DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
doi:10.3934/dcds.2021129

REALIZING ARBITRARY d-DIMENSIONAL DYNAMICS BY RENORMALIZATION OF C^d-PERTURBATIONS OF IDENTITY

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(Communicated by Jairo Bochi)

ABSTRACT. Any C^d conservative map f of the d-dimensional unit ball $\mathbb{B}^d$, $d \geq 2$, can be realized by renormalized iteration of a C^d perturbation of identity: there exists a conservative diffeomorphism of $\mathbb{B}^d$, arbitrarily close to identity in the C^d topology, that has a periodic disc on which the return dynamics after a C^d change of coordinates is exactly f.

What kind of dynamics can be *realized by renormalized iteration* of some diffeomorphism F of the unit ball that is close to identity? Given a d-dimensional C^r-diffeomorphism F, its *renormalized iteration* is an iteration of F, restricted to a certain d-dimensional ball and taken in some C^r-coordinates in which the ball acquires radius 1.

This natural question can be traced back to a celebrated paper by Ruelle and Takens [7], where it appeared in connection to the mathematical notion of turbulence. From a subsequent paper by Newhouse, Ruelle and Takens [6] it can be seen that any dynamics of class C^d on the d-dimensional torus $\mathbb{T}^d$ can be realized by renormalized iteration of a C^d-small perturbation of the identity map on $\mathbb{T}^d$. In [8] the straightforward strategy of [7] that leads to the latter result is clearly explained. We will reproduce this sketch below. This result is specific to tori. As D.Turaev points out in [8], it implies that on an arbitrary manifold M of dimension $d \geq 2$, arbitrary d-dimensional dynamics can be implemented by iterations of C^{d-1}-close to identity maps of $\mathbb{B}^d$, but the construction gives no clue of whether the same can be said about the C^d-close to identity maps.

The present note shows that the construction of [6] can be enhanced by an application of a method in the spirit of Moser [5] (or Anosov-Katok [1]), to get realization by renormalized iteration of $C^{d+\varepsilon}$-close to identity maps of $\mathbb{B}^d$.

2020 *Mathematics Subject Classification.* Primary: 37C15.

Key words and phrases. Realization by iterations, renormalization, Anosov-Katok method.

B. Fayad was supported in part by Knut and Alice Wallenberg foundation, grant KAW 2016.0403, and by the ANR-15-CE40-0001. M.Saprykina was supported in part by the Swedish Research Council, VR 2015-04012.

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More precisely, let $\mathbb{B}^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ denote the unit ball, and μ stand for the Lebesgue measure on it; for $r \in \mathbb{N} \cup \{\infty\}$ we denote by $\mathrm{Diff}^r_\mu(\mathbb{B}^d)$ the set of diffeomorphisms of class C^r of $\mathbb{B}^d$, preserving the boundary and μ.

Theorem 1. *For any natural $d \geq 2$ there exists $\varepsilon_0 > 0$ (one can take $\varepsilon_0 = (d+1)^{-4}$) such that the following holds.*

For any $0 < \varepsilon < \varepsilon_0$, any $\delta > 0$, any $f \in \mathrm{Diff}^{d+\varepsilon}_\mu(\mathbb{B}^d)$ there exists $F \in \mathrm{Diff}^{d+\varepsilon}_\mu(\mathbb{B}^d)$ and a periodic sequence of balls $B, F(B), \dots, F^M(B) = B \subset \mathbb{B}^d$ such that

- *$\|F - \mathrm{Id}\|_{d+\varepsilon} \leq \delta$;*
- *$h \circ F^M \circ h^{-1} = f$, where h is the similarity that sends B to $\mathbb{B}^d$.*

It should be mentioned that this theorem does not hold for $d = 1$, see e.g., [8], Sec.1.

We note that in [6], the restriction on smoothness is removed, to the price of realizing d-dimensional dynamics by renormalizing $(d+1)$-dimensional maps on some d-dimensional embedded manifold.

