On manifolds supporting distributionally uniquely ergodic diffeomorphisms

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Abstract

A smooth diffeomorphism is said to be *distributionally uniquely ergodic* (*DUE* for short) when it is uniquely ergodic and its unique invariant probability measure is the only invariant distribution (up to multiplication by a constant). Ergodic translations on tori are classical examples of DUE diffeomorphisms. In this article we construct DUE diffeomorphisms supported on closed manifolds different from tori, providing some counterexamples to a conjecture proposed by Forni in [For08].

1 Introduction

When we study the dynamics of a homeomorphism $f\colon M \to M$ (for the time being we can suppose M is a just compact metric space), we can consider the induced linear automorphism $f^\star$ on $C^0(M,\mathbb{C})$ given by

$$f^\star\psi := \psi \circ f, \quad \forall \psi \in C^0(M,\mathbb{C}).$$

If we endow $C^0(M,\mathbb{C})$ with the C^0-uniform topology, $f^\star$ turns to be a continuous linear operator and hence, its adjoint $f_\star$ acts on the topological dual space $(C^0(M,\mathbb{C}))'$ which coincides, by Riesz representation theorem, with $\mathfrak{M}(M)$, the space of complex finite measures on M.

At certain extent we can say that Ergodic Theory consists in understanding the relation between the *"non-linear"* dynamics of f and the linear one of $f_\star\colon \mathfrak{M}(M) \to \mathfrak{M}(M)$. The fixed points of $f_\star$, the so called f-invariant measures, play a key role in this theory.

When M is a closed smooth manifold and $f\colon M \to M$ is a C^r-diffeomorphism, every linear subspace $C^k(M,\mathbb{C}) \subset C^0(M,\mathbb{C})$ (where $0 \leq k \leq r \leq \infty$) is $f^\star$-invariant. Moreover, when $C^k(M,\mathbb{C})$ is equipped with the C^k-uniform topology, $f^\star\colon C^k(M,\mathbb{C}) \to C^k(M,\mathbb{C})$ turns to be a continuous isomorphism and hence, its adjoint $f_\star$ acts on $\mathcal{D}'_k(M)$, i.e. the space of distributions up to order k. Of course, the fixed points of $f_\star$ are called *invariant distributions.*

As usual, we say f is uniquely ergodic when it exhibits a unique f-invariant probability measure. On the other hand, when f is C^∞ and there is only one

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(up to multiplication by constant) f-invariant distribution, we shall say that f is *distributionally uniquely ergodic*, *DUE* for short. Ergodic translations on tori are the archetypical examples of DUE diffeomorphisms. Recently, the first and third authors showed in [AK11] that every smooth circle diffeomorphism with irrational rotation number is also DUE.

In 2008, Forni conjectured in [For08] that tori are the only closed manifolds supporting DUE diffeomorphisms.

In this paper we construct some new DUE systems, providing some counterexamples to Forni's conjecture. In fact, our main purpose consists in showing the following

Theorem A. *Let P be either*

(a) a compact nilmanifold, i.e. $P = N/\Gamma$ with N a nilpotent connected and simply connected Lie group and $\Gamma < N$ a uniform lattice;

(b) or a homogeneous space of compact type, i.e. $P = G/H$ where G is compact Lie group and $H < G$ a closed subgroup.

Then, there exist DUE diffeomorphisms on $M := \mathbb{T} \times P$.

It is interesting to remark that so far the most powerful techniques to study invariant distributions (for dynamical systems which are not hyperbolic) come from harmonic analysis. However, in general it is very hard to apply these techniques to dynamical systems which do not exhibit certain *"homogeneity"* (e.g. they preserve a smooth Riemannian structure, or are induced by translations on homogeneous spaces).

On the other hand, it is well-known that any DUE diffeomorphism preserving a Riemannian structure is topologically conjugate to an ergodic torus translation, and after some works of Flaminio and Forni [FF03, FF07] it was expected that there were no DUE homogeneous systems supported on homogeneous spaces different from tori. In fact, we recently learned that Flaminio, Forni and F. Rodríguez-Hertz [RH12] have shown indeed the validity of Forni's conjecture for homogeneous systems. So the main difficulty to prove our result consists in overcoming this apparent obstruction to apply harmonic analysis tools.

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2 Preliminaries and Notations

2.1 Manifolds, functional spaces and topology

All along this paper, M will denote a compact orientable smooth manifold without boundary. Given any $r \in \mathbb{N}_0 \cup \{\infty\}$, we write $\mathrm{Diff}^r(M)$ for the group of C^r-diffeomorphisms. The subgroup of C^r-diffeomorphisms which are isotopic to the identity shall be denoted by $\mathrm{Diff}_0^r(M)$.

If N denotes any other smooth manifold, we write $C^r(M,N)$ for the space of C^r-maps from M to N. For the sake of simplicity, we shall just write $C^r(M)$ instead of $C^r(M,\mathbb{C})$.

Let us recall that, when r is finite, the *uniform C^r-topology* turns $C^r(M)$ into a Banach space and $C^r(M,N)$ turns to be a Banach manifold.

The space $C^\infty(M)$ will be endowed with its usual Fréchet topology which can be defined as the projective limit of the family of Banach spaces $(C^r(M))_{r\in\mathbb{N}}$. In this case, $C^\infty(M,N)$ is endowed with a Fréchet manifold structure.

Of course, for any $r\in\mathbb{N}_0\cup\{\infty\}$, we assume $\mathrm{Diff}^r(M)$ equipped with the C^r-uniform topology inherited from its inclusion in $C^r(M,M)$.

Finally, if X is an arbitrary topological space and $z\in X$, we shall write $\mathrm{cc}(X,z)$ to denote the connected component of X containing z.

2.2 Some arithmetical notations

Given any natural number $q\in\mathbb{N}$, we write $q\mathbb{N}:=\{qn : n\in\mathbb{N}\}$.

Whenever we write a single rational number in the form p/q we always assume the integers p and q are coprime, i.e. 1 is the greatest common divisor of p and q. On the other hand, writing a vector with rational coordinates $(p_1/q,\dots,p_n/q)\in\mathbb{Q}^n$, we shall simply assume $\gcd(p_1,\dots,p_n,q)=1$.

Given any $x\in\mathbb{R}$, we write $\lfloor x\rfloor$ to denote the largest integer not greater than x. Analogously, $\lceil x\rceil$ denotes the smallest integer greater or equal than x.

For any $\boldsymbol{\alpha}\in\mathbb{R}^d$ we write

$$\|\boldsymbol{\alpha}\|_{\mathbb{T}^d}:=\mathrm{dist}(\boldsymbol{\alpha},\mathbb{Z}^d).$$

Notice that, since $\|\boldsymbol{\alpha}+\mathbf{n}\|_{\mathbb{T}^d}=\|\boldsymbol{\alpha}\|_{\mathbb{T}^d}$ for every $\mathbf{n}\in\mathbb{Z}^d$, we can naturally consider $\|\cdot\|_{\mathbb{T}^d}$ as defined on $\mathbb{T}^d$, too.

We say $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_d)\in\mathbb{R}^d$ is *irrational* when for every $(n_1,\dots,n_d)\in\mathbb{Z}^d$

$$\Big\|\sum_{i=1}^d n_i\alpha_i\Big\|_{\mathbb{T}^d}=0 \implies n_i=0,\ \text{for } i=1,\dots,d. \tag{1}$$

An irrational vector $\boldsymbol{\alpha}$ is said to be *Diophantine* if there exist constants $C,\tau>0$ satisfying

$$\Big\|\sum_{i=1}^d \alpha_i q_i\Big\|_{\mathbb{T}^d}\geq\frac{C}{\max_i|q_i|^\tau},$$

for every $(q_1,\dots,q_d)\in\mathbb{Z}^d\setminus\{0\}$. On the other hand, an irrational element of $\mathbb{R}^d$ which is not Diophantine is called *Liouville*.

2.3 Lie groups

2.3.1 Generalities

In this work we shall only deal with real connected Lie groups. As usual, if G denotes an arbitrary Lie group, its identity element is denoted by 1_G[1], its Lie algebra by $\mathfrak{g}$ and we write $\exp\colon\mathfrak{g}\to G$ for the exponential map.

[1] Except when G is abelian. In that case, we just write 0 to denote its identity element.

A smooth manifold is called a *homogeneous space* when it can be written as G/H, where G denotes a (real, connected) Lie group and $H < G$ a closed subgroup. We say H is *cocompact* when G/H is compact, and we say that G/H is of *compact type* when G is compact itself.

Clearly, the group G acts naturally (on the left) on G/H and it is well known that in such a case there exists at most one G-invariant Borel probability measure on G/H. When such a measure does exist, we will call it the *Haar measure of G/H*. A discrete cocompact subgroup will be called a *uniform lattice.* Let us recall that the existence of the Haar measure on G/H is guaranteed whenever either G is compact, or H is a uniform lattice.

