On ergodicity of cylindrical transformation given by the logarithm

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Abstract

Given $\alpha \in [0,1]$ and $\varphi : \mathbb{T} \to \mathbb{R}$ measurable, the *cylindircal cascade* $S_{\alpha,\varphi}$ is the map from $\mathbb{T}\times\mathbb{R}$ to itself given by $S_{\alpha,\varphi}(x,y) = (x+\alpha, y+\varphi(x))$ that naturally appears in the study of some ordinary differential equations on $\mathbb{R}^3$. In this paper, we prove that for a set of full Lebesgue measure of $\alpha \in [0,1]$ the cylindrical cascades $S_{\alpha,\varphi}$ are ergodic for every smooth function φ with a logarithmic singularity, provided that the average of φ vanishes.

Closely related to $S_{\alpha,\varphi}$ are the special flows constructed above R_α and under $\varphi+c$ where $c \in \mathbb{R}$ is such that $\varphi+c > 0$. In the case of a function φ with an asymmetric logarithmic singularity our result gives the first examples of ergodic cascades $S_{\alpha,\varphi}$ with the corresponding special flows being mixing. Indeed, when the latter flows are mixing the usual techniques used to prove the *essential value criterion* for $S_{\alpha,\varphi}$, that is equivalent to ergodicity, fail and we device a new method to prove this criterion that we hope could be useful in tackling other problems of ergodicity for cocycles preserving an infinite measure.

1 From flows to skew products

Let (M, x_t, ν) be a smooth dynamical system with continuous time and assume it has a global section (Σ, T, μ). For $\psi \in C^1(M,\mathbb{R})$ one can consider the flow on $M \times \mathbb{R}$ given by coupling x_t and the differential equation on $\mathbb{R}$

(1) $$\frac{dz}{dt} = \psi(x_t), \quad z \in \mathbb{R}.$$

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The flow determined by the coupling has a skew product form and it is given by the formula

$$(x_0, z_0) \mapsto (x_t, \int_0^t \psi(x_s)\, ds + z_0). \tag{2}$$

It has also a section, $\Sigma \times \mathbb{R}$, on which the dynamics writes as a skew product over T, namely

$$(\theta, z) \to (T\theta, z + \varphi(\theta)), \tag{3}$$

where φ is obtained by integrating ψ along flow segments of x_t: $\varphi(\theta) = \int_0^t \psi(x_s)\, ds$, where $t = t(\theta)$ is the first return time of $x_0 = \theta$ to Σ. In view of (2) the flow in (1) preserves the measure $\nu \times \lambda$, where λ denotes Lebesgue measure on the line. When (x_t, ν), or equivalently (T, μ), is ergodic, it is natural to ask whether the flow given by (1) is ergodic for $\nu \times \lambda$[1]. This is equivalent to ergodicity of the skew product in (3) for the measure $\mu \times \lambda$.

Remark 1 A necessary condition for ergodicity of (3) is that $\int_\Sigma \varphi(s)\, d\mu(s) = 0$, which by the Kac theorem we may always assume to hold by adding the constant $C = -\int_\Sigma \phi(\theta)\, d\mu / \int_\Sigma t(\theta)\, d\mu$ to ψ.

The study of skew products goes back to Poincaré and his work on differential equations on $\mathbb{R}^3$ (see §1.1 below where T is a minimal circular rotation and φ is smooth) and was later undertaken in the general context, where on the first coordinate, T is an arbitrary ergodic automorphism of a standard probability space $(X, \mathcal{B}, \mu)$, and on the second, φ is merely measurable (see monographs [1] and [28]).

In this note, we will prove the ergodicity of (3) when T is a minimal circular rotation R_α, α belongs to a set of full Lebesgue measure, and φ is a smooth function over the circle except for an asymmetric logarithmic singularity (cf. the precise Definition 1 below). But first, we will discuss the problems arising in the study of the ergodicity of (1) in the simplest case where x_t is a smooth area preserving flow on a surface and see how our result fits in this context.

Note that when x_t has only isolated fixed points od saddle type, the global section Σ exists and the return map T will not be defined at the last points where Σ intersects the incoming separatrices of the fixed points and moreover the return time function is asymptotic to infinity at these points. Further, if ψ does not vanish at a given fixed point, the function φ in (3) will have a singularity above the corresponding point where T is not defined and this singularity will have the same nature as the one for the return time function. It is not hard to see that a non-degenerate fixed point of the saddle type of the flow x_t yields a singularity of the logarithmic type for the return time function.

[1] Ergodicity for an infinite measure means that an invariant set either has zero measure or its complement has zero measure.

Definition 1 We will say that a real function φ defined over $\mathbb{T}$ has a *logarithmic singularity* at a point x_0 if φ is of class C^2 in $\mathbb{T} \setminus \{x_0\}$ and there exist $A, B \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{aligned} \lim_{x \to x_0^-} \varphi''(x)(x - x_0)^2 &= A, \\ \lim_{x \to x_0^+} \varphi''(x)(x - x_0)^2 &= B. \end{aligned}$$

We say that the singularity is *asymmetric* if $A + B \neq 0$.

1.1 The case of linear flows on the torus

When x_t is an irrational flow on the torus $\mathbb{T}^2$, it has a global section $\mathbb{T}$ on which the Poincaré return map is a minimal translation R_α. The resulting skew products

(4) $$S_{\alpha,\varphi}(\theta, z) = (\theta + \alpha, z + \varphi(\theta)),$$

were intensively studied (for both $z \in \mathbb{T}$ and $z \in \mathbb{R}$) since they have been first introduced by Poincaré in [27].

