Topological weak mixing and diffusion at all times for a class of Hamiltonian systems

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To our great friend and mentor Anatoly Katok

Abstract

We present examples of nearly integrable analytic Hamiltonian systems with several strong diffusion properties: topological weak mixing and diffusion at all times. These examples are obtained by AbC constructions with several frequencies.

Introduction

KAM theory (after Kolmogrov, Arnol'd, and Moser) states that, under mild *non-degeneracy assumptions*, Hamiltonian systems close to integrable have their phase space almost completely filled by invariant quasi-periodic tori. Starting from three degrees of freedom for autonomous Hamiltonians the existence of such tori on an energy surface does not prevent the orbits from *circulating* between the tori inside the surface. Indeed, it was conjectured by Arnol'd that a "general" Hamiltonian should have a dense orbit on a "general" energy surface [A]. A great amount of work has been dedicated to proving this conjecture (giving a precise meaning to the word "general"), but the picture is not yet completely clear, especially when it comes to real analytic Hamiltonians (see for example [BKZ] and references therein).

In his ICM list of problems [H], M. Herman asks: Can one find an example of a C^∞-Hamiltonian H in a small C^k-neighborhood, $k \geq 2$, of $H_0 = \|r\|^2/2$ such that on the energy surface $\{H = 1\}$ the Hamiltonian flow has a dense orbit?

A remarkable result in this direction is due to [KZZ]: they present an example of a Hamiltonian H of the form $H(\theta, r) = \frac{\langle r,r\rangle}{2} + h(r, \theta) \in C^\infty$ with a trajectory dense in a subset of the energy surface of large measure.

Here we present examples, with a degenerate integrable part, but with even more chaotic behavior for the perturbed system. Namely, we show that in a class of perturbations of rotators, the generic Hamiltonian is topologically weakly mixing on every energy surface. We also give examples of perturbed rotators for which the dynamics is diffusive at all times.

*Supported in part by Knut and Alice Wallenberg foundation, grant KAW 2016.0403, and by the ANR-15-CE40-0001.

†Supported in part by the Swedish Research Council, VR 2015-04012.

We now give the exact definitions of these properties and state our results precisely.

Definition 1 (Topological weak mixing). *We say that the flow Φ_H^t is topologically weakly mixing if there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that for any two open sets A, B on the same (arbitrarily chosen) energy surface $E_H(c) = \{(\theta, r) \mid H(\theta, r) = c\}$, there exists $N = N(A, B)$ such that $\Phi_H^{t_n}(A) \cap B \neq \emptyset$ for all $n \geq N$.*

Definition 2 (Diffusion at all times). *We say that the flow Φ_H^t exhibits diffusion at all times if for any open set $A \subset \mathbb{T}^d \times \mathbb{R}^d$ and any $R \geq 0$ there exists $T = T(A, R)$ such that $|\pi_r(\Phi_H^t(A))| > R$ for all $t \geq T$. Here π_r stands for the projection onto the r-variables.*

Given $\rho > 0$, denote by C_ρ^ω the space of bounded real functions on $\mathbb{R}^d \times \mathbb{R}^d$, that are $\mathbb{Z}^d$-periodic in the first d-vector of components, and can be extended to holomorphic functions on $D_\rho = \{(\theta, r) \in (\mathbb{C}^d, \mathbb{C}^d) \mid \max_j\{|\theta_j|, |r_j|, j = 1, \ldots d\} \leq \rho\}$ as $\mathbb{Z}^d$-periodic with respect to the real part, with the norm $\|f\|_\rho = \sup\{|f(\theta, r)|,\ (\theta, r) \in D_\rho\}$.

Let $\mathcal{O}_{\rho,\delta}^\omega = \{f \in C_\rho^\omega \mid \|f\|_\rho < \delta\}$ be a neighborhood of zero in the space of analytic functions with the above norm. This is a Baire space. Fixing $\rho = 1$ without loss of generality, we will write $\|f\| = \|f\|_1$, and $\mathcal{O}_\delta^\omega$ for $\mathcal{O}_{1,\delta}^\omega$.

Theorem A. *There exists a dense set $Y \subset \mathbb{R}^d$, $d \geq 3$, (Y continuum) such that for any $\omega \in Y$, for any $\delta > 0$, there exists a real analytic function $h : \mathbb{T}^d \to \mathbb{R}$, such that $h \in \mathcal{O}_\delta^\omega$, with the following property: the Hamiltonian flow $\Phi_H^t(\theta, r)$ defined by the Hamiltonian*

$$H = \langle r, \omega \rangle + h(\theta)$$

exhibits diffusion at all times.

By a *generic set* in this paper we mean a dense G_δ set.

Theorem B. *For any $d \geq 3$, for a generic $\omega \in \mathbb{R}^d$, any $\delta > 0$ and generic $h_1(\theta)$, $h_2(\theta) \in \mathcal{O}_\delta^\omega$, the flow Φ_H^t of the Hamiltonian*

$$H(\theta, r) = \frac{1}{1 + h_1(\theta)}(\langle r, \omega \rangle + h_2(\theta))$$

is topologically weakly mixing.

Note that the energy surface in the statement above is unbounded, which drove us to treat the topological version of weak mixing rather than the classical notion.

Diffusion at all times shows that the rigidity property of the rotator (convergence of the dynamics to identity along a subsequence of times) can be destroyed on all energy surfaces by a small perturbation. However, our examples are not topologically mixing, and it would be very interesting to produce examples that are topologically mixing on energy surfaces.

An interesting question concerns the possibility of similar examples in a neighborhood of an elliptic equilibrium. If the components of the frequency vector at the equilibrium are all of the same sign, the energy surfaces are bounded. We hope that the methods of [FS] can be used to construct smooth examples of topologically weakly mixing Hamiltonians in this context as well.