Making some abuse of notation, we will use the brackets $[\cdot,\cdot]$ to denote the Lie brackets on $\mathfrak{g}$, as well as the commutator operator in G, i.e. $[g,h] := ghg^{-1}h^{-1}$, for any $g,h \in G$.

More generally, if $A, B \subset G$ we define $[A,B] := \langle [a,b] : a \in A,\ b \in B\rangle$, i.e. the (abstract) subgroup of G generated by commutators of the set subsets A and B, respectively. And analogously, if $\mathfrak{h},\mathfrak{k} \subset \mathfrak{g}$, we define $[\mathfrak{h},\mathfrak{k}] := \operatorname{span}_{\mathbb{R}}\{[v,w] : v \in \mathfrak{h},\ w \in \mathfrak{k}\}$.

The *centers* of G and $\mathfrak{g}$ are defined by $Z(G) := \{g \in G : [g,h] = 1_G,\ \forall h \in G\}$ and $Z(\mathfrak{g}) := \{v \in \mathfrak{g} : [v,w] = 0,\ \forall w \in \mathfrak{g}\}$, respectively.

2.3.2 Tori

The d-dimensional torus will be denoted by $\mathbb{T}^d$ and will be identified with $\mathbb{R}^d/\mathbb{Z}^d$. The canonical quotient projection will be denoted by $\pi\colon \mathbb{R}^d \to \mathbb{T}^d$. For simplicity, we shall simply write $\mathbb{T}$ for the 1-torus, i.e. the circle.

The symbol Leb_d will be used to denote the Lebesgue measure on $\mathbb{R}^d$, as well as the Haar measure on $\mathbb{T}^d$. Once again, for the sake of simplicity, we just write Leb, and also $\mathrm{d}x$, instead of Leb_1.

For each $\alpha \in \mathbb{T}^d$, let $R_\alpha\colon \mathbb{T}^d \to \mathbb{T}^d$ be the rigid translation $R_\alpha : x \mapsto x + \alpha$.

2.3.3 Homogeneous skew-products

Given an arbitrary Lie group G and any closed subgroup $H < G$, for any $\alpha \in \mathbb{T}$ and any $\gamma \in C^r(\mathbb{T}, G)$, we define the *homogeneous skew-product* $H_{\alpha,\gamma} \in \mathrm{Diff}^r(\mathbb{T} \times G/H)$ by

$$H_{\alpha,\gamma} : (t, gH) \mapsto (t + \alpha, \gamma(t)gH), \quad \forall (t, gH) \in \mathbb{T} \times G/H.$$

The space of C^r homogeneous skew-products on $\mathbb{T} \times G/H$ shall be denoted by $\mathrm{SW}^r(\mathbb{T} \times G/H)$. If G/H admits a Haar measure, and we denote it by ν, then we clearly have $\mathrm{SW}^r(\mathbb{T} \times G/H) \subset \mathrm{Diff}^r_\mu(\mathbb{T} \times G/H)$, where $\mu := \mathrm{Leb} \otimes \nu$.

2.3.4 Nilmanifolds and Mal'cev theory

Given an arbitrary Lie group G, the *central descending series of* G can be recursively defined by $G_0 := G$ and

$$G_n := [G, G_{n-1}], \quad \forall n \geq 1.$$

We say G is *nilpotent* when $G_k = \{1_G\}$, for certain $k \in \mathbb{N}$. The *degree of nilpotency* of G is defined as the maximal natural number n such that $G_n \neq \{1_G\}$.

From now on, N shall denote a (connected) simply connected nilpotent Lie group admitting a uniform lattice $\Gamma < N$. As usual, the (compact) homogeneous space N/Γ is called a (compact) *nilmanifold.*

It is important to recall that in such a case the exponential map $\exp\colon \mathfrak{n} \to N$ is a real-analytic diffeomorphism (see Theorem 1.2.1 in [CG90]). Hence, in this case N as well as $\mathfrak{n}$ can be identified with the universal cover of N/Γ.

After Mal'cev [Mal49], a basis $\{v_1, v_2, \dots, v_d\}$ of the Lie algebra $\mathfrak{n}$ is called a *Mal'cev basis* whenever $\mathfrak{n}_{(i)} := \mathrm{span}_{\mathbb{R}}\{v_1, \dots, v_i\}$ is an ideal in $\mathfrak{n}$, for each $i \in \{1, \dots, d\}$. Moreover, such a basis is said to be a *strongly based on* Γ when

$$\Gamma = \exp(\mathbb{Z}v_1)\exp(\mathbb{Z}v_2)\dots\exp(\mathbb{Z}v_d). \tag{2}$$

In [Mal49] (see also [CG90]) it is proved that there always exists a Mal'cev basis strongly based on Γ when N and Γ are as above.

Since each $\mathfrak{n}_{(i)}$ is an ideal in $\mathfrak{n}$, $N_{(i)} := \exp(\mathfrak{n}_{(i)}) \subset N$ turns to be a (closed) normal subgroup of N, and the quotient $N^{(i)} := N/N_{(i)}$ a nilpotent connected and simply connected Lie group, itself.

On the other hand, as a consequence of (2) we have

$$\Gamma_{(i)} := \exp\Big(\mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \dots \oplus \mathbb{Z}v_i\}\Big) \subset N_{(i)}$$

is a discrete subgroup of Γ. Hence, $\Gamma^{(i)} := \Gamma/\Gamma_{(i)}$ can be naturally identified with a uniform lattice of $N^{(i)}$.

2.4 Distributions and distributional unique ergodicity

Given any $k \in \mathbb{N}_0$, the space of *distribution on M up to order k* is defined as the topological dual space of $C^k(M)$ and will be denoted by $\mathcal{D}'_k(M)$. When $k = 0$, by Riesz representation theorem $\mathcal{D}'_0(M)$ can be identified with the space of finite complex measures on M and so, it will also be denoted by $\mathfrak{M}(M)$.

On the other hand, as it is usually done, the topological dual space of $C^\infty(M)$ will be simply denoted by $\mathcal{D}'(M)$ and its elements are just called distributions.

Since all the inclusions $C^{k+1}(M) \hookrightarrow C^k(M)$ and $C^\infty(M) \hookrightarrow C^k(M)$ are continuous, making some abuse of notation we can consider the following chain of inclusions (modulo restrictions):

$$\mathfrak{M}(M) = \mathcal{D}'_0(M) \subset \mathcal{D}'_1(M) \subset \mathcal{D}'_2(M) \subset \ldots \subset \mathcal{D}'(M).$$

Moreover, since we are assuming M is compact, it is well-known that

$$\mathcal{D}'(M) = \bigcup_{k\geq 0} \mathcal{D}'_k(M).$$

Now, as it was already mentioned in §1, any $f \in \mathrm{Diff}^k(M)$ acts linearly on $C^k(M)$ by pull-back, and the adjoint of this action is the linear operator $f_\star\colon \mathcal{D}'_k(M) \to \mathcal{D}'_k(M)$ given by

$$\langle f_\star T, \psi\rangle := \langle T, f^\star\psi\rangle = \langle T, \psi \circ f\rangle, \quad \forall\, T \in \mathcal{D}'_k(M),\ \forall\, \psi \in C^k(M).$$

The space of *f-invariant distributions up to order k* is defined by

$$\mathcal{D}'_k(f) := \{T \in \mathcal{D}'_k(M) : f_\star T = T\}.$$

Of course, when f is C^∞ we write $\mathcal{D}'(f) := \bigcup_{k\geq 0} \mathcal{D}'_k(f)$.

Given any measure $\mu \in \mathfrak{M}(M)$ and $r \geq 0$, we define

$$\mathrm{Diff}^r_\mu(M) := \{f \in \mathrm{Diff}^r(M) : f_\star\mu = \mu\},$$

and

$$C^r_\mu(M) := \left\{\phi \in C^r(M) : \int_M \phi \,\mathrm{d}\mu = 0\right\}.$$

As usual, we say that f is uniquely ergodic when $\mathfrak{M}(f) := \mathcal{D}'_0(f)$ is one-dimensional. We will say that f is *distributionally uniquely ergodic* (or just *DUE* for short) when f is C^∞ and $\mathcal{D}'(f)$ has dimension one.

2.4.1 Coboundaries and distributions

Given any $f \in \mathrm{Diff}^r(M)$, any $\psi\colon M \to \mathbb{C}$ and $n \in \mathbb{Z}$, the *Birkhoff sum* is defined by

$$\mathcal{S}^n\psi = \mathcal{S}^n_f\psi := \begin{cases} \sum_{i=0}^{n-1} \psi \circ f^i & \text{if } n \geq 1; \\ 0 & \text{if } n = 0; \\ -\sum_{i=1}^{-n} \psi \circ f^{-i} & \text{if } n < 0. \end{cases}$$

We say that ψ is a C^ℓ-*coboundary* for f (with $0 \leq \ell \leq r$) whenever there exists $u \in C^\ell(M)$ solving the following cohomological equation:

$$u \circ f - u = \psi.$$

Observe that in this case it holds $\mathcal{S}^n\psi = uf^n - u$, for any $n \in \mathbb{Z}$.