Unlike the case $z \in \mathbb{T}$ where $S_{\alpha,\varphi}$ is ergodic (for the Haar measure of $\mathbb{T}^2$) if φ equals a constant β as soon as $1, \alpha$, and β are independent over $\mathbb{Q}$, a necessary condition for ergodicity in the case $z \in \mathbb{R}$ is that $\int_{\mathbb{T}} \varphi(\theta) d\theta = 0$. In this case, the existence of ergodic skew products was first discovered by Krygin in [20]. There exist elegant categorical proofs [12, 13] of the fact that the set of (α, φ) such that $S_{\alpha,\varphi}$ is ergodic forms a residual set (for the product topology) in the product of the circle with the space $C_0^r(\mathbb{T}, \mathbb{R})$ of functions of class C^r with zero mean value (and this is true for any finite regularity $r \in \mathbb{N}$ or for $r = \infty$ or for the space $C_{\delta,0}^\omega(\mathbb{T}, \mathbb{R})$ of real analtyic functions with zero mean value, analytically extendable in a fixed annular neighborhood of $\mathbb{T}$ of size δ, continuous on its boundary, which is a Baire space if considered with the topology of uniform convergence). Further, it actually holds that for a given Liouvillean α, i.e. an $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$\limsup_{p/q \in \mathbb{Q}} \frac{-\log |\alpha - \frac{p}{q}|}{\log q} = \infty,$$

the set of $\varphi \in C_0^\infty(\mathbb{T}, \mathbb{R})$ such that $S_{\alpha,\varphi}$ is ergodic is residual (for the C^∞ topology), and that for α satisfying

$$\limsup_{p/q \in \mathbb{Q}} \frac{-\log |\alpha - \frac{p}{q}|}{q} \geq \delta > 0,$$

then the set of $\varphi \in C^\omega_{\frac{\delta}{2\pi},0}(\mathbb{T}, \mathbb{R})$ such that $S_{\alpha,\varphi}$ is ergodic is residual (for the topology described above) (cf. e.g. [5]).

In specific situations however, proving ergodicity for skew products preserving an infinite measure may become a delicate task (cf. for example the problem of ergodicity raised in [9]). Ergodicity of $S_{\alpha,\varphi}$ was proved in several situations, e.g.: [2], [5], [7], [10], [20], [25], [26], [30].

1.2 The case of time changed linear flows on the torus with a stopping point

The easiest case of a flow with a section where the Poincaré map is not defined at an isolated point is a reparametrized irrational flow (multiply the constant vector field by a smooth scalar function) on the torus $\mathbb{T}^2$ where the orbit is stopped at an isolated point (isolated zero for the reparametrizing function). But this procedure is not interesting from the ergodic point of view because the flow thus obtained is uniquely ergodic with respect to the Dirac measure supported by the fixed point. The dynamics at the stopping point is too slow (note that the inverse of the reparametrizing function is not integrable, hence the flow preserves an infinite measure which is equivalent to Lebesgue measure). This problem can be bypassed by plugging in the phase space of the minimal linear flow a weaker isolated singularity coming from a Hamiltonian flow in $\mathbb{R}^2$. The so called Kochergin flows thus obtained preserve beside the Dirac measure at the singularity a measure that is equivalent to Lebesgue measure. These flows still have $\mathbb{T}$ as a global section with a minimal rotation for the return map, but the slowing down near the fixed point produces a singularity for the return time function above the last point where the section intersects the incoming separatrix of the fixed point. Again, if ψ does not vanish at the fixed point, this results in a singularity of the same nature for the function we obtain in the system (4). The strength of the singularity depends on how abruptly the linear flow is slowed down in the neighborhood of the fixed point. A mild slowing down is typically represented by the logarithm (e.g. when $\varphi(x) = -\log x - \log(1-x) - 2$). In this case ergodicity of (3) was proved in [11]. In the case of power like singularities, that were actually the ones considered by Kochergin, no $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is known for which we have ergodicity in (3).

The second case is indeed sensitively different from the first one for the following reason that we will further comment in the next subsection: the special flow over R_α and under a smooth function with at least one power like singularity is mixing [16, 8] while the one under a smooth function with symmetric logarithmic singularities is not [17, 22].

1.3 The case of a multi-valuated Hamiltonian on $\mathbb{T}^2$

In [4], Arnol'd investigated Hamiltonian flows corresponding to multi-valued Hamiltonians on a two dimensional torus for which the phase space decomposes into cells that are filled up by periodic orbits and one open ergodic component.

On this component, the flow can be represented as a special flow over a minimal rotation of the circle and under a ceiling function that is smooth except for some logarithmic singularities. The singularities are asymmetric since the coefficient in front of the logarithm is twice as big on one side of the singularity as the one on the other side, due to the existence of homoclinic saddle connections.

It follows that if x_t in (1) is such a flow, the system we obtain in (3), once we restrict our attention to the open ergodic component of x_t, is a skew product over a minimal rotation of the circle with in the second coordinate a function having asymmetric logarithmic singularities. In this paper we prove the following.

Theorem 1 *For a.e. $\alpha \in \mathbb{T}$, the cylindrical transformation $S_{\alpha,\varphi} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, $(x, y) \mapsto (x + \alpha, \varphi(x) + y)$ is ergodic for any function φ of class C^2 on $\mathbb{T} \setminus \{x_0\}$ with a logarithmic singularity at x_0 and with zero average.*

We do not know whether ergodicity holds for every irrational α, except for the special case when the singularity is symmetric [11]. Note that, unlike the symmetric case, the special flows over irrational rotations and under smooth functions with asymmetric logarithmic singularities are mixing [14, 18, 19]. We will explain now why this fact makes the usual proof of ergodicity of the skew product (4) fail. We first need to introduce the *essential value criterion* which is necessary and sufficient for the ergodicity of skew products.

