LYAPUNOV UNSTABLE ELLIPTIC EQUILIBRIA

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ABSTRACT. A new diffusion mechanism from the neighborhood of elliptic equilibria for Hamiltonian flows in three or more degrees of freedom is introduced. We thus obtain explicit real entire Hamiltonians on $\mathbb{R}^{2d}$, $d \geq 4$, that have a Lyapunov unstable elliptic equilibrium with an arbitrary chosen frequency vector whose coordinates are not all of the same sign. For non-resonant frequency vectors, our examples all have divergent Birkhoff normal form at the equilibrium.

On $\mathbb{R}^4$, we give explicit examples of real entire Hamiltonians having an equilibrium with an arbitrary chosen non-resonant frequency vector and a divergent Birkhoff normal form.

Introduction

An equilibrium $(p, q) \in \mathbb{R}^{2d}$ of an autonomous Hamiltonian flow is said to be Lyapunov stable or topologically stable if all nearby orbits remain close to (p, q) for all forward time.

The topological stability of equilibria of Hamiltonian flows is one of the oldest problems in mathematical physics. The important contributions to the understanding of this problem, dating back to the 18th century, form a fundamental part of the foundation and of the evolution of the theory of dynamical systems and celestial mechanics up to our days.

The goal of this note is to give examples of real analytic Hamiltonians that have a Lyapunov unstable non-resonant elliptic equilibrium.

A C^2 function $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ such that $DH(0) = 0$ defines a Hamiltonian vector field $X_H(x, y) = (\partial_y H(x, y), -\partial_x H(x, y))$ whose flow ϕ_H^t preserves the origin.

Naturally, to study the stability of the equilibrium at the origin, one has first to investigate the stability of the linearized system at the origin. By symplectic symmetry, the eigenvalues of the linearized system come by pairs $\pm\lambda$, $\lambda \in \mathbb{C}$. It follows that if the linearized system has an eigenvalue with a non zero real part, it also has an eigenvalue with positive real part and this implies instability of the origin for the linearized system as well as for the non-linear flow.

When all the eigenvalues of the linearized system are on the imaginary axis the stability question is more intricate. In the non-degenerate case where the eigenvalues are simple, we say that the origin is an *elliptic* equilibrium. The linear system is then symplectically conjugated to a direct product of planar rotations. The arguments of the eigenvalues are called the frequencies of the equilibrium since they correspond to angles of rotation of the linearized system. In this paper, we focus our attention on real analytic Hamiltonians $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ of the form

$$(*) \qquad H(x,y) = H_\omega(x,y) + O^3(x,y),$$
$$H_\omega(x,y) = \sum_{j=1}^{3} \omega_j I_j, \quad I_j = \frac{1}{2}(x_j^2 + y_j^2).$$

where $\omega \in \mathbb{R}^d$ has rationally independent coordinates. We say that $f \in O^l(x,y)$ when $\partial_z f(0) = 0$ for any multi-index z on the x_i and y_i of size less or equal to $l-1$. The elliptic equilibrium at the origin of the flow of X_H is then said to be *non-resonant*.

The phenomenon of averaging out of the non-integrable part of the nonlinearity effects at a non-resonant frequency is responsible for the long time effective stability around the equilibrium : the points near the equilibrium remain in its neighborhood during a time that is greater than any negative power of their distance to the equilibrium. This can be formally studied and proved using the Birkhoff Normal Forms (BNF) at the equilibrium, that introduce action-angle coordinates in which the system is integrable up to arbitrary high degree in its Taylor series (see Section 4 for some reminders about the BNF, and [Bi66] or [SM71], for example, for more details). Moreover, it was proven in [MG95, BFN15] that a typical elliptic fixed point is doubly exponentially stable in the sense that a neighboring point of the equilibrium remains close to it for an interval of time which is doubly exponentially large with respect to some power of the inverse of the distance to the equilibrium point.

In addition to the long time effective stability of non-resonant equilibria, KAM theory (after Kolmogorov Arnold and Moser), asserts that a non-resonant elliptic fixed point is in general accumulated by quasi-periodic invariant Lagrangian tori whose relative measurable density tends to one in small neighborhoods of the fixed point. This can be viewed as stability in a probabilistic sense, and is usually coined *KAM stability*. In classical KAM theory, KAM stability is established when the BNF has a non-degenerate Hessian. Further development of the theory allowed to relax the non degeneracy condition and [EFK13] proved KAM-stability of a non-resonant elliptic fixed point under the non-degeneracy condition of the BNF (see Section 4).

Despite the long time effective stability, and despite the genericity of KAM-stability, Arnold conjectured that apart from two cases, the case of a sign-definite quadratic part of the Hamiltonian, and generically for $d = 2$, an elliptic equilibrium point of a generic real analytic Hamiltonian system is Lyapunov unstable [Arn94, Section 1.8].

REMARK 1. When the quadratic part of the Hamiltonian is sign-definite, which corresponds to all the ω_i having the same sign in $(*)$, the invariance of the Hamiltonian under the motion forces the Lyapunov stability of the equilibrium. In the case $d = 2$, the Arnold's iso-energetic non-degenracy condition, that is generic, forces Lyapunov stability due to the existence of KAM tori in every energy surface.

Although a rich literature in the direction of proving this conjecture exist in the C^∞ smoothness (we mention [KMV04] below, but to give a list of contributions would exceed the scope of this introduction), the conjecture is still wide open in the real analytic category. For instance, not a single example of real analytic Hamiltonians was known that has an unstable non-resonant elliptic equilibrium. The main goal of this work is to give the first examples of real analytic Hamiltonians having an unstable non-resonant elliptic equilibrium, with an arbitrary frequency vector for $d \geq 4$, under the condition that not all the coordinates have the same sign. In our constructions, we can guarantee that the BNF at the unstable elliptic equilibrium is non-degenerate, so that Lyapunov instability coexists for such examples with KAM stability (see Section 1 below).

We can also give examples of unstable elliptic equilibria where we guarantee that the BNF is divergent. Inspired by these constructions, we also obtain explicit examples of real entire Hamiltonians having an elliptic equilibrium at the origin and a divergent BNF, for *any* degree of freedom and for *any* non-resonant frequency vector (no sign condition is required here, since we do not claim any dynamical properties). As consequence of the existence of such examples and of the alternative proved in [PM97], we obtain the generic divergence of the BNF at a non resonant elliptic equilibrium.

The genericity of divergence of the BNF was recently obtained by Krikorian for symplectomorphisms with an elliptic fixed point [K19] by a completely different method than ours (see Section 2.1 below).

We start with a reminder on Birkhoff normal forms at a non-resonant elliptic equilibrium, and of the KAM theorem proved in [EFK15]. More insight on the way the BNF is obtained will be given in Section 4.1.

1. Birkhoff normal forms at a non-resonant elliptic equilibrium

For H as in $(*)$, ω non-resonant, for all $N \geq 1$, there exists an exact symplectic transformation $\Phi_N = \mathrm{Id} + O^2(x,y)$, and a polynomial B_N of degree N in the variables $I_1, \dots, I_d$, such that

$$H \circ \Phi_N(x,y) = B_N(I) + O^{2N+1}(x,y).$$

There also exists a formal exact symplectic transformation $\Phi_\infty = \mathrm{Id} + O^2(x,y)$, where $O^2(x,y)$ is formal power series in x and y with no affine terms such that

$$H \circ \Phi_\infty(x,y) = B_\infty(I)$$

where B_∞ is a uniquely defined formal power series of the action variables I_j, called the Birkhoff Normal Form (BNF) at the origin.

For more details on the Birkhoff Normal Form at a Diophantine, and more generally at any non-resonant elliptic equilibrium, one can consult for example [SM71].

Divergent Birkhoff Normal Forms. When the domain of convergence of the formal power series $B_\infty(\cdot)$ does not contain any ball around zero, we say that the BNF diverges. We use the same terminology for a general formal power series in several variables.

Non-degenerate Birkhoff Normal Forms. Following [Rüs01], this definition was given in [EFK15] :

DEFINITION 1. *We say that the BNF at a non resonant elliptic fixed point B_∞ is Rüssmann non-degenerate or simply non-degenerate if there does not exist any vector γ such that for every I in some neighborhood of 0*

$$\langle \nabla B_\infty(I), \gamma\rangle = 0.$$

DEFINITION 2. *We say that a non resonant equilibrium point of a real analytic Hamiltonian H is KAM-stable if, in any neighborhood of the equilibrium, the set of real analytic KAM-tori for X_H has Lebesgue density one at 0.*

Real analytic KAM-tori are invariant Lagrangian tori on which the flow generated by H is real analytically conjugated to a minimal translation flow (of Diophantine frequency vector) on the torus $\mathbb{R}^d/\mathbb{Z}^d$. In [EFK15] the following was proven

THEOREM 1 ([EFK15]). – *Let $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ be a real analytic function of the form $(*)$ and assume that ω is non-resonant. If the BNF of H at the origin is non-degenerate, then the origin is KAM-stable.*

2. Unstable equilibria and divergent Birkhoff normal forms. General statements

2.1. Lyapunov unstable equilibria.

We start with the existence of real entire Hamiltonians with unstable non-resonant equilibria.

THEOREM A. – *There exists a non-resonant $\omega \in \mathbb{R}^3$ and a real entire Hamiltonian $H : \mathbb{R}^6 \to \mathbb{R}$, such that the origin is a Lyapunov unstable elliptic equilibrium with frequency ω of the Hamiltonian flow Φ_H^t of H.*

For any $\omega \in \mathbb{R}^d, d \geq 4$, such that not all its coordinates are of the same sign, there exists a real entire Hamiltonian $H : \mathbb{R}^{2d} \to \mathbb{R}$ such that the origin is a Lyapunov unstable elliptic equilibrium with frequency ω of the Hamiltonian flow Φ_H^t of H.

Moreover, in all our examples, for non-resonant frequencies ω, the Birkhoff normal form at the origin is divergent.

Finally, for non-resonant frequencies ω, it is possible to choose the Hamiltonians H such that the origin is KAM stable.

Detailed statements with an explicit definition of the Hamiltonians that prove Theorem A, will be given in Section 3.

Note that we do not obtain the existence of an orbit that accumulates on the origin. Based on a different diffusion mechanism, [FMS17] gives examples of smooth symplectic diffeomorphisms of $\mathbb{R}^6$ having a non-resonant elliptic fixed point that attracts an orbit.

Note also that the question of Lyapunov instability in two degrees of freedom remains open. The question remains open also in the case of three degrees of freedom and Diophantine frequency vectors.

As explained earlier, Lyapunov instability of an elliptic fixed point of frequency vector ω is only possible when not all the coordinates of ω are of the same sign.

2.2. **The case of quasi-periodic tori**. The same constructions that yield Theorem A can be carried out on $\mathbb{R}^d \times \mathbb{T}^d$ to get examples, starting from $d = 3$, of real analytic Hamiltonians with an invariant quasi-periodic torus $\{0\} \times \mathbb{T}^d$ that is Lyapunov unstable. Moreover, in that case, the condition that the coordinates of the frequency vector of the quasi-periodic torus are not all of the same sign is not anymore required. We will explain this in Section 6 after the explicit form of the Hamiltonians with Lyapunov unstable equilibria is given. We do not pursue the constructions on $\mathbb{R}^d \times \mathbb{T}^d$ in detail in this paper, because the work [FF19] provides many examples of real analytic Hamiltonians with invariant quasi-periodic tori $\{0\} \times \mathbb{T}^d$ that are Lyapunov unstable. The construction method of [FF19] is quite different from the one introduced here. There, the constructions are limits of successive conjugacies of integrable Hamiltonians and as such, they have a convergent Birkhoff normal form at the invariant quasi-periodic torus.