The space of C^ℓ-coboundaries will be denoted by $B(f, C^\ell(M))$. Following Katok [Kat01], we say f is *cohomologically* C^ℓ*-stable* whenever $B(f, C^\ell(M))$ is a closed in $C^\ell(M)$.

Finally, notice that as a straightforward consequence of Hahn-Banach theorem we get

Proposition 2.1. *Given any* $f \in \mathrm{Diff}^k(M)$, *with* $k \in \mathbb{N}_0 \cup \{\infty\}$, *it holds*

$$\mathrm{cl}_k(B(f, C^k(M))) = \bigcap_{T\in\mathcal{D}'_k(f)} \ker T,$$

where $\mathrm{cl}_k(\cdot)$ *denotes the closure in* $C^k(M)$.

2.4.2 Unique ergodicity vs. DUE

There are many well-known examples of uniquely ergodic systems which are not DUE. Maybe, the simplest one is given by the *parabolic* map

$$\mathbb{T}^2 \ni (x, y) \mapsto (x + \alpha, y + x),$$

with $\alpha \in \mathbb{T} \setminus (\mathbb{Q}/\mathbb{Z})$ (see [Kat01] for details). Horocycle flows on constant negatively curved closed surfaces and minimal homogeneous flows on closed nilmanifolds different from tori are more elaborated examples [FF03, FF07].

On the other hand, a classical result due to Kronecker affirms that a translation $R_\alpha\colon \mathbb{T}^d \to \mathbb{T}^d$ is uniquely ergodic (Leb_d is the only R_α-invariant probability measure) if and only $\alpha = (\alpha_1, \dots, \alpha_d)$ is irrational.

Moreover, we have the following result which belongs to the *folklore*:

Proposition 2.2. *R_α is DUE, and it is cohomologically C^∞-stable if and only if α is a Diophantine vector.*

Proof. See Proposition 2.3 in [AK11] for a proof. □

Recently, the first and third authors extended this result in [AK11] showing that any minimal circle diffeomorphism is also DUE (see [NT12] for a much simpler proof of non-existence of invariant distributions up to order 1).

As it was already mentioned in §1, the main aim of this paper consists in constructing DUE diffeomorphism which are not topologically conjugate to ergodic translations on tori.

3 Proof of Theorem A: general strategy

As it was already mentioned in §2.3, in both cases considered in Theorem A the homogeneous space P admits a Haar measure that will be denoted by ν_P. Then, the Haar measure of M, which is a homogeneous space itself, is given by $\mu := \mathrm{Leb}_1 \otimes \nu_P$.

Let us recall that any homogeneous skew-product on $M = \mathbb{T} \times P$ (see §2.3.3) preserves the measure μ. In other words, $\mathrm{SW}^r(\mathbb{T} \times P) \subset \mathrm{Diff}^r_\mu(M)$.

3.1 The Anosov-Katok space

Let us consider the *horizontal* $\mathbb{T}$-action $T \colon \mathbb{T} \times M \to M$ given by

$$T_\alpha(t,p) = T\big(\alpha,(t,p)\big) = (t+\alpha,p), \quad \forall (t,p) \in M = \mathbb{T} \times P.$$

Then we define the *Anosov-Katok space*

$$\mathcal{AK}^\infty(T) := \mathrm{cl}_\infty \left\{ H \circ T_\alpha \circ H^{-1} : \alpha \in \mathbb{T},\ H \in \mathrm{SW}^\infty(\mathbb{T} \times P) \right\}. \qquad (3)$$

Observe that each $T_\alpha \in \mathrm{SW}^\infty(\mathbb{T} \times P)$ and hence, $\mathcal{AK}^\infty(T) \subset \mathrm{SW}^\infty(\mathbb{T} \times P)$.

To prove Theorem A we will show

Theorem 3.1. *Generic diffeomorphisms in $\mathcal{AK}^\infty(T)$ are DUE. More precisely, the set*

$$\mathrm{DUE}(T) := \{ f \in \mathcal{AK}^\infty(T) : \dim \mathcal{D}'(f) = 1 \}$$

contains a dense G_δ-subset of $\mathcal{AK}^\infty(T)$.

Let us now describe the general strategy to prove Theorem 3.1:

A family $(V_n)_{n \geq 1}$ will be called a *filtration* of $C^\infty_\mu(M)$ whenever it satisfies:

- for every $n \geq 1$, $V_n \subset C^\infty_\mu(M)$ is a closed linear subspace,
- $V_n \subset V_{n+1}$, for every $n \geq 1$;
- $\bigcup_{n \geq 1} V_n$ is dense in $C^\infty_\mu(M)$.

Sections 4 and 5 are dedicated to the proof of the following lemma, in the nilpotent and the compact cases, respectively:

Lemma 3.2. *If P is as in Theorem A, then there exists a filtration of $C^\infty_\mu(M)$, called (V_k), satisfying the following condition:*

For every $n \in \mathbb{N}$ and every $q_0 \in \mathbb{N}$, there exists $\bar{q} \in \mathbb{N}$ and a homogeneous skew-product $H_{0,\gamma} \in \mathrm{SW}^\infty(\mathbb{T} \times P)$ such that:

(i) $H_{0,\gamma} \circ T_{1/q_0} = T_{1/q_0} \circ H_{0,\gamma}$*;*

(ii) $V_n \subset B\big(H_{0,\gamma} \circ T_{\bar{p}/\bar{q}} \circ H_{0,\gamma}^{-1}, C^\infty(M)\big)$*, for every $\bar{p} \in \mathbb{Z}$ coprime with $\bar{q}$.*

To prove Lemma 3.2, we shall need the following elementary

Lemma 3.3. *Let M be an arbitrary manifold, $f\colon M \to M$ a periodic C^r-map (i.e. there exists $q \in \mathbb{N}$ such that $f^q = id_M$) and $\phi \in C^k(M)$, with $0 \le k \le r \le \infty$. Then, $\phi \in B(f, C^k(M))$ iff*

$$\mathcal{S}^q_f\phi(x) = \sum_{j=0}^{q-1} \phi(f^j(x)) = 0, \quad \forall x \in M. \tag{4}$$

Proof of Lemma 3.3. If $\phi \in B(f, C^k(M))$, then there exists $u \in C^k(M)$ such that $\phi = u \circ f - u$. Hence, $\mathcal{S}^q_f\phi(x) = u(f^q(x)) - u(x) = 0$, for every $x \in M$.

Reciprocally, let us suppose (4) holds. Then, using a formula we learned from [MOP77][2], we write

$$v(x) := -\frac{1}{q}\sum_{j=1}^{q} \mathcal{S}^j_f\phi(x), \quad \forall x \in M.$$

It clearly holds $v \in C^k(M)$, and

$$\begin{aligned} v(f(x)) - v(x) &= -\frac{1}{q}\left(\sum_{j=1}^{q} \big(\mathcal{S}^j_f\phi(f(x)) - \mathcal{S}^j_f\phi(x)\big)\right) \\ &= -\frac{1}{q}\Big(\mathcal{S}^q_f\phi(f(x)) - q\phi(x)\Big) = \phi(x), \end{aligned}$$

for every $x \in M$. Thus, $\phi \in B(f, C^k(M))$. □

Now, assuming Lemma 3.2, we can prove Theorem 3.1, and henceforth, Theorem A, too:

Proof of Theorem 3.1. Let $(\phi_m)_{m\in\mathbb{N}}$ be a dense sequence in $C^\infty_\mu(M)$ and define

$$A_m := \left\{ f \in \mathcal{AK}^\infty(T) : \exists u \in C^\infty(M),\ \|uf - u - \phi_m\|_{C^m} < \frac{1}{m} \right\}.$$

Each set A_m is clearly open in $\mathcal{AK}^\infty(T)$, and by Proposition 2.1, it holds

$$\mathrm{DUE}(T) = \bigcap_{m \ge 1} A_m$$

Thus, we have to show each A_m is dense in $\mathcal{AK}^\infty(T)$. To do that, consider a fixed set A_m, any rational number p_0/q_0 and an arbitrary homogeneous skew-product $H \in \mathrm{SW}^\infty(\mathbb{T} \times P)$.

[2]We thank A. Navas for bringing this equation to our attention.

