants from Lemma 5.6. Then the diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converge in the Diff$^\infty$ *topology to a measure preserving diffeomorphism f, and*

$$d_\infty(f, R_\alpha) < 3\varepsilon.$$

Moreover, the sequence of diffeomorphisms

$$\hat{f}_n := H_n \circ R_\alpha \circ H_n^{-1} \in \mathcal{A}_\alpha \tag{5.10}$$

also converges to f in the Diff$^\infty$ *topology, hence $f \in \mathcal{A}_\alpha$.*
Furthermore, if for a sequence of positive integers m_n we have for all n:

$$|\alpha - \alpha_n| < \frac{1}{2^{n+1} m_{n-1} \|\!|H_n|\!\|_1}, \tag{5.11}$$

then for any $m \le m_n$ we have

$$d_0(f^m, f_n^m) \le \frac{1}{2^n}. \tag{5.12}$$

Proof. By construction we have: $h_n \circ R_{\alpha_n} = R_{\alpha_n} \circ h_n$. Hence,

$$f_{n-1} = H_{n-1} \circ R_{\alpha_n} \circ H_{n-1}^{-1} = H_n \circ R_{\alpha_n} \circ H_n^{-1}.$$

By Lemma 5.6, for all k and n,

$$\begin{aligned} d_k(f_n, f_{n-1}) =& d_k(H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}, H_n \circ R_{\alpha_n} \circ H_n^{-1}) \\ \le& C_k |||H_n|||_{k+1}^{k+1} |\alpha_{n+1} - \alpha_n|. \end{aligned}$$

Estimating $|\alpha_{n+1} - \alpha_n| \le 2|\alpha - \alpha_n|$, and using assumption (5.9), we get for any $k \le k_n$:

$$d_k(f_n, f_{n-1}) \le d_{k_n}(f_n, f_{n-1}) \le \frac{2C_{k_n} |||H_n|||_{k_n+1}^{k_n+1}}{2k_n C_{k_n} |||H_n|||_{k_n+1}^{k_n+1}} \le \frac{1}{k_n}.$$

Hence, for any fixed k, the sequence (f_n) converges in Diff^k, and therefore, in Diff^∞. Moreover, one easily computes (using the definition of the d_∞-metric) that

$$d_\infty(f, R_\alpha) \le |\alpha - \alpha_1| + \sum_{n=1}^{\infty} d_\infty(f_n, f_{n-1}) < 3\varepsilon;$$

(here we denoted $f_0 = R_{\alpha_1}$).

To prove that $f \in \mathcal{A}_\alpha$, we show that the sequence of functions $\hat{f}_n \in \mathcal{A}_\alpha$ converge to f. For this it is enough to note that, for any n and $k \le k_n$, Lemma 5.6 and assumption (5.9) imply:

$$\begin{aligned} d_k(f_n, \hat{f}_n) =& d_k(H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}, H_n \circ R_\alpha \circ H_n^{-1}) \\ \le& C_{k_n} |||H_n|||_{k_n+1}^{k_n+1} |\alpha_{n+1} - \alpha| \le \frac{1}{k_n}. \end{aligned}$$

To prove the third statement of the lemma, note that for any $m \le m_{n-1}$,

$$\begin{aligned} d_0(f_n^m, f_{n-1}^m) =& d_0(H_n \circ R_{m\alpha_{n+1}} \circ H_n^{-1}, H_n \circ R_{m\alpha_n} \circ H_n^{-1}) \\ \le& |||H_n|||_1 2m|\alpha - \alpha_n| \le \frac{1}{2^n}. \end{aligned}$$

Then $d_0(f^m, f_{n-1}^m) \le \sum\limits_{i=n}^{\infty} d_0(f_i^m, f_{i-1}^m) = \frac{1}{2^{n-1}}$. □

Let a Liouville number α be fixed. Here we show that, for any given sequence k_n, the sequence of convergents α_n of α can be chosen so that (5.9) holds, and for any $m_{n-1} \le q_n$, (5.11) holds.

Lemma 5.8. *Fix an increasing sequence k_n of natural numbers, satisfying $\sum_{n=1}^{\infty} 1/k_n < \infty$, and let the constants C_n be as in Lemma 5.6. For any Liouville number α, there exists a sequence of convergents $\alpha_n = p_n/q_n$, such that the diffeomorphisms H_n, constructed as in (5.1) with these α_n and with ϕ_n given by (5.4), satisfy (5.9) and (5.11) with any $m_{n-1} \le q_n$. Further, we can choose α_n so that in addition (3.5)) holds.*