We note that the method of [FF19] is for the moment inapplicable to equilibrium points and the question of having unstable elliptic equilibrium with a convergent BNF is still completely open in the real analytic setting.

2.3. Generic divergence of Birkhoff normal forms for all $d \geq 2$. Inspired by the constructions of Theorem A, it is possible to obtain explicit examples of real entire Hamiltonians having an elliptic equilibrium at the origin and a divergent BNF for *any* degree of freedom including $d = 2$, and for any non-resonant frequency vector, including vectors whose coordinates are all of the same sign. The difference with the examples of Theorem A is that the divergence of the BNF in the case $d = 2$ and the case all the coordinates of the frequency vector are of the same sign, do not give much informations about the asymptotic dynamics in the neighborhood of the origin.

THEOREM B. – *For any non-resonant $\omega \in \mathbb{R}^d, d \geq 2$, there exists a real entire Hamiltonian $H : \mathbb{R}^{2d} \to \mathbb{R}$ such that the origin is an elliptic equilibrium with frequency ω of the Hamiltonian flow Φ_H^t, and such that the Birkhoff normal form at the origin is divergent.*

The proof of Theorem B is inspired from the proof of the divergence of the BNF in the examples of Theorem A. However, to include the two degrees of freedom case requires a substantial difference on which we will comment in the beginning of Section 4.4. In the end of Section 4.4, we explain the slight modification required to prove Theorem B in the case of non-resonant vectors whose coordinates are all of the same sign.

Note that, due to the result of Perez-Marco of [PM03], the existence for any non-resonant $\omega \in \mathbb{R}^d$ of just one example of a real analytic Hamiltonian with divergent BNF, implies that divergence of the BNF is typical for this frequency. Denote by $\mathcal{H}_\omega$, the set of analytic Hamiltonians having an elliptic fixed point of frequency ω at the origin. As a consequence of Theorem B and of [PM03, Theorem 1] we get

COROLLARY. – *For any non-resonant $\omega \in \mathbb{R}^d, d \geq 2$, the generic Hamiltonian in $\mathcal{H}_\omega$ has a divergent BNF at the origin.*

More precisely, all Hamiltonians in any complex (resp. real) affine finite-dimensional subspace V of $\mathcal{H}_\omega$ have a divergent BNF except for an exceptional pluripolar set.

This answers, for all frequency vectors the question of Eliasson on the typical behavior of the BNF (see for example [E89, E90, EFK15] and the discussion around this question in [PM03]). What was known up to recently, was the generic divergence of the normalization, proved by Siegel in 1954 [Si54] in $\mathcal{H}_\omega$ for any fixed ω. Examples of analytic Hamiltonians with non-resonant elliptic fixed points and divergent BNF were constructed by Gong [Go12] on $\mathbb{R}^{2d}$ for arbitrary $d \geq 2$, but only for some class of Liouville frequency vectors.

The generic divergence of the BNF was recently obtained by Krikorian for symplectomorphisms of the plane with an elliptic fixed point at the origin [K19]. The method of Krikorian is completely different from ours and does not rely on the dichotomy proved by Perez-Marco. He has an indirect proof that gives a more refined result than the generic divergence of the BNF. Indeed, he proves that the convergence of the BNF, combined with torsion (a generic condition), implies the existence of a larger measure set of invariant curves in small neighborhoods of the origin than what actually holds for a generic symplectomorphism.

2.4. About the diffusion mechanism that will be used. In the C^∞ category, examples of unstable elliptic equilibria can be obtained *via* the successive conjugation method, the Anosov-Katok method. They can be obtained in two degrees of freedom or for $\mathbb{R}^2$ symplectomorphisms, provided the frequency at the elliptic equilibrium is not Diophantine ([AK66, FS05, FS17]). In three or more degrees of freedom, smooth examples with Diophantine frequencies can be obtained through a more sophisticated version of the successive conjugation method (see [EFK15, FS17]). The Anosov-Katok examples are infinitely tangent to the rotation of frequency ω at the fixed point and as such are very different in nature from our construction. In particular, KAM stability is in general excluded in these constructions.

Again in the C^∞ class but in the non-degenerate case, R. Douady gave examples in [Dou88] of Lyapunov unstable elliptic points for symplectic diffeomorphisms on $\mathbb{R}^{2d}$ for any $d \geq 2$. Douady's examples can have any chosen Birkhoff Normal Form at the origin provided its Hessian at the fixed point is non-degenerate. Douady's examples are modeled on the Arnold diffusion mechanism through chains of heteroclinic intersections between lower dimensional partially hyperbolic invariant tori that accumulate toward the origin. The construction consists of a countable number of compactly supported perturbations of a completely integrable flow, and as such was carried out only in the C^∞ category.

In [KMV04], the authors admit Mather's proof of Arnold diffusion for a cusp residual set of nearly integrable convex Hamiltonian systems in 2.5 degrees of freedom, and deduce from it that generically, a convex resonant totally elliptic point of a symplectic map in 4 dimensions is Lyapunov unstable, and in fact has orbits that converge to the fixed point.

A third diffusion mechanism, closely related to Arnold diffusion mechanism, is Herman's synchronized diffusion, and is due to Herman, Marco and Sauzin [MS02]. It is based on the following coupling of two twist maps of the annulus (the second one being integrable with linear twist): at exactly one point p of a well chosen periodic orbit of period q on the first twist map, the coupling consists of pushing the orbits in the second annulus up on some fixed vertical Δ by an amount that sends an invariant curve whose rotation number is a multiple of $1/q$ to another one having the same property. The dynamics of the coupled maps on the line $\{p\} \times \Delta$ will thus drift at a linear speed.

The diffusion mechanism that underlies our constructions is inspired by all these three mechanisms described above but is quite different from each. In 3 degrees of freedom, we start with a product of rotators of frequencies $\omega_1, \omega_2, \omega_3$, where $\omega_1\omega_2 < 0$ and then perturb this integrable Hamiltonian by adding a monomial of the 4 coordinates (x_1, y_1, x_2, y_2) that has a diffusive multi-saddle at the origin. The perturbation almost commutes with the rotators, provided $\bar{\omega} = (\omega_1, \omega_2)$ is very well approached by resonant vectors. The perturbed system has then an orbit that starts very close to the origin and that diffuses in the first four coordinates as is the case for the resonant system. We use the third action, $I_3 = x_3^2 + y_3^2$, that is invariant by the whole flow, as a coupling parameter.

To get diffusion from arbitrary small neighborhoods of the origin, one has to add successive couplings that commute with increasingly better resonant approximations of $\bar{\omega}$. The use of the third action as a coupling parameter, allows to isolate the effect of each successive coupling from the other ones. Indeed, to isolate the effect of each individual coupling from all the successive couplings is easy because these terms can be chosen to be extremely small compared to it. On the other hand, if we look at adequately small values of the third action, the effect of the prior coupling terms is tamed out due to Birkhoff averaging (we refer to Section 5.2 for a more precise description of the diffusion mechanism).

In the case of 4 degrees of freedom (or more) we can take the frequency vector of the equilibrium to be arbitrary, provided all the coordinates are not of the same sign, assuming for definiteness $\omega_1\omega_2 < 0$. The idea is that if ω_1 is replaced by $\omega_1 + I_4$ then the vector $(\omega_1 + I_{4,n}, \omega_2)$ will be resonant, for a sequence $I_{4,n} \to 0$, which allows to adopt the three degrees of freedom diffusion strategy.

3. Explicit constructions

In this section, we give the explicit constructions that yield Theorems A and B.

Starting from 4 degrees of freedom, it is possible to give examples with arbitrary frequency vectors, in particular Diophantine. Recall that ω is said to be Diophantine if there exists $\gamma, \tau > 0$ such that $|\langle k, \omega \rangle| \geq \gamma |k|^{-\tau}$, for all $k \in \mathbb{Z}^d - \{0\}$, with $\langle \cdot \rangle$ being the canonical scalar product and $|\cdot|$ its associated norm.

In his ICM talk of 1998 [He98], Herman conjectured that a real analytic elliptic equilibrium with a Diophantine frequency vector must be accumulated by a set of positive measure of KAM tori. This conjecture is still open. However, our examples can be chosen such that the Birkhoff Normal Form is non-degenerate, which implies KAM-stability as established in [EFK13] (see Theorem 1 above).

In all the sequel, we denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^{2d}$, indifferently on the value of d that will be clear from the context. We also denote indifferently B_r the Euclidean ball of radius r in $\mathbb{R}^{2d}$ for any value of d. For $k \in \mathbb{N}$, we denote by $\|H\|_{C^k(B_R)}$ the C^k norm of H on the ball B_R.

3.1. Lyapunov unstable elliptic equilibrium in three degress of freedom.

We suppose $\omega \in \mathbb{R}^3$ is such that there exists a sequence $\{(k_n, l_n)\} \in \mathbb{N}^* \times \mathbb{N}^*$ satisfying $k_0 + l_0 > 10$ and

$$(\mathcal{L}) \qquad |k_n\omega_1 + l_n\omega_2| < e^{-e^{n^4(k_n+l_n)}}.$$

The set of vectors satisfying $(\mathcal{L})$[1] is clearly a G_δ-dense set, since resonant vectors form a dense set in $\mathbb{R}^2$. Since we assume that ω is non-resonant we can, up to extracting, assume in addition that

$$(\mathcal{N}R) \quad k_n \geq \max_{0<k+l\leq k_{n-1}+l_{n-1}} e^{\frac{1}{|k\omega_1+l\omega_2|}}.$$

To simplify the presentation, we introduce the complex variables

$$(1) \qquad \xi_j = \frac{1}{\sqrt{2}}(x_j + iy_j), \quad \eta_j = \frac{1}{\sqrt{2}}(x_j - iy_j).$$

Note that in these variables H_ω as in $(*)$ reads as $\sum \omega_j \xi_j \eta_j$.

For $n \in \mathbb{N}$ we define on $\mathbb{R}^4$ the following real polynomial Hamiltonians

$$(2) \qquad F_n(x_1, x_2, y_1, y_2) = a_n(\xi_1^{k_n}\xi_2^{l_n} + \eta_1^{k_n}\eta_2^{l_n}), \quad a_n = e^{-n(k_n+l_n)}.$$

We finally define a real entire Hamiltonian on $\mathbb{R}^6$

$$(3) \qquad H(x,y) = H_\omega(x,y) + \sum_{n\in\mathbb{N}} I_3 F_n(x_1, x_2, y_1, y_2)$$

THEOREM 2. – *The origin is a Lyapunov unstable equilibrium of the Hamiltonian flow Φ_H^t of H. More precisely, for every $n \geq 1$, there exists $z_n \in \mathbb{R}^6$, such that $|z_n| \leq \frac{1}{n}$, and $\tau_n \geq 0$ such that $|\Phi_H^{\tau_n}(z_n)| \geq n$.*

Moreover, the Birkhoff normal form of H at the origin is divergent.