Since $(V_n)_{n\geq 1}$ is a filtration, there exists $n \in \mathbb{N}$ and $\phi \in V_n$ such that

$$\left\|\phi \circ H^{-1} - \phi_m\right\|_{C^m} < \frac{1}{m}. \tag{5}$$

Now, invoking Lemma 3.2, from n and q_0 we obtain a natural number $\bar{q}$ and a homogeneous skew-product $H_{0,\gamma}$ satisfying (i) and (ii). Then, for each $\ell \in \mathbb{N}$, let us define

$$\begin{aligned}
\hat{p}_\ell &:= \bar{q}p_0\ell + 1,\\
\hat{q}_\ell &:= \bar{q}q_0\ell,\\
p_\ell &:= \frac{\hat{p}_\ell}{\gcd(\hat{p}_\ell, \hat{q}_\ell)},\\
q_\ell &:= \frac{\hat{q}_\ell}{\gcd(\hat{p}_\ell, \hat{q}_\ell)}.
\end{aligned}$$

Notice that, for each ℓ, p_ℓ and q_ℓ are coprime, q_ℓ is multiple of $\bar{q}$ and $\frac{p_\ell}{q_\ell} \to \frac{p_0}{q_0}$, as $\ell \to +\infty$.

Then observe that, for every $\ell \in \mathbb{N}$ and any $(t,x) \in \mathbb{T} \times P$ it holds:

$$\begin{aligned}
\mathcal{S}^{q_\ell}_{H_{0,\gamma}T_{p_\ell/q_\ell}H^{-1}_{0,\gamma}}\phi(t,x) &= \sum_{j=0}^{q_\ell-1} \phi\Big(t + \frac{jp_\ell}{q_\ell}, \gamma\Big(t+\frac{jp_\ell}{q_\ell}\Big)\gamma(t)^{-1}x\Big)\\
&= \sum_{j=0}^{q_\ell-1} \phi\Big(t + \frac{j}{q_\ell}, \gamma\Big(t+\frac{j}{q_\ell}\Big)\gamma(t)^{-1}x\Big)\\
&= \sum_{r=0}^{q_\ell/\bar{q}-1}\sum_{s=0}^{\bar{q}-1} \phi\Big(t + \frac{s}{\bar{q}} + \frac{r}{q_\ell}, \gamma\Big(t+\frac{s}{\bar{q}} + \frac{r}{q_\ell}\Big)\gamma(t)^{-1}x\Big)\\
&= 0,
\end{aligned} \tag{6}$$

where las equality is consequence of condition (ii) of Lemma 3.2 and Lemma 3.3. Thus, we conclude that $\phi \in B(H_{0,\gamma}T_{p_\ell/q_\ell}H^{-1}_{0,\gamma}, C^\infty(M))$.

Henceforth,

$$\phi \circ H^{-1} \in B\left(HH_{0,\gamma}T_{p_\ell/q_\ell}H^{-1}_{0,\gamma}H^{-1}, C^\infty(M)\right), \tag{7}$$

for every $\ell \in \mathbb{N}$.

On the other hand, $T_{p_\ell/q_\ell} \to T_{p_0/q_0}$ in $\mathrm{Diff}^\infty(M)$, as $\ell \to \infty$. Hence, from (i) of Lemma 3.2, we get

$$HH_{0,\gamma}T_{p_\ell/q_\ell}H^{-1}_{0,\gamma}H^{-1} \xrightarrow{C^\infty} HT_{p_0/q_0}H^{-1}, \quad \text{as } \ell \to \infty. \tag{8}$$

Now, putting together (5), (7) and (8) we conclude $HT_{p_0/q_0}H^{-1} \in \mathrm{cl}_\infty(A_m)$, as desired. □

3.2 Real-analytic DUE diffeomorphisms

Before starting with the proof of Lemma 3.2, it is interesting to remark that using the techniques we applied in §3.1 it is possible to prove the existence of real-analytic DUE diffeomorphisms on $M = \mathbb{T} \times P$.

In fact, for the time being let us suppose G is an arbitrary Lie group and let us write $G^{\mathbb{C}}$ for the complexification of G.

Then, for each $\Delta > 0$, let us define $C^{\omega}_{\Delta}(\mathbb{T}, G)$ as the set of real-analytic functions $\gamma\colon \mathbb{T} \to G$ that admit a holomorphic extension from the complex band $A_{\Delta} := \{z \in \mathbb{C} : |\mathrm{Im}\, z| \leq \Delta\}/\mathbb{Z}$ to $G^{\mathbb{C}}$. Let us consider $C^{\omega}_{\Delta}(\mathbb{T} G)$ endowed with the distance dunction d_{Δ} given by

$$d_{\Delta}(\gamma_0, \gamma_1) := \sup_{z \in A_{\Delta}} d_{G^{\mathbb{C}}}(\gamma_0(z), \gamma_1(z)), \quad \forall \gamma_0, \gamma_1 \in C^{\omega}_{\Delta}(\mathbb{T}, G),$$

where $d_{G^{\mathbb{C}}}$ denotes a left invariant distance on $G^{\mathbb{C}}$.

Then, taking into account that, for any $\Delta > 0$, $C^{\omega}_{\Delta}(\mathbb{T}, G)$ is dense in $C^{\infty}(\mathbb{T}, G)$, repeating the same argument used in the Proof of Theorem 3.1, we can easily show that the set

$$\mathrm{DUE}^{\omega}_{\Delta}(T) := \left\{(\alpha, \gamma) \in C^{\omega}_{\Delta}(\mathbb{T}, G) : H_{0,\gamma} \circ T_{\alpha} \circ H_{0,\gamma}^{-1} \in \mathrm{SW}^{\omega}(\mathbb{T} \times P) \text{ is DUE}\right\}$$

is generic in $\mathbb{T} \times C^{\omega}_{\Delta}(\mathbb{T}, G)$, and in particular, non-empty.

4 The nilpotent case

All along this section, let us assume P is a compact nilmanifold equal to N/Γ, where N is a (connected) simply connected nilpotent Lie group and $\Gamma < N$ is a uniform lattice.

4.1 The filtration in the nilpotent case

Observe that any complex function on P can be lifted to its universal covering, which can be identify with N itself, getting a Γ-invariant complex function[3]. So, we can naturally identify $C^{\infty}(P)$ with

$$C^{\infty}_{\Gamma}(N) := \{\phi \in C^{r}(N) : \phi(xg) = \phi(x),\ \forall x \in N,\ \forall g \in \Gamma\}.$$

Moreover, since the exponential map $\exp\colon \mathfrak{n} \to N$ is a real-analytic diffeomorphism, we can identify $C^{\infty}(N)$ with $C^{\infty}(\mathfrak{n})$, and henceforth, $C^{\infty}(P)$ with a closed linear subspace of $C^{\infty}(N) = C^{\infty}(\mathfrak{n})$.

Let $\mathcal{V} = \{v_1, \ldots, v_d\}$ be a Mal'cev basis of $\mathfrak{n}$ strongly based on Γ (see §2.3.4 for details). Fixing this basis, we identify $C^{\infty}(\mathfrak{n})$ with $C^{\infty}(\mathbb{R}^d)$ simply writing

$$\phi\left(\sum_{i=1}^{d} x_i v_i\right) = \phi(x_1, x_2, \ldots, x_d), \quad \forall (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Thus, making some abuse of notation, we shall assume that

$$C^{\infty}(P) \subset C^{\infty}(N) = C^{\infty}(\mathfrak{n}) = C^{\infty}(\mathbb{R}^d) \tag{9}$$

Now let us analyze some of the equivariant conditions a function $\phi \in C^{\infty}(\mathbb{R}^d)$ must satisfy to belong to $C^{\infty}(P)$. First, since $v_1 \in Z(\mathfrak{n})$ (and $\exp(\mathbb{Z}v_1) \in \Gamma$),

[3]Consider the Γ-action on N given by right translations.

we conclude that, if $\phi \in C^\infty(P) \subset C^\infty(\mathbb{R}^d)$, then it is $\mathbb{Z}$-periodic in its first variable. Hence, we can consider the *Fourier-like* development

$$\phi(x_1, x_2, \dots x_d) = \sum_{k\in\mathbb{Z}} \hat{\phi}_k^{(1)}(x_2, \dots, x_d) e^{2\pi i k x_1}, \tag{10}$$

where each $\hat{\phi}_k^{(1)} \in C^\infty(\mathbb{R}^{d-1})$. Here, the 0^{th} Fourier-function $\hat{\phi}_0^{(1)}$ has a particularly nice interpretation: it can be naturally considered as defined on the nilpotent Lie group $N^{(1)} := N/N_{(1)}$, or more precisely, on the the compact nilmanifold $N^{(1)}/\Gamma^{(1)}$ (see §2.3.4 for these notations).

On the other hand, observe that the basis $\{v_2 + \mathfrak{n}_{(1)}, v_3 + \mathfrak{n}_{(1)}, \dots, v_d + \mathfrak{n}_{(1)}\}$ is a Mal'cev one for $\mathfrak{n}/\mathfrak{n}_{(1)}$ strongly based on the lattice $\Gamma^{(i)} = \Gamma/\Gamma_{(i)}$.