Following up on [7, 6] and his own work on universality, Turaev in [8] asks whether an arbitrary d-dimensional dynamics can be realized by iterations of a C^r-close to identity map of B^d and a large r? Theorem 1 does not say anything for $r > d + \varepsilon$. While a more careful application of the same tools of its proof may yield the same statement in $C^{d+1-\varepsilon}$ regularity, dealing with higher regularities will certainly require new ideas.

Related results. In the present paper we are concerned with *exactly realizing* and not just *approximating* any given map. Approximating any given dynamics by renormalization (on almost periodic discs) is a very interesting topic with a vast list of related results. For instance, D. Turaev in [8], shows that for any $r \geq 1$ the renormalized iterations of C^r-close to identity maps of an n-dimensional unit ball $\mathbb{B}^n$ ($n \geq 2$) form a residual set among all orientation-preserving C^r-diffeomorphisms $\mathbb{B}^n \to \mathbb{R}^n$. As an application he shows that any generic n-dimensional dynamical phenomenon can be arbitrarily closely (in C^r) approximated by iterations of C^r-close to identity maps, with the same dimension of the phase space. We refer to Turaev's paper for an account and for references. We just mention that recently, a highlight of the universality approach was the construction by Berger and Turaev in [2] of smooth conservative disc diffeomorphisms that are arbitrarily close to identity in any regularity and that have positive metric entropy, thus solving a conjecture made by Herman in [4].

An application to universality in the neighborhood of an elliptic fixed point of a area preserving surface diffeomorphisms. Anatole Katok (personal communication) observed, that any area preserving surface diffeomorphism that has an elliptic periodic point can be perturbed in C^r topology (arbitrary r) so that the perturbed diffeomorphism is area preserving and has a periodic disc on which the return dynamic is identity. Hence, we obtain the following consequence of Theorem 1.

Corollary 1. Any $C^{2+\varepsilon}$ area preserving diffeomorphism g of a surface that has an elliptic periodic point can be perturbed in the $C^{2+\varepsilon}$ topology into an area preserving diffeomorphism $\tilde{g}$ that has a periodic disc on which the renormalized dynamics is equal to any prescribed $F \in \mathrm{Diff}^{2+\varepsilon}_\mu(\mathbb{B}^2)$.

In other words, arbitrarily close (in the sense of $C^{2+\varepsilon}$) to any area preserving diffeomorphism with an elliptic periodic point one can find any prescribed dynamics.

We do not know whether every area preserving surface diffeomorphism that is not uniformly hyperbolic can be perturbed in the C^2 topology into one that has elliptic periodic points (this is known to hold in C^1 topology). Would this be proved, Corollary 1 would imply that any area preserving surface diffeomorphism that is not uniformly hyperbolic can be perturbed in C^2 topology into one that contains any prescribed dynamics.

Proofs. Our construction follows in part the straightforward approach of [7, 6] of fragmenting the target dynamics into a composition of a large number of close to identity maps that are then reproduced (up to rescaling) on a sequence of pairwise disjoint small balls. Thus, we start by presenting the constructions from [7, 6], and then introduce and explain the changes we had to make to carry out the construction in slightly higher regularity.

1. Fragmentation. The first ingredient of the proof is the following proposition of [7]; see [6] or Turaev [8], page 2, for a comprehensive illustration.

Proposition 1 (Fragmentation)**.** *Given $r \in \mathbb{N}$, $f \in \mathrm{Diff}^r_\mu(\mathbb{B})$, for any $M \in \mathbb{N}$ there exists $f_0, \dots, f_{M-1} \in \mathrm{Diff}^r_\mu(\mathbb{B})$ and a constant $C(r, f) > 0$ such that*

- $\|f_i - \mathrm{Id}\|_r \le C(r, f)M^{-1}$,
- $f = f_0 \circ \dots \circ f_{M-1}$.