Assume that T is an ergodic automorphism of a standard probability Borel space $(X, \mathcal{B}, \mu)$. Let $\varphi : X \to \mathbb{R}$ be a measurable map. Denote by $\varphi^{(\cdot)}(\cdot) : \mathbb{Z} \times X \to \mathbb{R}$ the cocycle generated by φ, i.e. given by the formula

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{n-1}x) & \text{if} \quad n > 0 \\ 0 & \text{if} \quad n = 0 \\ -(\varphi(T^n x) + \ldots + \varphi(T^{-1}x)) & \text{if} \quad n < 0 \end{cases} \tag{5}$$

Denote by T_φ the transformation of $(X \times \mathbb{R}, \mathcal{B} \otimes \mathcal{B}(\mathbb{R}), \mu \otimes \lambda)$ given by

$$T_\varphi(x, y) = (Tx, \varphi(x) + y).$$

Note that $(T_\varphi)^n(x, y) = (T^n x, \varphi^{(n)}(x) + y)$ for each $n \in \mathbb{Z}$.

Following [28] a number $a \in \mathbb{R}$ is called an *essential value* of φ if for each $A \in \mathcal{B}$ of positive measure, for each $\varepsilon > 0$ there exists $N \in \mathbb{Z}$ such that

$$\mu(A \cap T^{-N}A \cap [|\varphi^{(N)}(\cdot) - a| < \varepsilon]) > 0.$$

Denote by $E(\varphi)$ the set of essential values of φ. Then the essential value criterion states as follows

Proposition 1 *[[28],[1]] We have*
1. $E(\varphi)$ is a closed subgroup of $\mathbb{R}$.
2. $E(\varphi) = \mathbb{R}$ iff T_φ is ergodic.

Usual methods of proving ergodicity of $S_{\alpha,\varphi}$ take into consideration a sequence of distributions

$$\left(\varphi^{(n_k)}\right)_*(\mu),\ k \geq 1 \tag{6}$$

(along some rigid sequence $\{n_k\}$, i.e. $n_k\alpha \to 0 \pmod 1$ when $k \to \infty$) as probability measures on the one-point compactification of $\mathbb{R}$. As shown in [23] each point in the topological support of a "rigid" limit point of (6) is an essential value of the cocycle φ, hence contributing to ergodicity of $S_{\alpha,\varphi}$. This method is especially well adapted to those φ whose Fourier transform satisfies $\hat{\varphi}(n) = \mathrm{O}(1/|n|)$, hence in particular for φ of bounded variation. The log symmetric φ also enjoys this property, see [11], and indeed ergodicity in this case holds over every irrational rotation. However the method fails in the case of an asymmetric logarithmic function (or for functions with power like singularities, no matter whether they are symmetric or not) since the distributions (6) tend to Dirac measure at infinity. The latter is indeed a necessary condition for mixing of the corresponding special flows, cf. [22] or [29] for a more general case.

In the present note, in order to prove ergodicity of φ, we will apply a different method which rather resembles Aaaronson's abstract essential value condition (EVC) from [3].

To be more precise, the problem we face is the following: given $a \in \mathbb{R}$ and a rigidity sequence $\{q_n\}_{n\in\mathbb{N}}$ of R_α, the sets $A_n(a,\epsilon)$ of points $x \in \mathbb{T}$ where $\varphi^{(q_n)}(x) \in [a-\epsilon, a+\epsilon]$ have their measure tending to zero as n goes to infinity; and if we ask that q_n be a very strong rigidity sequence (α well approximated by rationals) so as to force $R_\alpha^{q_n}A_n(a,\epsilon)$ to self-intersect, we will not be able to have good lower bounds on the measure of the sets A_n and it will be impossible therefore to show that a is an essential value. If to the contrary we consider badly approximated numbers α, $R_\alpha^{q_n}A_n(a,\epsilon)$ will be disjoint from $A_n(a,\epsilon)$ making the usual proof of the essential value fail. However, we stick to these numbers and prove for some rigidity sequence $\{q_n\}_{n\in\mathbb{N}}$, that the sets $A_n(a,\epsilon)$ are not too small (although their measure goes to zero), i.e. that $\sum \mu(A_n) = \infty$[2], then we use the structure of these sets on the circle and their almost independence for different values of n to deduce, using a generalized version of the Borel-Cantelli lemma, that any measurable set can be measurably approximated by a union of A_n's. We conclude after observing that the same holds for the sets $B_n = R_\alpha^{q_n}A_n$.

1.4 Open problem: The general case of transitive area preserving flows with isolated singularities

On surfaces of higher genus the presence of fixed points is unavoidable for index reasons. For area preserving flows with only isolated singularities, the return

[2]This condition fails when we consider functions with power like singularities and, in the case of asymmetric logarithmic singularities, it holds only under some arithmetic restrictions of Diophantine type on α. For technical reasons, we do assume however that, along a sequence of integers with positive density, the partial quotients of α are "large enough".

map to any transversal is conjugate to an interval exchange map. Furthermore, if the flow is transitive then it is quasi-minimal, i.e. every semi-orbit other than a fixed point or a point on a separatrix of a saddle is dense. In general, the closure of any transitive component is a surface with a quasi-minimal flow. If in addition the fixed points are non-degenerate saddles then the singularities of the return time function at the discontinuities of the interval exchange map are of logarithmic type. These singularities are usually symmetric but asymmetric situations similar to the one treated in the present paper may appear, if for instance there is a saddle point with one of its separatrices forming a homoclinic saddle connection. In this general setting, ergodicity of the underlying systems (1) is unknown:

Problem Let $T : I \to I$ be an ergodic interval exchange map. Let φ be a smooth function defined over I with logarithmic singularities at the discontinuity points of T. Assuming that $\int_I \varphi(\theta)d\theta = 0$, is $S : I \times \mathbb{R} \to I \times \mathbb{R}, (\theta, z) \mapsto (T\theta, z + \varphi(\theta))$ ergodic?