That means we can repeat our previous argument to prove that $\hat{\phi}_0^{(1)}$ is $\mathbb{Z}$-periodic on its first variable, and hence, we can consider the Fourier-like development

$$\hat{\phi}_0^{(1)}(x_2, \dots, x_d) = \sum_{k\in\mathbb{Z}} \hat{\phi}_k^{(2)}(x_3, \dots, x_d) e^{2\pi i k x_2}.$$

Once again, the Fourier-coefficient function $\hat{\phi}_0^{(2)}$ can be considered as an element of $C^\infty(N^{(2)}/\Gamma^{(2)})$ and the set $\{v_3 + \mathfrak{n}_{(2)}, \dots, v_d + \mathfrak{n}_{(d)}\}$ is a Mal'cev basis strongly based on $\Gamma^{(2)}$.

By induction, we get a family of *Fourier-like coefficients*

$$\hat{\phi}_k^{(j)} \in C^\infty(\mathbb{R}^{d-j}), \quad \forall j \in \{1, \dots, d\}, \ \forall k \in \mathbb{Z},$$

where each $\hat{\phi}_0^{(j)} \in C^\infty(N^{(i)}/\Gamma^{(i)}) \subset C^\infty(\mathbb{R}^{d-j})$ and satisfies

$$\hat{\phi}_0^{(j)}(x_{j+1}, \dots x_d) = \sum_{k\in\mathbb{Z}} \hat{\phi}_k^{(j+1)}(x_{j+2}, \dots, x_d) e^{2\pi i k x_{j+1}}.$$

Now we proceed to define the *pseudo-polynomials* on P: we shall say that $\phi \in C^\infty(P)$ is a *pseudo-polynomial* (with respect to $\mathcal{V}$) of degree less or equal than $n \in \mathbb{N}$ iff

$$\hat{\phi}_k^{(j)} \equiv 0, \quad \text{for every } j \in \{1, \dots, d\} \text{ and } |k| > n.$$

The linear space of pseudo-polynomials on P will be denoted by $\mathfrak{Pol}(P)$ and we shall write $\mathfrak{Pol}_n(P)$ for the subspace of pseudo-polynomials with degree at most n.

Notice that $M = \mathbb{T} \times P$ is a compact nilmanifold itself, hence we can talk about pseudo-polynomials on M. In this case, we shall add the vector $v_0 := \partial_t$ (which generated the Lie algebra of $\mathbb{R}$) to the basis $\mathcal{V}$, and hence any function $\phi \in C^\infty(M)$ will be written in coordinates $(t, x_1, \dots, x_d)$, being ϕ $\mathbb{Z}$-periodic on its first coordinate, too. So, we can also consider the Fourier-like development

$$\phi(t, x_1, \dots, x_d) = \sum_{k\in\mathbb{Z}} \hat{\phi}_k^{(0)}(x_1, \dots, x_d) e^{2\pi i k t},$$

with each $\hat{\phi}_k^{(0)} \in C^\infty(P)$. Of course, by analogy with our definition of pseudo-polynomials on P, we define

$$\mathfrak{Pol}_n(M) := \{\phi \in C^\infty(M) : \hat{\phi}_0^{(0)} \in \mathfrak{Pol}_n(P), \ \hat{\phi}_k^{(0)} \equiv 0, \ \forall |k| > n\}.$$

Combining an inductive argument on the dimension of M with classical Fourier theory one can easily show

Proposition 4.1. *The linear space*

$$\mathfrak{Pol}(M) = \bigcup_{n\geq 0} \mathfrak{Pol}_n(M)$$

is dense in $C^\infty(M)$*. In particular, this implies that the family* (V_n) *given by*

$$V_n := \mathfrak{Pol}_n(M) \cap C^\infty_\mu(M), \quad \forall n \geq 1$$

is a filtration of $C^\infty_\mu(M)$*.*

Now we will prove Lemma 3.2 assuming the filtration (V_n) is given by Proposition 4.1:

Proof of Lemma 3.2 in the nilpotent case. Let us write $d := \dim N$. We will recursively define, for $k \in \{0, 1, \dots, d\}$, two sequences $\gamma_k \in C^\infty(\mathbb{T}, N)$ and $(\bar{q}_k) \subset \mathbb{N}$ satisfying the following condition: there exists a constant $C_k \in \mathbb{R}$ such that for every $p \in \mathbb{Z}$ coprime with $\bar{q}_k$ and every $\phi \in V_n$,

$$\mathcal{S}^q_{H_k T_{p/\bar{q}_k} H_k^{-1}} \phi(t,x) = C_k \hat{\phi}^{(k)}_0(\gamma_k(t)^{-1} x N^{(k)}), \quad \forall (t,x) \in \mathbb{T} \times N, \tag{11}$$

where $H_k = H_{0,\gamma_k} \subset \mathrm{SW}^\infty(\mathbb{T} \times P)$ and considering $\hat{\phi}^{(k)}_0$ as a complex function on $N^{(k)} = N/N_{(k)}$ (see §2.3.4 for notations).

At this point it is important to notice that, since $\phi \in V_n \subset C^\infty_\mu(M)$, then

$$\int_P \hat{\phi}^{(k)}_0 \, \mathrm{d}\nu = 0, \quad \text{for every } 0 \leq k \leq d.$$

Thus, in particular, the complex number $\phi^{(d)}_0$ is equal to zero, and so, condition (11) for $k = d$ means that the Birkhoff sum vanishes. By Lemma 3.3, this is equivalent to $\phi \in B(H_d T_{p/\bar{q}_d} H_d^{-1}, C^\infty)$.

Now let us start with the case $k = 0$. Observe that without lost of generality we can assume $n < q_0$. Let us define $\gamma_0 \equiv 1_N$ (so, $H_0 = id_M$) and $\bar{q}_0 := q_0$. Hence, for every $\phi \in V_n$, and every $p \in \mathbb{Z}$ coprime with $\bar{q}_0$, we have

$$\begin{aligned} \mathcal{S}^{\bar{q}_0}_{H_0 T_{p/\bar{q}_0} H_0^{-1}} \phi(t,x) &= \sum_{j=0}^{\bar{q}_0 - 1} \phi\left(t + j\frac{p}{\bar{q}_0}, x\right) \\ &= \sum_{j=0}^{\bar{q}_0 - 1} \sum_{|\ell| \leq n} \hat{\phi}^{(0)}_\ell(x) e^{2\pi i \ell (t + \frac{j}{\bar{q}_0})} = \bar{q}_0 \phi^{(0)}_0(x), \end{aligned} \tag{12}$$

for every $(t,x) \in \mathbb{R} \times \mathbb{R}^d$. So, condition (11) is verified for $k = 0$.

Now, suppose we have already defined $\gamma_{k-1} \in C^\infty(\mathbb{T}, N)$ (and then, $H_{k-1} = H_{0,\gamma_{k-1}} \in \mathrm{SW}^\infty(\mathbb{T} \times G/H)$) and $\bar{q}_{k-1} \in \mathbb{N}$, with $1 \leq k \leq d$, then let us construct γ_k and $\bar{q}_k$.

To do this, we start considering an auxiliary C^∞-function $\rho \colon [0,1] \to [0,1]$ satisfying the following conditions:

(*i*) $\rho(t) = 0$, for every t in a neighborhood of 0;

(*ii*) $\rho(t) = 1$, for every t in a neighborhood of 1;

(*iii*) $\rho'(t) \geq 0$, for all t;

(*iv*) $\rho(1-t) = 1 - \rho(t)$, for all t.

Then, we use ρ to define a new auxiliary function $\eta\colon [0,1] \to \mathbb{R}$ as follows:

$$\eta(t) := \begin{cases} \rho\big(2q_0 t - \lfloor 2q_0 t \rfloor\big) + \frac{\lfloor 2q_0 t\rfloor}{q_0}, & \text{if } t \in \big[0, \frac{1}{2}\big); \\ \rho\big(2q_0(1-t) - \lfloor 2q_0(1-t)\rfloor\big) + \frac{\lfloor 2q_0(1-t)\rfloor}{q_0}, & \text{if } t \in \big[\frac{1}{2}, 1\big] \end{cases}$$

In this way, η turns to be smooth and it vanishes in some neighborhoods of 0 and 1, so we can consider it as an element of $C^\infty(\mathbb{T}, \mathbb{R})$. Observe that for any $m \in \mathbb{Z} \setminus \{0\}$ with $|m| < q_0$ and any $t \in \mathbb{T}$, it holds

$$\sum_{\ell=0}^{2q_0-1} e^{2\pi i m \eta(t+\ell/2q_0)} = \sum_{\ell=0}^{q_0-1} \Big(e^{2\pi i m(\eta(t)+\ell/q_0)} + e^{-2\pi i m(\eta(t)+\ell/q_0)} \Big) = 0. \tag{13}$$

Then we define

$$\bar{q}_k := 2\bar{q}_{k-1} q_0,$$

and $\gamma_k \in C^\infty(\mathbb{T}, N)$ by

$$\gamma_k(t) := \gamma_{k-1}(t) \exp(\eta(\bar{q}_{k-1} t) v_k), \quad \forall t \in \mathbb{T}.$$