Proof. Consider a Lipschitz isotopy $\psi : [0, 1] \to \mathrm{Diff}^r_\mu(\mathbb{B})$ (Lipschitz in $t \in [0, 1]$) such that $\psi_0 = \mathrm{Id}$ and $\psi_1 = f$. Let $f_i = \psi^{-1}_{i/M} \circ \psi_{(i+1)/M}$ for $i = 0, \dots, M - 1$. The estimate is straightforward. □

2. Idea of the proof by [7, 6]. Given $\delta > 0$, fix an integer $A > 2/\delta$, and consider an A^{d-2}-periodic translation on a torus $\mathbb{T}^{d-1}$, defined by

$$S^t(X) = X + t(1, \frac{1}{A}, \dots, \frac{1}{A^{d-2}}).$$

Let γ be the closed invariant curve of S^t passing through the origin. The turbular neighbourhood of γ of radius $\frac{1}{3A}$ does not intersect itself. Let D be the $(d-1)$-dimensional ball of radius $\rho := \frac{1}{4A}$ centred at the origin, and let $D_i = S^{\frac{i}{A}}(D)$ for $i = 0, \dots, A^{d-1} - 1$. Then S^t is an isometry, $S^{A^{d-2}}(D) = D$, and all D_i for $i = 0, \dots, A^{d-1} - 1$ are disjoint. In other words, D is the base of a periodic tower of discs for the map $S^{\frac{1}{A}}$. The height of the tower is A^{d-1}.

To prove the theorem of [6], given $\varepsilon > 0$ and a map $f \in \mathrm{Diff}^{d-\varepsilon}_\mu(\mathbb{T}^{d-1})$, one applies Proposition 1 with $M = A^{d-1}$ in order to obtain $f_0 \dots, f_{M-1}$ with $\|f_i - \mathrm{Id}\|_d \le C(d, f)M^{-1}$ such that $f = f_0 \circ \dots \circ f_{M-1}$. Let h_i be a similarity sending D_i into the unit ball $\mathbb{B}^{d-1}$ (i.e., h_i expands linearly by $1/\rho$), and define the desired map by $F|_{D_i} = S^{\frac{1}{A}} \circ h_i^{-1} \circ f_i \circ h_i$ for $i = 0, \dots, M - 1$; extend F by $S^{\frac{1}{A}}$ to the whole $\mathbb{T}^{d-1}$. One easily estimates

$$\|F - \mathrm{Id}\|_{d-\varepsilon} \le \|F - S^{\frac{1}{A}}\|_{d-\varepsilon} + \delta/2 \le$$
$$\rho^{-(d-1-\varepsilon)} \max_i \|f_i - \mathrm{Id}\|_{d-\varepsilon} \le (4A)^{(d-1-\varepsilon)} M^{-1} + \delta/2 = c_0 A^{-\varepsilon} + \delta/2,$$

which is smaller than δ for large A.

In order to prove the analogous result on $\mathbb{B}^d$ instead of $\mathbb{T}^{d-1}$, one can embed the set $[0,1] \times \mathbb{T}^{d-1}$ into $\mathbb{B}^d$ and do the same construction (extending the balls to d-dimensional ones). Then on $\mathbb{B}^d$ one gets a realization $\tilde{F}$ such that $\|\tilde{F} - \mathrm{Id}\|_{d-\varepsilon}$ is small, exactly as for the torus $\mathbb{T}^{d-1}$.

3. Permutation map. To increase the smoothness of the realization for $\mathbb{B}^d$, we will find a longer sequence of d-dimensional balls B_i, $i = 0, \dots, M-1$, of radius ρ, with

$$\rho = \frac{1}{4A}, \quad M = qA^{d-1},$$

where q is of order A^{ε_0} for a certain $\varepsilon_0 = \varepsilon_0(d)$, and a transformation T mapping each B_i into B_{i+1} with $T^M(B_0) = B_0$ (the transformation T plays the role of $S^{\frac{1}{A}}$ in the argument above). The increase of M together with a relatively tame estimate for the norm of T permits us to estimate the closeness of approximation in the $C^{d+\varepsilon}$-norm. We are able to ensure that T maps B_i into B_{i+1} isometrically not for all, but for *a large proportion* of the iterates $i = 1, \dots M$, and this is enough for our purposes.