From the form of the perturbations of the rotator in Theorems A and B, we can see that the Hamiltonians that we construct are in fact in the closure of Hamiltonians that are conjugate to the rotator. Just like reparametrizations of linear flows on the torus, or abelian skew products above them, our constructions can thus be viewed *a posteriori* as particular instances of the AbC method (Approximation by Conjugation) [AK], that is also called AK-method in reference to Dmitry Anosov and Anatoly Katok who first introduced the method. The method was already used in the Hamiltonian context by Katok in [K2] to show the existence of integrable degenerate Hamiltonians with some particular Liouville frequencies and bounded energy surfaces that can be smoothly perturbed to become ergodic on the energy surfaces. Subsequent constructions that use the AbC method with several frequencies appeared in [EFK, FS, FF] to discuss the stability of elliptic equilibria and invariant tori, in particular those with Diophantine frequency vectors. The examples of [K2] were constructed following the usual AK-method with successive conjugations of a circle action, which gives C^∞ flows that are rigid in the sense that the dynamics converge to Identity along a subsequence of times. In our constructions, we bypass the smoothness limitation and the rigidity of the perturbed dynamics by resorting to the reparametrization technique of translation flows in dimension larger than 3 used in [F]. This technique exploits the Liouville phenomenon in several directions [Y] to avoid the Denjoy-Koksma cancellations that appear in dimension two [K1, Koc].

1 Notations and definitions

To alleviate the notations, we will give the proofs for $d = 3$ since there is no difference at all in the proof of the general case.

1.1 General notations

- For a vector r, its components are denoted by r_j, $j = 1, 2, 3$; for a vector $r_0 \in \mathbb{R}^3$, we denote its components by $r_{0,j}$, $j = 1, 2, 3$.

- For $a \in \mathbb{R}$ and $l > 0$, we denote

$$I(a, l) = a + [0, l] = [a, a + l].$$

- For a set S in the phase space, let $\pi_r(S)$ stand for the orthogonal projection of S onto the space of actions (r-space). In the similar way introduce notations $\pi_\theta(S)$, $\pi_{r_j}(S)$, $\pi_{\theta_j}(S)$ for $j = 1, \dots, d$.

- When we write that $\frac{p}{q}$ is a rational number or $\frac{p}{q} \in \mathbb{Q}$, we assume that $q \in \mathbb{N}$, $q \geq 1$, $p \in \mathbb{Z}$, and the numbers p and q are relatively prime.

- Let $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ be a rationally independent vector. Then, in particular, $\omega_3 \neq 0$, and we can rewrite $\omega = \omega_3(\frac{\omega_1}{\omega_3}, \frac{\omega_2}{\omega_3}, 1)$. Without loss of generality, the constructions are performed under the assumption that ω is normalised: $\omega = (\alpha, \alpha', 1)$.

- Denote $H_0(\theta, r) = \langle \omega, r \rangle = \sum_{j=1}^{3} \omega_j r_j$;

- $\Phi^t_H(\cdot)$ stands for the flow map at time t, defined by the Hamiltonian H;

- Fix $c \in \mathbb{R}$ and consider an energy surface $E_H(c) = \{(\theta, r) \mid H(\theta, r) = c\}$. In our constructions, $\|H - H_0\|_1$ is small, so $E_H(c)$ is uniformly close to $E_{H_0}(c) = \{r \mid \langle r, \omega \rangle = c\} \times \mathbb{T}^3$.

1.2 Arithmetic reminders. Yoccoz pairs of frequencies

Denote

$$|||k\alpha||| = \inf_{p \in \mathbb{Z}} |k\alpha + p|.$$

For an irrational number α there exists a sequence of rational numbers $(\frac{p_n}{q_n})_{n \geq 1}$, called the *convergents* of α such that

$$|||q_{n-1}\alpha||| < |||k\alpha||| \quad \text{for all } k < q_n,$$

and for any n

$$\frac{1}{q_n + q_{n+1}} \leq (-1)^n (q_n\alpha - p_n) \leq \frac{1}{q_{n+1}}. \tag{1.1}$$

Definition 3. *Let $\omega = (\alpha, \alpha', 1) \in \mathbb{R}^3$ where α and α' are irrational real numbers with the corresponding sequences of convergents $(\frac{p_n}{q_n})_{n \geq 1}$ and $(\frac{p'_n}{q'_n})_{n \geq 1}$. We say that $\omega = (\alpha, \alpha', 1) \in Y$ if for all $n = 0, \ldots, \infty$, the denominators of the convergents of α and α', respectively, satisfy:*

$$e^{q_n} \leq q'_n/4, \quad e^{q'_n} \leq q_{n+1}/4. \tag{1.2}$$

By [Y], the set Y is nonempty, of cardinality continuum. This is the set of frequencies used in Theorem A.

1.3 Intervals and rectangles in an energy surface

Here we describe the standard sets used in the construction. In particular, *intervals* are defined to be small one dimensional curves that lie in a given energy surface and whose projection onto the 5-dimensional space (θ, r_1, r_2) is a linear segment parallel either to the θ_1 axis or to the θ_2 axis. The coordinate r_3 is defined by the requirement that the curve lies in the energy surface. More precisely:

- Given $s_0 = (\theta_0, r_0)$ and $l > 0$, define the *intervals*

$$J^{(1)}(s_0, l) =$$
$$\{(\theta, r) \in I(\theta_{0,1}, l) \times \{\theta_{0,2}\} \times \{\theta_{0,3}\} \times \{r_{0,1}\} \times \{r_{0,2}\} \times \mathbb{R} \mid H(\theta, r) = H(\theta_0, r_0)\},$$

and

$$J^{(2)}(s_0, l) =$$
$$\{(\theta, r) \in \{\theta_{0,1}\} \times I(\theta_{0,2}, l) \times \{\theta_{0,3}\} \times \{r_{0,1}\} \times \{r_{0,2}\} \times \mathbb{R} \mid H(\theta, r) = H(\theta_0, r_0)\},$$

- For any $s_0 = (\theta_0, r_0)$, $l_1 > 0$ and $l_2 > 0$, define the *rectangle*

$$R(s_0, l_1, l_2) =$$
$$\{(\theta, r) \in I(\theta_{0,1}, l_1) \times I(\theta_{0,2}, l_2) \times \{\theta_{0,3}\} \times \{r_{0,1}\} \times \{r_{0,2}\} \times \mathbb{R} \mid H(\theta, r) = H(\theta_0, r_0)\}.$$

As before, the projection of $R(\theta_0, r_0, l_1, l_2)$ onto the space (θ, r_1, r_2) is a flat rectangle parallel to (θ_1, θ_2)-plane; r_3 is chosen so that $R(\theta_0, r_0, l_1, l_2) \subset E_H(c)$.