Assuming the inductive hypothesis, let us prove condition (11) holds for k, too. Let $p \in \mathbb{Z}$ be any number coprime with $\bar{q}_k$, $\phi \in V_n$ arbitrary and (t,x) be any point in $\mathbb{T} \times N$. Then we have:

$$\begin{aligned} \mathcal{S}^{\bar{q}_k}_{H_k T_{p/\bar{q}_k} H_k^{-1}} \phi(t,x) &= \sum_{j=0}^{\bar{q}_k - 1} \phi\bigg(t + \frac{j}{\bar{q}_k}, \gamma_k\Big(t + \frac{j}{\bar{q}_k}\Big) \gamma_k(t)^{-1} x \bigg) \\ &= \sum_{\ell=0}^{2q_0-1} \sum_{j=0}^{\bar{q}_{k-1}-1} \phi\bigg(\Big(t + \frac{\ell}{\bar{q}_k}\Big) + \frac{j}{\bar{q}_{k-1}}, \gamma_k\bigg(\Big(t + \frac{\ell}{\bar{q}_k}\Big) + \frac{j}{\bar{q}_{k-1}}\bigg)\gamma_k(t)^{-1} x \bigg) \\ &= C_{k-1} \sum_{\ell=0}^{2q_0-1} \hat{\phi}_0^{(k-1)} \bigg(\exp\bigg(\eta\Big(\bar{q}_{k-1} t + \frac{\ell}{2q_0}\Big) v_k \bigg) \gamma_k(t)^{-1} x N^{(k-1)} \bigg) \\ &= C_{k-1} \sum_{\ell=0}^{2q_0-1} \hat{\phi}_0^{(k-1)} \bigg(\tilde{x}_k + \eta\Big(\bar{q}_{k-1} t + \frac{\ell}{2q_0}\Big), \tilde{x}_{k+1}, \dots, \tilde{x}_d \bigg) \\ &= C_{k-1} \sum_{\ell=0}^{2q_0-1} \sum_{|m|\leq n} \hat{\phi}_m^{(k)}(\tilde{x}_{k+1}, \dots, \tilde{x}_d) e^{2\pi i m(\tilde{x}_k + \eta(\bar{q}_{k-1}t + \ell/2q_0))} \\ &= C_{k-1} \sum_{|m|\leq n} \hat{\phi}_m^{(k)}(\tilde{x}_{k+1}, \dots, \tilde{x}_d) e^{2\pi i m \tilde{x}_k} \sum_{\ell=0}^{2q_0-1} e^{2\pi i m \eta(t+\ell/2q_0)} \\ &= q_0 C_{k-1} \hat{\phi}_0^{(k)}(\tilde{x}_{k+1}, \dots, \tilde{x}_d) = q_0 C_{k-1} \hat{\phi}_0^{(k)}\Big(\gamma_k(t)^{-1} x N^{(k)}\Big), \end{aligned} \tag{14}$$

where the sixth equality is consequence of (13) and where $(\tilde{x}_k, \tilde{x}_{k+1}, \dots, \tilde{x}_d)$ denotes the "coordinates" of the point $\gamma_{k+1}(t)^{-1} x N^{(k-1)}$ in the Lie algebra $\mathfrak{n}^{(k-1)}$, i.e. they satisfy the following equation:

$$\tilde{x}_k v_k + \tilde{x}_{k+1} v_{k+1} + \ldots + \tilde{x}_d v_d + \mathfrak{n}_{(k-1)} = \exp^{-1}_{N^{(k-1)}}(\gamma_k(t)^{-1} x N^{(k-1)}).$$

In this way, (14) shows condition (11) holds for k, finishing the proof of the lemma. □

5 The compact case

We start this section with a geometric construction which is completely independent of the homogeneous structure of the supporting manifold. So, for the time being, let us suppose M is an arbitrary smooth connected closed manifold and μ any Borel probability measure on M.

5.1 Equidistributed loops

Given a finite dimensional subspace $E \subset C^\infty_\mu(M)$, a smooth loop $\gamma \in C^\infty(\mathbb{T}, M)$ is said to be *E-equidistributed* when there exists $m \in \mathbb{N}$ such that

$$\sum_{j=0}^{m-1} \phi\Big(\gamma\Big(t + \frac{j}{m}\Big)\Big) = 0, \quad \forall t \in \mathbb{T},\ \forall \phi \in E. \tag{15}$$

The number m will be called the *period of the loop.*

Theorem 5.1. *For every finite dimensional subspace $E \subset C^\infty_\mu(M)$, there exists a smooth E-equidistributed loop $\theta \in C^\infty(\mathbb{T}, M)$.*

Proof. Let $N := \dim E$ and $(\phi_1)_{i=1}^N$ be a basis of E, and define $\Phi\colon M \to \mathbb{R}^N$ by $\Phi(x) := (\phi_1(x), \dots, \phi_N(x))$, for every $x \in M$. For each $m \in M$, let us write

$$\Phi^{(m)}(x_1, \dots, x_m) := \sum_{j=1}^{m} \Phi(x_j) \in \mathbb{R}^N, \quad \forall (x_1, \dots, x_m) \in M^m.$$

Let us consider the sets $Y^{(m)}, Z^{(m)} \subset M^m$ given by

$$Y^{(m)} := \left\{\bar{x} \in M^m : D\Phi_m(\bar{x})\colon T_{\bar{x}}M^m \to \mathbb{R}^N \text{ is surjective}\right\}, \tag{16}$$

$$Z^{(m)} := \left\{\bar{x} \in M^m : \Phi_m(\bar{x}) = 0 \in \mathbb{R}^N\right\}, \tag{17}$$

and then define $X^{(m)} := Y^{(m)} \cap Z^{(m)}$.

We divide the rest of the proof in several lemmas:

Lemma 5.2. *For every $n \geq N$, the set $Y^{(n)}$ is non-empty.*

Proof of Lemma 5.2. First observe that, given any $n \geq 1$,

$$D\Phi^{(n)}{}_{(x_1,\dots,x_n)}(v_1, \dots, v_n) = \sum_{j=1}^{n} D\Phi_{x_j}(v_j), \tag{18}$$

for every $(x_1, \dots, x_n) \in M^n$ and every $(v_1, \dots, v_n) \in T_{(x_1,\dots,x_n)}M^n$. This lets us affirm that for any $k, n \in \mathbb{N}$ and any *"forget-some-coordinate"* projection $\mathrm{pr}\colon M^{n+k} \to M^n$, it holds

$$\mathrm{pr}^{-1}(Y^{(n)}) \subset Y^{(n+k)}. \tag{19}$$

That means it is enough to show $Y^{(N)}$ is non-empty. Reasoning by contradiction, suppose this is not the case. By (18), this implies the set

$$\{D\Phi_x(v) : x \in M,\ v \in T_xM\}$$

is contained in a proper linear sub-space of $\mathbb{R}^N$, and therefore, we can find a non-identically zero linear functional $\mathcal{L}\colon \mathbb{R}^N \to \mathbb{R}$ such that $D(\mathcal{L}\circ\Phi)_x = 0$, for every $x \in M$. Of course, since we are assuming M is connected, this implies $\mathcal{L}\circ\Phi\colon M \to \mathbb{R}^N$ is a constant function. Since the coordinate functions of Φ (i.e. functions $\phi_1, \ldots, \phi_N$) have zero integral with respect to μ, we conclude that $\mathcal{L}\circ\Phi \equiv 0$, contradicting the linear independence of the set $(\phi_i)_{i=1}^N$. So, $Y^{(N)} \neq \emptyset$, and by (19), we get $Y^{(n)} \neq \emptyset$, for every $n \geq N$. ☐

Lemma 5.3. *There exists $m \in \mathbb{N}$ such that $X^{(m)}$ is non-empty.*

Proof of Lemma 5.3. Consider the set

$$C_\Phi := \bigcup_{n\geq 1} \left\{ \sum_{j=1}^n \lambda_j \Phi(x_j) \in \mathbb{R}^N : x_j \in M,\ \lambda_j > 0,\ \forall j \in \{1, \ldots, n\} \right\}.$$

Observe C_Φ is a convex cone in $\mathbb{R}^N$. We claim $C_\Phi = \mathbb{R}^N$. In fact, if this would not be the case, then there should exist a non-null linear functional $\mathcal{L}\colon \mathbb{R}^N \to \mathbb{R}$ such that $\mathcal{L}(y) \geq 0$ for every $y \in C_\Phi \subset \mathbb{R}^N$ and, in particular, $\mathcal{L}(\Phi(x)) \geq 0$, for every $x \in M$. But since the coordinate functions of Φ belong to $C_\mu^\infty(M)$, it holds

$$\int_M \mathcal{L}(\Phi(x))\, \mathrm{d}\mu = 0.$$

Hence, $\mathcal{L}\circ\Phi$ should be identically equal to zero, contradicting the linear independence of the coordinate functions $(\phi_i)_{i=1}^N$. Thus, $C_\Phi = \mathbb{R}^N$.