Definition 0.1 (Funny periodic tower of balls)**.** For $z \in \mathbb{B}^d$, $\eta > 0$ we denote by $B(z, \eta)$ the ball of radius η around z. For $T \in \mathrm{Diff}^r_\mu(\mathbb{B}^d)$, $N \in \mathbb{N}$, $\eta, \gamma > 0$, we say that $B = B(z, \eta)$ is a base of an (η, N, γ)-funny periodic tower of discs for T if

- All $B_i := T^i B$, $i = 1, \dots, N-1$, are disjoint;
- $T^N B = B$, and $T^N : B \to B$ is the Identity map,
- at least $[\gamma N]$ integers $n_i \in [1, N]$ are isometry times for T in the sense that T^{n_i} is an isometry from B to the disc $B_i := T^{n_i} B$.

The terminology *funny* periodic towers is borrowed from the notion of funny rank one introduced by J.-P. Thouvenot to weaken the notion of rank one systems (see [3]).

Proposition 2 (Permutation map)**.** *For any $d \geq 2$, $r \geq 1$, $A, q \in \mathbb{N}$ such that $q^{r^4} \ll A$, there exists $T \in \mathrm{Diff}^\infty_\mu(\mathbb{B}^d)$ such that*

- $\|T - \mathrm{Id}\|_r \leq \frac{q^{r^4}}{A}$,
- *T has a $(\frac{1}{4A}, \frac{1}{2}qA^{d-1}, 1/2)$-funny periodic tower of discs.*

Proof. Embedding a "fat torus" $\mathbb{F}$ into $\mathbb{B}^d$. For a fixed $d \geq 2$, denote by $\mathbb{F}$ a "fat torus":

$$\mathbb{F} = [0,1] \times \mathbb{T}^{d-1},$$

where $\mathbb{T}^{d-1} = \mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$. We will denote the natural Haar-Lebesgue measure on $\mathbb{F}$ by μ, and the coordinates in $\mathbb{F}$ by $(x, y, z) \in [0,1] \times \mathbb{T} \times \mathbb{T}^{d-2}$. (In two dimensions a "fat torus" $\mathbb{F}^2$ is an annulus with coordinates (x, y).)

Given $q > 1$, consider an open set $P \in \mathbb{B}^d$ such that

- P contains a cube $Q_q + \vec{s}$, where $Q_q := (0, 1 - \frac{1}{q})^d$ is a standard cube, and $\vec{s} \in \mathbb{R}^d$ is a constant vector;
- P is C^∞-diffeomorphic to $\mathbb{F}$;
- $\mu(P) = 1$.

Let $h : P \to \mathbb{F}$ be a C^∞ diffeomorphism, transforming the Lebesgue measure on $\mathbb{B}^d$ to that on $\mathbb{F}$, and such that h is an isometry on the cube Q_q. Such a volume-preserving map exists by [5] or [1].

PARTITION OF $\mathbb{F}$. Let

$$R_q = [0,1] \times \left[0, \frac{1}{q}\right] \times \mathbb{T}^{d-2} \subset \mathbb{F},$$

$$\Delta_q = [0.1\,, 0.9] \times \left[\frac{1}{10q}\,, \frac{9}{10q}\right] \times \mathbb{T}^{d-2} \subset R_q.$$

Extend this construction $\frac{1}{q}$-periodically in y to the whole $\mathbb{F}$ to obtain a partition of $\mathbb{F}$ into thin rectangular blocks.

SWITCHING TO A "LONGER" FAT TORUS $\tilde{\mathbb{F}}_q$. Consider another fat torus

$$\tilde{\mathbb{F}}_q = \left[0, \frac{1}{q}\right] \times (\mathbb{R}/(q\mathbb{Z})) \times \mathbb{T}^{d-2}$$

with coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ corresponding to the above splitting. Let

$$\tilde{R}_q = \left[0, \frac{1}{q}\right] \times [0,1] \times \mathbb{T}^{d-2} \subset \tilde{\mathbb{F}}_q,$$

$$\tilde{\Delta}_q = \left[\frac{1}{10q}\,, \frac{9}{10q}\right] \times [0.1\,, 0.9] \times \mathbb{T}^{d-2} \subset \tilde{R}_q.$$

Notice that $\tilde{\Delta}_q$ can be mapped into Δ_q by an isometry. Extend this construction 1-periodically in $\tilde{y}$ to the whole $\tilde{\mathbb{F}}_q$.