2 Notations. Properties of the sums $\varphi^{(q_n)}$

Throughout this text, X will denote the additive circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ identified with [0,1) (mod 1). Recall (see e.g. [15]) that each irrational number $\alpha \in [0, 1)$ admits a development into the continued fraction expansion

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{\cdots}}},$$

(a_i are positive integers) and a_i are called the *partial quotients* of α, $i \geq 1$. We have

$$\frac{1}{2q_i q_{i+1}} < |\alpha - \frac{p_i}{q_i}| < \frac{1}{q_i q_{i+1}},$$

where

$$q_0 = 1, q_1 = a_1, q_{i+1} = a_{i+1} q_i + q_{i-1}$$

$$p_0 = 0, p_1 = 1, p_{i+1} = a_{i+1} p_i + p_{i-1}.$$

Recall also (e.g. [15]) that there exists a constant $c > 1$ such that for n large enough

$$q_n \geq c^n. \tag{7}$$

Untill the last section φ will be

$$\varphi(x) = -1 - \log(1 - x), \quad x \in [0, 1).$$

Note that $\varphi \in L^1(\mathbb{T})$ and that $\int \varphi \, d\mu = 0$.

If $f:\mathbb{T}\to\mathbb{R}$ is of bounded variations, the following Denjoy-Koksma inequality holds for the Birkhoff sums of f along R_α

$$\left|\frac{1}{q_n}f^{(q_n)}(x)-\int_0^1 f\,d\mu\right|\le\frac{1}{q_n}\mathrm{Var}\,f$$

for each $x\in[0,1)$ (see e.g. the proof of the Koksma inequality in [21]).

Assume that $\alpha\in\mathbb{T}$ is irrational. Put

$$H(\alpha)=\{n\ge 0:\ q_{n+1}\ge 100q_n \text{ and } \alpha<\frac{p_n}{q_n}\}.$$

Denote

$$\overline{I}_{n,l}=\left[\frac{l}{q_n}+\frac{1}{50q_n},\frac{l+1}{q_n}-\frac{1}{50q_n}\right],$$

$l=0,1,\dots,q_n-1$.

Lemma 1 *Assume that $H(\alpha)$ is infinite. Then for any $a\in\mathbb{R}$, for all sufficiently large $n\in H(\alpha)$ we have:*

(8) $\varphi^{(q_n)}$ *is continuous and strictly increasing on each* $\overline{I}_{n,l}$,

(9) $$\left|\left(\varphi^{(q_n)}\right)'(x)-q_n\log q_n\right|<\frac{1}{\sqrt{n}}q_n\log q_n \text{ for every } x\in\overline{I}_{n,l},$$

(10) $$\varphi^{(q_n)}\left(\frac{l}{q_n}+\frac{3}{4q_n}\right)\ge a+1,$$

(11) $$\varphi^{(q_n)}\left(\frac{l}{q_n}+\frac{1}{4q_n}\right)\le a-1,$$

$l=0,1,\dots,q_n-1$.

Proof.

Denote

$$\overline{\varphi}(x)=\left(1-\chi_{[1-\frac{1}{50q_n},1]}(x)\right)\varphi(x),\ x\in[0,1).$$

Assume that $n\in H(\alpha)$. We have

$$\left|\alpha-\frac{p_n}{q_n}\right|\le\frac{1}{100q_n^2}.$$

Moreover, since $\alpha<\frac{p_n}{q_n}$, no point $x,x+\alpha,\dots,x+(q_n-1)\alpha$ belongs to $[1-\frac{1}{50q_n},1)$ whenever $x\in\overline{I}_{n,l}$, $l=0,1,\dots,q_n-1$ (indeed, $x+s\alpha=x+s\frac{p_n}{q_n}+s(\alpha-\frac{p_n}{q_n})$). It follows that

(12) $$\overline{\varphi}^{(q_n)}(x)=\varphi^{(q_n)}(x)\ \text{ for }\ x\in\bigcup_{l=0}^{q_n-1}\overline{I}_{n,l}.$$

Moreover,

$$\operatorname{Var} \overline{\varphi} = 2\log(50q_n) - 1. \tag{13}$$

Integrating by parts the integral $\int_0^{1-\frac{1}{50q_n}} \log(1-x)\,dx$ we find that

$$\int_0^1 \overline{\varphi}(x)\,dx = -\frac{\log q_n}{50q_n}. \tag{14}$$

We also have

$$\operatorname{Var} \overline{\varphi}' = 100q_n - 1 \tag{15}$$

and

$$\int_0^1 \overline{\varphi}'(x)\,dx = \log(50q_n). \tag{16}$$

In view of (12) we have to show that the properties (8)-(11) hold for $\overline{\varphi}^{(q_n)}(x)$, $x \in \overline{I}_{n,l}$. Since no point $x, x+\alpha, \ldots, x+(q_n-1)\alpha$ belongs to $[1-\frac{1}{50q_n}, 1)$ and $\overline{\varphi}'$ is strictly positive on $[0, 1-\frac{1}{50q_n})$, (8) directly follows. Now, from (15) and the Denjoy-Koksma inequality we obtain that

$$\left|\left(\overline{\varphi}^{(q_n)}\right)'(x) - q_n \int_0^1 \overline{\varphi}'\,d\mu\right| \le 100q_n - 1. \tag{17}$$

Hence using (16) and (7),

$$\left|\left(\overline{\varphi}^{(q_n)}\right)'(x) - q_n \log q_n\right| \le \frac{1}{\sqrt{n}} q_n \log q_n \tag{18}$$

for n large enough. Put

$$I_{n,l} = \left[\frac{l}{q_n}, \frac{l+1}{q_n}\right],$$

$l = 0, 1, \ldots, q_n - 1$ and

$$\widetilde{\overline{\varphi}}^{(q_n)}(x) = \overline{\varphi}(x) + \overline{\varphi}(x + \frac{1}{q_n}) + \ldots + \overline{\varphi}(x + \frac{q_n - 1}{q_n}),$$