Proof. By Lemma 5.4, we have: $\|\phi_n\|_k \le c_1(n,k)q_n^k$. Then for h_n as in (5.1), we get:

$$\|h_n\|_k \le c_2(n,k)q_n^{2k}.$$

With the help of the Faa di Bruno's formula (that gives an explicit equation for the n-th derivative of the composition), we estimate:

$$\|H_n\|_k \le \|H_{n-1} \circ h_n\|_k \le c_3(n,k)q_n^{2k^2},$$

where $c_3(n,k)$ depends on the derivatives of H_{n-1} up to order k, which do not depend on q_n. Suppose that, for each n, q_n is chosen so that

$$q_n \ge c_3(n,n+1).$$

Then $\|H_n\|_{k_n+1} \le q_n^{2(k_n+1)^2+1} \le q_n^{3(k_n+1)^2}$. We choose the sequence of convergents of α satisfying

$$|\alpha - \alpha_n| = |\alpha - p_n/q_n| < \frac{1}{2^{n+1}k_n C_{k_n} q_n^{3(k_n+1)^3+1}};$$

the latter is possible since α is Liouville. Then

$$|\alpha - \alpha_n| < \frac{1}{2^{n+1}q_n k_n C_{k_n}\|H_n\|_{k_n+1}^{k_n+1}},$$

which implies both (5.9) and (5.11). As for (3.5), i.e. $\|DH_{n-1}\|_0 \le \ln q_n$, it is possible to have it just by choosing q_n large enough. □

5.4. **Proof of weak mixing.**

5.4.1. *Choice of the mixing sequence m_n.* We shall assume that for all n we have:

$$q_{n+1} \ge 10n^2 q_n. \tag{5.13}$$

Define, as in the analytic case,

$$m_n = \min\left\{ m \le q_{n+1} \mid \inf_{k\in\mathbb{Z}} \left| m\frac{q_n p_{n+1}}{q_{n+1}} - 1/2 + k \right| \le \frac{q_n}{q_{n+1}} \right\}.$$

Let $a_n = (m_n\alpha_{n+1} - \frac{1}{2q_n}) \mod \frac{1}{q_n}$. Then the choice of m_n and the growth condition (5.13) imply:

$$|a_n| \le \frac{1}{q_{n+1}} \le \frac{1}{10n^2 q_n}. \tag{5.14}$$

Hence, if we use the notation

$$\overline{D}^1_{n,j} = I_{n,j} \times [0,1] \subset D^1_{n,j},$$

we have

$$R^{m_n}_{\alpha_{n+1}}(\overline{D}^1_{n,j}) \subset D^2_{n,j'} \tag{5.15}$$

for some $j' \in \mathbb{Z}$.

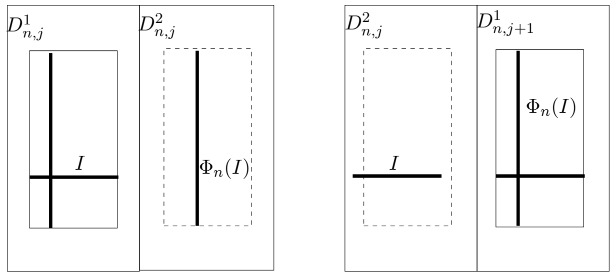

FIGURE 2. Action of Φ_n

5.4.2. *Choice of the decompositions η_n.* We define η_n to be the partial decomposition of M consisting of the horizontal intervals $I_{n,j} \times \{r\} \subset D^1_{n,j}$, where $r \in [1/(3n), 1-1/(3n)]$, defined by (5.5) and of the intervals $\overline{I}_{n,j} \times \{r\}$ with $r \in [1/(3n), 1-1/(3n)]$ and

$$\overline{I}_{n,j} = \left[\frac{j}{q_n} + \frac{1}{2q_n} + \frac{1}{6nq_n} - a_n, \frac{j+1}{q_n} - \frac{1}{6nq_n} - a_n\right].$$

It follows form (5.14) that the intervals $\overline{I}_{n,j} \times \{r\}$ are in $D^2_{n,j}$.

LEMMA 5.9. *The mapping $\Phi_n = \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ transforms the atoms of the decomposition η_n into vertical intervals of the form $\{\theta\} \times [1/(3n), 1-1/(3n)]$ for some θ.*

The proof is illustrated on Figure 2.

Proof. Consider first an interval I_n of the type $I_n = I_{n,j} \times \{r\}$, $r \in [1/(3n), 1-1/(3n)]$. By construction of ϕ_n (see §5.2.4), we have that $\phi_n^{-1}(I_n)$ is a vertical segment of the form $\{\theta\} \times [1/(3n), 1-1/(3n)]$ for some $\theta \in I_{n,j}$. From (5.15) we deduce that $R_{\alpha^{m_n}_{n+1}} \circ \phi_n^{-1}(I_n) = \{\theta'\} \times [1/(3n), 1-1/(3n)] \subset D^2_{n,j'}$, for some $\theta' \in \mathbb{T}$ and $j' \in \mathbb{Z}$ and we conclude using that ϕ_n acts as the identity on $D^2_{n,j'}$.