We can modify the definitions of the Hamiltonians H_ω and H on $\mathbb{R}^6$ as follows

$$\widetilde{H}_\omega(x,y) = (\omega_1 + I_3^3)I_1 + (\omega_2 + I_3^4)I_2 + \omega_3 I_3,$$
$$(4) \qquad \widetilde{H}(x,y) = \widetilde{H}_\omega(x,y) + \sum_{n\in\mathbb{N}} I_3 F_n(x_1, x_2, y_1, y_2).$$

Since we took $k_0 + l_0 > 10$, $\widetilde{H}_\omega$ gives the BNF of $\widetilde{H}$ at the origin up to order 5 in the action variables. But $\nabla \widetilde{H}_\omega(I) = (\omega_1 + I_3^3, \omega_2 + I_3^4, \omega_3 + 3I_3^2 I_1 + 4I_3^3 I_2)$ is clearly non-degenerate, and this implies that the BNF of $\widetilde{H}$ is non-degenerate. We then have KAM stability of the origin as a consequence of Theorem 1. Since $\widetilde{H} - H = O^3(I_3)$, we will see from the proof of Theorem 2 that the Lyapunov instability of the origin and the divergence of the BNF also hold for $\widetilde{H}$. Thus we have the following.

THEOREM 3. – *The origin is a Lyapunov unstable equilibrium of the Hamiltonian flow $\Phi_{\widetilde{H}}^t$ of $\widetilde{H}$. The Birkhoff normal form of $\widetilde{H}$ at the origin is non-degenerate, hence the equilibrium is KAM-stable. Moreover, the Birkhoff normal form of H at the origin is divergent.*

[1]The requirement of double exponential approximations is not uncommon in instability results in real analytic and holomorphic dynamics as is the case for example in [PM97].

3.2. Lyapunov unstable elliptic equilibrium in four degrees of freedom. In 4 degrees of freedom (or more), our method yields unstable elliptic equilibria for any frequency vector, provided its coordinates are not all of the same sign. Suppose for instance that $\omega = (\omega_1, \dots, \omega_4)$ is such that $\omega_1\omega_2 < 0$. Without loss of generality, we will also assume that $\max(|\omega_1|, |\omega_2|) \le 1$.

We assume (ω_1, ω_2) non-resonant (the resonant case follows from Corollary 2 below). By Dirichlet principle, there exists a sequence $(k_n, l_n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that

$$|k_n\omega_1 + l_n\omega_2| < \frac{1}{k_n}. \tag{5}$$

WLOG, we assume that $k_n\omega_1 + l_n\omega_2 < 0$. Then, for $I_{4,n} = -(k_n\omega_1 + l_n\omega_2)/k_n \in (0, \frac{1}{k_n^2})$, it holds that

$$k_n(\omega_1 + I_{4,n}) + l_n\omega_2 = 0. \tag{$\mathcal{R}$}$$

Since (ω_1, ω_2) is non-resonant, we can, up to extracting, additionally ask that for all $(k, l) \in \mathbb{N}^2 \setminus \{0, 0\}$ such that $k + l \le k_{n-1} + l_{n-1}$, we have $k(\omega_1 + I_{4,n}) + l\omega_2 \ne 0$ and[2]

$$k_n \ge \max_{0<k+l\le k_{n-1}+l_{n-1}} e^{\frac{1}{|k(\omega_1+I_{4,n})+l\omega_2|}}, \quad k_n \ge e^{e^{e^{n^4(k_{n-1}+l_{n-1})}}}. \tag{$\mathcal{N}R'$}$$

We define the following real entire Hamiltonians on $\mathbb{R}^8$

$$H_\omega(x, y) = (\omega_1 + I_4)I_1 + \sum_{j=2}^{4} \omega_j I_j,$$

$$H(x, y) = H_\omega(x, y) + \sum_{n\in\mathbb{N}} I_3 F_n(x_1, x_2, y_1, y_2) \tag{6}$$

THEOREM 4. – *The origin is a Lyapunov unstable equilibrium for the Hamiltonian flow of H. More precisely, for every $n \ge 1$, there exists $z_n \in \mathbb{R}^8$, such that $|z_n| \le \frac{1}{n}$, and $\tau_n \ge 0$ such that $|\Phi_H^{\tau_n}(z_n)| \ge n$.*

Moreover, the Birkhoff normal form of H at the origin is divergent.

We can modify the definition of the Hamiltonian on $\mathbb{R}^8$ similarly to what was done in the construction of Theorem 4 as follows. Take

$$\widetilde{H}_\omega(x, y) = (\omega_1 + I_4)I_1 + (\omega_2 + I_3^3)I_2 + \omega_3 I_3 + \omega_4 I_4,$$

$$\widetilde{H}(x, y) = \widetilde{H}_\omega(x, y) + \sum_{n\in\mathbb{N}} I_3 F_n(x_1, x_2, y_1, y_2), \tag{7}$$

where, as in the construction of Theorem 4, we suppose that $I_{4,n} = -(k_n\omega_1 + l_n\omega_2)/k_n \in (0, \frac{1}{k_n^2})$, satisfies ($\mathcal{R}$) and ($\mathcal{N}R'$).

From the definition of $\widetilde{H}$ in (7), it is clear that

$$\nabla \widetilde{H}_\omega(I) = (\omega_1 + I_4, \omega_2 + I_3^3, \omega_3 + 3I_3^2 I_2, \omega_4 + I_1)$$

is non-degenerate. Hence, KAM stability of the origin for the flow of $\widetilde{H}$ follows from Theorem 1.

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[2]We are using that (ω_1, ω_2) is non-resonant and that, for $k + l \le k_{n-1} + l_{n-1}$ we have $k(\omega_1 + I_{4,n}) + l\omega_2) \sim k\omega_1 + l\omega_2$ if k_n is sufficiently large.

Since $\widetilde{H} - H = O^3(I_3)$, exactly the same proof of Theorem 4 implies the Lyapunov instability of the origin and the divergence of the BNF for $\widetilde{H}$. We thus have the following.

THEOREM 5. – *The origin is a Lyapunov unstable equilibrium of the Hamiltonian flow $\Phi^t_{\widetilde{H}}$ of $\widetilde{H}$. The Birkhoff normal form of $\widetilde{H}$ at the origin is non-degenerate, hence the equilibrium is KAM-stable. Moreover, the Birkhoff normal form of H at the origin is divergent.*

3.3. Divergent Birkhoff normal forms with arbitrary frequencies for all $d \geq 2$.

In this section, we give the explicit examples that prove Theorem B. Of course, we can treat just the case $d = 2$ since for the case $d \geq 3$ it is sufficient to add to the Hamiltonians defined in $d = 2$ a trivial integrable part $\sum_{j=3}^{d} \omega_j I_j$.

We suppose $\omega_1\omega_2 < 0$ and will explain later, at the end of Section 4.4, what modifications should be applied to treat the case $\omega_1\omega_2 > 0$. Without loss of generality, we will also assume that $\max(|\omega_1|, |\omega_2|) \leq 1$.

WLOG, we can assume that $|\omega_2| = \theta|\omega_1|$ for some $\theta > 1$ that will be fixed in all the sequel.

Since (ω_1, ω_2) is non resonant, Dirichlet principle allows to define a sequence $(k_n, l_n) \in \mathbb{N}^2$ such that for all $n \in \mathbb{N}$

$$|k_n\omega_1 + l_n\omega_2| < \frac{1}{k_n}, \quad k_n \geq 10e^{e^n}. \tag{8}$$

The difference with the examples of Theorems (2) to (5) is that when $d = 2$ we do not have the extra action variables I_3 and I_4 that were instrumental in the diffusion mechanism as well as in the proof of divergence of the BNF in these constructions. Instead, we intend to give to I_2 a double role that includes the role of I_4 in the proof of divergence of the BNF of Theorems 4 and 5. Introduce the integrable Hamiltonian

$$H_\omega(x, y) = (\omega_1 + I_2)I_1 + \omega_2 I_2,$$

In the construction we will use a sequence of numbers $\zeta_n \in [0, 1]$ that we will fix inductively in the proof. For any choice of the sequence $(\zeta_n)_{n\in\mathbb{N}} \in [0, 1]^{\mathbb{N}}$, we define a real entire Hamiltonian on $\mathbb{R}^4$ as follows

$$H(x, y) = H_\omega(x, y) + \sum_{n\in\mathbb{N}} \zeta_n F_n(x_1, x_2, y_1, y_2), \tag{9}$$

where F_n are as in (2).

THEOREM 6. *For ω and (k_n, l_n) as in* (8)*, there exists $(\zeta_n)_{n\in\mathbb{N}} \in \{0, \frac{1}{2}, 1\}^{\mathbb{N}}$ such that the Hamiltonian H as in* (9) *has an elliptic equilibrium at the origin with a divergent Birkhoff Normal form.*

4. Birkhoff Normal Forms. Proofs of divergence.

4.1. Calculating the BNF : Resonant and non-resonant terms.

To simplify the computations, we prefer to use the complex variables ξ_j and η_j introduced in (1).

Note that in these variables H_ω as in $(*)$ reads as $\sum \omega_j \xi_j \eta_j$. We easily verify that, in these variables, the Poisson bracket is given by

$$\{F, G\} = i \sum_j \frac{\partial F}{\partial \xi_j} \frac{\partial G}{\partial \eta_j} - \frac{\partial F}{\partial \eta_j} \frac{\partial G}{\partial \xi_j},$$

while the Hamiltonian equations are given by

$$\begin{cases} \dot{\xi}_j = -i \partial_{\eta_j} H(\xi, \eta) \\ \dot{\eta}_j = i \partial_{\xi_j} H(\xi, \eta) \end{cases}$$

We will say that a function F defined in the variables ξ and η is real when $F(\xi, \bar{\xi})$ is real, which means that in the original variables (x, y), F is real valued.

Monomials. For $\underline{u} = (u_1, \ldots, u_k)$, $\underline{v} = (v_1, \ldots, v_{k'})$, we use the notation

$$\xi_{\underline{u}} \eta_{\underline{v}} := \xi_{u_1} \ldots \xi_{u_k} \eta_{v_1} \ldots \eta_{v_{k'}}.$$

and for $c_{\underline{u},\underline{v}} \in \mathbb{R}$ we call $c_{\underline{u},\underline{v}} \xi_{\underline{u}} \eta_{\underline{v}}$ a monomial.

It is very simple to detect in the variables ξ and η, the monomials that only depend on the actions, since these are exactly the monomials for which $k = k'$ and $\{u_1, \ldots, u_k\} = \{v_1, \ldots, v_k\}$. We call such monomials *resonant monomials*, and we call the other monomials *non resonant*.

Elimination of non resonant terms by canonical conjugacies. For a Hamiltonian H as in $(*)$, and since ω is non-resonant, it is easy to eliminate by conjugacy a non resonant monomial $c_{\underline{u},\underline{v}} \xi_{\underline{u}} \eta_{\underline{v}}$ from the expression of H. Indeed, if we take

$$\chi = i c_{\underline{u},\underline{v}} \frac{1}{\omega_{u_1} + \ldots + \omega_{u_k} - \omega_{v_1} - \ldots - \omega_{v_{k'}}} \xi_{\underline{u}} \eta_{\underline{v}} \tag{10}$$

we get for the time one map Φ_χ^1 of the Hamiltonian flow of χ, also called the Lie transform associated to χ,

$$H \circ \Phi_\chi^1 = H + \{H_\omega, \chi\} + \{H - H_\omega, \chi\} + \frac{1}{2!} \{\{H, \chi\}, \chi\} + \ldots \tag{11}$$

and observe that $\{H_\omega, \chi\} = -c_{\underline{u},\underline{v}} \xi_{\underline{u}} \eta_{\underline{v}}$, while the terms enclosed in the other brackets that are introduced by the composition by Φ_χ^1 are all of degree strictly higher than $k + k'$ (because the $\deg(\{f, g\})$ is either 0 or $\deg(f) + \deg(g) - 2$). The reduction to the BNF is done progressively by eliminating non-resonant monomials of higher and higher degree. A useful observation is that any term with degree higher or equal to $2N$ does not affect the BNF terms of order strictly less than N.