We define the (generalized) circle actions on $\mathbb{F}$ and $\tilde{\mathbb{F}}_q$, respectively:

$$S_t(x, y, z_1 \dots z_{d-2}) = (x, y, z_1 \dots z_{d-2}) + t\left(0, 1, \frac{1}{A}, \dots, \frac{1}{A^{d-2}}\right),$$

$$\tilde{S}_t(\tilde{x}, \tilde{y}, \tilde{z}_1 \dots \tilde{z}_{d-2}) = (\tilde{x}, \tilde{y}, \tilde{z}_1 \dots \tilde{z}_{d-2}) + t\left(0, 1, \frac{1}{qA}, \dots, \frac{1}{qA^{d-2}}\right).$$

We now define h_q to be a volume-preserving map from $\mathbb{F}$ to $\tilde{\mathbb{F}}_q$ such that

- h_q maps the set $\mathbb{F}|_{y=0}$ to $\tilde{\mathbb{F}}_q|_{\tilde{y}=0}$;
- h_q acts as identity on the last $d-2$ components;
- $h_q(R_q) = \tilde{R}_q$, $h_q \circ S_{\frac{1}{q}} = \tilde{S}_1 \circ h_q$;
- $h_q(\Delta_q) = \tilde{\Delta}_q$, and h_q acts as an isometry in restriction to Δ_q;
- $\|h_q\|_r \leq q^{2r}$.

DEFINITION OF THE DESIRED MAP T.

Let $\varphi : (0, q^{-1}) \to \mathbb{R}$ of class C^r be such that

$$\varphi(\tilde{x}) = \begin{cases} 1/A, & \text{for } \tilde{x} \in [0.3q^{-1}, 0.7q^{-1}] \\ 0 & \text{for } \tilde{x} \in (0, 0.2q^{-1}] \cup [0.8q^{-1}, q^{-1}). \end{cases}$$

It is easy to construct such a function satisfying $\|\varphi\|_r \leq q^{2r}/A$.

Define on $\tilde{\mathbb{F}}_q$ the shear map

$$\tilde{T}_\varphi(\tilde{x}, \tilde{y}, \tilde{z}_1 \dots \tilde{z}_{d-2}) = (\tilde{x}, \tilde{y}, \tilde{z}_1 \dots \tilde{z}_{d-2}) + \varphi(\tilde{x}) \cdot \left(0, 1, \frac{1}{qA}, \dots, \frac{1}{qA^{d-2}}\right).$$

Finally, let $T : P \mapsto P$ be defined by

$$T = h^{-1} \circ h_q^{-1} \circ \tilde{T}_\varphi \circ h_q \circ h.$$

The fact that $\tilde{T}_\varphi$ equals identity close to the boundary of $\tilde{\mathbb{F}}_q$ implies that T equals identity close to the boundary of P, and can thus be extended by identity to the whole $\mathbb{B}^d$.

The obtained transformation $T : \mathbb{B}^d \mapsto \mathbb{B}^d$ satisfies the conclusion of Proposition 2. Indeed, we have that

$$\|T - \mathrm{Id}\|_r \leq$$
$$\left(\|h_q\|_{r+1}\|h_q^{-1}\|_{r+1}\|h\|_{r+1}\|h^{-1}\|_{r+1}\right)^{r+1} \|\tilde{T}_\varphi - \mathrm{Id}\|_{r+1} \leq \frac{q^{r^4}}{A}.$$