Similarly, for $r \in [1/(3n), 1-1/(3n)]$ and an interval $I_n = \overline{I}_{n,j} \times \{r\} \in D^2_{n,j}$, we have that

$$\phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}(I_n) = \phi_n \circ R^{m_n}_{\alpha_{n+1}}(I_n) = \phi_n(I_{n,j'} \times \{r\}) = \{\theta\} \times [1/(3n), 1-1/(3n)],$$

for some $j' \in \mathbb{Z}$ and $\theta \in \mathbb{T}$. □

5.4.3. *Proof of Theorem 5.1.* Let the diffeomorphisms f_n be constructed as in (5.1), following Lemma 5.7 and Lemma 5.8, so that convergence of f_n, closeness to Identity of their limit f, as well as (3.3) and (3.5), hold. We want to apply Proposition 3.9 to get weak mixing. Since the sequence of decompositions $\eta_n \to \epsilon$ by construction, and since it consists of intervals with length less than $1/q_n$, to finish it is enough to show that for any interval I_n of the decomposition η_n, and for $\Phi_n = \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$, we have: $\Phi_n(I_n)$ is $(0, 2/(3n), 0)$-distributed. The conditions of the definition follow immediately from the construction and Lemma 5.9. Indeed, the projection of $\Phi_n(I_n)$ to the r-axis is the interval $[1/(3n), 1-1/(3n)]$, hence, in the definition of $(\gamma, \delta, \varepsilon)$-distribution (Definition 3.6) we can take $\delta = 2/(3n)$. Furthermore, since the image of any interval I_n is vertical, γ can be taken equal to 0. Finally, the restriction of Φ_n to I_n being affine, one verifies that for any interval $\tilde{J}_n \subset J_n$:

$$\lambda(I \cap \Phi_n^{-1}(\tilde{J}))\lambda(J) = \lambda(I)\lambda(\tilde{J}).$$

Hence, we take $\varepsilon = 0$.

We have verified the conditions of Proposition 3.9. This implies weak mixing of the limit diffeomorphism f. □

References

[1] D.V. Anosov and A.B. Katok; *New examples in smooth ergodic theory. Ergodic diffeomorphisms*, Trans. Moscow Math. Soc. **23** (1970), p. 1–35.

[2] B. Fayad; *Weak mixing for reparametrized linear flows on the torus*, Ergodic Theory and Dynam. Systems, volume **22** (2002), p. 187–201.

[3] B. Fayad; *Analytic mixing reparametrizations of irrational flows*, Ergodic Theory Dynam. Systems, **22** (2002), p. 437–468.

[4] B. Fayad and A. Katok; *Constructions in elliptic dynamics*, Ergodic Theory and Dynam. Systems **24**, volume dedicated to the memory of Michael Herman, (2004), p. 1477–1520.

[5] P. R. Halmos; *Lectures in ergodic theory*, Japan Math. Soc. (1956).

[6] A. Katok and E. Robinson Jr.; *Cocycles, cohomology and combinatorial constructions in ergodic theory*, Proc. Symp. Pure Math 69, Smooth ergodic theory and its applications (Seattle, WA, 1999), 107–173, Amer. Math. Soc., Providence, RI, 2001.

[7] A. N. Kolmogorov; *On dynamical systems with an integral invariant on the torus*, Doklady Akad. Nauk SSSR (N.S.), **93** (1953), p. 763–766.

[8] J. Moser; *On the volume elements on a manifold,* Trans. Amer. Math. Soc. **120** (1965), p. 286–294.

[9] M. Saprykina; *Analytic non-linearizable uniquely ergodic diffeomorphisms on* $\mathbb{T}^2$, Ergodic Theory and Dynam. Systems **23** (2003), no. 3, p. 935–955.

[10] M. D. Šklover; *Classical dynamical systems on the torus with continuous spectrum*, Izv. Vysš. Učebn. Zaved. Matematika **10** (1967), p. 113–124.

[11] A. Windsor; *Minimal but not uniquely ergodic diffeomorphisms*, in *Smooth Ergodic Theory and its applications*, Proc. Symp. Pure Math, AMS, (2001), p. 809–824.

Bassam Fayad, Université Paris 13, CNRS 7539, Villetaneuese 93430
E-mail address: fayadb@math.univ-paris13.fr

Maria Saprykina, Department of Mathematics, University of Toronto, Toronto, ON, Canada M5S 3G3
E-mail address: masha@math.kth.se