We say that the *size of the rectangle* $R(\theta_0, r_0, l_1, l_2)$ is $l_1 \times l_2$.

- Given n and $s_0 = (\theta_0, r_0)$, a *box* $B_n(s_0) \subset E_H(c)$ is defined by

$$\begin{aligned} B_n(s_0) = &\{(\theta, r) \mid \theta_j \in I\left(\theta_{0,j}, \frac{1}{n}\right), j = 1, 2, 3, \\ &r_j \in I\left(r_{0,j}, \frac{1}{n}\right), j = 1, 2,\ H(\theta, r) = H(\theta_0, r_0)\}. \end{aligned} \tag{1.3}$$

These sets, having full dimension in $E_H(c)$, will be used as test sets: in particular, to prove Theorem B, we will show that at certain times t_n, the image of any rectangle $R_n \subset E_H(c)$ intersects each box $B_n \subset E_H(c)$. To do so, we will need the notion of stretching:

Definition 4. *Given positive l, L and t, we say that the flow map Φ_H^t is $(1, l, L)$-stretching if for any interval $J^{(1)} = J^{(1)}(\theta_0, r_0, l)$ with $|r_0| \leq L/10$ we have:*

$$\pi_{r_1}(\Phi_H^t(J^{(1)})) \supset [-L, L],$$

and the map $(\theta, r) \mapsto \pi_{r_1}(\Phi_H^t(\theta, r))$ is independent of θ_2 where $\theta = (\theta_1, \theta_2, \theta_3)$. Analogously, we say that the flow map Φ_H^t is $(2, l, L)$-stretching if for any interval $J^{(2)} = J^{(2)}(\theta_0, r_0, l)$ with $|r_0| \leq L/10$ we have:

$$\pi_{r_2}(\Phi_H^t(J^{(2)})) \supset [-L, L],$$

and the map $(\theta, r) \mapsto \pi_{r_2}(\Phi_H^t(\theta, r))$ is independent of θ_1.

2 Proofs of the main theorems

Here we prove the main theorems modulo the technical statements, whose demonstration is deferred to the next section.

2.1 The construction for Theorem A

Let us fix an arbitrary vector $(\alpha, \alpha', 1) \in Y$ with $(\frac{p_n}{q_n})_{n\geq 1}$ and $(\frac{p'_n}{q'_n})_{n\geq 1}$ being the corresponding sequences of convergents. Let

$$\begin{aligned} h(\theta) &= -\sum_{n=1}^{\infty} h_n(\theta) - \sum_{n=1}^{\infty} h'_n(\theta), \\ h_n(\theta) &= e^{-q_n}\cos 2\pi(q_n\theta_1 - p_n\theta_3), \quad h'_n(\theta) = e^{-q'_n}\cos 2\pi(q'_n\theta_2 - p'_n\theta_3). \end{aligned} \tag{2.1}$$

Theorem A follows from

Theorem 1. *For any $\omega \in Y$, for h as in (2.1), the Hamiltonian flow $\Phi_H^t(\theta, r)$ defined by the Hamiltonian*

$$H = \langle r, \omega\rangle + h(\theta)$$

exhibits diffusion at all times.

Remark 1. *This Hamiltonian can be seen as a limit of an Anosov-Katok type construction, i.e., it has the form:*

$$H = \lim H^{(n)}, \quad H^{(n)} = H_0 \circ \Psi_n \circ \cdots \circ \Psi_1,$$

where Ψ_i are symplectic analytic coordinate changes.

The proof of Theorem 1 relies on the following proposition that is proved in Section 3.1.

Proposition 1. *For any $\omega \in Y$, for h as in (2.1), the Hamiltonian flow $\Phi_H^t(\theta, r)$ defined by the Hamiltonian*

$$H = \langle r, \omega \rangle + h(\theta)$$

satisfies for each n:

(a) *For all $t \in [e^{q_n}, q_{n+1}/4]$, Φ_H^t is $(1, 1/q_n, q_n)$-stretching.*

(b) *For all $t \in [e^{q'_n}, q'_{n+1}/4]$, Φ_H^t is $(2, 1/q'_n, q'_n)$-stretching.*

Here we show how this proposition implies Theorem 1.

Proof of Theorem 1. Since the sequences (q_n) and (q'_n) satisfy (1.2), the union of the intervals $\cup_{n>N}[e^{q_n}, q_{n+1}/4] \cup [e^{q'_n}, q'_{n+1}/4]$ contains the half-line $t > e^{q_N}$. Proposition 1 implies that for each $t \in [e^{q_n}, q_{n+1}/4]$, Φ_H^t stretches small rectangles in the direction of r_1 with a large factor, and for each $t \in [e^{q'_n}, q'_{n+1}/4]$, Φ_H^t stretches small rectangles in the direction of r_2.

Hence Φ_H^t exhibits stretching with an increasingly strong factor as $t \to \infty$, in at least one of the two directions r_1 and r_2. This implies the conclusion of Theorem 1. □

2.2 The construction for Theorem B

Consider $\omega = (\alpha, \alpha', 1)$, and suppose that there exist sequences $(p_n/q_n)_{n\geq 1}$ and $(p'_n/q'_n)_{n\geq 1}$ such that

$$q_n^4 \leq q'_n, \quad |q_n\alpha - p_n| \leq e^{-2q'_n}, \quad |q'_n\alpha' - p'_n| \leq e^{-q'_n}. \tag{2.2}$$

We start by observing that the set S of pairs (α, α') with this assumption contains a generic set in $\mathbb{R}^2$. This implies, of course, that the set of numbers $\omega = (\omega_1, \omega_2, \omega_3) = \omega_3\,(\alpha, \alpha', 1)$ such that $(\alpha, \alpha') \in S$ and $\omega_3 \in \mathbb{R}$, is generic in $\mathbb{R}^3$.