As for the funny periodic tower, let $\tilde{B}$ be the disc of radius $\rho = \frac{1}{4A}$ centred at $(\tilde{x}, \tilde{y}, \tilde{z}) = (\frac{1}{2q}, 0, 0)$. Take for the base of the tower of T the ball $B := h^{-1} \circ h_q^{-1}(\tilde{B})$. Under $\tilde{T}_\varphi$, the center of $\tilde{B}$ is translated by $1/A$ along the trajectory of the shift $\tilde{S}$ passing through this point. It is easy to see that the tubular neighborhood of radius $1/(3A)$ of this trajectory does not intersect itself, and $\tilde{S}^{qA^{d-1}} = \mathrm{Id}$. Moreover, $\tilde{T}_\varphi$ is an isometry on this neighborhood. Therefore, $\tilde{B}$ is the base of a tower by discs of height qA^{d-1} for $\tilde{T}_\varphi$ on $\tilde{\mathbb{F}}$. Moreover the times $n_i \in [0, A^{d-1}q - 1]$ such that $\tilde{T}_\varphi^{n_i}(\tilde{\mathbb{F}}) \in \cup_{\ell=0}^{q-1} \tilde{S}_\ell \tilde{\Delta}_q$ represent a proportion of around $8/10$ of $A^{d-1}q$, hence clearly more than $1/2$ of $n \in [0, A^{d-1}q]$ are isometric times for T. For such times $T^{n_i}|_B = h^{-1} \circ h_q^{-1} \circ \tilde{T}_\varphi^{n_i} \circ h_q \circ h$ is a composition of several isometries and is thus an isometry. □

4. Proof of Theorem 1. Before we proceed to the proof, we need a short calculation.

Lemma 0.2. *Let t and g be two maps in $\mathrm{Diff}^r_\mu(\mathbb{B}^d)$. Suppose that*

$$\|g - \mathrm{Id}\|_r \leq \alpha < 1, \quad \|t - \mathrm{Id}\|_r \leq \beta < 1.$$

Then there exists a constant $C_1 = C_1(r, d)$ such that

$$\|t \circ g - \mathrm{Id}\|_r \leq \alpha + C_1\beta.$$

Proof. Since $\|g - \mathrm{Id}\|_r \leq \alpha < 1$, we have $\|g\|_1 \leq \|g - \mathrm{Id}\|_r + \|\mathrm{Id}\|_r \leq 2$, and for $1 < k \leq r$ any partial derivative $g^{(\vec{k})}$ of g of order $k = |\vec{k}|$ we have $\|g^{(\vec{k})}\|_0 \leq \|g - \mathrm{Id}\|_r \leq \alpha$. Notice that

$$\|t \circ g - \mathrm{Id}\|_r \leq \|g - \mathrm{Id}\|_r + \|(t - \mathrm{Id}) \circ g\|_r \leq \alpha + \|(t - \mathrm{Id}) \circ g\|_r.$$

To estimate the last term, notice that any partial derivative of the map $(t - \mathrm{Id}) \circ g$ is a sum (with universal constants as coefficients) of terms of the following type:

$$(t - \mathrm{Id})^{(\vec{k})} \circ g \cdot \prod_j (g^{(\vec{l}_j)})^{p_j},$$

where $g^{(\vec{l}_j)}$ denotes a partial derivative of order $l_j := |\vec{l}_j| \in \mathbb{Z}^d$, and p_j stands for the power; here we have $l_j \leq r$ and $p_j \leq r$. It is easy to see by induction that, in each product, the number of derivatives of g of order one (counted with the power) is less than r. Since, by assumption, $\|t - \mathrm{Id}\|_r \leq \beta$, $\|g\|_1 \leq 2$ and the 0-norm of all the derivatives of order higher than one is not larger than 1, each term can be estimated by $2^r\|t-\mathrm{Id}\|_r$. Thus, there exists a universal constant $C_1 = C_1(d,r)$ such that

$$\|(t-\mathrm{Id})\circ g\|_r \le C_1\|t-\mathrm{Id}\|_r \le C_1\beta.$$

□

Proof of Theorem 1. Given $d \ge 2$ and $\delta > 0$, fix any $0 < \varepsilon < \varepsilon_0 := \frac{1}{(d+1)^4}$, let $r := d+\varepsilon$, and assume that $f \in \mathrm{Diff}^r_\mu(\mathbb{B})$.