Since the monomials F_j defined in (2) play a crucial role in all our constructions, we list here some facts related to their elimination by conjugacies. Recall that $a_j = e^{-j(k_j + l_j)}$ and define

$$\chi_j = -i E_j, \quad E_j = b_j (\xi_1^{k_j} \xi_2^{l_j} - \eta_1^{k_j} \eta_2^{l_j}), \quad b_j = \frac{a_j}{k_j \omega_1 + l_j \omega_2}, \tag{12}$$

and observe that

$$\{H_\omega, \chi_j\} = -F_j. \tag{13}$$

When we conjugate a Hamiltonian as in (3) or (6) by Φ_χ^1, we see from the bracket (13) and from (11) that F_j will disappear from the expression of $H \circ \Phi_\chi^1$. However, the conjugacy will create *many* new terms that come from all the other brackets in (11). We will need to keep track of these terms as we aim at some control on the

BNF. The following elementary computation will be very useful. For $i, j \in \mathbb{N}$ we have that

$$\frac{1}{a_i b_j}\{F_i, E_j\} = -ik_ik_j(\xi_1^{k_i-1}\xi_2^{l_i}\eta_1^{k_j-1}\eta_2^{l_j} + \eta_1^{k_i-1}\eta_2^{l_j}\xi_1^{k_j-1}\xi_2^{l_i}) - il_il_j(\xi_1^{k_i}\xi_2^{l_i-1}\eta_1^{k_j}\eta_2^{l_j-1} + \eta_1^{k_i}\eta_2^{l_i-1}\xi_1^{k_j}\xi_2^{l_j-1}). \tag{14}$$

In particular, we see from (14) that $\{F_i, E_j\}$ will contain only non-resonant terms if $i \neq j$ and that the only resonant terms that appear in $\{F_n, E_n\}$ are

$$-2ia_nb_n(k_j^2 I_1^{k_j-1}I_2^{l_j} + l_j^2 I_1^{k_j}I_2^{l_j-1})$$

that will play a crucial role all through the sequel.

REMARK 2. In the case of a resonant frequency vector ω one cannot formally conjugate the Hamiltonian to an action dependent formal power series, because monomial terms $\xi_{u_1}\ldots\xi_{u_k}\eta_{v_1}\ldots\eta_{v_{k'}}$ that do not only depend on the actions may be *resonant with* ω, namely when $\omega_{u_1} + \ldots + \omega_{u_k} - \omega_{v_1} - \ldots - \omega_{v_{k'}} = 0$. Such resonant frequencies and resonant monomials that do not only depend on the actions will be instrumental in our constructions.

4.2. Divergence of the BNF in the Lyapunov unstable construction on $\mathbb{R}^6$.

The goal of this section is to prove the divergence of the BNF in Theorems 2 and 3. We let H be as in (3) (exactly the same proof applies for $\widetilde{H}$ as in (4) since $H - \widetilde{H} = O^3(I_3)$). We denote by B_∞ the Birkhoff normal form of H at 0. We introduce $\upsilon(I_1, I_2)$ and $\varphi(I_1, I_2)$ to be the formal power series such that

$$B_\infty(I) = H_\omega(I) + I_3\upsilon(I_1, I_2) + I_3^2\varphi(I_1, I_2) + O^3(I_3).$$

The notation $O^3(I_3)$ stands for a power series of the form $I_3^3\psi(I_1, I_2, I_3)$ where ψ is a formal power series. For the divergence of B_∞ it suffices to see that φ is a divergent power series. We will explicitly compute υ and φ.

PROPOSITION 1. *We have* $\upsilon \equiv 0$ *and*

$$\varphi(I_1, I_2) = -\sum_{j=1}^{\infty} a_j b_j \left(k_j^2 I_1^{k_j-1}I_2^{l_j} + l_j^2 I_1^{k_j}I_2^{l_j-1}\right).$$

The divergence of φ is now immediate from our Liouville hypothesis ($\mathcal{L}$) that implies $|k_j\omega_1 + l_j\omega_2| < e^{-e^{j^4(k_j+l_j)}}$, hence $|a_jb_j| \geq e^{0.5e^{j^4(k_j+l_j)}}$. Observe that, in fact, the super Liouville condition $|k_j\omega_1 + l_j\omega_2| < e^{-j^2(k_j+l_j)}$ is sufficient for the divergence of φ and thus of the BNF.

Proof of Proposition 1. Recall that H is given for $(x, y) \in \mathbb{R}^6$ by

$$H(x, y) = H_\omega(x, y) + \sum_{j\in\mathbb{N}} I_3F_n(x_1, x_2, y_1, y_2).$$

Since all the terms F_n are non resonant, it follows from Section 4.1 that the linear part in I_3 of the BNF of H reduces to $\omega_3 I_3$, that is, $\upsilon \equiv 0$.

We fix $n \in \mathbb{N}$. Recall the definition (12) of χ_j, E_j of Section 4.1, that satisfy $\{H_\omega, \chi_j\} = -F_j$. Define the following Hamiltonian on $\mathbb{R}^6$

$$\widehat{\chi} = \sum_{j\leq n-1} I_3\chi_j \tag{15}$$

Next, we conjugate the flow of H with the time one map of $\widehat{\chi}$. Recall that

$$H \circ \Phi^1_{\widehat{\chi}} = H + \{H, \widehat{\chi}\} + \frac{1}{2!}\{\{H, \widehat{\chi}\}, \widehat{\chi}\} + \frac{1}{3!}\{\{\{H, \widehat{\chi}\}, \widehat{\chi}\}, \chi\} + \dots$$

Hence

$$H \circ \Phi^1_{\widehat{\chi}} = H + \{H, \widehat{\chi}\} + \frac{1}{2!}\{\{H_\omega, \widehat{\chi}\}, \widehat{\chi}\} + O^3(I_3)$$

where $O^3(I_3)$ denotes a Hamiltonian of the form $I_3^3 W(x_1, x_2, y_1, y_2, I_3)$. Observe that $O^3(I_3)$ does not affect $\theta(I_1, I_2)$ nor $\varphi(I_1, I_2)$. From (13) we get that

$$\{H, \widehat{\chi}\} = -\sum_{j \le n-1} I_3 F_j - I_3^2 \{\sum_{j \ge 1} F_j, \sum_{j \le n-1} iE_j\}.$$

Using (13) again to compute $\{\{H_\omega, \widehat{\chi}\}, \widehat{\chi}\}$ we get

$$H \circ \Phi^1_{\widehat{\chi}} = H_\omega + I_3 \sum_{j \ge n} F_j + B_n I_3^2 + O^3(I_3) \tag{16}$$

$$B_n = -\frac{1}{2}\{\sum_{j \le n-1} F_j, \sum_{j \le n-1} iE_j\} - \{\sum_{j \ge n} F_j, \sum_{j \le n-1} iE_j\}$$

To compute $\varphi(I_1, I_2)$, we first separate in B_n the resonant monomials from the non-resonant ones. Recall that ω is assumed to be non-resonant, and that the non-resonant monomials are thus of the form $c_{u_1,u_2,v_1,v_2} \xi_1^{u_1} \xi_2^{u_2} \eta_1^{v_1} \eta_2^{v_2}$ with $u_1 \ne v_1$ or $u_2 \ne v_2$. From (14), we see that $\{F_i, E_j\}$ is a sum of non resonant monomials except when $i = j$ and that

$$H \circ \Phi^1_{\widehat{\chi}} = H_\omega - \sum_{j \le n-1} a_j b_j I_3^2 \left(k_j^2 I_1^{k_j - 1} I_2^{l_j} + l_j^2 I_1^{k_j} I_2^{l_j - 1}\right) + I + II + III \tag{17}$$

with

$$I = I_3 \sum_{j \ge n} F_j(\xi_1, \xi_2, \eta_1, \eta_2)$$

$$II = I_3^2 \mathcal{N}$$

$$III = O^3(I_3)$$

where $\mathcal{N}$ is a sum of non-resonant terms

$$\mathcal{N} = \sum_{u_1 \ne v_1 \text{or } u_2 \ne v_2} c_{u_1,u_2,v_1,v_2} \xi_1^{u_1} \xi_2^{u_2} \eta_1^{v_1} \eta_2^{v_2}.$$

Observe that $\sum_{j \le n-1} a_j b_j \left(k_j^2 I_1^{k_j-1} I_2^{l_j} + l_j^2 I_1^{k_j} I_2^{l_j-1}\right)$ equals to the truncation of φ of Proposition 1 to $j \le n-1$. Since (17) holds for every n, we will finish if we prove that the terms I, II and III do not contribute to B_∞ any term of the form $I_3^a I_1^k I_2^l$ with $a \le 2$ and $k + l \le k_{n-1} + l_{n-1} - 1$. The degree of the monomials in I is indeed too high, and the terms in III will only contribute terms of order 3 in I_3. For the term II we need one more elimination by conjugacy that we will now do.

Define

$$\mathcal{A}_n := \big\{(u_1, u_2, v_1, v_2) \in \mathbb{N}^4 : u_1 \ne v_1 \text{or } u_2 \ne v_2 \\ \text{and } u_1 + u_2 < k_{n-1} + l_{n-1}, v_1 + v_2 < k_{n-1} + l_{n-1}\}$$

and

$$\psi = I_3^2 \sum_{\mathcal{A}_n} \frac{-ic_{u_1,u_2,v_1,v_2}}{(u_1 - v_1)\omega_1 + (u_2 - v_2)\omega_2} \xi_1^{u_1} \xi_2^{u_2} \eta_1^{v_1} \eta_2^{v_2} \tag{18}$$

and observe that since

$$\{H_\omega, \psi\} = -I_3^2 \sum_{\mathcal{A}_n} c_{u_1,u_2,v_1,v_2} \xi_1^{u_1} \xi_2^{u_2} \eta_1^{v_1} \eta_2^{v_2}$$

then (17) gives

$$\begin{aligned} H \circ \Phi_\chi^1 \circ \Phi_\psi^1 = H_\omega - \sum_{j \leq n-1} a_j b_j I_3^2 \left(k_j^2 I_1^{k_j - 1} I_2^{l_j} + l_j^2 I_1^{k_j} I_2^{l_j - 1} \right) \\ + I' + II' + III' \end{aligned} \tag{19}$$

where I', II', III' are real analytic Hamiltonians around the origin of the form

$$\begin{aligned} I' &= I = I_3 \sum_{j \geq n} F_j(\xi_1, \xi_2, \eta_1, \eta_2) \\ II' &= I_3^2 \sum_{u_1+u_2 \geq k_{n-1}+l_{n-1} \text{ or } v_1+v_2 \geq k_{n-1}+l_{n-1}} c_{u_1,u_2,v_1,v_2} \xi_1^{u_1} \xi_2^{u_2} \eta_1^{v_1} \eta_2^{v_2} \\ III' &= I_3^3 W'(\xi_1, \xi_2, \eta_1, \eta_2, I_3). \end{aligned}$$

Again, the terms in III' do not contribute to the $O^2(I_3)$ part of the BNF of H at 0. Since the order of the $(\xi_1, \xi_2, \eta_1, \eta_2)$-terms in I' and II' are higher than $k_{n-1} + l_{n-1}$, we see that they do not contribute to B_∞ any term of the form $I_3^a I_1^k I_2^l$ with $a < 3$ and $k + l \leq k_{n-1} + l_{n-1} - 1$. The proof of Proposition 1 is thus completed. □

4.3. Divergence of the BNF in the Lyapunov unstable construction on $\mathbb{R}^8$.

We want to prove the divergence of the BNF at the origin for the Hamiltonians of Theorems 4 and 5. Take H as in (6) (exactly the same proof applies for $\widetilde{H}$ as in (7) since $H - \widetilde{H} = O^3(I_3)$). We proceed along the same lines as in the case of $\mathbb{R}^6$, to this difference that we replace everywhere ω_1 by $\omega_1 + I_4$, in particular in the definition of b_j in (12) that becomes

$$b_j(I_4) = \frac{a_j}{k_j(\omega_1 + I_4) + l_j \omega_2}.$$

Observe that $b_j(I_4)$ is a convergent power series in the neighborhood of 0. However its radius of convergence tends to 0 as j tends to infinity since $|k_{j-1}\omega_1 + l_{j-1}\omega_2|/k_{j-1} \to 0$.