Lemma 1. *There exists a generic (dense G_δ) set $\hat{S} \subset S \subset \mathbb{R}^2$ of pairs (α, α') satisfying the following: there exist sequences $(p_n/q_n)_{n\geq 1}$ and $(p'_n/q'_n)_{n\geq 1}$ of rational numbers such that estimate (2.2) holds for all n.*

Proof. We want to describe the set S of pairs (α, α') such that for any N there exist p/q and $p'/q' \in \mathbb{Q}$ such that $q > N$, $q' \geq q^4$, $p, p' \in \mathbb{Z}$, and

$$|\alpha - p/q| \leq e^{-2q'}/q, \quad |\alpha' - p'/q'| \leq e^{-q'}/q'.$$

The set S contains the following set $\hat{S}$:

$$\bigcap_{N\geq 1} \left(\bigcup_{q\geq N} \bigcup_{p\in\mathbb{Z}} \bigcup_{q'\geq q^4} \bigcup_{p'\in\mathbb{Z}} \left(\frac{p}{q} - \frac{e^{-2q}}{q}, \frac{p}{q} + \frac{e^{-2q}}{q} \right) \times \left(\frac{p'}{q'} - \frac{e^{-q'}}{q'}, \frac{p'}{q'} + \frac{e^{-q'}}{q'} \right) \right),$$

which is a countable intersection (in N) of open dense sets. $\square$

Here we present an explicit example of a Hamiltonian whose flow is topologically weakly mixing. It is easy to see that such examples can be produced arbitrarily close to H_0. From this, the genericity of Hamiltonians with the weak mixing property is obtained in the standard way.

Having fixed $\omega \in \mathbb{R}^3$ as in (2.2), let

$$\phi(\theta) = 1 + \sum_{n=1}^{\infty} q_n e^{-q_n'} \cos 2\pi(q_n\theta_1 - p_n\theta_3), \tag{2.3}$$

and for $\kappa_n := q_n^2$, introduce

$$\begin{aligned} \tilde{h}(\theta) &= -\sum_{n=1}^{\infty} \tilde{h}_n(\theta) - \sum_{n=1}^{\infty} \tilde{h}_n'(\theta), \\ \tilde{h}_n(\theta) &= \kappa_n e^{-q_n'} \cos 2\pi\kappa_n(q_n\theta_1 - p_n\theta_3), \quad \tilde{h}_n'(\theta) = e^{-q_n'} \cos 2\pi(q_n'\theta_2 - p_n'\theta_3). \end{aligned} \tag{2.4}$$

Notice that since $\|\tilde{h}_n\|_1 \le \kappa_n e^{-q_n'} e^{2\pi\kappa_n(q_n+p_n)} \le q_n^2 e^{4\pi q_n^3 - q_n'}$, assumption $q_n^4 \le q_n'$ implies that $\|\sum_{n=1}^{\infty} \tilde{h}_n(\theta)\|_1 < \infty$. Clearly, this implies that $\|\tilde{h}(\theta)\|_1 < \infty$.

Theorem 2. *For ω as in (2.2), ϕ as in (2.3), $\tilde{h}$ as in (2.4), the Hamiltonian flow $\Phi^t_{\tilde{H}}(\theta, r)$ defined by*

$$\tilde{H} = \frac{1}{\phi(\theta)} \left(\langle r, \omega \rangle + \tilde{h}(\theta) \right) \tag{2.5}$$

is topologically weakly mixing.

More precisely, for $t_n = e^{q_n'}$, $n \ge 1$, we have: for any two open sets A and B on the same energy surface there exists $N = N(A, B)$ such that

$$\Phi^{t_n}_{\tilde{H}}(A) \cap B \ne \emptyset \quad \textit{for all} \quad n \ge N.$$

Assuming Theorem 2, we show how it yields Theorem B.

Proof of Theorem B.

It follows from classical arguments (Cf. [Ha]) that weak mixing for the flows as in Theorem B holds for a G_δ-set of functions $(h_1, h_2) \in \mathcal{O}^\omega_\delta(0)^2$. It is left to show the density of weak mixing for $(h_1, h_2) \in \mathcal{O}^\omega_\delta(0)^2$ for a fixed δ. To do this, notice that for any $\epsilon > 0$, both $(\phi(\theta) - 1)$ and $\tilde{h}(\theta)$ can be chosen ϵ-close to zero in the fixed norm: it is enough to choose q_1 large enough. Moreover, from the proof of Theorem 2 it follows that the same result holds true if we change $\phi(\theta)$ and $\tilde{h}(\theta)$ by $\phi(\theta) + P(\theta) \in \mathcal{O}^\omega_\delta(0)$ and $\tilde{h}(\theta) + Q(\theta) \in \mathcal{O}^\omega_\delta(0)$ with P and Q trigonometric polynomials. This implies the density of the weak mixing property. $\square$

The proof of Theorem 2 relies on the following two propositions that are proved in Sections 3.1 and 3.2, respectively.

Proposition 2. *For ω as in* (2.2)*, ϕ as in* (2.3)*, $\tilde{h}$ as in* (2.4)*, $\tilde{H}$ as in* (2.5)*, and $t_n = e^{q'_n}$, we have:*

(a) *$\Phi_{\tilde{H}}^{t_n}$ is $(1, \frac{1}{q_n^3}, q_n)$-stretching;*

(b) *$\Phi_{\tilde{H}}^{t_n}$ is $(2, \frac{1}{q'_n}, q'_n)$-stretching.*

Proposition 3. *For ω as in* (2.2)*, ϕ as in* (2.3)*, $\tilde{h}$ as in* (2.4)*, the Hamiltonian flow $\Phi_{\tilde{H}}^t(\theta, r)$ defined by*

$$\tilde{H} = \frac{1}{\phi(\theta)} \Big(\langle r, \omega \rangle + \tilde{h}(\theta) \Big)$$

satisfies for $t_n = e^{q'_n}$ the following. For any rectangle $R_n := R(\theta_0, r_0, 1/q_n, 1/q_n)$ with $|r_0| \leq n$ and any box B_n (see notations in Sec. 1.3*) there exists a rectangle $R'_n \subset R_n$ of size $1/q_n^3 \times 1/q_n^3$ such that*

$$\pi_\theta(\Phi_{\tilde{H}}^{t_n}(R'_n)) \subset \pi_\theta(B_n).$$

Proof of Theorem 2. Fix R_n and B_n as above.