Let $C = C(r,d,f)$ be the constant from Proposition 1, and let $C_0 = 4^r C$. Let $C_1 = C_1(d,r)$ be the constant from Lemma 0.2. Since $\varepsilon < \frac{1}{(d+\varepsilon)^4} = \frac{1}{r^4}$, for sufficiently large integer A there exists an integer q such that

$$2C_0A^{\varepsilon}(\delta)^{-1} < q < (\delta A/(2C_1))^{1/r^4}.$$

Then we have

$$C_0 q^{-1}A^{\varepsilon} \le \delta/2, \quad C_1\frac{q^{r^4}}{A} \le \delta/2. \tag{1}$$

Let $M = \frac{1}{2}qA^{d-1}$ and apply Proposition 1 to get $f_0,\dots,f_{M-1} \in \mathrm{Diff}^{d+\varepsilon}_\mu(\mathbb{B}^d)$ such that $\|f_i-\mathrm{Id}\|_r \le C(d,r,f)M^{-1}$ and $f = f_0\circ\ldots\circ f_{M-1}$.

Let T be as in Proposition 2 with d, r, A and q as above. We let $B_i = T^{n_i}B$, where $B = B_0$ is the base of the $(1/(4A), 2M, 1/2)$-funny periodic tower of discs, and $0 = n_0,\dots,n_M = 2M$ are isometric times for T. By Proposition 2, $T^{n_M} = T^{2M}$ is the Identity from B to B, and

$$\|T-\mathrm{Id}\|_r \le \frac{q^{r^4}}{A} \le \frac{\delta}{2C_1}. \tag{2}$$

Let $h_i : B_i \to \mathbb{B}^d$ be similarities that send B_i onto $\mathbb{B}^d$ such that for $i = 0,\dots,M-1$ we have

$$h_i = h_0\circ T^{-n_i},$$

or equivalently

$$h_{i+1} = h_i\circ T^{-(n_{i+1}-n_i)}. \tag{3}$$

Observe that since $T^{n_M} : B_0 \to B_0$ is the Identity map, it follows that

$$h_{n_M} = h_0\circ T^{-n_M} = h_0. \tag{4}$$

We define $\bar F \in \mathrm{Diff}^{d+\varepsilon}_\mu(\mathbb{B}^d)$ by the following two conditions:

- $\bar F|_{B_i} = h_i^{-1}\circ f_i\circ h_i, \forall i = 0,\dots,M-1$;
- $\bar F$ equals Identity on the complementary of $B_0\sqcup\ldots\sqcup B_{M-1}$.

Since h_i are similarities that expand by $1/\rho = 4A$, we have that

$$\begin{aligned}\|\bar F-\mathrm{Id}\|_r \le& 4^{r-1}A^{r-1}\max_i\|f_i-\mathrm{Id}\|_r \le 4^{r-1}C(r,d,f)\,A^{r-1}M^{-1}\\ =& C_0q^{-1}A^{r-d} = C_0q^{-1}A^{\varepsilon} \le \delta/2.\end{aligned} \tag{5}$$

Define finally $F \in \mathrm{Diff}^{d+\varepsilon}_\mu(\mathbb{B}^d)$ by

$$F := T\circ\bar F.$$

Then $F^{n_M}(B_0) = B_0 = B$ and by (3) and (4) we have

$$h_0\circ F^{n_M}|_B\circ h_0^{-1} = f,$$

hence F is a realisation of f as desired.

It is left to estimate $\|F - \mathrm{Id}\|_r$ with $r = d + \varepsilon$. Using Lemma 0.2 with $t = T$, $g = \bar{F}$, $\alpha = \delta/2$, $\beta = \delta/(2C_1)$, together with estimates (2) and (5), we get:

$$\|F - \mathrm{Id}\|_r \leq \|\bar{F} - \mathrm{Id}\|_r + \|(T - \mathrm{Id}) \circ \bar{F}\|_r \leq \delta/2 + C_1\delta/(2C_1) = \delta.$$

Thus we have $\|F - \mathrm{Id}\|_{d+\varepsilon} < \delta$, as desired. This completes the proof of Theorem 1. □

Acknowledgments. The authors would like to thank P. Berger, J.-P. Thouvenot and D. Turaev for fruitful discussions.

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Received February 2021; revised July 2021; early access September 2021.

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