We introduce $\upsilon(I_1, I_2, I_4)$ and $\varphi(I_1, I_2, I_4)$ so that

$$B_\infty(I) = H_\omega(I) + I_3 \upsilon(I_1, I_2, I_4) + I_3^2 \varphi(I_1, I_2, I_4) + O^3(I_3),$$

where υ and φ are formal power series in the variables I_1, I_2 and I_4 given by the following.

PROPOSITION 2. *We have $\upsilon \equiv 0$ and*

$$\varphi(I_1, I_2, I_4) = -\sum_{j=1}^{\infty} a_j b_j(I_4) \left(k_j^2 I_1^{k_j - 1} I_2^{l_j} + l_j^2 I_1^{k_j} I_2^{l_j - 1} \right).$$

Moreover, as a formal power series in the variables (I_1, I_2, I_4), φ is divergent.

Proof. The proof of Proposition 2 is exactly similar to that of Proposition 1 with $\omega_1 + I_4$ in place of ω_1 everywhere. In particular, we define χ_j, $\widehat{\chi}$ and ψ as in (12), (15) and (18), with $\omega_1 + I_4$ in place of ω_1. Due to the hypothesis ($\mathcal{N}R'$), we get in a sufficiently small neighborhood of the origin:

$$H \circ \Phi^1_{\widehat{\chi}} \circ \Phi^1_{\psi} = H_\omega - \sum_{j \leq n-1} a_j b_j(I_4) I_3^2 \left(k_j^2 I_1^{k_j-1} I_2^{l_j} + l_j^2 I_1^{k_j} I_2^{l_j-1} \right) + I' + II' + III' \tag{20}$$

where I', II', III' are real analytic Hamiltonians around the origin (for this, we restrict to $I_4 \ll 1$) and are of the same form as in (19) with an additional dependence on I_4, and do not contribute to B_∞ any term of the form $I_4^b I_3^a I_1^k I_2^l$ with $b \in \mathbb{N}$, $a < 3$ and $k + l \leq k_{n-1} + l_{n-1} - 1$. Proposition 2 is thus proved. □

Finally, the divergence of the power series $\varphi(I_1, I_2, I_4)$ is an immediate consequence of the fact that $b_n(I_4)$ has a radius of convergence that tends to 0 as n tends to infinity. This finishes the proof of divergence of the BNF at 0 of the Hamiltonian (6).

4.4. Divergence of the BNF for arbitrary frequencies on $\mathbb{R}^4$.

For a Hamiltonian as in (9), we want to make an inductive choice of $(\zeta_n)_{n\in\mathbb{N}} \in \{0, \frac{1}{2}, 1\}^{\mathbb{N}}$ that guarantees the divergence of the BNF at the origin. Similarly to the proof of divergence of the BNF of Theorem 5, the main ingredient is the appearance in the computation of the BNF of terms that include a pole close to $I_2 = 0$. Unlike I_4, the term I_2 in $(\omega_1 + I_2)I_1$ is not decoupled from the nonlinearities F_n. For that reason, the computations of the BNF will be more involved.

The key to the proof is an explicit formal computation of some coefficients of the BNF of H. Recall the definition of $\theta > 0$ such that $|\omega_2| = \theta|\omega_1|$, and that we assumed WLOG $\theta > 1$. Fix $\varepsilon \in (0, 0.01)$ such that $1 + 2\varepsilon < \theta$ and consider the integer

$$\hat{k}_n = [(1+\varepsilon)l_n] < k_n \tag{21}$$

PROPOSITION 3. *The coefficient Γ_n of $I_1^{k_n-1} I_2^{\hat{k}_n}$ in the BNF of H at the origin is given by*

$$\Gamma_n = \zeta_n^2 \gamma_n + \zeta_n P_n(\zeta_0, \dots, \zeta_{n-1}) + Q_n(\zeta_0, \dots, \zeta_{n-1}), \tag{22}$$

where P_n and Q_n are polynomials (that depend on ω and (k_j, l_j) for $j \leq n$), and

$$\gamma_n = (-1)^{\hat{k}_n - l_n} a_n^2 k_n \left(\frac{k_n}{k_n\omega_1 + l_n\omega_2} \right)^{\hat{k}_n - l_n + 1}. \tag{23}$$

In particular, (8) *and* (21) *imply that $|\gamma_n| \geq e^{nk_n}$.*

PROOF THAT PROPOSITION 3 IMPLIES THEOREM 6. Take $\zeta_0 = 0$. Once $\zeta_0, \dots, \zeta_{n-1}$ are fixed, for each choice of ζ_n, we denote by $\Gamma_n(\zeta_n)$ the value of Γ_n given by Proposition 3. We then have from (22) that

$$\Gamma_n(1) + \Gamma_n(0) - 2\Gamma_n(1/2) = \frac{\gamma_n}{2},$$

hence we get that

$$\max\left(|\Gamma_n(1)|, |\Gamma_n(0)|, |\Gamma_n(1/2)|\right) \geq |\gamma_n|/12 \geq e^{nk_n}/12.$$

We thus choose $\zeta_n \in \{0, 1/2, 1\}$ that realizes the latter maximum. Doing so for all $n \in \mathbb{N}$, implies that the BNF of H at the origin is divergent and finishes the proof of Theorem 6.[3] □

PROOF OF PROPOSITION 3. The rest of this section is dedicated to the proof of Proposition 3. As usual, the computation of the BNF, and in our case of just one part of it, is done by successive eliminations by conjugacy of non-resonant monomials. Each conjugacy introduces terms of higher degree that one needs to keep track of in the rest of the procedure. We first outline some of the guiding principals of our strategy for the computation of the coefficient Γ_n of the BNF of H that will be made explicit in the proof.

- Each term in $\sum_{m\geq n+1} \zeta_m F_m(x_1, x_2, y_1, y_2)$ has degree strictly higher than $2k_n + 2\hat{k}_n$ and thus this sum does not affect the coefficient Γ_n.
- The contribution of $\sum_{m\leq n} \zeta_m F_m(x_1, x_2, y_1, y_2)$ to Γ_n are polynomials in the variables $(\zeta_0, \zeta_1, \ldots, \zeta_n)$.
- The terms that appear with powers $d \geq 3$ of ζ_n have degree higher than $3(k_n + l_n) - 4$. By (21), $3(k_n + l_n) - 4 \geq 2k_n + 2\hat{k}_n$, hence these terms will not have any effect on the coefficient Γ_n.
- The terms that appear with a degree strictly larger than $2k_n+2\hat{k}_n-2$ or with a degree in the combined variables ξ_1 and η_1 strictly larger than $2k_n - 2$ do not have any effect on the coefficient Γ_n. The latter is due to the fact that all terms in $H - H_\omega$ have a degree in the combined variables ξ_1 and η_1 strictly larger than 2.

These ideas will yield in particular that the coefficient Γ_n has a quadratic form in ζ_n as in (22) but some additional attention is needed to estimate the coefficient γ_n that comes with ζ_n^2.

Before we start the proof, we introduce some notations that will alleviate the presentation.

DEGREES. For any Hamiltonian $G = \sum_k g_k$ where each g_k is a monomial we define

- $d(G)$ to be the minimal degree of the monomials g_k;
- $d_1(G)$ to be the minimal degree in the combined variables ξ_1 and η_1.
- $\bar{d}(G)$ to be the minimal degree of the non-resonant monomials g_k ($\bar{d}(G) = \infty$ if G has only resonant terms).

It will be useful to organize the terms of the Hamiltonians obtained by successive conjugacies in various classes that we now introduce. All the Hamiltonians that we will meet in the explicit computations of the BNF at a finite order will be analytic in small neighborhoods of the origin. However, we will not need this fact in reality, and all the definitions and calculations below can be viewed as operations on merely formal power series.

SOME SPECIAL TYPES OF HAMILTONIANS. We define three types of Hamiltonians $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{R}$

[3] Note that, whatever values are taken by $P_n(\zeta_0, \ldots, \zeta_{n-1})$ and $Q_n(\zeta_0, \ldots, \zeta_{n-1})$, only a very small measure of $\zeta_n \in [0, 1]$ would give $|\Gamma_n| \leq e^{\frac{n}{2}k_n}$. Hence, the prevalent choice of the sequence $\{\zeta_n\}$ will give a divergent BNF. This however, follows from the existence of just one example by the dichotomy result of Perez-Marco.

- A Hamiltonian is of type $\mathcal{R}$ if it is a sum (possibly 0) of monomials p such that for each p at least one of the two following condition holds : $d_1(p) \geq 2k_n$ or $d(p) \geq 2k_n + 2\hat{k}_n$.
- A Hamiltonian is of type $\mathcal{G}_1$ if it is not of type $\mathcal{R}$ and if it is a sum (possibly 0) of monomials p that do not depend on ζ_n and depend polynomially on $(\zeta_0, \ldots, \zeta_{n-1})$ and such that $d_1(p) > 2$.
- A Hamiltonian is of type $\mathcal{G}_2$ if it is not of type $\mathcal{R}$ and if it is a sum (possibly 0) of monomials p that do not depend on ζ_n and depend polynomially on $(\zeta_0, \ldots, \zeta_{n-1})$ and such that $d_1(p) > k_n$.

Note that we put no condition on the dependence on $(\zeta_0, \ldots, \zeta_n)$ of the terms of a Hamiltonian in $\mathcal{R}$.

We fix for all the sequel the function

$$\phi_n := -a_n^2 k_n^2 I_1^{k_n-1} I_2^{l_n} U_n, \quad U_n = \frac{1}{k_n(\omega_1 + I_2) + l_n\omega_2}.$$

Definition 1. *We say that a Hamiltonian h is of admissible if*

$$h = g_1 + \zeta_n g_2 + r$$

with $(g_1, g_2, r) \in \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{R}$.

Definition 2. *We say that a Hamiltonian G is in good form if*

$$G = H_\omega + h + \zeta_n^2 \phi_n$$

with h admissible.