$x \in [0,1)$. We have $\int_{I_{n,l}} \widetilde{\overline{\varphi}}^{(q_n)}\,d\mu = \int_0^1 \overline{\varphi}\,d\mu$, so by (16),

$$\int_{I_{n,l}} \widetilde{\overline{\varphi}}^{(q_n)}\,d\mu = -\frac{\log q_n}{50q_n}. \tag{19}$$

In a similar manner as we proved (8) and (9) we have that $\widetilde{\overline{\varphi}}^{(q_n)}$ is continuous and strictly increasing on each $I_{n,l}$ and

$$\left|\left(\widetilde{\overline{\varphi}}^{(q_n)}\right)'(x) - q_n \log q_n\right| < \frac{1}{\sqrt{n}} q_n \log q_n \tag{20}$$

for n large enough. Moreover,

$$\left|\overline{\varphi}^{(q_n)}(x) - \widetilde{\overline{\varphi}}^{(q_n)}(x)\right| \leq \frac{q_n \log q_n}{q_{n+1}} \left(1 + \frac{1}{\sqrt{n}}\right) \tag{21}$$

for n large enough (and $x \in \overline{I}_{n,l}$). Indeed, for $x \in \overline{I}_{n,l}$, using the fact that $\varphi' \geq 0$ and that $i\frac{p_n}{q_n} > i\alpha$ for $i = 0, 1, \ldots, q_n - 1$, we have

$$\left|\overline{\varphi}^{(q_n)}(x) - \widetilde{\overline{\varphi}}^{(q_n)}(x)\right| = \sum_{i=0}^{q_n-1} \overline{\varphi}'(\xi_{x,i}) \left(i\frac{p_n}{q_n} - i\alpha\right)$$

for some $\xi_{x,i} \in \left[x + i\alpha, x + i\frac{p_n}{q_n}\right]$, $i = 0, 1, \ldots, q_n - 1$. Since $0 \leq \overline{\varphi}'(\xi_{x,i}) \leq \overline{\varphi}'(x + i\frac{p_n}{q_n})$, we obtain that

$$\left|\overline{\varphi}^{(q_n)}(x) - \widetilde{\overline{\varphi}}^{(q_n)}(x)\right| \leq \frac{q_n}{q_n q_{n+1}} \sum_{i=0}^{q_n-1} \overline{\varphi}'(x + i\frac{p_n}{q_n}) =$$

$$\frac{1}{q_{n+1}} \left(\widetilde{\overline{\varphi}}^{(q_n)}\right)'(x) \leq \frac{q_n}{q_{n+1}}(1 + \frac{1}{\sqrt{n}}) \log q_n$$

and (21) follows.

In order to prove (10) it is hence enough to show that

$$\widetilde{\overline{\varphi}}^{(q_n)}\left(\frac{l}{q_n} + \frac{3}{4q_n}\right) \geq \frac{q_n \log q_n}{q_{n+1}} \left(1 + \frac{1}{\sqrt{n}}\right) + a + 1. \tag{22}$$

To show (22), in view of (20) and the fact that $q_{n+1} \geq 100 q_n$, it is enough to show that

$$\widetilde{\overline{\varphi}}^{(q_n)}\left(\frac{l}{q_n} + \left(\frac{3}{4} - \frac{1}{5}\right)\frac{1}{q_n}\right) \geq 0$$

(because the derivative of $\widetilde{\overline{\varphi}}^{(q_n)}$ is of order $q_n \log q_n$, hence on the interval of length $\frac{1}{5}\frac{1}{q_n}$ the difference of the values of the function at the endpoints is at least of order $q_n \log q_n \cdot \frac{1}{5q_n} = \frac{1}{5}\log q_n$ which is bounded from below by the sequence of order $\frac{q_n}{q_{n+1}}(1 + \frac{1}{\sqrt{n}}) \log q_n$). Suppose to the contrary that

$$\widetilde{\overline{\varphi}}^{(q_n)}\left(\frac{l}{q_n} + \left(\frac{3}{4} - \frac{1}{5}\right)\frac{1}{q_n}\right) \leq 0.$$

Using (20) consecutively for intervals $\left[\frac{l}{q_n}, \frac{l}{q_n} + (\frac{3}{4} - \frac{1}{5})\frac{1}{q_n}\right]$ of length $(\frac{3}{4} - \frac{1}{5})\frac{1}{q_n}$ and $\left[\frac{l}{q_n} + \left(\frac{3}{4} - \frac{1}{5}\right)\frac{1}{q_n}, \frac{l+1}{q_n}\right]$ of length $\left(\frac{1}{4} + \frac{1}{5}\right)\frac{1}{q_n}$ we find that

$$\int_{I_{n,l}} \widetilde{\overline{\varphi}}^{(q_n)}\, d\mu \leq -\left(\frac{3}{4} - \frac{1}{5}\right)^2 \frac{1}{q_n^2}\left(1 - \frac{1}{\sqrt{n}}\right) q_n \log q_n +$$

$$\left(\frac{1}{4}+\frac{1}{5}\right)^2\frac{1}{q_n^2}\left(1+\frac{1}{\sqrt{n}}\right)q_n\log q_n\leq-\frac{1}{11}\frac{\log q_n}{q_n},$$

when n is large enough, which is a contradiction with (19).

In order to complete the proof it is enough to show that

$$\tilde{\overline{\varphi}}^{(q_n)}\left(\frac{l}{q_n}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{q_n}\right)\leq 0.$$

Suppose the contrary. Then

$$\int_{I_{n,l}}\tilde{\overline{\varphi}}^{(q_n)}\,d\mu\geq\left(1-\frac{1}{\sqrt{n}}\right)\left(\frac{3}{4}-\frac{1}{5}\right)^2\frac{1}{q_n^2}q_n\log q_n-$$

$$\left(1+\frac{1}{\sqrt{n}}\right)\left(\frac{1}{4}+\frac{1}{5}\right)^2\frac{1}{q_n^2}q_n\log q_n\geq 0$$

for n large enough – contradiction with (19). □

Remark 2 It is clear that small modifications in the proof of Lemma 1 will give us a similar result also in case $\alpha>\frac{p_n}{q_n}$.