By Proposition 3, there exists a rectangle $R'_n \subset R_n$ of size $1/q_n^3 \times 1/q_n^3$ such that

$$\pi_\theta(\Phi_{\tilde{H}}^{t_n}(R'_n)) \subset \pi_\theta(B_n).$$

By Proposition 2, we can find a rectangle $\bar{R}_n \subset R'_n$ such that

$$\pi_r(\Phi_{\tilde{H}}^{t_n}(\bar{R}_n)) \subset \pi_r(B_n).$$

Hence

$$\Phi_{\tilde{H}}^{t_n}(\bar{R}_n)) \subset B_n$$

and the proof is finished. □

3 Stretching

3.1 Stretching in the action directions.

Lemma 2. *Let $\frac{p}{q}, \frac{p'}{q'} \in \mathbb{Q}$ and $\omega = (\alpha, \alpha', 1)$ and $a_q, b_{q'}$ satisfy*

$$e^{-q} \geq a_q \geq 4|q\alpha - p|, \quad e^{-q'} \geq b_{q'} \geq 4|q'\alpha' - p'|.$$

Define

$$H(\theta, r) = \langle r, \omega \rangle - a_q \cos 2\pi(q\theta_1 - p\theta_3) - b_{q'} \cos 2\pi(q'\theta_2 - p'\theta_3).$$

Then the following holds:

(a) *For each $t \in [a_q^{-1}, 1/(4|q\alpha - p|)]$ the flow map Φ_H^t is $(1, 1/q, 2q)$ stretching.*

(b) *For each $t \in [b_{q'}^{-1}, 1/(4|q'\alpha' - p'|)]$, the flow map Φ_H^t is $(2, 1/q', 2q')$ stretching.*

Proof of Lemma 2. The Hamiltonian H defines the following system of equations (we omit r_3 from the considerations):

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{r}_1 = -2\pi q a_q \sin 2\pi(q\theta_1 - p\theta_3), \\ \dot{r}_2 = -2\pi q' b_{q'} \sin 2\pi(q'\theta_2 - p'\theta_3). \end{cases} \tag{3.1}$$

This system can be integrated explicitly: the solution with initial conditions $(\theta(0), r(0)) = (\theta_0, r_0)$ satisfies

$$\begin{cases} \theta = \theta_0 + t\omega, \\ r_1(t) = c_1 + \frac{q a_q}{q\alpha - p} \cos 2\pi((q\theta_{0,1} - p\theta_{0,3}) + t(q\alpha - p)), \\ r_2(t) = c_2 + \frac{q' b_{q'}}{q'\alpha' - p'} \cos 2\pi((q'\theta_{0,2} - p'\theta_{0,3}) + t(q'\alpha - p')), \end{cases} \tag{3.2}$$

where c_k is a constant such that $r_k(0) = r_{0,k}$, $k = 1, 2$. Notice that $r_1(t) = \pi_{r_1}(\Phi_H^t(r_0, \theta_0))$ is independent of $\theta_{0,2}$, and $r_2(t) = \pi_{r_2}(\Phi_H^t(r, \theta)))$ is independent of $\theta_{0,1}$.

Fix an arbitrary $t \in [a_q^{-1}, 1/(4(q\alpha - p))]$ and $s_0 = (r_0, \theta_0)$ with $|r_0| < q/10$, and consider the interval $J^{(1)} = J^{(1)}(s_0, \frac{1}{q})$, see notations in Section 1.3. Assume $q\alpha - p > 0$ (the opposite case is similar). There exists a point $s^+ = (\theta^+, r_0) \in J^{(1)}$ with $\theta^+ = (\theta_1^+, \theta_{0,2}, \theta_{0,3})$ such that

$$q\theta_1^+ - p\theta_{03} = -1/4 \mod 1.$$

Then $\cos 2\pi(q\theta_1^+ - p\theta_{03}) = 0$. Consider the trajectory of the above flow with the initial condition $(\theta(0), r(0)) = (\theta^+, r_0)$. For the first action component this reads: $r_1(0) = c_1 = r_{0,1}$, which implies

$$|c_1| = |r_{0,1}| \le |r_0| \le q/10.$$

Assumption $t \in [a_q^{-1}, 1/(4(q\alpha - p))]$ implies, in particular, $(2\pi t(q\alpha - p)) \in [0, \pi/2]$. Using the fact that $\sin(x) \ge x/2$ for all $x \in [0, \pi/2]$, we get

$$\frac{q a_q}{q\alpha - p} \cos 2\pi((q\theta_1^+ - p\theta_{03}) + t(q\alpha - p)) = \\ \frac{q a_q}{q\alpha - p} \sin 2\pi t(q\alpha - p) \ge \pi t q a_q \ge 3q.$$

Since $|c_1| < q/10$, we get $\pi_{r_1}(\Phi_H^t(s^+)) > 2q$.

In the same way, there is a point $s^- \in J^{(1)}$ such that $\pi_{r_1}(\Phi_H^t(s^-)) < -2q$. The result follows by continuity. □

Here we prove Proposition 1 that was used for the proof of Theorem 1.

Proof of Proposition 1.

Let us prove statement (a), the second statement is similar. Since q_n/p_n is a convergent of α, by (1.1) we have for all n that $q_{n+1} \le \frac{1}{|q_n\alpha - p_n|}$. Condition (1.2) implies that $e^{q_n} \le q_{n+1}/4$, so we have

$$[\, e^{q_n}, q_{n+1}/4 \,] \subset [\, e^{q_n}, 1/(4|q_n\alpha - p_n|) \,].$$

Consider $H_n(\theta,r)=\langle r,\omega\rangle-h_n(\theta)-h'_n(\theta)$, where h_n, h'_n are defined by (2.1). Fix $r=r_0$ with $|r_0|<q/10$. The first component of $\Phi^t_{H_n}(\theta,r_0)$, i.e., $r_{n,1}(\theta,r_0,t):=\pi_{r_1}(\Phi^t_{H_n}(\theta,r_0))$, is given by