A crucial fact about admissible Hamiltonians is their stability under addition (obvious) and under Poisson brackets :

Lemma 1. *If h and f are admissible Hamiltonians , we have that $\{h, f\}$ is admissible.*

Proof. Take two admissible Hamiltonians : $g = g_1 + \zeta_n g_2 + r$ and $g = g_1' + \zeta_n g_2' + r'$. That $\{f, g\}$ is admissible follows immediately from the following elementary facts

$\{g_1, g_1'\} \in \mathcal{G}_1$;
$\{g_1, g_2'\}, \{g_2, g_1'\} \in \mathcal{G}_2$;
$\{g_2, g_2'\} \in \mathcal{R}$;
$\{u, v\} \in \mathcal{R}$ for any $u \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{R}$ and any $v \in \mathcal{R}$

□

A consequence of Lemma 1 is the following stability result of Hamiltonians in a *good form*.

Lemma 2. *If G is a Hamiltonian in good form and if χ is an admissible Hamiltonian, then $G \circ \Phi_\chi^1$ is in good form.*

Proof. We have that

$$G \circ \Phi_\chi^1 = G + \{G, \chi\} + \frac{1}{2!}\{\{G, \chi\}, \chi\} + \ldots$$

By definition, the addition of an admissible Hamiltonian to a Hamiltonian in good form leaves it in good form. Hence, by Lemma 1, it suffices to show that $\{G, \chi\}$ is admissible. Now, $\{G,\chi\} = \{H_\omega,\chi\} + \{G - H_\omega,\chi\}$. But $\{H_\omega,\chi\}$ is clearly admissible, and Lemma 1 and the fact that $\{u,\phi_n\} \in \mathcal{R}$ for any $u \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{R}$ imply that $\{G - H_\omega,\chi\}$ is admissible. □

We can now state the following corollary that opens the way to the proof of Proposition 3.

COROLLARY 1. *If a Hamiltonian G is in good form, then the coefficient Γ_n of $I_1^{k_n-1}I_2^{\hat{k}_n}$ in the BNF of G at the origin is as described in Proposition 3.*

Proof. Take G in good form. Suppose $\bar{d}(G) < 2k_n + 2\hat{k}_n$. We let h be the sum of the non-resonant monomials of minimal degree of G. By definition h is admissible. We take g to be the corresponding Hamiltonian such that $\{H_\omega, g\} = -h$. The definition of g was given in (10) and it implies that g is admissible. Next, we have that

$$G \circ \Phi_g^1 = G - h + \{G - H_\omega, g\} + \frac{1}{2!}\{\{G,g\},g\} + \dots$$

Corollary 1 tells us that $G \circ \Phi_g^1$ is in good form. Since $G - H_\omega$ is admissible, we know that all the terms in $G - H_\omega$ have degree strictly higher than 2. Hence $\bar{d}\left(\{G - H_\omega, g\} + \frac{1}{2!}\{\{G,g\},g\} + \dots\right) > d(g) = d(h) = \bar{d}(G)$. By definition of h we have that $\bar{d}(G-h) > \bar{d}(G)$. Hence $\bar{d}(G \circ \Phi_g^1) > \bar{d}(G)$.

Iterating this procedure, we arrive after a finite number of steps to a canonical change of coordinates Ψ such that $G \circ \Psi$ is in good form and $\bar{d}(G) \geq 2k_n + 2\hat{k}_n$.

$$G \circ \Psi = H_\omega + \varphi_1 + \zeta_n \varphi_2 + \zeta_n^2 \phi_n + r$$

with $\varphi_1 \in \mathcal{G}_1$, $\varphi_2 \in \mathcal{G}_2$, both resonant and $r \in \mathcal{R}$. Since $d(r) \geq 2k_n + 2\hat{k}_n$, r has no effect on the term Γ_n. The fact that $\varphi_1 \in \mathcal{G}_1$ and $\varphi_2 \in \mathcal{G}_2$ implies that they depend polynomially in $(\zeta_0, \dots, \zeta_{n-1})$ and do not depend on ζ_n.

Since $U_n = \frac{1}{k_n(\omega_1 + I_2) + l_n\omega_2}$, the expansion of U_n in a power series in I_2 gives that the coefficient of $I_1^{k_n-1}I_2^{\hat{k}_n}$ coming from ϕ_n is exactly γ_n given by (23). Hence, we get the expression of Γ_n of Proposition 3. Finally, it is straightforward that (8) and (21) imply that $|\gamma_n| \geq e^{nk_n}$. □

With Corollary 1 at hand, to conclude the proof of Proposition 3, we just need to conjugate H as in (9) to put it in *good form*. We will see now that this is realizable *via* the natural conjugacy that kills the term $\zeta_n F_n$ of H. We adapt the definition of the χ_n of (12) as follows

$$\bar{\chi}_n = -i\zeta_n E_n U_n, \quad E_n = a_n(\xi_1^{k_n}\xi_2^{l_n} - \eta_1^{k_n}\eta_2^{l_n}).$$

Note that U_n has a pole as a function of I_2 near 0. However, it is still a nice analytic function in a tiny neighborhood of 0 and we will use the power series expansion of U_n later.

Using Leibniz's product rule for Poisson brackets, we have that $\{H_\omega, E_n U_n\} = U_n\{H_\omega, E_n\}$ since H_ω and U_n commute. But

$$\{H_\omega, iE_n\} = (k_n\omega_1 + l_n\omega_2)F_n + (k_n I_2 + l_n I_1)F_n,$$

hence, multiplying by $\zeta_n U_n$ and regrouping gives

$$\{H_\omega, \bar{\chi}_n\} = -\zeta_n F_n - \zeta_n l_n I_1 F_n U_n. \tag{24}$$

This will be used to show that if we denote by $\Phi^1_{\bar{X}_n}$ the time one map of the Hamiltonian flow of $\bar{X}_n$, then we get the following.

LEMMA 3. *We have that $H \circ \Phi^1_{\bar{X}_n}$ is in good form.*

Proof of Proposition 3. Lemma 3 gives that $G := H \circ \Phi^1_{\bar{X}_n}$ is in good form. Corollary 1 allows then to conclude. □

We now turn to the proofs of the lemma.

PROOF OF LEMMA 3. We have

$$H \circ \Phi^1_{\bar{X}_n} = H + \{H, \bar{X}_n\} + \frac{1}{2!}\{\{H, \bar{X}_n\}, \bar{X}_n\} + \dots \tag{25}$$
$$= H + \{H, \bar{X}_n\} + \frac{1}{2!}\{\{H, \bar{X}_n\}, \bar{X}_n\} + r, \quad r \in \mathcal{R}$$
$$= H + \{H, \bar{X}_n\} + \frac{1}{2!}\{\{H_\omega, \bar{X}_n\}, \bar{X}_n\} + r', \quad r' \in \mathcal{R}.$$

Next, recalling (24), we get

$$\{H, \bar{X}_n\} = \{H_\omega, \bar{X}_n\} + \sum_{j\in\mathbb{N}} \zeta_j \{F_j, \bar{X}_n\}$$
$$= -\zeta_n F_n - \zeta_n l_n F_n I_1 U_n - i\zeta_n^2 \{F_n, E_n U_n\} - \sum_{j\neq n} i\zeta_j\zeta_n \{F_j, E_n U_n\} \tag{26}$$

and

$$\{\{H_\omega, \bar{X}_n\}, \bar{X}_n\} = i\zeta_n^2 \{F_n, E_n U_n\} + i\zeta_n^2 l_n \{F_n I_1 U_n, E_n U_n\}. \tag{27}$$

Putting together (25), (26) and (27) implies

$$H \circ \Phi^1_{\bar{X}_n} = \sum_{j\leq n-1} \zeta_j F_j - \zeta_n l_n F_n I_1 U_n \tag{28}$$
$$- \frac{i}{2}\zeta_n^2 \{F_n, E_n U_n\} - \sum_{j\neq n} i\zeta_j \zeta_n \{F_j, E_n U_n\} + \frac{i}{2}\zeta_n^2 l_n \{F_n I_1 U_n, E_n U_n\} + \mathcal{R}$$

We classify all these terms as follows.

CLAIM. *With $\phi_n = -a_n^2 k_n^2 I_1^{k_n - 1} I_2^{l_n} U_n$, we have that*

- (C1) $\sum_{j\leq n-1} \zeta_j F_j \in \mathcal{G}_1$
- (C2) $l_n F_n I_1 U_n + \sum_{j\neq n} i\zeta_j \{F_j, E_n U_n\} \in \mathcal{G}_2$
- (C3) $\{F_n, E_n U_n\} - 2i\phi_n \in \mathcal{R}$
- (C4) $\{F_n I_1 U_n, E_n U_n\} \in \mathcal{R}$

Proof of the claim. $(C1)$ is obvious. $(C2)$ follows from the fact that $d_1(F_j) > 2$ for every j. To see $(C3)$ and $(C4)$ observe that (14) implies that

$$\{F_n, E_n\} U_n + 2i\phi_n = -2ia_n^2 l_n^2 I_1^{k_n} I_2^{l_n - 1} U_n \in \mathcal{R} \tag{29}$$

$$\{F_n, U_n\} E_n = a_n^2 l_n k_n \left(\xi_1^{2k_n} \xi_2^{2l_n} + \eta_1^{2k_n} \eta_2^{2l_n} - 2I_1^{k_n} I_2^{l_n}\right) U_n^2 \in \mathcal{R} \tag{30}$$

$$\{I_1, E_n\} F_n = -k_n F_n^2 \in \mathcal{R}. \tag{31}$$

□

Plugging $(C1) - (C4)$ in (28) we get that

$$H \circ \Phi^1_{\chi_n} \in \mathcal{G}_1 + \zeta_n \mathcal{G}_2 + \zeta_n^2 \phi_n + \mathcal{R}$$

as required in Lemma 3. □

PROOF OF THEOREM B. To finish the proof of Theorem B, we need to consider the case where $\omega_1\omega_2 > 0$.

We then have a sequence $(k_n, l_n) \in \mathbb{N}^2$ such that

$$|k_n\omega_1 - l_n\omega_2| < \frac{1}{k_n}, \quad k_n \geq 10e^{e^n}. \tag{32}$$

Next, we replace the definition of the Hamiltonian in (9) by

$$H(x,y) = H_\omega(x,y) + \sum_{n\in\mathbb{N}} \zeta_n(\xi_1^{k_n}\eta_2^{l_n} + \eta_1^{k_n}\xi_2^{l_n}). \tag{33}$$

The proof that the BNF of the Hamiltonian in (33) is divergent, under the condition (32), follows then exactly the same lines as that for the Hamiltonian (9) under the condition (8). □

We turn now to the proof of Lyapunov instability in the various examples.

5. Lyapunov instability. Proofs.

5.1. Lyapunov unstable resonant equilibria on $\mathbb{R}^4$.

In case ω is resonant, it is known that instabilities are more likely to happen. Algebraic examples were known since long time ago [LC1901, Ch26] (see [MS02, §31]). Our construction is actually based on the existence in two degrees of freedom, for resonant frequencies, of polynomial Hamiltonians that have invariant lines that go through the origin such that any point on such a line converges to the origin for negative times and goes to infinity in finite time in the future.