The lemma below will be essential in the proof of ergodicity of φ.

Lemma 2 *For any $a\in\mathbb{R}$, any $0<\varepsilon<1$, for any $n\in H(\alpha)$ sufficiently large there exists an interval*

$$J_{n,l}(a,\varepsilon)\subset\left[\frac{l}{q_n}+\frac{1}{4q_n},\frac{l}{q_n}+\frac{3}{4q_n}\right]$$

$(l=0,1,\ldots,q_n-1)$ *such that for each $x\in J_{n,l}(a,\varepsilon)$,*

$$\varphi^{(q_n)}(x)\in[a-\varepsilon,a+\varepsilon]\tag{23}$$

and

$$|J_{n,l}(a,\varepsilon)|=\frac{2\varepsilon}{q_n\log q_n}+o\left(\frac{1}{q_n\log q_n}\right)\tag{24}$$

.

Proof.

In view of (8), (10) and (11) of Lemma 1,

$$\varphi^{(q_n)}\left(\left[\frac{l}{q_n}+\frac{1}{4q_n},\frac{l}{q_n}+\frac{3}{4q_n}\right]\right)\subset[a-1,a+1],$$

while the estimation (24) follows from (9). □

3 Borel-Cantelli lemma and the Essential Value Criterion

We will assume now that α satisfies:

$$n \in H(\alpha) \text{ for all } n \geq n_0, \tag{25}$$

$$\sum_{n=1}^{\infty} \frac{1}{\log q_n} = +\infty. \tag{26}$$

Fix $a \in \mathbb{R}$ and $\varepsilon > 0$. Denote

$$A_n = A_n(a, \varepsilon) = \bigcup_{l=0}^{q_n - 1} J_{n,l}(a, \varepsilon).$$

Lemma 3 *For each* $k \geq 1$,

$$\sum_{n \geq k} \mu \left(A_n | \bigcap_{j=k}^{n-1} A_j^c \right) = +\infty.$$

Proof.

First let us notice that the set A_k^c is obtained from $[0, 1)$ by discarding q_k intervals $J_{k,l}(a, \varepsilon)$, $l = 0, 1, \ldots, q_k - 1$, next the set $(A_k \cup J_{k+1})^c$ we obtain from A_k^c by discarding q_{k+1} intervals $J_{k+1,l}(a, \varepsilon)$, $l = 0, 1, \ldots, q_{k+1} - 1$, and so on. At each step $s = 0, 1, \ldots, n - 1$ the set $\bigcap_{j=k}^{k+s} A_j^c$ is hence a union of at most $q_k + q_{k+1} + \ldots + q_{k+s} + 1$ consecutive, pairwise disjoint intervals which we will call *s-holes.* Call an s-hole *good* if its length is at least $\frac{6}{q_{s+1}}$, otherwise it is called *bad.* Assume now that (a, b) is a good s-hole. At step $s + 1$ we first divide $[0, 1)$ into q_{s+1} intervals of equal length $\frac{1}{q_{s+1}}$. Since (a, b) is a good s-hole, we find

$$0 \leq r_1 < r_2 \leq q_{s+1} - 1, r_2 - r_1 \geq 5 \text{ and } \left[\frac{r_1 + i}{q_{s+1}}, \frac{r_1 + i + 1}{q_{s+1}} \right] \subset (a, b)$$

for each $i = 0, 1, \ldots, r_2 - r_1 - 1$. We take r_1 and r_2 extremal with the above properties. For each $i = 0, 1, \ldots r_2 - r_1 - 1$ we then consider $J_{k+s+1,r_1+i}(a, \varepsilon)$. We have

$$J_{k+s+1,r_1+i}(a, \varepsilon) \subset \left[\frac{r_1 + i}{q_{s+1}} + \frac{1}{4} \frac{1}{q_{s+1}}, \frac{r_1 + i}{q_{s+1}} + \frac{3}{4} \frac{1}{q_{s+1}} \right], \tag{27}$$

$i = 0, 1, \ldots, r_2 - r_1 - 1$. Since $q_{n+1} \geq 100q_n$, it follows that (a, b) is producing at least $r_2 - r_1 - 1$ good $(s + 1)$-holes. Notice also that (27) and the inequality $q_{n+1} \geq 100q_n$ imply that any (either good or bad) s-hole cannot produce more that two bad $(s+1)$-holes. With these observations in hands we will show that for each $s \geq 0$,

$$G_{k+s} \geq B_{k+s} \tag{28}$$

where G_{k+s} (resp. B_{k+s}) stands for the number of good (resp. bad) s-holes. Indeed, for $s=0$, $B_{k+s}=0$. Assume that (28) holds for some $s \geq 0$. Since each good s-hole produces at least $r_2 - r_1 - 1$ good $(s+1)$-holes, we have $G_{s+k+1} \geq 4G_{k+s}$. The number B_{k+s+1} is bounded by $2G_{k+s} + 2B_{k+s}$, whence $G_{k+s+1} \geq B_{k+s+1}$ and (28) follows.