By Lemma 5.2, $Y^{(N)}$ is non-empty, so we can consider an arbitrary point $(z_1, \ldots, z_N) \in Y^{(N)}$. By our previous assertion about C_Φ, there exist $n \in \mathbb{N}$, positive numbers $\lambda_1, \ldots, \lambda_n$ and points $x_1, \ldots, x_n \in M$ such that

$$\sum_{j=1}^n \lambda_j \Phi(x_j) = -\Phi_N(z_1, \ldots, z_N) = -\sum_{j=1}^N \Phi(z_j). \tag{20}$$

Now, since $\Phi_N(Y^{(N)})$ is open in $\mathbb{R}^N$, we can assume (up to an arbitrary small perturbation of the points $z_1, \ldots, z_N$) that each $\lambda_j \in \mathbb{Q}$, and hence we can find $p_1, \ldots, p_n, q \in \mathbb{N}$ such that $\lambda_j = p_j/q$, for each $1 \leq j \leq n$. Now, we define $m := qN + \sum_{1\leq j\leq n} p_j$ and we claim $X^{(m)} \neq \emptyset$. In fact, if we define $(w_1, \ldots, w_m) \in M^m$ by

$$w_j := \begin{cases} z_{\lceil j/q \rceil}, & \text{if } 1 \leq j \leq qN, \\ x_k, & \text{if } qN < j \leq qN + \sum_{\ell=1}^k p_\ell, \end{cases}$$

from (20) we easily conclude $(w_1, \ldots, w_m) \in X^{(m)}$. ☐

Now, for each $n \geq 2$, let us consider the diffeomorphism $\sigma_n\colon M^n \to M^n$ given by

$$\sigma_n(x_1, x_2, \ldots, x_n) := (x_2, \ldots, x_n, x_1), \quad \forall (x_1, \ldots, x_n) \in M^n.$$

And we shall prove our last

Lemma 5.4. *There exist $m \geq 1$ and $\bar{z} \in X^{(m)}$ such that $\sigma_m(\bar{z}) \in \mathrm{cc}(X^{(m)}, \bar{z})$.*

Proof of Lemma 5.4. Let m_0 be a natural number such that $X^{(m_0)}$ is non-empty, and let $\bar{z} = (x_1, \dots, x_{m_0})$ be any point in $X^{(m_0)}$. For each $q \in \mathbb{N}$, let us consider the point $\bar{z}^{(q)} = (z_1^{(q)}, z_2^{(q)}, \dots, z_{qm_0}^{(q)}) \in X^{(qm_0)}$ given by

$$z_j^{(q)} := x_{\lceil j/q \rceil}, \quad \text{for } 1 \le j \le qm_0.$$

We claim that $\sigma_{qm_0}(\bar{z}^{(q)}) \in \mathrm{cc}(X^{(qm_0)}, \bar{z}^{(q)})$, provided q is sufficiently large. To prove this, we shall construct a continuous curve $\rho\colon [0,1] \to X^{(qm_0)} \subset M^{qm_0}$, with $\rho(0) = \bar{z}^{(q)}$ and $\rho(1) = \sigma_{qm_0}(\bar{z}^{(qm_0)})$.

For each $1 \le j \le qm_0$, we write $\rho_j\colon [0,1] \to M$ for the j^{th}-coordinate function of ρ, i.e. $\rho(t) = (\rho_1(t), \rho_2(t), \dots, \rho_{qm_0}(t)) \in M^{qm_0}$.

We start defining ρ on the interval $[0,1/2]$. To do that, fitst let us consider continuous paths $\alpha = (\alpha_1, \dots, \alpha_{m_0})\colon [0,1] \to M^{m_0}$ such that

$$\begin{aligned} \alpha_i(0) &= x_i, & \alpha_i(1) &= x_{i+1}, & &\text{for } 1 \le i < m_0, \\ \alpha_{m_0}(0) &= x_{m_0}, & \alpha_{m_0}(1) &= x_1. \end{aligned}$$

Now, we choose a (small) neighborhood U of $\bar{z}$ in M^{m_0} such that $U \subset Y^{(m_0)}$ and $U \cap X^{(m_0)}$ is connected. Since $\Phi^{(m_0)}$ is a submersion on $Y^{(m_0)}$, we can find a continuous path $\beta\colon [0,1] \to U \subset M^{m_0}$ satisfying $\beta(0) = \bar{z}$ and

$$\Phi^{(m_0)}\big(\beta(t)\big) = -\frac{\Phi^{(m_0)}\big(\alpha(t)\big)}{q}, \quad \forall t \in [0,1], \tag{21}$$

provided q is sufficiently large. Notice that, since $\alpha(1) \in X^{(m_0)}$, then $\beta(1)$ also belongs to $X^{(m_0)}$ itself.

Then, we define each coordinate function of the path ρ on $[0,1/2]$ by

$$\rho_j(t) := \begin{cases} \beta_{\lceil j/q \rceil}(2t), & \text{if } 1 \le j < m_0 q, \text{ and } j \notin q\mathbb{Z} \\ \alpha_{j/q}(2t), & \text{if } j \in \{q, 2q, \dots, m_0 q\}, \end{cases}$$

and every $t \in [0,1/2]$. Notice that, as a consequence of (21), $\rho(t) \in X^{(qm_0)}$, for every $t \in [0,1/2]$.

In order to define path ρ on $[1/2,1]$, let us consider a continuous path $\gamma\colon [0,1] \to X^{(m_0)}$ joining $\beta(1)$ to $\bar{z}$. Such a path γ does exist because both points $\beta(1)$ and $\bar{z}$ belong to $U \cap X^{(m_0)}$, which is a connected open set of the smooth manifold $X^{(m_0)}$, and hence, it is arc-wise connected.

Finally, we define ρ on $[1/2,1]$ by

$$\rho_j(t) := \begin{cases} \gamma_{\lceil j/q \rceil}(2t-1), & \text{if } j \notin q\mathbb{Z} \\ x_{j+1}, & \text{if } j \in q\mathbb{Z} \text{ and } 1 \le j < qm_0, \\ x_1, & j = qm_0. \end{cases}$$

In this way, ρ is clearly a continuous path contained in $X^{(qm_0)}$ and joins $\rho(0) = \bar{z}$ to $\rho(1) = \sigma_{qm_0}(\bar{z})$, as desired. □

Finally, let m and $\bar{z} \in X^{(m)}$ as in Lemma 5.4. Since $X^{(m)} \subset M^m$ is a σ_m-invariant embedded submanifold, and σ_m is an m-periodic diffeomorphism, we can find a smooth loop $\tilde{\theta} \in C^\infty(\mathbb{T}, X^{(m)})$ satisfying

$$\tilde{\theta}\left(t + \frac{1}{m}\right) = \sigma_m(\tilde{\theta}(t)), \quad \forall t \in \mathbb{T}. \tag{22}$$

Then, for any $t \in \mathbb{T}$, if we write $\theta(t) = (\theta_1(t), \ldots, \theta_m(t)) \in M^m$, it holds

$$\mathbb{R}^N \ni 0 = \Phi^{(m)}(\theta_1(t), \ldots, \theta_m(t)) = \sum_{j=1}^{m} \Phi(\theta_j(t)) = \sum_{j=1}^{m} \Phi\bigg(\theta_1\Big(t + \frac{j}{m}\Big)\bigg),$$

where last equality is consequence of (22).

Thus, $\theta = \theta_1$ is a smooth E-equidistributed loop, as desired. □

5.2 The filtration in the compact case

In this section we construct the filtration of $C^\infty_\mu(\mathbb{T} \times P)$ in order to prove Lemma 3.2.

To do this, we return to our homogeneous setting, assuming G is a compact (connected) Lie group, $H < G$ a closed subgroup, $P = G/H$ and $M = \mathbb{T} \times P$. For the sake of simplicity of the exposition, we start assuming $H = \{1_G\}$. The general case will be easily gotten from this particular one.

If ν_G denotes the Haar (probability) measure on G, there are two unitary representations of G on $L^2_0(G, \nu_G) := \{\phi \in L^2(G, \nu_G) : \int \phi \nu_G = 0\}$ given by

$$\begin{aligned} &(L_g\phi)(x) := \phi(g^{-1}x), \\ &(R_g\phi)(x) := \phi(xg), \quad \forall g, x \in G, \ \forall \phi \in L^2_0(G, \nu_G). \end{aligned} \tag{23}$$

By the classical Peter-Weyl theorem, we know left action L decomposes in a direct sum of finite-dimensional irreducible sub-representations, i.e. there exists a family $(E_n)_{n\geq 1}$ of finite-dimension subspaces of $L^2_0(G, \nu_G)$ such that $\bigoplus_n E_n$ is dense in $L^2_0(G, \nu_G)$ and each E_n is L-invariant, with no proper L-invariant subspace contained in E_n. Moreover, these spaces satisfy $E_n \subset C^\infty_{\nu_G}(G) = C^\infty(G) \cap L^2_0(G, \nu_G)$, for every $n \geq 1$ and they are also R-invariant (for instance, see §3.3 in [Sep07] for details).