For $k, l \in \mathbb{N}^* \times \mathbb{N}^*, k + l > 2$, define the following real Hamiltonians

$$F_{k,l}(x_1, x_2, y_1, y_2) = \xi_1^k\xi_2^l + \eta_1^k\eta_2^l.$$

We have

PROPOSITION 4. – *For any $n \in \mathbb{N}^*$, there exist $t_n \in [0, (2n)^{k+l-2}]$ such that*

$$\Phi^{t_n}_{F_{k,l}}(B_{\frac{1}{2n}}) \bigcap B^c_{2n} \neq \emptyset.$$

If ω_1 and ω_2 are such that $k\omega_1 + l\omega_2 = 0$, then the Hamiltonian flow of $F_{k,l}$ commutes with that of $\omega_1 I_1 + \omega_2 I_2$ (since $\{\omega_1 I_1 + \omega_2 I_2, F_{k,l}\} = 0$). Hence we get the following direct consequence of Proposition 4 :

COROLLARY 2. – *If ω_1 and ω_2 are such that $k\omega_1 + l\omega_2 = 0$ for some $k, l \geq 1$ and $k + l > 2$, then for any $a \in \mathbb{R}^*$, the flow of $H(x_1, x_2, y_1, y_2) = \omega_1 I_1 + \omega_2 I_2 + aF_{k,l}$ has an elliptic fixed point with frequency (ω_1, ω_2) that is Lyapunov unstable.*

Proof of Proposition 4. We let $u = \sqrt{l/k}$, $\alpha = k + l - 1$. We assume $\alpha \geq 2$. WLOG, we suppose that $u \geq 1$.

Pick and fix $\nu, \nu' \in (0, 1)$ such that

$$-\frac{1}{4} + k\nu + l\nu' = 1.$$

Define a subset of $\mathbb{R}^4$,

$$\Delta := \left\{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 : (\xi_1, \xi_2) = \left(re^{i2\pi\nu}, ure^{i2\pi\nu'}\right), r \in \mathbb{R}\right\}.$$

The Hamiltonian equations of $F_{k,l}$ give

$$\dot{\xi}_1 = -ik\eta_1^{k-1}\eta_2^l, \quad \dot{\xi}_2 = -il\eta_1^k\eta_2^{l-1} \tag{34}$$

Since the Hamiltonian $F_{k,l}$ is real, if we start with a real initial condition, the solutions stay real, that is the relations $\eta_i = \bar{\xi}_i$ are conserved during the motion. This allows to compute the right hand sides of (34) for $(x_1, x_2, y_1, y_2) \in \Delta$, to get

$$\dot{\xi}_1 = ku^l r^\alpha e^{i2\pi\nu}, \quad \dot{\xi}_2 = ku^{l+1} r^\alpha e^{i2\pi\nu'}$$

which shows that for $(x_1, x_2, y_1, y_2) \in \Delta$, we have $\dot{\xi}_2 = ue^{i2\pi(\nu'-\nu)}\dot{\xi}_1$. Hence Δ is invariant by the flow $\Phi^t_{F_n}$, and moreover, the restriction of the vector field on Δ is given by

$$\dot{r} = ku^l r^\alpha.$$

Hence if we start with $r_0 = \frac{1}{2n}$ we see that

$$r(t)^{\alpha-1} = \frac{1}{(2n)^{\alpha-1} - (\alpha-1)ku^l t}.$$

Define then t_n such that $r(t_n) = 2n+1$. Note that $0 \leq t_n \leq T_n := (2n)^{\alpha-1}/(ku^l(\alpha-1)) < (2n)^{\alpha-1}$ since T_n is an explosion time of $r(t)$ with the initial condition $r_0 = \frac{1}{2n}$. □

5.2. Description of the diffusion mechanism.

We first describe the proof of Theorem 2, that is, diffusion in 3 degrees of freedom near a close to resonant elliptic equilibrium.

We want to exhibit diffusive orbits for the flow of
$H_\omega(x, y) + \sum_{n\in\mathbb{N}} I_3 F_n(x_1, x_2, y_1, y_2)$, where F_n is given by (2).

- From Corollary 2, we know that if $k_n\bar{\omega}_1 + l_n\bar{\omega}_2 = 0$ then the flow of $\bar{\omega}_1 I_1 + \bar{\omega}_2 I_2 + F_n(x_1, x_2, y_1, y_2)$ is unstable
- Due to $(\mathcal{L})$, an approximation argument (section 5.3) will show that, for fixed $I_3 = \mathbf{I} := e^{-e^{n^3(k_n+l_n)}}$, the flow of $H_\omega(x, y) + \mathbf{I}F_n(x_1, x_2, y_1, y_2)$, has a point satisfying $I_1, I_2 \sim 1/n, I_3 = \mathbf{I}$, that escapes after a time much smaller than $\mathbf{I}^{-1.1}$.
- The terms $F_l, l > n$ are too small and do not disrupt the diffusion at this time scale
- The terms $F_l, l < n$ average out to an I_3^2 term that contributes with $\mathcal{O}(\mathbf{I}^2)$ magnitude at this level of I_3 and do not disrupt the diffusion at this time scale that is much smaller than $\mathbf{I}^{-1.1}$.

In four degrees of freedom, we replace the almost resonance condition on ω by the use of the fourth action variable that is also invariant along the flow. Indeed, when we fix the value of this variable to $I_{4,n}$ such that $k_n(\omega_1+I_{4,n})+l_n\omega_2=0$, we find our Hamiltonian exactly in the form to which we can apply the Corollary 2. The variable I_3 plays then the same role as in the precedent case, of isolating the effect of a single F_n in the diffusion, for various values of $I_3\to 0$.

5.3. Approximation by resonant systems and diffusive orbits.

LEMMA 4. – *There exists a constant $C_d>0$ such that the following holds. Suppose $F,G\in C^2(\mathbb{R}^{2d},\mathbb{R})$, $\omega\in\mathbb{R}^d$, $A,r,R,a,T>0$ such that $r\le R$, $C_d a e^{C_d aAT}\le 1/4$, and*

- $H(x,y)=\sum_{j=1}^d \omega_j I_j + aF(x,y)$
- $h(x,y)=\sum_{j=1}^d \omega_j I_j + aF(x,y)+a^2G(x,y)$
- $\|F\|_{C^2(B_{R+1})}\le A,\quad \|G\|_{C^1(B_{R+1})}\le A$
- *For all* $s\in[0,T]:\Phi_H^s(B_r)\subset B_R$

Then, for all $s\in[0,T]$ and for all $z\in B_r$:

$$|\Phi_H^s(z)-\Phi_h^s(z)|\le C_d a e^{C_d aAT}. \tag{35}$$

Proof. Let $(X(s),Y(s)):=\Phi_H^s(z)$ and $(x(s),y(s)):=\Phi_h^s(z)$. Define the matrices $U_j=\begin{pmatrix}0 & -\omega_j\\ \omega_j & 0\end{pmatrix}$, and introduce the variables

$$\begin{pmatrix}u_j(s)\\ v_j(s)\end{pmatrix}=e^{sU_j}\begin{pmatrix}x_j(s)-X_j(s)\\ y_j(s)-Y_j(s)\end{pmatrix}.$$

Let $\xi(s)=(u_1(s),v_1(s),\dots,u_d(s),v_d(s))$. Since e^{sU_j} is a Euclidean isometry matrix, (35) is equivalent to proving that for all $s\in[0,T]$ and for all $z\in B_r$

$$|\xi(s)|\le C_d a e^{C_d aAT}.$$

The Hamiltonian equations give that

$$\begin{pmatrix}\dot u_j(s)\\ \dot v_j(s)\end{pmatrix}=e^{sU_j}\begin{pmatrix}ad_{x_j}F(\Phi_h^s(z))-ad_{x_j}F(\Phi_H^s(z))+a^2d_{x_j}G(\Phi_h^s(z))\\ ad_{y_j}F(\Phi_h^s(z))-ad_{y_j}F(\Phi_H^s(z))+a^2d_{y_j}G(\Phi_h^s(z))\end{pmatrix}.$$

Since, as long as $\Phi_H^s(z)$, and $\Phi_h^s(z)$ are in B_{R+1}, we have that

$$\begin{aligned}|d_{x_j}F(\Phi_h^s(z))-d_{x_j}F(\Phi_H^s(z))| &\le \|F\|_{C^2(B_{R+1})}\sum\left(|x_j(s)-X_j(s)|+|y_j(s)-Y_j(s)|\right)\\ &\le \sqrt{2d}\|F\|_{C^2(B_{R+1})}|\xi(s)|,\end{aligned}$$

and a similar bound for the y_j derivatives, the bounds on F and G then yield

$$|\dot\xi(s)|\le 2daA|\xi(s)|+\sqrt{2d}a^2A,\quad \xi(0)=0.$$

Gronwall's inequality then implies that for some constant $C_d>0$, and as long as $\Phi_H^s(z)$, and $\Phi_h^s(z)$ are in B_{R+1} we have

$$|\xi(s)|\le C_d a e^{C_d aAs}.$$

Finally the condition $C_d a e^{C_d a A T} \le 1/4$ allows to conclude, since it also makes sure that $\Phi_h^s(B_r) \subset B_{R+1}$ for $s \in [0,T]$, from the fact that $\Phi_H^s(B_r) \subset B_R$. □

COROLLARY 3. – *Let* $a \in (e^{-2e^{n^3(k_n+l_n)}}, e^{-e^{n^3(k_n+l_n)}})$. *Let* $H \in C^2(\mathbb{R}^{2d}, \mathbb{R})$ *be such that*

$$H(x,y) = H_\omega(x,y) + aK_n(x_1,x_2,y_1,y_2) + a^2 G_n(x,y)$$

with $\|G_n\|_{C^1(B_{2n})} \le e^{4n(k_n+l_n)}$, *and where* $K_n(x_1,x_2,y_1,y_2) = \xi_1^{k_n}\xi_2^{l_n} + \eta_1^{k_n}\eta_2^{l_n} = a_n^{-1}F_n$ *where* F_n *is given by* (2).

If $(\mathcal{L})$ *holds, there exist* $t_n \in [0,(2n)^{k_n+l_n-2}]$ *and* $z_n \in \mathbb{R}^{2d}$ *such that* $|z_n| = \frac{1}{2n}$ *and* $|\Phi_H^{\frac{t_n}{a}}(z_n)| \ge n$.

Proof. From $(\mathcal{L})$, there exists ω_1' such that $|\omega_1' - \omega_1| < e^{-e^{n^4(k_n+l_n)}}$ and $|k_n\omega_1' + l_n\omega_2| = 0$. Then, $\{\omega_1'\xi_1\eta_1 + \omega_2\xi_2\eta_2, K_n\} = 0$. Hence if we define $\omega' = (\omega_1', \omega_2, \dots, \omega_d)$ and

$$H'(x,y) = H_{\omega'}(x,y) + aK_n(x_1,x_2,y_1,y_2),$$

we get that

$$|\Phi_{H'}^{\frac{t}{a}}(z)| = \left|\Phi_{\omega_1' I_1 + \omega_2 I_2}^{\frac{t}{a}}\left(\Phi_{aK_n}^{\frac{t}{a}}(z)\right)\right| = |\Phi_{aK_n}^{\frac{t}{a}}(z)| = |\Phi_{K_n}^{t}(z)|.$$

Hence, by Proposition 4, there exists $t_n \in [0,(2n)^{k_n+l_n-2}]$ and $z_n \in \mathbb{R}^{2d}$ such that $|z_n| \le \frac{1}{2n}$, $|\Phi_{H'}^{\frac{t_n}{a}}(z_n)| = n+1$ and $\Phi_{H'}^{\frac{s}{a}}(B_{\frac{1}{n}}) \subset B_{n+1}$ for every $s \le t_n$.