Fix $s \geq 0$ and consider the trace of A_{k+s+1} on a good s-hole (a,b). There exists an absolute constant $c_1 > 0$ such that

$$\mu\left(A_{k+s+1} \cap (a,b)\right) \geq c_1 \mu(A_{k+s+1}) \mu(a,b)$$

(indeed, $\mu(A_{k+s+1})$ is of order $\frac{2\varepsilon}{\log q_{k+s+1}}$, $\mu(a,b)$ is of order $(r_2-r_1)\frac{1}{q_{k+s+1}}$ and $\mu(A_{k+s+1}) \cap (a,b)$ is of order $(r_2-r_1)\frac{2\varepsilon}{q_{k+s+1}\log q_{k+s+1}}$). Taking into account (28), it follows that

$$\mu\left(A_{k+s+1} \cap \bigcap_{j=0}^{s-1} A_{k+j}^c\right) \geq \frac{c_1}{2}\mu(A_{k+s+1})\mu(\bigcap_{j=0}^{s-1} A_{k+j}^c).$$

Hence

$$\sum_{n \geq k} \mu(A_n | \bigcap_{j=k}^{n-1} A_j^c) \geq c_2 \sum_{n \geq k} \mu(A_n) \geq c_2 \varepsilon \sum_{n \geq k} \frac{1}{\log q_n} = +\infty$$

and the lemma follows. □

In what follows we will make use of the following variant of the Borel-Cantelli lemma (see [24], Prop. IV-4.4):

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{C_n\} \subset \mathcal{F}$. Suppose that for each $k \geq 0$

$$\sum_{n=k}^{\infty} P(C_n | \bigcap_{j=k}^{n-1} C_j^c) = +\infty.$$

Then

$$\limsup_{n \to \infty} C_n = \Omega \pmod{P}.$$

Directly from this and from Lemma 3 we obtain the following.

Lemma 4 *Under the above assumptions,* $\limsup_{n\to\infty} A_n(a,\varepsilon) = \mathbb{T}$ *(mod* μ*).* □

Denote $B_n(a,\varepsilon) = T^{q_n} A_n(a,\varepsilon)$, $n \geq n_0$.

Lemma 5 *Under the above assumptions,* $\limsup_{n\to\infty} B_n(a,\varepsilon) = \mathbb{T}$ *(mod μ).*

Proof.

Note that $T^{q_n} J_{n,l}(a,\varepsilon)$ is an interval of the same length as $J_{n,l}(a,\varepsilon)$ and due to the condition $|\alpha - \frac{p_n}{q_n}| < \frac{1}{100q_n^2}$ its position with respect to $J_{n,l}(a,\varepsilon)$ is not essentially changed. Therefore we see that the arguments that lead to the proof of Lemma 4 work well also in case of the sequence $B_n(a,\varepsilon)$, $n \geq n_0$. $\square$

We are now able to prove that each real number is an essential value of φ under some restriction on α.

Proposition 2 *If α satisfies (25) and (26) then the logarithmic cylindrical transformation is ergodic.*

Proof.

Take $a \in \mathbb{R}$. We will show that $a \in E(\varphi)$. Fix $0 < \varepsilon < 1$. By Lemmas 4 and 5, for any $s \geq 1$ we have (in measure)

$$\bigcup_{n=s}^{\infty} A_n = \mathbb{T} = \bigcup_{n=s}^{\infty} B_n,$$

where $A_n = A_n(a,\varepsilon)$, $B_n = B_n(a,\varepsilon)$. Fix an interval I. We have as l goes to infinity,

$$\mu(T^{q_l} I \triangle I) = \mu((I + q_l\alpha)\triangle I) \to 0. \tag{29}$$

Take an interval $\overline{I}$ that is strictly included in I and such that $|\overline{I}| \geq \frac{99}{100}|I|$. For s large enough the set $A_s = \bigcup_{n\geq s}\bigcup_{0\leq l\leq q_n-1} J_{n,l} \cap \overline{I}$ satisfies $A_s \subset I$ and

$$\mu(A_s) > \frac{3}{4}|I|, \tag{30}$$

likewise, using (29), the set $B_s = \bigcup_{n\geq s}\bigcup_{0\leq l\leq q_n-1} T^{q_n} J_{n,l} \cap \overline{I}$ satisfies $B_s \subset I$ and

$$\mu(B_s) > \frac{3}{4}|I|. \tag{31}$$

Note that if $x \in A_s$, say $x \in J_{n,l}(a,\varepsilon)$, then $T^{q_n}x \in B_s \subset I$ and $|\varphi^{(q_n)}(x) - a| < \varepsilon$.

Finally, take any Borel set $C \subset [0,1)$ of positive measure. Let x_0 be a point of density. Take a small $\delta > 0$ and let $I \ni x_0$ be an interval so that

$$\mu(C \cap I) \geq (1-\delta)\mu(I). \tag{32}$$

Taking into account (30), (31) and (32), and choosing δ sufficiently small we obtain a pair (n, l) such that the set

$$\{x \in C : \ x \in J_{n,l} \text{ and } T^{q_n}x \in C\}$$

is of positive measure and hence $a \in E(\varphi)$. □

4 Proof of Theorem 1

In order to formulate the main result of this note, first notice that to prove the assertion of Proposition 2 we only need the conditions (25) and (26) both to hold along a common subsequence of denominators (indeed, in the proof of Lemma 3, and hence of Lemmas 4 and 5, we will consider the sets $J_{n,l}(a, \varepsilon)$ for n belonging to the subsequence and the relevant condition of independence needed to use the Borel-Cantelli lemma also holds). Hence we have proved the following.

Proposition 3 *Assume that for α irrational there exists a subsequence $\{n_k\}$ such that*

$$q_{n_k+1} \geq 100 q_{n_k}, \tag{33}$$

$$\sum_{k=1}^{\infty} \frac{1}{\log q_{n_k}} = +\infty. \tag{34}$$

Then the cylindrical transformation $(x, y) \mapsto (x + \alpha, -1 - \log(1 - x) + y)$ is ergodic. □

Let us notice that the conditions (26) and (34) are almost equivalent in the following precise sense: (26) holds if and only if (34) holds along an arbitrary subsequence $\{n_k\}$ of positive lower density. (Indeed, positive lower density of $\{n_k\}$ means that there exists a constant $M > 0$ such that $n_k \leq Mk$ for each $k \geq 1$; write $\{1, 2, \ldots, Mn\} = \bigcup_k D_k$, where $D_k = \{kM, kM + 1, \ldots, (k + 1)M - 1\}$ and notice that given k, $\sum_{s \in D_k} \frac{1}{\log q_s} \leq M \cdot \frac{1}{\log q_{n_k}}$ since the sequence $\{\frac{1}{\log q_n}\}$ is decreasing.) The condition (26) is satisfied for any α with bounded partial quotients. We hence proved the following.