$$\begin{aligned}r_{n,1}(\theta,r_0,t)=&c_n(\theta,r_0)+\frac{q_ne^{-q_n}}{q_n\alpha-p_n}\cos 2\pi((q_n\theta_1-p_n\theta_3)+t(q_n\alpha-p_n)):=\\&c_n(\theta,r_0)+f_n(\theta,t),\end{aligned}$$

where $c_n(\theta,r_0)$ is such that $r_{n,1}(\theta,r_0,0)=r_0$. By Lemma 2, for each $t\in[e^{q_n},q_{n+1}/4]$ the flow map $\Phi^t_{H_n}$ is $(1,1/q_n,2q_n)$ stretching, i.e., in any interval $J^{(1)}(r_0,\theta_0,1/q_n)$ with $|r_0|\le q_n/10$ there are points $s^+=(r_0,\theta^+)$ and $s^-=(r_0,\theta^-)$ such that

$$r_{n,1}(s^+,t)\ge 2q_n,\quad r_{n,1}(s^-,t)\le -2q_n.$$

Hence, in particular,

$$f_n(\theta^+,t)-f_n(\theta^+,t)=r_{n,1}(s^+,t)-r_{n,1}(s^+,0)\ge 2q_n-q_n/10=1.9q_n.$$

Let us show that for these t, Φ^t_H has the same stretching properties as $\Phi^t_{H_n}$ (with $2q_n$ replaced by q_n). The first component of $\Phi^t_H(\theta,r_0)$, i.e., $r_1(\theta,r_0,t):=\pi_{r_1}(\Phi^t_H(\theta,r_0)$, is given by the formula

$$\begin{aligned}r_1(\theta,r_0,t)=&c(\theta,r_0)+\sum_{k=1}^{\infty}\frac{q_ke^{-q_k}}{q_k\alpha-p_k}\cos 2\pi((q_k\theta_1-p_k\theta_3)+t(q_k\alpha-p_k)):=\\&c(\theta,r_0)+\sum_{k=1}^{\infty}f_k(\theta,t)=c(\theta,r_0)+f_n(\theta,t)+C_n(\theta,t)+D_n(\theta,t),\end{aligned}$$

where $C_n(\theta,t)=\sum_{k=1}^{n-1}f_n(\theta,t)$, $D_n(\theta,t)=\sum_{k=n+1}^{\infty}f_k(\theta,t)$, and $c(\theta,r_0)$ is a constant such that $r_1(\theta,r_0,0)=r_{0,1}$.

Consider $C_n(\theta,t)$. By (1.1), $1/|q_k\alpha-p_k|\le 2q_{k+1}$ for all $k\ge 1$, so for any θ we have

$$|C_n(\theta,t)-C_n(\theta,0)|\le 2\sum_{k=1}^{n-1}\left|\frac{q_ke^{-q_k}}{q_k\alpha-p_k}\right|\le 4\sum_{k=1}^{n-1}q_ke^{-q_k}q_{k+1}\le q_n/100$$

due to the growth condition on q_k.

Now consider $D_n(t)$. By (1.1), for any $k\ge 1$ we have: $|q_k\alpha-p_k|\le\frac{1}{q_{k+1}}$. Hence, for any $t\le q_{n+1}$ we have:

$$\begin{aligned}|D_n(\theta,t)-&D_n(\theta,0)|\le t\,\sup|D'_n(\theta,\tilde t)|\le\\&2\pi t\sum_{k=n+1}^{\infty}\left|\frac{q_ke^{-q_k}}{q_k\alpha-p_k}\right||q_k\alpha-p_k|\le 2\pi q_{n+1}\sum_{k=n+1}^{\infty}q_ke^{-q_k}<1.\end{aligned}$$

This implies that for any r_0 with $|r_0|\le q/10$ we have:

$$\begin{aligned}r_1(s^+,t)-&r_1(s^+,0)\ge f_n(\theta^+,t)-f_n(\theta^+,t)-|C_n(\theta,t)-C_n(\theta,0)|-\\&|D_n(\theta,t)-D_n(\theta,0)|\ge 1.9q_n-0.01q_n-1>1.5q_n.\end{aligned}$$

Since, by assumption, $|r_1(s^+,0)| = |r_0| \le q_n/10$, we get

$$r_1(s^+,t) \ge q_n.$$

By the same argument, $r_1(s^-,t) \le -q_n$. Thus, Φ_H^t is $(1, 1/q_n, q_n)$ stretching. □

Below we prove an analog of Proposition 1 for the Hamiltonian $\tilde{H}$ of Theorem 2. To begin with, notice that our choice of ϕ and $\tilde{h}$ implies that the Hamiltonian system of $\tilde{H}$ has a particularly simple form.

Lemma 3. *For ω as in (2.2), ϕ as in (2.3), $\tilde{h}$ as in (2.4), the Hamiltonian flow $\Phi^t_{\tilde{H}}(\theta, r)$ defined by*

$$\tilde{H} = \frac{1}{\phi(\theta)} \left(\langle r, \omega \rangle + \tilde{h}(\theta) \right)$$

satisfies

$$\begin{cases} \dot{\theta} = \frac{1}{\phi(\theta)}\omega, \\ \dot{r}_1 = \frac{-2\pi}{\phi(\theta)} \sum_{n=1}^{\infty} q_n \left(C q_n e^{-q'_n} \sin 2\pi(q_n\theta_1 - p_n\theta_3) + \kappa_n^2 e^{-q'_n} \sin 2\pi\kappa_n(q_n\theta_1 - p_n\theta_3) \right), \\ \dot{r}_2 = \frac{-2\pi}{\phi(\theta)} \sum_{n=1}^{\infty} q'_n e^{-q'_n} \sin 2\pi(q'_n\theta_2 - p'_n\theta_3), \end{cases}$$

where $C = \tilde{H}(\theta, r)$. We omit the expression for r_3 since it is not used below.