Now since $|\omega_1' - \omega_1| < e^{-e^{n^4(k_n+l_n)}} \le a^2$, we have that

$$H(x,y) = H_{\omega'}(x,y) + aK_n(x_1,x_2,y_1,y_2) + a^2 G_n'(x,y)$$

with $\|G_n'\|_{C^1(B_{2n})} \le e^{4n(k_n+l_n)} + 1$. Note also that $\|K_n\|_{C^2(B_{2n})} \le e^{n(k_n+l_n)}$.

Let $A = e^{4n(k_n+l_n)} + 1$. Observe that for $T = \frac{t_n}{a}$, and C_d as in Lemma 4, we have that $C_d a e^{C_d a A T} = C_d a e^{C_d A t_n} \le \frac{1}{4}$. We can thus apply Lemma 4, with $r = \frac{1}{2n}$, $R = n+1$, and deduce that for all $s \in [0,t_n]$ and for all $z \in B_{\frac{1}{2n}}$:

$$\left|\Phi_H^{\frac{s}{a}}(z) - \Phi_{H'}^{\frac{s}{a}}(z)\right| \le aAt_n e^{dAt_n} \le \frac{1}{4}$$

and the conclusion of the corollary thus holds if we apply the latter inequality to $z = z_n$ and $s = t_n$. □

5.4. PROOF OF THEOREMS 2 AND 3.

We first take H as in (3). We fix $n \in \mathbb{N}$ large, and want to show that there exists $z_n \in \mathbb{R}^6$, such that $|z_n| \le \frac{1}{n}$, and $\tau_n \ge 0$ such that $|\Phi_H^{\tau_n}(z_n)| \ge n$.

Note that for any value $\mathbf{I} \in \mathbb{R}_+$, the set $\{(x,y) \in \mathbb{R}^6 : I_3 = \xi_3\eta_3 = \mathbf{I}\}$ is invariant under all the flows we consider in this construction.

We restrict from here on our attention to

$$I_3 = \mathbf{I} := e^{-e^{n^3(k_n+l_n)}}. \tag{36}$$

For $r > 0$, we denote

$$\hat{B}(r) := \{(x_1,x_2,y_1,y_2,x_3,y_3) : (x_1,x_2,y_1,y_2) \in B(r), I_3 = \mathbf{I}\}.$$

In all this section, the norms $\|\cdot\|_{C^k(\hat{B}_r)}$ will refer to the C^k norm with respect to the variables 1 and 2 and not 3. Recall the definitions of b_j, χ_j and $\widehat{\chi}$ of (12) and (15). Since, by the assumption $(\mathcal{N}R)$, $k_n \geq e^{\frac{1}{|k_j\omega_1+l_j\omega_2|}}$ for any $j \leq n-1$, we have for sufficiently large n

$$b_j \leq a_j \ln k_n. \tag{37}$$

It follows from (15) and (37) that for sufficiently large n

$$\|\widehat{\chi}\|_{C^1(\hat{B}_{2n})} \leq \mathbf{I} n \ln k_n \leq \mathbf{I}^{0.9}, \tag{38}$$

from the definition of $\mathbf{I}$ in (36). We thus get for $z \in \hat{B}(2n)$

$$\left|\Phi^1_{\widehat{\chi}}(z) - z\right| \leq \mathbf{I}^{0.8}. \tag{39}$$

Next, we recall that the conjugation of the flow of H by the time one map of $\widehat{\chi}$ was computed in (16) as [4]

$$H \circ \Phi^1_{\widehat{\chi}} = H_\omega + I_3 F_n + \sum_{j \geq n+1} I_3 F_j + B_n I_3^2 + \mathcal{R} \tag{40}$$

$$B_n = -\frac{1}{2}\{\sum_{j\leq n-1} F_j, \sum_{j \leq n-1} iE_j\} - \{\sum_{j\geq n} F_j, \sum_{j\leq n-1} iE_j\}$$

where

$$\mathcal{R} = \frac{1}{2}\{\{H - H_\omega, \widehat{\chi}\}, \widehat{\chi}\} + \frac{1}{3!}\{\{\{H, \widehat{\chi}\}, \widehat{\chi}\}, \widehat{\chi}\} + \dots$$

Hence due to (36) and (38), we have that $\mathcal{R}$ is a real analytic Hamiltonian that is of order 3 in I_3 that satisfies

$$\|\mathcal{R}\|_{C^1(\hat{B}(2n))} \leq \mathbf{I}^{\frac{5}{2}}. \tag{41}$$

Now, (37) gives that

$$\|B_n\|_{C^1(\hat{B}(2n))} \leq n \ln k_n. \tag{42}$$

Since $k_{n+1} \geq \mathbf{I}^{-1}$ we have from the definition of $a_j = e^{-j(k_j+l_j)}$ that

$$\|\sum_{j\geq n+1} F_j\|_{C^1(\hat{B}(2n))} \leq \mathbf{I} \tag{43}$$

Since I_3 is invariant by the Hamiltonian flow of all the functions we are considering, we now fix $I_3 = \mathbf{I}$ and consider the flow of $H \circ \Phi^1_{\widehat{\chi}}$ in restriction to the (x_1, x_2, y_1, y_2) variables. Introduce $a := \mathbf{I} a_n$ and recall the definition of $K_n = a_n^{-1} F_n$. We then have from (40), (41), (42) and (43),

$$H \circ \Phi^1_{\widehat{\chi}} = H_\omega + a K_n(x_1, x_2, y_1, y_2) + a^2 G(x, y)$$

with $\|G\|_{C^1(\hat{B}_{2n})} \leq a_n^{-2}(1 + n \ln k_n + \mathbf{I}^{1/2}) \leq e^{3n(k_n+l_n)}$.

Observe that from the definitions of $\mathbf{I}$ and a_n in (36) and (12), we have that $a = \mathbf{I} a_n \in (e^{-2e^{n^3(k_n+l_n)}}, e^{-e^{n^3(k_n+l_n)}})$. Since $(\mathcal{L})$ holds by hypothesis, we can thus apply Corollary 3 and get that there exist $t_n < (2n)^{k_n+l_n-2}$ and $w_n \in \mathbb{R}^4$ such that $|w_n| = \frac{1}{2n}$ and $|\Phi^{\frac{t_n}{a}}_{H\circ\Phi^1_{\widehat{\chi}}}(w_n)| \geq n$. To finish we pick $z_n = \Phi^{-1}_{\widehat{\chi}}(w_n, x_3, y_3)$, where $(x_3, y_3) \in \mathbb{R}^2$ are such that $I_3 = \mathbf{I}$. Thus, $|\Phi^{\frac{t_n}{a}}_H(z_n)| \geq n$, while (39) implies that

[4] In the computations that will follow, it is helpful to keep in mind that $a_j F_j$ and $a_j E_j$ are bounded in analytic norm, and that even if b_j is large, it remains negligible compared to $\mathbf{I}^{-0.1}$.

$|z_n| \le \frac{1}{n}$. This completes the proof of Lyapunov instability of the equilibrium at the origin for H as in (3). Since for $\widetilde{H}$ as in (4), we have that $H - \widetilde{H} = O^3(I_3)$, the same proof of topological instability for H applies to $\widetilde{H}$.

The divergence of the BNF of H and $\widetilde{H}$ at the origin was obtained in Section 4.2. Thus, the proof of Theorems 2 and 3 is completed. □

5.5. PROOF OF THEOREMS 4 AND 5. We take H as in (6). Note that for any value of $(\mathbf{I},\mathbf{J}) \in \mathbb{R}_+ \times \mathbb{R}_+^*$, the set $\{(x,y) \in \mathbb{R}^8 : I_3 = \xi_3\eta_3 = \mathbf{I}, \xi_4\eta_4 = \mathbf{J}\}$ is invariant under all the flows we consider in this construction.

If we fix now $I_4 = \mathbf{J} := I_{4,n}$ and $I_3 = \mathbf{I} := e^{-e^{n^3(k_n+l_n)}}$, the restriction of the flow of H to the (x_1,x_2,y_1,y_2) space takes the form:

$$H(x,y) = (\omega_1 + \mathbf{J})I_1 + \omega_2 I_2 + \omega_3 \mathbf{I} + \omega_4 \mathbf{J} + \sum_{n\in\mathbb{N}} \mathbf{I} F_n(x_1,x_2,y_1,y_2)$$

which has the same flow as in the proof of Theorem 2 with this difference that ω_1 is replaced by $\omega_1 + \mathbf{J}$. Moreover, the hypothesis $(\mathcal{R})$ and $(\mathcal{N}R')$ of Theorem 4 imply hypotheses $(\mathcal{L})$ and $(\mathcal{N}R)$ of Theorem 2, so the existence of the diffusive orbit for H as in (6) follows from Theorem 2. Since for $\widetilde{H}$ as in (7), it holds that $H - \widetilde{H} = O^3(I_3)$, the same proof of the topological instability of the equilibrium at the origin for H applies to $\widetilde{H}$.

The divergence of the BNF of H and $\widetilde{H}$ at the origin was obtained in Section 4.3. Thus, the proof of Theorems 4 and 5 is completed.

□

6. The case of $\mathbb{R}^d \times \mathbb{T}^d$. Lyapunov unstable quasi-periodic tori

To finish, we briefly describe in this section how the constructions of Theorems 2 to 5 can be carried to the case of Hamiltonians on $\mathbb{R}^d \times \mathbb{T}^d$, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. We will only discuss the case of Theorem 4, the others being similar. Closely related to Hamiltonians as in $(*)$ are the Hamiltonians on $\mathbb{R}^d \times \mathbb{T}^d$, $d \ge 4$, expressed in action-angle variables by

$$(**) \qquad H(r,\theta) = H_\omega(r) + \mathcal{O}^2_\theta(r),$$

$$H_\omega(r) = (\omega_1 + r_4)r_1 + \sum_{j=2}^{4} \omega_j r_j,$$

where $\mathcal{O}^2_\theta(r)$ denotes a real analytic Hamiltonian on $\mathbb{R}^d \times \mathbb{T}^d$ that is of order 2 in the r coordinates. For these Hamiltonians, the torus $\{0\} \times \mathbb{T}^d$ is invariant by the flow of $X_H = (\partial_\theta H, -\partial_r H)$, and the restricted dynamics on this torus is a translation flow of frequency ω.

With ω non-resonant, we again take $(k_n, l_n) \in \mathbb{Z}^2$ such that

$$|k_n\omega_1 + l_n\omega_2| < \frac{1}{k_n^2}.$$

Since we do not assume that $\omega_1\omega_2 < 0$ it is possible that $k_n l_n < 0$. Similarly to F_n of (2), we introduce

$$\bar{F}_n(r_1,r_2,\theta_1,\theta_2) = e^{-n(k_n+l_n)} r_1^{|k_n|} r_2^{|l_n|} \cos(2\pi(k_n\theta_1 + l_n\theta_2))$$

and the real entire Hamiltonians

$$H(r,\theta) = H_\omega(r) + \sum_{n\in\mathbb{N}} r_3 \bar{F}_n(r_1, r_2, \theta_1, \theta_2)$$

that satisfy $(**)$ and for which one can check similar results as those proved in Theorems 2 and 4.

Observe that in this action-angle setting, the fact that k_n or l_n may be negative does not constitute any restriction to the construction, and this is the reason why the condition $\omega_1\omega_2 < 0$ is not needed.

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