Corollary 1 *Assume that α has bounded partial quotients. Assume that there exists a subsequence $\{n_k\}$ of positive lower density such that (33) is satisfied along this subsequence. Then the cylindrical logarithmic transformation is ergodic.* □

Remark 3 Let us notice that (inductively, using the formula $q_{n+1} = a_{n+1}q_n + q_{n-1}$) we have

$$a_1 \dots a_n \le q_n \le a_1 \dots a_n \cdot 2^n.$$

It follows from this estimation that

$$\sum_{n=1}^{\infty} \frac{1}{\log q_n} = +\infty \text{ iff } \sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} \log a_i} = +\infty.$$

Indeed, all we need to show is that

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} \log a_i} = +\infty \text{ iff } \sum_{n=1}^{\infty} \frac{1}{n + \sum_{i=1}^{n} \log a_i} = +\infty.$$

This equivalence holds because as we have already noticed:
a series of positive decreasing frequencies is divergent iff it is divergent along a subsequence of positive lower density, and moreover, given two increasing sequences a positive real numbers, $\{b_n\}$, $\{c_n\}$ such that the series $\sum 1/b_n$ and $\sum 1/c_n$ diverge, also the series $\sum 1/(b_n + c_n)$ diverges, for either on a set of positive lower density we have

$$\frac{1}{b_n + c_n} \ge \frac{1}{2b_n} \text{ or } \frac{1}{b_n + c_n} \ge \frac{1}{2c_n}.$$

We claim now that the assumptions of Proposition 3 are satisfied for a.e. $\alpha \in \mathbb{T}$. Indeed, we have that for a.e. irrational number $\alpha \in \mathbb{T}$,

$$\lim_{n\to\infty} \frac{\log q_n}{n} = \frac{\pi^2}{12 \log 2}$$

(see e.g. [6], Chapter 7), so the condition (26) is satisfied for a.e. irrational α. Then, consider the Gauss transformation $x \mapsto Tx := \{\frac{1}{x}\}$, $x \in (0,1)$ which preserves the finite absolutely continuous measure $dm = \frac{1}{1+x}\,dx$ with respect to which T is mixing. We also have $T^n x \in [1/(k+1), 1/k)$ if and only if $a_n(x) = k$. Consider $f(x) = \chi_{[1/(k+1),1/k)}(x)$. By the ergodic theorem, for a.e. $x \in (0,1)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(a_n(x)) = m(\frac{1}{k+1}, \frac{1}{k})$$

and in particular the set of n's such that $a_n(x) = k$ has positive density. We hence proved

Proposition 4 *The cylindrical transformation* $(x, y) \mapsto (x + \alpha, -1 - \log(1 - x) + y)$ *is ergodic for a.e.* $\alpha \in \mathbb{T}$. □

Note that all the calculations that were made for $\varphi(x) = -1 - \log(1-x)$ in view of Lemma 1 are also valid for any function of class C^2 on $\mathbb{T} \setminus \{x_0\}$ having a logarithmic singularity at $x_0 \in \mathbb{T}$ (as in Definition 1) with $A = 0$ and $B \neq 0$, and with zero average.

Note also that Lemma 1 will hold for $\varphi_1 = \varphi + f$ whenever $f^{(q_n)} \to 0$ uniformly, in particular when f is absolutely continuous and has zero mean (the uniform convergence to zero follows from the Denjoy-Koksma inequality). Similarly, consider the case of a function φ_1 having an asymmetric logarithmic singularity at 0. Then for some $D > 0$ we have that $\varphi_1 = \tilde{\varphi} + f$ where $f(x) = -D \log x - D \log(1-x), x \in (0,1)$, and $\tilde{\varphi}$ has a logarithmic singularity at 0 (as in definition 1) with $A = 0$ and $B \neq 0$. Fix $0 < \eta < 1$ and let

$$\overline{f}_n(x) = f(x) \cdot \chi_{[\eta/q_n, 1-\eta/q_n]}.$$

We have $\int_0^1 \overline{f}'_n \, d\mu = 0$, hence by the Denjoy-Koksma inequality

$$|(\overline{f}'_n)^{(q_n)}(x)| \leq 2q_n/\eta \tag{35}$$

for each but finitely many $x \in \mathbb{T}$ (and for $n \geq n_0$). It follows that there exists a constant $c = c(\eta)$ such that if we put

$$\tilde{I}_{n,l} = \left[\frac{l}{q_n} + \frac{c}{q_n}, \frac{l+1}{q_n} - \frac{c}{q_n}\right] \quad (l = 0, 1, \ldots, q_n - 1)$$

then by the proof of Lemma 1 we will obtain (8)-(11) to hold on each $\tilde{I}_{n,l}$ if we replace φ by φ_1 and the RHS in the estimate (9) by $o(q_n \log q_n)$. It then follows that also Lemma 2 holds and by repeating all the other other arguments we end up by proving the following.

Theorem 2 *For a.e. $\alpha \in \mathbb{T}$, the cylindrical transformation $(x, y) \mapsto (x + \alpha, \varphi(x) + y)$ is ergodic for any function φ of class C^2 on $\mathbb{T} \setminus \{x_0\}$ with an asymmetric logarithmic singularity at x_0 and with zero average.* □

Theorem 1 then follows from this and the result of [11] in the symmetric case.

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