In particular, this implies that, if $\gamma \colon \mathbb{T} \to G$ an E_n-equidistributed with period m, then

$$\sum_{j=0}^{m-1} \phi\bigg(\gamma\Big(t + \frac{j}{m}\Big)x\bigg) = \sum_{j=0}^{m-1} (R_x\phi)\bigg(\gamma\Big(t + \frac{j}{m}\Big)\bigg) = 0, \tag{24}$$

for every $t \in \mathbb{T}$, every $x \in G$ and any $\phi \in E_n$.

Now, for each $\phi \in C^\infty(M)$ and every $k \in \mathbb{Z}$, we define $\hat{\phi}_k \in C^\infty(G)$ by

$$\hat{\phi}_k(x) := \int_{\mathbb{T}} \phi(t, x) e^{-2\pi i k t} \, \mathrm{d}t, \quad \forall x \in G, \tag{25}$$

and

$$V_n := \left\{ \phi \in C^\infty_\mu(M) : \hat{\phi}_0 \in \bigoplus_{j \leq n} E_j, \ \hat{\phi}_k \equiv 0, \ \forall\, |k| > n \right\}. \tag{26}$$

By Peter-Weyl theorem and classical Fourier series arguments we have

Lemma 5.5. *The family (V_n) given by (26) is a filtration for $C^\infty_\mu(M)$.*

Proof of Lemma 3.2 in the compact case. Let us consider the filtration $(V_j)_{j\geq 1}$ given by (26). As we did in the nilpotent case, without lost of generality we can assume $n < q_0$.

Let $\tilde{\gamma} \in C^\infty(\mathbb{T}, G)$ be a $\left(\bigoplus_{j\leq k} E_j\right)$-equidistributed loop in G, and let $\tilde{m}$ be its period. Then let us define $\gamma\colon \mathbb{T} \to G$ by $\gamma(t) := \tilde{\gamma}(q_0 t)$ and write $\bar{q} := q_0\tilde{m}$. Notice $H_{0,\gamma} \in \mathrm{SW}^\infty(\mathbb{T}\times G)$ clearly commutes with T_{1/q_0}. Let us show that condition (ii) of Lemma 3.2 also holds.

To do that, let p be any integer coprime with $\bar{q}$ and let us consider an arbitrary $\phi \in V_n$. Once again we consider the Fourier-like development of ϕ:

$$\phi(t,x) = \sum_{|\ell|\leq n} \hat{\phi}_\ell(x) e^{2\pi i \ell t}, \quad \forall (t,x) \in \mathbb{T}\times G,$$

where each $\hat{\phi}_\ell \in C^\infty(G)$ is given by (25) and $\hat{\phi}_0 \in \bigoplus_{j\leq n} E_j$.

Then we have,

$$\begin{aligned}
\mathcal{S}^{\bar{q}}_{H_{0,\gamma}T_{p/\bar{q}}H_{0,\gamma}^{-1}}\phi(t,x) &= \sum_{j=0}^{\bar{q}-1} \phi\left(t+\frac{j}{\bar{q}}, \gamma\Big(t+\frac{j}{\bar{q}}\Big)x\right) \\
&= \sum_{j=0}^{\bar{q}-1} \phi\left(t+\frac{j}{q}, \tilde{\gamma}\Big(q_0 t+\frac{j}{\tilde{m}}\Big)x\right) \\
&= \sum_{j=0}^{\bar{q}-1} \sum_{|\ell|\leq n} \hat{\phi}_\ell\left(\tilde{\gamma}\Big(q_0 t+\frac{j}{\tilde{m}}\Big)x\right) e^{2\pi i \ell (t+\frac{j}{\bar{q}})} \\
&= \sum_{j=0}^{q_0-1} \sum_{k=0}^{\tilde{m}-1} \sum_{|\ell|\leq n} \hat{\phi}_\ell\left(\tilde{\gamma}\Big(q_0 t+\frac{k}{\tilde{m}}\Big)x\right) e^{2\pi i \ell (t+\frac{j}{q_0}+\frac{k}{\tilde{m}})} \qquad (27)\\
&= \sum_{|\ell|\leq n} e^{2\pi i \ell t} \sum_{k=0}^{\tilde{m}-1} \hat{\phi}_\ell\left(\tilde{\gamma}\Big(q_0 t+\frac{k}{\tilde{m}}\Big)x\right) e^{2\pi i \ell \frac{k}{\tilde{m}}} \sum_{j=0}^{q_0-1} e^{2\pi i \ell \frac{j}{q_0}} \\
&= q_0 \sum_{k=0}^{\tilde{m}-1} \hat{\phi}_0\left(\tilde{\gamma}\Big(q_0 t+\frac{k}{\tilde{m}}\Big)x\right) \\
&= q_0 \sum_{k=0}^{\tilde{m}-1} R_x\hat{\phi}_0\left(\tilde{\gamma}\Big(q_0 t+\frac{k}{\tilde{m}}\Big)\right) = 0
\end{aligned}$$

for every $t \in \mathbb{T}$ and every $x \in G$, and where the last equality is a consequence (24) and invariance by R_x of $\bigoplus_{j\leq n} E_j$.

Thus, by Lemma 3.3, it follows from (27) that $V_n \subset B(H_{0,\gamma}T_{p/q}H_{0,\gamma}^{-1})$, as desired. □

5.3 The case $H \neq \{1_G\}$

Now, let us suppose $H < G$ is a proper closed subgroup. Since G and H are both compact, they admit unique Haar probability measures, which will be denoted by ν_G and ν_H, respectively. The Haar measure on G/H will be simply denoted by ν.

We will write $\pi_H\colon G\to G/H$ for the canonical projection and we can define the linear operator $\Pi_H\colon C^\infty(G)\to C^\infty(G/H)$ by

$$\Pi_H\psi(gH) := \int_H \psi(gx)\,\mathrm{d}\nu_H(x), \quad \forall\psi\in C^\infty(G),$$

Let us remark that Π_H is continuous, closed and surjective (in fact, the pull-back by π_H is a section of Π_H) and satisfies $\Pi_H(C^\infty_{\nu_G}(G)) = C^\infty_\nu(G/H)$. In particular, the family $(\Pi_H(E_j))_{j\geq 1}$, where spaces E_j are defined as in §5.2, turns to be a filtration of $C^\infty_\nu(G/H)$, where each $\Pi_H(E_j)$ has finite dimension.

Then, we have the following

Lemma 5.6. *If $\gamma\in C^\infty(\mathbb{T},G)$ is an E_k-equidistributed loop (with $k\in\mathbb{N}$ arbitrary), then $\pi_H\circ\gamma$ is a $\Pi_H(E_k)$-equidistributed loop on G/H.*

Proof. Let m denote the period of γ and $\phi\in E_k$ be arbitrary. Then we have

$$\begin{aligned}
\sum_{j=0}^{m-1}\Pi_H(\phi)\Big(\pi_H\circ\gamma\Big(t+\frac{j}{m}\Big)\Big) &= \sum_{j=0}^{m-1}\int_H \phi\Big(\gamma\Big(t+\frac{j}{m}\Big)y\Big)\,\mathrm{d}\nu_H(y)\\
&= \int_H\left(\sum_{j=0}^{m-1}\phi\Big(\gamma\Big(t+\frac{j}{m}\Big)y\Big)\right)\mathrm{d}\nu_H(y)\\
&= \int_H\left(\sum_{j=0}^{m-1}(R_y\phi)\Big(\gamma\Big(t+\frac{j}{m}\Big)\Big)\right)\mathrm{d}\nu_H(y) = 0,
\end{aligned}$$

where last equality is consequence of the R-invariance of space E_k. □

Now, using Lemma 5.6 we can easily extend our proof of §5.2 to the case where H is a proper subgroup. In fact, given any $\phi\in C^\infty(\mathbb{T}\times G/H)$ and any $k\in\mathbb{Z}$, once again we can define $\hat\phi_k\in C^\infty(G/H)$ by

$$\hat\phi_k(gH) := \int_{\mathbb{T}}\phi(t,gH)e^{2\pi ikt}\,\mathrm{d}t, \quad \forall gH\in G/H,$$

and so (re)define the filtration $(V_n)_{n\in\mathbb{N}}$ of $C^\infty_\mu(\mathbb{T}\times G/H)$ analogously to (26):

$$V_n := \left\{\phi\in C^\infty_\mu(\mathbb{T}\times G/H) : \hat\phi_0\in\bigoplus_{j\leq n}\Pi_H(E_j),\ \hat\phi_k\equiv 0,\ \forall\,|k|>n\right\}, \quad \forall n\in\mathbb{N},$$

Then, invoking Lemma 5.6 and the above filtration, *mutatis mutandis* we can extend the proof of Lemma 3.2 we did in §5.2 in the case $H=\{1_G\}$ to the general one.

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