Proof of Lemma 3. Recall that the value of $\tilde{H}(\theta, r) = \frac{1}{\phi(\theta)}(\langle r, \omega\rangle + \tilde{h}(\theta)) := C$ is constant on the solutions of the corresponding system of equations. For $j = 1, 2$ we have: $\dot{r}_j = -\partial_{\theta_j}\tilde{H}$, where

$$\partial_{\theta_j}\tilde{H} = -\frac{1}{\phi^2}\,\partial_{\theta_j}\phi\,(\langle r, \omega\rangle + \tilde{h}) + \frac{1}{\phi}\,\partial_{\theta_j}\tilde{h} = -\frac{1}{\phi}\,(C\,\partial_{\theta_j}\phi - \partial_{\theta_j}\tilde{h}).$$

Explicit substitution finishes the proof. □

In the next lemma (which is an analog of Lemma 2) we study the action components of the above system in a simplified form: we consider only the n-th term in the sums above. The study of the angle components is postponed to Proposition 3.

Lemma 4. *Let $\omega = (\alpha, \alpha', 1)$, where α and α' are irrational. Assume that there exist rational numbers p/q and p'/q' satisfying (2.2) with q_n and q'_n replaced by q and q', respectively. Let $\phi(\theta) : \mathbb{T}^3 \mapsto \mathbb{R}$ be a smooth function satisfying $3/4 \le |\phi(\theta)| \le 2$ for all $\theta \in \mathbb{T}^3$. Denote $\kappa = q^2$, take any $C \in \mathbb{R}$ with $|C| \le q^{1/2}$, and $r_0 \in \mathbb{R}^3$ with $|r_0| \le q/10$, and let $\Phi_S^t(\theta_0, r_0)$ be the flow of the system*

$$\begin{cases} \dot{\theta} = \frac{1}{\phi(\theta)}\omega, \\ \dot{r}_1 = \frac{-2\pi}{\phi(\theta)} q \left(C q e^{-q'} \sin 2\pi(q\theta_1 - p\theta_3) + \kappa^2 e^{-q'} \sin 2\pi\kappa(q\theta_1 - p\theta_3) \right), \\ \dot{r}_2 = \frac{-2\pi}{\phi(\theta)} q' e^{-q'} \sin 2\pi(q'\theta_2 - p'\theta_3) \end{cases} \tag{3.3}$$

with initial conditions $\Phi_S^0(\theta_0, r_0) = (\theta_0, r_0)$. Then for $t = e^{q'}$ the following holds:

(a) Φ_S^t *is $(1, \frac{1}{q\kappa}, 2q)$-stretching;*

(b) Φ_S^t *is $(2, \frac{1}{q'}, 2q')$-stretching.*

Proof of Lemma 4. The proof is analogous to that of Lemma 2. The only difference is that in this case $\dot{\theta}(t)$ is not constant. Let us study the r_1-component of $\Phi_S^t(\theta_0, r_0)$ (the analysis of $r_2(t)$ is similar). Fix an arbitrary (θ_0, r_0) with $|r_0| \leq q/10$ and let $\tilde{J}^{(1)} = J^{(1)}(\theta_0, r_0, \frac{1}{\kappa q})$.

Since $3/4 \leq |\phi(\theta)| \leq 2$, the Mean Value theorem implies that the angle variables satisfy

$$\theta(\theta_0; t) = \theta_0 + t\,\xi(\theta_0, t)\omega,$$

where $3/4 \leq |\xi(\theta, t)| \leq 2$ for all θ, t.

Given an initial condition $r_1(\theta, r_0, 0) = r_{0,1}$, the system defines

$$\begin{aligned} r_1(\theta, r_0; t) = c_1 + \frac{Cq^2 e^{-q'}}{q\alpha - p} \cos 2\pi(q\theta_1(\theta; t) - p\theta_3(\theta; t)) + \\ \frac{q\kappa e^{-q'}}{q\alpha - p} \cos 2\pi\kappa(q\theta_1(\theta; t) - p\theta_3(\theta; t)) := c_1(\theta, r_0) + g_1(\theta, r_0; t) + g_2(\theta, r_0; t), \end{aligned}$$

where $c_1(\theta, r_0)$ is the constant such that $r_1(\theta, r_0, 0) = r_{0,1}$. Notice that $g_2(\theta_0, r_0, t)$ is the leading term in the above expression. Assume that $q\alpha - p > 0$, the opposite case is similar. Clearly, there exists a point $s^+ = (\theta^+, r_0) \in \tilde{J}^{(1)}$ with $\theta^+ = (\theta_1^+, \theta_{02}, \theta_{03})$ such that

$$\kappa(q\theta_1^+ - p\theta_{03}) = -1/4 \bmod 1.$$

We will show that $r_1(\theta^+, r_0; e^{q'}) > 2q$.

First consider the term g_2. Notice that $g_2(\theta^+, r_0, 0) = 0$. For $\theta(\theta^+; t) = \theta^+ + t\,\xi(\theta^+; t)\omega$ we have:

$$\kappa(q\theta_1(\theta^+; t) - p\theta_3(\theta^+; t)) = \big(-1/4 + \kappa t\xi(\theta^+; t)(q\alpha - p)\big) \bmod 1.$$

Then for $t = e^{q'}$ we have:

$$\begin{aligned} g_2(\theta^+, r_0; t) = &\frac{q\kappa e^{-q'}}{q\alpha - p} \cos 2\pi\kappa(q\theta_1(\theta^+; t) - p\theta_3(\theta^+; t)) = \frac{q\kappa e^{-q'}}{q\alpha - p} \sin\big(2\pi\kappa t\xi(\theta^+, t)(q\alpha - p)\big) \\ &\geq \frac{q\kappa e^{-q'}}{q\alpha - p} \pi\kappa t\xi(\theta^+, t)(q\alpha - p) \geq q\kappa^2 e^{-q'} t = q^5. \end{aligned}$$

We used the evident estimate $\sin(x) \geq x/2$ for $x \in [0, \pi/2]$, and

$$2\pi\kappa t\xi(\theta^+, t)\,(q\alpha - p) \leq 4\pi\kappa e^{q'} e^{-2q'} = 4\pi q^2 e^{-q'} \in [0, \pi/2].$$

To estimate the other terms in $r_1(\theta^+, r_0; t)$, notice that $|c_1 + g_1(\theta^+, r_0; 0)| = |r_{0,1}| \leq q/10$. By (3.3), the derivative $\dot{g}_1(\theta^+, r_0, t)$ satisfies: $|\dot{g}_1(\theta^+, r_0, t)| \leq 2\pi Cq^2 e^{-q'} \leq 2\pi q^3 e^{-q'}$. For $t = e^{q'}$ we have: $\Delta g_1(\theta^+, r_0, t) := |g_1(\theta^+, r_0, t) - g_1(\theta^+, r_0, 0)| \leq 2\pi t q^2 e^{-q'} = 2\pi q^2$, and finally:

$$\begin{aligned} r_1(\theta^+, &r_0; t) = c_1 + g_1(\theta^+, r_0, t) + g_2(\theta^+, r_0, t) = \\ &r_{0,1} + \Delta g_1(\theta^+, r_0, t) + g_2(\theta^+, r_0, t) \geq g_2(\theta^+, r_0, t) - |r_{0,1}| - |\Delta g_1(\theta^+, r_0, t)| \geq \\ &q^5 - q/10 - 2\pi q^3 > 2q. \end{aligned}$$

In the same way, there is a point $(\theta^-, r_0) \in \tilde{J}^{(1)}$ such that the solution $r_1(\theta^-, r_0;\, t)$ with the initial condition $r_1(\theta^-, r_0;\, 0) = r_{0,1}$ satisfies: $r_1(\theta^-, r_0;\, e^{q'}) \leq -2q$. This implies that Φ_S^t is $(1, \frac{1}{\kappa q}, 2q)$ stretching. In the same way one verifies that Φ_S^t is $(2, \frac{1}{q'}, 2q')$ stretching.

$\square$

Proof of Proposition 2. The proof of Proposition 2 follows from Lemma 4 exactly as Proposition 1 followed from Lemma 2. $\square$

3.2 Stretching in the angle directions

In this section we prove Proposition 3. Namely, we study the behavior of $\pi_\theta(\Phi^t_{\tilde{H}})$, which is the solution of the equation

$$\dot{\theta} = \frac{1}{\phi(\theta)}\omega. \tag{3.4}$$

Since this flow does not depend on r, we fix an arbitrary r_0 and omit it from the notations. To shorten the notations from Sec. 1.3, we denote

$$\Phi^t_\phi(\theta_0) := \pi_\theta\left(\Phi^t_H(r_0,\theta_0)\right),$$

$$B_{\theta,n}(\theta_0) := \pi_\theta\left(B_n(r_0,\theta_0)\right),$$

$$R_\theta(\theta_0,l,l') := \pi_\theta\left(R(r_0,\theta_0,l,l')\right).$$

Notice that $R_\theta(\theta_0,l,l')$ is indeed a flat rectangle.

Proof of Proposition 3. Since $\tilde{h}(\theta)$ does not depend on the r variables, the restriction of the flow of $\tilde{H} = \frac{1}{\phi(\theta)}\left(\langle r,\omega\rangle + \tilde{h}(\theta)\right)$ onto $\mathbb{T}^3$ is the same as that of $\bar{H} = \frac{1}{\phi(\theta)}\langle r,\omega\rangle$. We will study the latter one.

First observe that, considering the global section $\{\theta_3 = 0\}$, one can see the flow of $\bar{H}$ to be equivalent to a special flow $T^t_{(\alpha,\alpha'),\varphi}$ above the translation $T_{(\alpha,\alpha')}$ on $\mathbb{T}^2$ and under a ceiling function of the form

$$\varphi(\theta) = 1 + \sum_{n=1}^{\infty} q_n e^{-q_n'}\cos 2\pi(q_n\theta_1). \tag{3.5}$$

The phase space $M_{(\alpha,\alpha'),\varphi}$ of $T^t_{(\alpha,\alpha'),\varphi}$ is $\mathbb{T}^2\times\mathbb{R}$ with the identification $(\theta_1,\theta_2,s+\varphi(\theta_1,\theta_2)) \sim (\theta_1+\alpha,\theta_2+\alpha',s)$. These flows were studied in [F] and the proof of the proposition follows from [F]. For completeness, we sketch the proof here. Observe that, as proved in Proposition 3 of [F], for intervals $I_n \subset \mathbb{R}$ of length $(1/2-2/n)q_n^{-1}$ of the form $|||q_n\theta_1||| \in [\frac{1}{n},\frac{1}{2}-\frac{1}{n}]$ or $|||q_n\theta_1||| \in [\frac{1}{2}+\frac{1}{n},1-\frac{1}{n}]$, and for $m \in [t_n/2, 2t_n]$, it holds for some constant $C>0$ that for every $\theta_1 \in I_n$

$$C^{-1}\frac{q_n^2}{n} \le |\partial_{\theta_1}\varphi_m(\theta)| \le Cq_n^2, \tag{3.6}$$

where φ_m stands for the m-th Birkhoff sum of the function φ. The latter estimate follows from the very good rational approximation of ω_1, since (1.1) implies that $|||q_n\omega_1||| \le e^{-q_n'}/4$. Now, the LHS of (3.6) implies that $T^{t_n}_{(\alpha,\alpha'),\varphi}(I_n\times\{\theta_2\}\times\{s\})$, for any $\theta_2\in\mathbb{T}$ and any $s \le C$, is a union of more than $\sqrt{q_n}$ almost vertical strips that follow the orbit of I_n under the base translation $T_{(\alpha,\alpha')}$. Since $\sqrt{q_n} \gg \exp\circ\exp(n)$, we get that $T^{t_n}_{(\alpha,\alpha'),\varphi}(I_n\times\{\theta_2\}\times\{s\})$ is e^{-2n} dense in the space $M_{(\alpha,\alpha'),\varphi}$. This fact, plus the RHS of (3.6), together with the fact that there is no shear in the θ_2 direction (the ceiling function depends only on θ_1), imply that for any box $B_{\theta,n}$, and for any $s \in \mathbb{R}$, there exists a rectangle R'_n of size $1/q_n^3 \times 1/q_n^3$ such that

$$T^{t_n}_{(\alpha,\alpha'),\varphi}(R'_n \times \{s\}) \subset B_{\theta,n}.$$

Going back to the original flow on $\mathbb{T}^d$, this implies the requirement of the proposition.

□

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